*Ergod. Th. & Dynam. Sys.* (2018), **38**, 1168–1200 © *Cambridge University Press, 2016*
doi:10.1017/etds.2016.65

# Lyapunov spectrum for Hénon-like maps at the first bifurcation

HIROKI TAKAHASI

*Department of Mathematics, Keio University, Yokohama, 223-8522, Japan*
*(e-mail: hiroki@math.keio.ac.jp)*

(*Received* 30 *March* 2015 *and accepted in revised form* 7 *April* 2016)

*Abstract*. For a strongly dissipative Hénon-like map at the first bifurcation parameter at which the uniform hyperbolicity is destroyed by the formation of tangencies inside the limit set, we effect a multifractal analysis, i.e. decompose the set of non-wandering points on the unstable manifold into level sets of an unstable Lyapunov exponent, and give a partial description of the Lyapunov spectrum which encodes this decomposition. We derive a formula for the Hausdorff dimension of the level sets in terms of the entropy and unstable Lyapunov exponent of invariant probability measures, and show the continuity of the Lyapunov spectrum. We also show that the set of points for which the unstable Lyapunov exponents do not exist carries the full Hausdorff dimension.

## 1. *Introduction*

In the study of chaotic dynamical systems, one often encounters invariant sets with complicated geometric structures. The multifractal analysis treats the so-called multifractal decomposition of these sets, and the associated multifractal spectrum which encodes the decomposition. The goal is to relate the spectrum to other characteristics of the system, such as entropy and Lyapunov exponents of invariant measures, and to study the regularity of the spectrum, for instance convexity, smoothness and analyticity. With this study one tries to get more refined descriptions of the dynamics than purely stochastic considerations (see, e.g., [**20**] for general accounts of this subject).

The case of conformal uniformly expanding systems is well understood [**2**, **19**–**21**, **35**], and a complete picture is emerging. For one-dimensional maps, some progress has been made to relax this assumption: allowing parabolic fixed points [**12**, **14**, **18**]; allowing critical points [**7**, **8**, **11**, **13**, **22**]. Nevertheless, little is known on higher dimensional systems. Indeed, one can mention interesting recent developments [**1**, **32**] on two-dimensional *parabolic horseshoes*. In these papers, however, the existence of global continuous invariant foliations is assumed, which allows one to reduce a considerable part of the analysis to one-dimensional dynamics. To our knowledge, there is no previous

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

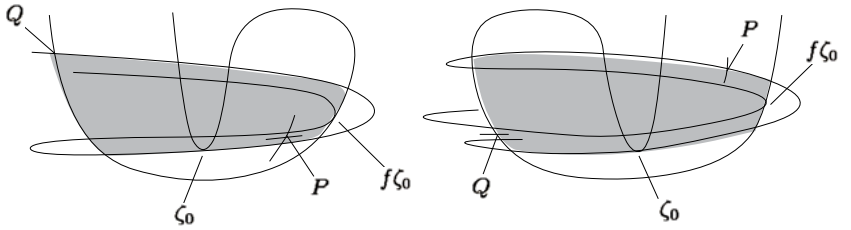



FIGURE 1. Organization of the invariant manifolds for $a = a^*$: there are two fixed saddles $P$ and $Q$ near $(1/2, 0)$ and $(-1, 0)$, respectively. In the orientation-preserving case (left), $W^u(Q)$ and $W^s(Q)$ meet each other tangentially. In the orientation-reversing case (right), $W^u(P)$ and $W^s(Q)$ meet each other tangentially. The point of tangency near $(0, 0)$ is denoted by $\zeta_0$. The shaded domains represent the rectangle $R$ (see §2.2) containing the non-wandering set $\Omega$.

result on the multifractal analysis of two-dimensional maps having tangencies of invariant manifolds. This type of maps admit no global continuous invariant foliation, and so new arguments and ideas are necessary to reduce to one-dimensional dynamics.

In this paper we are concerned with a family of planar diffeomorphisms

$$f_a : (x, y) \in \mathbb{R}^2 \mapsto (1 - ax^2, 0) + b \cdot \Phi(a, b, x, y), \quad a \in \mathbb{R}, 0 < b \ll 1, \tag{1}$$

where $\Phi$ is bounded continuous in $(a, b, x, y)$ and $C^2$ in $(a, x, y)$. We assume† that there exists a constant $C > 0$ such that for all $a$ near 2 and small $b$,

$$\|D \log |\det Df_a|\| \le C. \tag{2}$$

This family of diffeomorphisms has a fundamental importance in the creation of the theory of non-uniformly hyperbolic strange attractors [**4**, **17**, **34**]. A relevant problem is to study the dynamics at a *first bifurcation parameter* $a^* = a^*(b) \in \mathbb{R}$. This parameter does not belong to the parameter sets of positive Lebesgue measure constructed in [**4**, **17**, **34**], and satisfies the following properties [**3**, **6**, **9**, **30**]:

- $a^* \to 2$ as $b \to 0$;
- the non-wandering set of $f_a$ is a uniformly hyperbolic horseshoe for $a > a^*$;
- for $a = a^*$, there is a single orbit of homoclinic or heteroclinic tangency involving (one of) the two fixed saddles. The tangency is quadratic, and the family $\{f_a\}_{a\in\mathbb{R}}$ unfolds this tangency generically.

Let $P$, $Q$ denote the fixed saddles of $f$ near $(1/2, 0)$, $(-1, 0)$, respectively. The orbit of tangency intersects a small neighborhood of the origin exactly at one point, denoted by $\zeta_0$ (Figure 1). If $f_{a^*}$ preserves orientation, then $\zeta_0 \in W^s(Q) \cap W^u(Q)$. If $f_{a^*}$ reverses orientation, then $\zeta_0 \in W^s(Q) \cap W^u(P)$. The map $f_{a^*}$ falls into the class of *non-uniformly hyperbolic systems*. The sole obstruction to the uniform hyperbolicity is the point of the tangency $\zeta_0$.

The aim of this paper is to perform the multifractal analysis of $f_{a^*}$, in particular to study its *Lyapunov spectrum*. Although some aspects of the dynamics of $f_{a^*}$ resemble the horseshoe before the first bifurcation, the presence of tangency is an intrinsic hurdle for understanding the global dynamics.

† Condition (2) is used exclusively in the proof of Lemma 4.1. See [**25**].

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

We state our settings in more precise terms. Write $f$ for $f_{a^*}$. At a point $z \in \mathbb{R}^2$, define a one-dimensional subspace $E^u_z$ of $T_z\mathbb{R}^2$ which is exponentially contracted by backward iterates:

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Df^{-n}|E^u_z\| < 0.$$

Since $f^{-1}$ expands area, the one-dimensional subspace of $T_z\mathbb{R}^2$ with this property is unique, when it makes sense. We call $E^u_z$ an *unstable direction* at $z$, and define an *unstable Jacobian* at $z$ by $J^u(z) = \|Df|E^u_z\|$. Let $\Omega$ denote the non-wandering set of $f$, which is a compact set. By a result of [**24**], $E^u_z$ makes sense for any $z \in \Omega$, and $z \mapsto E^u_z$ is continuous on $\Omega$ except at $Q$ where it is merely measurable.

For $z \in \Omega$, define

$$\underline{\lambda}^u(z) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log J^u(f^i z) \quad \text{and} \quad \bar{\lambda}^u(z) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log J^u(f^i z).$$

If both values coincide, then call this common value an *unstable Lyapunov exponent at* $z$ and denote it by $\lambda^u(z)$. Since the (non-uniform) expansion along the unstable direction is responsible for the chaotic behavior, the distribution of the unstable Lyapunov exponent is important for understanding the dynamics of $f$.

If $f$ preserves orientation, let $W^u = W^u(Q)$. Otherwise, let $W^u = W^u(P)$. A good deal of information is contained in the unstable slice

$$\Omega^u = \Omega \cap W^u.$$

For each $\beta \in \mathbb{R}$, consider the level set

$$\Omega^u(\beta) = \{z \in \Omega^u : \lambda^u(z) \text{ is defined and } \lambda^u(z) = \beta\}.$$

The first question to ask is what are the values of $\beta$ for which $\Omega^u(\beta) \neq \emptyset$. For uniformly hyperbolic systems as in the case $a > a^*$, such values are all positive and form a compact interval. One can easily see that this is not the case for $f = f_{a^*}$, because $\lambda^u(\zeta_0) < 0$.

Let $\mathcal{M}(f)$ denote the set of $f$-invariant Borel probability measures. With a slight abuse of notation, an *unstable Lyapunov exponent* of a measure $\mu \in \mathcal{M}(f)$ is the number $\lambda^u(\mu)$ defined by

$$\lambda^u(\mu) = \int \log J^u \, d\mu.$$

Set

$$\lambda^u_m = \inf\{\lambda^u(\mu) : \mu \in \mathcal{M}(f)\} \text{ and } \lambda^u_M = \sup\{\lambda^u(\mu) : \mu \in \mathcal{M}(f)\}.$$

By a result of [**6**], $\lambda^u_m > 0$. Since any measure is supported on the compact set $\Omega$, $\lambda^u_M < \infty$. Set $I = [\lambda^u_m, \lambda^u_M]$.

THEOREM A. *Let $b > 0$ be sufficiently small and $f = f_{a^*(b)}$ as above. Then $\Omega^u(\beta) \neq \emptyset$ if and only if $\beta \in \{\lambda^u(\zeta_0)\} \cup I$.*

The number $\lambda^u(\zeta_0)$ equals the stable Lyapunov exponent of the Dirac measure at $Q$, and so $\lambda^u(\zeta_0) \to -\infty$ as $b \to 0$. The interval $I$ does not degenerate to a point as $b \to 0$, because the unstable Lyapunov exponents of the Dirac measures at $P$ and $Q$ converge to

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

log 2 and log 4, respectively. In fact, one can show that $\lambda_m^u \to \log 2$ and $\lambda_M^u \to \log 4$ as $b \to 0$.

A proof of Theorem A relies on the fact that $a^* \to 2$ as $b \to 0$, and so $f = f_{a^*}$ may be viewed as a singular perturbation of the endomorphism $(x, y) \mapsto (1 - 2x^2, 0)$. However, the multifractal picture is quite in contrast to that of the quadratic map $x \in [-1, 1] \to 1 - 2x^2$. The Lyapunov exponent of the quadratic map takes only three values: it is log 4 at the repelling fixed point $-1$ and its preimage 1, $-\infty$ at the preimages of 0 and log 2 at all other well-defined points.

Now consider a multifractal decomposition

$$\Omega^u = \left( \bigcup_{\beta \in \{\lambda^u(\zeta_0)\} \cup I} \Omega^u(\beta) \right) \cup \hat{\Omega}^u,$$

where $\hat{\Omega}^u$ denotes the set of those $z \in \Omega^u$ for which $\underline{\lambda}^u(z) \neq \bar{\lambda}^u(z)$, and so $\lambda^u(z)$ is undefined. This decomposition has an extremely complicated topological structure. One can show that if $\beta \in I$, then $\Omega^u(\beta)$ is dense in $\Omega^u$ with respect to the induced topology on $W^u$.

To evaluate the size of each level set, we adopt the Hausdorff dimension on $W^u$ defined as follows. Given $p \in (0, 1]$, the unstable Hausdorff $p$-measure of a set $A \subset W^u$ is defined by

$$m_p^u(A) = \lim_{\varepsilon \to 0} \left( \inf \sum_{U \in \mathcal{U}} \text{length}(U)^p \right),$$

where length$(\cdot)$ denotes the length on $W^u$ with respect to the induced Riemannian metric, and the infimum is taken over all countable coverings $\mathcal{U}$ of $A$ by open sets of $W^u$ with length $\leq \varepsilon$. The unstable Hausdorff dimension of $A$, denoted by $\dim_H^u$, is the unique number in [0, 1] such that

$$\dim_H^u(A) = \sup\{p : m_p^u(A) = \infty\} = \inf\{p : m_p^u(A) = 0\}.$$

Set

$$L^u(\beta) = \dim_H^u(\Omega^u(\beta)).$$

The object of our study is the function $\beta \mapsto L^u(\beta)$, called a *Lyapunov spectrum*. We give a formula for $L^u(\beta)$ in terms of the unstable Lyapunov exponents and entropy of invariant probability measures. The entropy of $\mu \in \mathcal{M}(f)$ is denoted by $h(\mu)$.

THEOREM B. *For any $\beta \in I$,*

$$L^u(\beta) = \lim_{\varepsilon \to 0} \sup \left\{ \frac{h(\mu)}{\lambda^u(\mu)} : \mu \in \mathcal{M}(f), |\lambda^u(\mu) - \beta| < \varepsilon \right\}.$$

Due to the existence of tangency, the unstable Lyapunov exponent as a function of measures may not be lower semi-continuous, and this is indeed the case if $f$ preserves orientation [**29**]. Hence, the limit in $\varepsilon$ is necessary. A formula similar to the one in Theorem B was obtained in [**8**] for a positive measure set of quadratic maps $x \in [-1, 1] \to 1 - ax^2$, but only for the time averages of continuous functions.

We now move on to properties of the Lyapunov spectrum. Let us recall the thermodynamic formalism of $f$ developed in [**24**, **25**]. For $t \in \mathbb{R}$, define

$$P(t) = \sup\{h(\mu) - t\lambda^u(\mu) : \mu \in \mathcal{M}(f)\}.$$

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

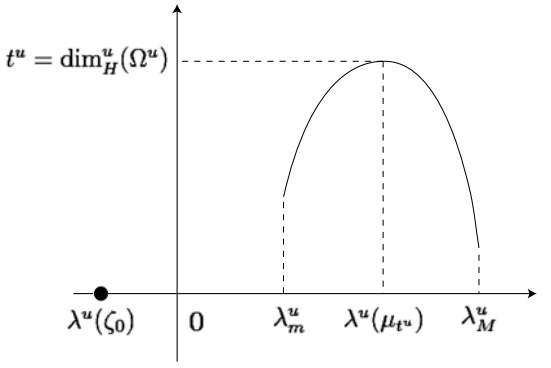



FIGURE 2. Schematic picture of the graph of the Lyapunov spectrum $L^u : \{\lambda^u(\zeta_0)\} \cup I \to \mathbb{R}$.

A measure which attains this supremum is called an *equilibrium measure* for $-t \log J^u$. The function $t \mapsto P(t)$ is convex. One has $P(0) > 0$, and Ruelle's inequality [**23**] gives $P(1) \le 0$. Since $f$ has no Sinaĭ–Ruelle–Bowen measure [**28**], $P(1) < 0$ holds. Hence, the equation $P(t) = 0$ has a unique solution in $(0, 1)$, denoted by $t^u$. There exists a unique equilibrium measure for $-t^u \log J^u$ (see [**25**, Theorem A]) denoted by $\mu_{t^u}$, and $t^u = \dim_H^u(\Omega^u)$, $t^u \to 1$ as $b \to 0$ (see [**25**, Theorem B]).

THEOREM C. *The following holds for the function $\beta \in I \mapsto L^u(\beta)$:*
(a) *it is continuous;*
(b) *it is increasing on $[\lambda_m^u, \lambda^u(\mu_{t^u})]$ and decreasing on $[\lambda^u(\mu_{t^u}), \lambda_M^u]$;*
(c) *it is strictly positive in the interior of $I$;*
(d) *$L^u(\beta) = t^u$ if and only if $\beta = \lambda^u(\mu_{t^u})$.*

Theorem C illustrates what is sometimes called a *multifractal miracle*. Even though the multifractal decomposition is topologically complicated, the Lyapunov spectrum which encodes the decomposition is continuous, and has several nice properties.

*Remark.* From Theorem C(b), the minimum of $L^u$ is attained at the boundary of $I$. It is not known if the minimum is strictly positive. The convexity of the Lyapunov spectrum is not known either (Figure 2 is a schematic).

The last theorem states that $\hat{\Omega}^u$ carries the full Hausdorff dimension. For the subshift of finite type, it is known [**2**] that the set of irregular points for which the time averages of a given continuous function do not converge carries the full dimension. Since $\log J^u$ is not continuous, the same argument does not work in our setting.

THEOREM D. $\dim_H^u(\hat{\Omega}^u) = t^u$.

Let us briefly outline main arguments used in proofs of the theorems. All the difficulties stem from orbits' return to a small neighborhood of the point $\zeta_0$ of tangency. Our strategy to treat these returns is to introduce critical points (see §2.4) in the spirit of Benedicks and Carleson [**4**]. The assumption that $f$ corresponds to the first bifurcation parameter $a^*$ implies that all critical points are non-recurrent, and in particular the derivatives along

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

their orbits grow exponentially fast. In order to compensate the loss of hyperbolicity due to returns to a small neighborhood of $\zeta_0$, we 'bind' generic orbits to orbits of critical points, and let them copy the derivatives along the critical orbits.

To perform the multifractal analysis for the two-dimensional map $f$, a basic idea is to use a (locally defined) stable foliation to identify points on the same leaf (called *long stable leaves* in our terms, see §3.4), and to recover the one-dimensional argument [**7**, **8**] as much as possible. Due to the lack of uniform hyperbolicity, the existence of such leaves is not guaranteed *a priori*. To work out this issue, we formulate a condition in terms of the speed of recurrence to the critical points, sufficient for the existence of long stable leaves (see §§3.3 and 3.4). It is true that points for which this recurrence condition fails do exist. However, the derivatives along these points oscillate, and as a result the unstable Lyapunov exponent is undefined (see §4.2). Hence, we may exclude these points from our consideration.

The rest of this paper consists of five sections. In §2, following [**24**], we introduce critical points and develop a local analysis near orbits of critical points. In §3, we push this analysis further to obtain global results on orbits which successively visit a critical region. In §4, following [**24**, **25**], we introduce a main technical ingredient, called *an inducing scheme*. In rough terms, this is an induced map with uniformly hyperbolic features (albeit with countably many branches). We will use the inducing scheme to construct coverings of each level set $\Omega^u(\beta)$ by arbitrarily short curves, and use them altogether for the upper estimate of $L^u(\beta)$. In §5, we develop a smooth ergodic theory for $f$, which will be used for the lower estimate of $L^u(\beta)$. All the theorems are proved in §6.

## 2. *Local analysis near the orbit of critical points*

In this section, after a few preliminary definitions and statements taken mainly from [**24**, **25**], we introduce the key concept 'critical points' and provide results related to them.

### 2.1. *Constants and notation.*

Throughout this paper we shall be concerned with positive constants $\lambda$, $\delta$, $b$, the purposes of which are as follows:

- $\lambda$ is used to evaluate the rate of expansion of derivatives away from the point $\zeta_0$ of tangency (see Lemma 2.1);
- $\delta$ determines the size of a neighborhood of $\zeta_0$ (see §2.3);
- $b$ determines the magnitude of the reminder term $b \cdot \Phi$ in (1).

The $\lambda$ is a fixed constant in $(0, \log 2)$. The $\delta$ and $b$ are small constants chosen in this order. The letter $C$ is used to denote any positive constant which is independent of $\delta$ or $b$.

A *slope* of a non-zero tangent vector $v = \binom{\xi}{\eta}$ at a point in $\mathbb{R}^2$ is defined by $\mathrm{slope}(v) = |\eta|/|\xi|$ if $\xi \neq 0$, and $\mathrm{slope}(v) = \infty$ if $\xi = 0$. The Euclidean length of a $C^1$ curve $\gamma$ in $\mathbb{R}^2$ is denoted by $\mathrm{length}(\gamma)$.

### 2.2. *Location of the tangency.*

By a *rectangle*, we mean any compact domain bordered by two compact curves in $W^u$ and two in the stable manifolds of $P$ or $Q$. By an *unstable side* of a rectangle, we mean any of the two boundary curves in $W^u$. A *stable side* is defined similarly.

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

By the results of [**24**], there exists a rectangle $R$ contained in the set $[-2, 2] \times [-\sqrt{b}, \sqrt{b}]$ with the following properties (see Figure 1):

- $\Omega = \{z \in R : f^n z \in R \text{ for every } n \in \mathbb{Z}\}$;
- one of the unstable sides of $R$ contains $\zeta_0$;
- one of the stable sides of $R$ contains $f\zeta_0$. This side is denoted by $\alpha_0^+$. The other side, denoted by $\alpha_0^-$, contains $Q$;
- $f\alpha_0^+ \subset \alpha_0^-$.

2.3. *Dynamics outside of critical region.* Set

$$I(\delta) = \{(x, y) \in R : |x| < \delta\}.$$

Observe that $\zeta_0 \in I(\delta)$. The next two lemmas state that the dynamics outside of $I(\delta)$ is 'uniformly hyperbolic' and no critical behavior occurs.

LEMMA 2.1. *For any $\lambda \in (0, \log 2)$ and $\delta \in (0, 1)$, there exists $b > 0$ such that the following holds for $f = f_{a^*(b)}$: if $n \geq 1$ and $z \in R$ are such that $z, fz, \ldots, f^{n-1}z \notin I(\delta)$, then for any unit tangent vector $v$ at $z$ with* $\text{slope}(v) \leq \sqrt{b}$*:*
(a) $\|Df^n v\| \geq \delta e^{\lambda n}$*. If, in addition $f^n z \in I(\delta)$, then $\|Df^n v\| \geq e^{\lambda n}$;*
(b) $\text{slope}(Df^n v) \leq \sqrt{b}$.

*Proof.* This comes from the fact that $f$ may be viewed as a small perturbation of the map $x \mapsto 1 - 2x^2$. □

LEMMA 2.2. [**27**, Lemma 2.3] *Let $\gamma$ be a $C^2$ curve in $R$ and $z \in \gamma$. For each $i \geq 0$, let $\kappa_i(z)$ denote the curvature of $f^i\gamma$ at $f^i z$. Then*

$$\kappa_i(z) \leq \frac{(Cb)^i}{\|Df^i|T_z\gamma\|^3}\kappa_0(z) + \sum_{j=1}^{i} \frac{(Cb)^j}{\|Df^j|T_{f^{i-j}z}f^{i-j}\gamma\|^3}.$$

By a *$C^2(b)$-curve*, we mean a compact $C^2$ curve in $R$ such that the slopes of its tangent directions are $\leq \sqrt{b}$ and the curvature is everywhere $\leq \sqrt{b}$.

LEMMA 2.3. *If $\gamma$ is a $C^2(b)$-curve in $R$ not intersecting $I(\delta)$, then $f\gamma$ is a $C^2(b)$-curve.*

*Proof.* This comes from Lemmas 2.1 and 2.2. □

2.4. *Critical points.* Returns to the inside of $I(\delta)$ are inevitable and must be treated with care. A key ingredient is the notion of critical points, i.e., points of tangencies between $C^2(b)$-curves in $W^u$ and preimages of leaves of a stable foliation. We quote results from [**24**] surrounding critical points, and develop them slightly further.

From the hyperbolicity of the saddle $Q$, there exist two mutually disjoint connected open sets $U^-$, $U^+$ independent of $b$ such that $\alpha_0^- \subset U^-$, $\alpha_0^+ \subset U^+$, $U^+ \cap fU^+ = \emptyset = U^+ \cap fU^-$ and a foliation $\mathcal{F}^s$ of $U = U^- \cup U^+$ by one-dimensional leaves such that:

- $\mathcal{F}^s(Q)$, the leaf of $\mathcal{F}^s$ containing $Q$, contains $\alpha_0^-$;
- if $z, fz \in U$, then $f(\mathcal{F}^s(z)) \subset \mathcal{F}^s(fz)$;
- let $e^s(z)$ denote the unit vector in $T_z\mathcal{F}^s(z)$ whose second component is positive. Then $z \mapsto e^s(z)$ is $C^1$, $\|Dfe^s(z)\| \leq Cb$ and $\|De^s(z)\| \leq C$;
- if $z, fz \in U$, then $\text{slope}(e^s(z)) \geq C/\sqrt{b}$.

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

We say that $\zeta \in W^u \cap I(\delta)$ is a *critical point* if $f\zeta \in U^+$ and $T_{f\zeta}W^u = T_{f\zeta}\mathcal{F}^s(f\zeta)$.

From the first two conditions on $\mathcal{F}^s$ and $f\alpha_0^+ \subset \alpha_0^-$, there is a leaf of $\mathcal{F}^s$ which contains $\alpha_0^+$. Since $f\zeta_0 \in \alpha_0^+$, we have $f\zeta_0 \in U^+$ and $T_{f\zeta_0}W^u = T_{f\zeta_0}\mathcal{F}^s(f\zeta_0)$, namely, $\zeta_0$ is a critical point. The next lemma tells about the location of all other critical points. Let $S$ denote the compact lenticular domain bounded by the parabola $R \cap f^{-1}\alpha_0$ and the unstable side of $R$ not containing $\zeta_0$. Note that $fS \subset U^+$.

LEMMA 2.4. *Let $\gamma$ be a $C^2(b)$-curve in $W^u$ stretching across $I(\delta)$. Then there exists a unique critical point $\zeta \in \gamma$. In addition, $\zeta \in S$. If $\zeta \neq \zeta_0$, then $\zeta$ is contained in the interior of $S$.*

*Proof.* We claim that any leaf of $\mathcal{F}^s$ at the right of the one containing $\alpha_0^+$ is tangent to $f\gamma$ and the tangency is quadratic, or else it intersects $f\gamma$ exactly at two points. This follows from [**27**, Lemma 2.2], the uniform boundedness of $\|De^s(z)\|$ and $\mathrm{slope}(e^s(z))$. Hence, there exists a critical point on $\gamma$. If $\zeta_1, \zeta_2$ are distinct critical points on $\gamma$, then the leaves $\mathcal{F}^s(f\zeta_1)$, $\mathcal{F}^s(f\zeta_2)$ must intersect each other, which is a contradiction. Hence, the uniqueness holds. Since the quadratic tangency occurs on or at the right of $\alpha_0^+$, the last two statements hold. □

By Lemma 2.4, any critical point other than $\zeta_0$ is contained in the interior of $S$, so that it is mapped to the outside of $R$, and then escapes to infinity under forward iteration. Hence, the critical orbits are contained in a region where the uniform hyperbolicity is apparent. By binding generic orbits which fall inside $I(\delta)$ to suitable critical points, and then copying the exponential growth along the critical orbits, one shows that the horizontal slopes and the expansion are restored after suffering from the loss due to the folding behavior near $I(\delta)$.

In the next lemma we assume that $\delta > 0$ is sufficiently small. Let $\zeta$ be a critical point and $z \in I(\delta)\backslash S$. We say that a unit tangent vector $v$ at $z$ is *in admissible position relative to* $\zeta$ if there exists a $C^2(b)$-curve which is tangent to both $T_\zeta W^u$ and $v$. Set

$$c(b) = -\frac{1}{\log b}. \tag{3}$$

Let us agree that for two positive real numbers $A$, $B$, $A \approx B$ indicates that both $A/B$, $B/A$ are bounded from above by a constant independent of $\delta$ or $b$.

LEMMA 2.5. *Let $\zeta$ be a critical point, $z \in (\Omega \cap I(\delta))\backslash S$ and let $v$ be a unit tangent vector at $z$ in admissible position relative to $\zeta$. There exist positive integers $p = p(\zeta, z)$, $q = q(\zeta, z)$ such that:*

(a) $q \le -c(b)\log|\zeta - z| \ll -(2/3)\log|\zeta - z| \le p$;
(b) $f^i\zeta,\ f^i z \in U$ *for every* $i \in \{1, 2, \ldots, p\}$;
(c) $\mathrm{slope}(Df^p v) \le \sqrt{b}$ *and* $\|Df^p v\| \ge e^{(\lambda/3)p}$;
(d) $\|Df^q v\| \le C|\zeta - z|^{1-c(b)}$;
(e) $\|Df^i v\| < 1$ *for every* $i \in \{1, 2, \ldots, q-1\}$ *and*
$\|Df^i v\| \approx |\zeta - z| \cdot \|D_{fz}f^{i-1}\binom{1}{0}\|$ *for every* $i \in \{q, q+1, \ldots, p\}$.

*Proof.* We only give a proof of (d). The rest of the items are contained in [**24**, Lemma 2.5]. Split $Dfv = A \cdot \binom{1}{0} + B \cdot e^s(fz)$, $A$, $B \in \mathbb{R}$. Since the forward orbit of $f\zeta$ does

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

not intersect $I(\delta)$, the tangent vector $\binom{1}{0}$ at $f\zeta$ grows exponentially in norm under forward iteration. Since the forward orbit of $fz$ shadows that of $f\zeta$, $\|D_{fz}f^{q-1}\binom{1}{0}\| \approx \|D_{f\zeta}f^{q-1}\binom{1}{0}\|$ holds. From the quadratic behavior near the critical point, we have $|A| \approx |\zeta - z|$. Then $q \ll p$ in Lemma 2.5(a) and the exponential contraction of $e^s(fz)$ implies that $|A| \cdot \|D_{fz}f^{q-1}\binom{1}{0}\| \gg |B| \cdot \|D_{fz}f^{q-1}e^s(fz)\|$. Hence, $\|Df^q v\| \approx |\zeta - z| \cdot \|D_{f\zeta}f^{q-1}\binom{1}{0}\| \le C|\zeta - z|^{1-c(b)}$, where the last inequality follows from the definition of $q$ in [**24**, §2.3]. □

## 3. *Global estimates and constructions*

In this section we use the results in the previous section to analyze orbits which successively visit the critical region $I(\delta)$. The idea is to associate to each return to $I(\delta)$ a suitable critical point for which Lemma 2.5 works. This can be done by using a nice geometry of $W^u$, which however is very particular to the first bifurcation parameter $a^*$. Having established this, we introduce a criterion for the existence of long stable leaves, in terms of the speed of recurrence to the point $\zeta_0$ of tangency. These leaves will later become necessary to quotient along stable directions and reduce to the one-dimensional case.

### 3.1. *Existence of binding points.*

Let $\alpha_1^+$ denote the connected component of $R \cap W^s(P)$ containing $P$, and $\alpha_1^-$ the connected component of $R \cap f^{-1}\alpha_1^+$ not containing $P$. Let $\Theta$ denote the rectangle bordered by $\alpha_1^-$, $\alpha_1^+$ and the unstable sides of $R$.

LEMMA 3.1. *Let $\gamma$ be a $C^2(b)$-curve in $I(\delta)$ and suppose that there exists a critical point on $\gamma$. If $n \ge 1$ is such that $\Theta \cap f^i\gamma = \emptyset$ for every $i \in \{0, 1, \dots, n-1\}$ and $\Theta \cap f^n\gamma \ne \emptyset$, then any connected component of $\Theta \cap f^n\gamma$ is a $C^2(b)$-curve.*

*Proof.* By Lemma 2.2, it suffices to show that for any $z \in \gamma$, $\|Df^{n-i}|T_{f^i z}f^i\gamma\| \ge \delta$ for every $i \in \{0, 1, \dots, n-1\}$. This follows from Lemmas 2.1 and 2.5(e). □

Let $\tilde{\Gamma}^u$ denote the collection of $C^2(b)$-curves in $\Theta \cap W^u$ which is maximal with respect to inclusion.

LEMMA 3.2. *Any element of $\tilde{\Gamma}^u$ is a $C^2(b)$-curve with end points in $\alpha_1^-$, $\alpha_1^+$.*

*Proof.* Let $\gamma$ denote the unstable side of $\Theta$ not containing $\zeta_0$. This is $C^2(b)$, and contains a fundamental domain in $W^u$. It suffices to show that for each $n \ge 0$, any connected component of $\Theta \cap \bigcup_{i=0}^n f^i\gamma$ is a $C^2(b)$-curve with end points in $\alpha_1^-$, $\alpha_1^+$. This holds for $n = 0$. If it holds for $n = k$, then by Lemma 3.1, any connected component of $\Theta \cap \bigcup_{i=0}^{k+1} f^i\gamma$ is $C^2(b)$. Since the end points of $\gamma$ are mapped to the stable sides of $\Theta$, the statement holds for $n = k + 1$. □

Define

$$\Gamma^u = \{\gamma^u : \gamma^u \text{ is the pointwise limit of the sequence in } \tilde{\Gamma}^u\}.$$

Since elements of $\tilde{\Gamma}^u$ are $C^2(b)$ by Lemma 3.2, the pointwise convergence is equivalent to the uniform convergence. Since curves in $\tilde{\Gamma}^u$ are pairwise disjoint, the uniform convergence is equivalent to the $C^1$ convergence. Hence, curves in $\Gamma^u$ are $C^1$ and the

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

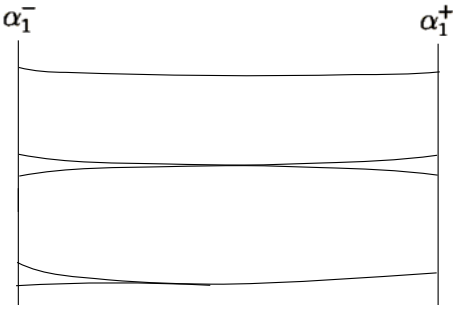



FIGURE 3. The long unstable leaves.

slopes of their tangent directions are $\leq \sqrt{b}$. Elements of $\Gamma^u$ are called *long unstable leaves*. Set

$$\mathcal{W}^u = \bigcup_{\gamma^u \in \Gamma^u} \gamma^u.$$

Several remarks are in order on the long unstable leaves:

- each leaf is the (strictly) monotone limit of curves in $\tilde{\Gamma}^u$, so that any connected component of $\mathcal{W}^u$ contains at most two leaves (see Figure 3);
- two intersecting leaves are tangent at every point of the intersection;
- if $z \in \mathcal{W}^u$, then $E_z^u = T_z\gamma^u$, where $\gamma^u$ denotes any leaf containing $z$ [**25**, Lemma 3.2(P2)];
- $\Omega \cap \Theta \subset \mathcal{W}^u$ [**24**, Lemma 2.8].

LEMMA 3.3. *If $z \in \Omega \cap I(\delta)$, then there exists a critical point relative to which any unit vector spanning $E_z^u$ is in admissible position.*

*Proof.* A long unstable leaf containing $z$ is accumulated by curves in $\tilde{\Gamma}^u$, each of which contains a critical point by Lemma 2.4. □

If $z \in \Omega \cap I(\delta)$, then critical points as in Lemma 3.3 are not unique. Let $\zeta(z)$ denote the one which is closest to the fixed saddle in $W^u$ with respect to the induced metric on $W^u$, and call it a *binding point* for $z$. Write $p(z) = p(\zeta(z), z)$ and $q(z) = q(\zeta(z), z)$ and call them the *bound* and *fold periods* of $z$.

3.2. *Bound–free structure.* To the forward orbit of $z \in \Omega$ we associate a sequence

$$0 \leq n_1 < n_1 + p_1 < n_2 < n_2 + p_2 < n_3 < \cdots$$

of integers which record the pattern of recurrence to $I(\delta)$ in the following manner. First, $n_1 = \min\{n \geq 0 : f^n z \in I(\delta)\}$ and $p_1 = p(f^{n_1} z)$. Given $n_k$ and $p_k$, set $n_{k+1} = \min\{n \geq n_k + p_k : f^n z \in I(\delta)\}$ and $p_{k+1} = p(f^{n_{k+1}} z)$. This decomposes the forward orbit of $z$ into segments corresponding to time intervals $(n_k, n_k + p_k)$ and $[n_k + p_k, n_{k+1}]$, during which we refer to the points in the orbit of $z$ as being 'bound' and 'free', respectively. The $\{n_k\}_k$ are the only return times to $I(\delta)$.

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

3.3. *Controlled points.* For $z \in \Omega$, define

$$d_{\mathrm{crit}}(z) = \begin{cases} |\zeta(z) - z| & \text{if } z \in I(\delta), \\ 1 & \text{otherwise,} \end{cases}$$

where $\zeta(z)$ is the binding point for $z$ determined in §3.1. We say that $z \in \Omega$ is *controlled* if $d_{\mathrm{crit}}(f^n z) \geq b^{n/8}$ holds for every integer $n \geq 0$.

The next lemma states that points without too deep returns become eventually controlled.

LEMMA 3.4. *Let $m \geq 1$. If $d_{\mathrm{crit}}(f^n z) \geq b^{n/9}$ for every integer $n \geq m$, then there exists $k \in \{0, 1, \ldots, m\}$ such that $f^k z$ is controlled.*

*Proof.* Suppose that $f^k z$ is not controlled for every $k \in \{0, 1, \ldots, m\}$. Then it is possible to define a sequence $\{k_i\}_{i=1}^s$ of non-negative integers inductively as follows: $k_1 = \min\{n \geq 0 : d_{\mathrm{crit}}(f^n z) < b^{n/8}\}$. From the assumption, we have $k_1 < m$. Given $k_1, \ldots, k_i$ with $k_1 + \cdots + k_i < m$ and $d_{\mathrm{crit}}(f^{k_1+\cdots+k_i} z) < b^{k_i/8}$, define $k_{i+1} = \min\{n > 0 : d_{\mathrm{crit}}(f^{k_1+\cdots+k_i+n} z) < b^{n/8}\}$. We have $k_1 + \cdots + k_{s-1} < m \leq k_1 + \cdots + k_s$. Since $b^{k_i/8} \cdot \|Df^{100k_i}\| \ll 1$, $f^{k_1+\cdots+k_i} z$ shadows the forward orbit of the binding point at least up to time $100k_i$, and so $100k_i < k_{i+1}$. This yields $k_1 + \cdots + k_s < (9/8)k_s$, and thus $d_{\mathrm{crit}}(f^{k_1+\cdots+k_s} z) < b^{k_s/8} < b^{k_1+\cdots+k_s/9}$. From the assumption on $z$ and $m \leq k_1 + \cdots + k_s$, we have $d_{\mathrm{crit}}(f^{k_1+\cdots+k_s} z) \geq b^{k_1+\cdots+k_s/9}$. These two inequalities yield a contradiction. □

3.4. *Long stable leaves.* By a *vertical $C^2(b)$-curve* we mean a compact $C^2$ curve in $R$ with end points in the unstable sides of $R$, and of the form

$$\{(x(y), y) : |x'(y)| \leq C\sqrt{b},\ |x''(y)| \leq C\sqrt{b}\}.$$

A vertical $C^2(b)$-curve $\gamma^s$ is called a *long stable leaf* if $|f^n z_1 - f^n z_2| \leq Cb^{n/2}|z_1 - z_2|$ holds for all $z_1, z_2 \in \gamma^s$ and every $n \geq 0$.

LEMMA 3.5. *If $z \in \Omega$ is controlled, then there exists a unique long stable leaf through $z$, denoted by $\gamma^s(z)$. In addition, the following holds:*

(a) *for all $z_1, z_2 \in \gamma^s(z) \cap \Omega$ and $n > 0$,*

$$\frac{\|Df^n | E^u_{z_2}\|}{\|Df^n | E^u_{z_1}\|} \leq 2;$$

(b) *if $z_1, z_2 \in \Omega$ are controlled, then the Hausdorff distance between $\gamma^s(z_1)$ and $\gamma^s(z_2)$ is $\leq e^{C\sqrt{b}}|z_1 - z_2|$.*

*Proof.* In view of the results in [**17**, §§6 and 7C], [**5**, Lemma 2.4] and [**25**, Sublemma A.2], it suffices to show the following expansion estimate:

$$\|Df^n | E^u_z\| \geq b^{n/10} \text{ for every } n \geq 0. \tag{4}$$

To show (4), we introduce the bound/free structure on the orbit of $z$. If $f^n z$ is free, then the orbit $z, \ldots, f^n z$ is decomposed into alternate bound and free segments. Applying the expansion estimates in Lemmas 2.1 and 2.5, we have $\|Df^n | E^u_z\| \geq \delta e^{(\lambda/3)n} \geq b^{n/10}$.

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

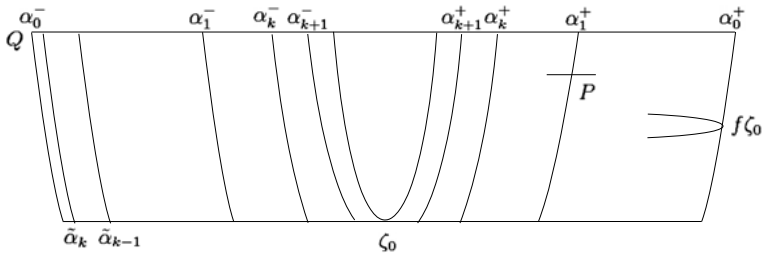



FIGURE 4. The rectangle $R$ and the curves $\{\tilde{\alpha}_k\}$, $\{\alpha_k^+\}$, $\{\alpha_k^-\}$. The $\{\tilde{\alpha}_k\}$ accumulates on the left stable side of $R$. Both $\{\alpha_k^+\}$ and $\{\alpha_k^-\}$ accumulate on the parabola $R \cap f^{-1}\alpha_0^+$ containing the point of tangency $\zeta_0$ near the origin.

If $f^n z$ is bound, then there exists an integer $0 < m < n$ such that $f^m z \in I(\delta)$ and $m < n < m + p$, where $p$ is the bound period of $f^m z$. Since $f^{m+p}z$ is free and $\|Df\| < 5$, we have $\|Df^n|E_z^u\| \geq 5^{-(m+p-n)}\|Df^{m+p}|E_z^u\| > 5^{-p}$. Since $z$ is controlled, $p \leq -(n/12)\log b$, and so $\|Df^n|E_z^u\| \geq b^{(\log 5/12)n} > b^{n/10}$. □

## 4. *Construction of inducing scheme*

This section consists of two explicit parts. In the first part (§4.1), following [**24**, **25**], we build an inducing scheme with uniformly hyperbolic features. For the rest of this section we perform more elaborate estimates and constructions, carefully avoiding points which return close to $\zeta_0$ infinitely many times.

4.1. *Inducing.* Define a sequence $\{\tilde{\alpha}_k\}_{k=0}^{\infty}$ of compact curves in $R \cap W^s(P)$ inductively as follows. First, set $\tilde{\alpha}_0 = \alpha_1^+$. Given $\tilde{\alpha}_{k-1}$, define $\tilde{\alpha}_k$ to be one of the two connected components of $R \cap f^{-1}\tilde{\alpha}_{k-1}$ which is at the left of $\zeta_0$. Observe that $\tilde{\alpha}_1 = \alpha_1^-$. By the inclination lemma, the Hausdorff distance between $\tilde{\alpha}_k$ and $\alpha_0^-$ converges to 0 as $k \to \infty$.

For each $k \geq 0$, let $\alpha_k$ denote the connected component of $R \cap f^{-1}\tilde{\alpha}_k$ which is not $\tilde{\alpha}_{k+1}$. The set $R \cap f^{-1}\alpha_k$ consists of two curves, one at the left of $\zeta_0$ and the other at the right. They are denoted by $\alpha_{k+1}^-$ and $\alpha_{k+1}^+$, respectively (see Figure 4). By definition, these curves obey the following diagram:

$$\{\alpha_{k+1}^-, \alpha_{k+1}^+\} \xrightarrow{f^2} \tilde{\alpha}_k \xrightarrow{f} \tilde{\alpha}_{k-1} \xrightarrow{f} \tilde{\alpha}_{k-2} \xrightarrow{f} \cdots \xrightarrow{f} \tilde{\alpha}_1 = \alpha_1^- \xrightarrow{f} \tilde{\alpha}_0 = \alpha_1^+.$$

Define $r : \Theta \to \mathbb{N} \cup \{\infty\}$ by

$$r(z) = \inf(\{n > 0 : f^n z \in \Theta\} \cup \{\infty\}),$$

which is the first return time of $z$ to $\Theta$. Note that:

- $r(z) = 1$ if and only if $z \in \alpha_1^- \cup \alpha_1^+$; $r(z) = k + 1$ $(n \geq 1)$ if and only if $z$ is sandwiched by $\alpha_k^+$ and $\alpha_{k+1}^+$, or by $\alpha_k^-$ and $\alpha_{k+1}^-$; $r(z) = \infty$ if and only if $z \in S$;
- each level set of $r$ except $S$ has exactly two connected components.

Let $\mathcal{P}$ denote the partition of the set $\Theta \backslash (S \cup \alpha_1^- \cup \alpha_1^+)$ into connected components of the level sets of the function $r$. The $\mathcal{P}$ is well defined because $\alpha_k$ and $\alpha_0^+$ are long stable leaves, and the Hausdorff distance between them converges to 0 as $k \to \infty$

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

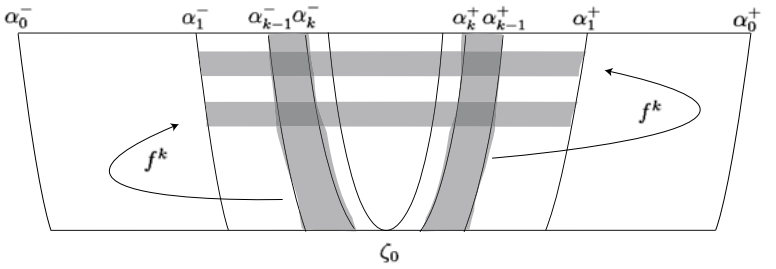



FIGURE 5. The proper rectangles (shaded) in $\mathcal{P}_1$ with inducing time $k$ and their $f^k$-images.

by Lemma 3.5(b). Set $\mathcal{P}_1 = \{\overline{\omega} : \omega \in \mathcal{P}\}$, where the bar denotes the closure operation. For each $n \geq 2$, define

$$\mathcal{P}_n = \left\{\omega_0 \cap \bigcap_{i=1}^{n-1} f^{-r(\omega_0)} \circ f^{-r(\omega_1)} \circ \cdots \circ f^{-r(\omega_{i-1})}\omega_i : \omega_0, \omega_1, \ldots, \omega_{n-1} \in \mathcal{P}_1\right\}.$$

Elements of $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ are called *proper rectangles.* It is straightforward to see the following:

- the unstable sides of a proper rectangle are formed by two curves contained in the unstable sides of $\Theta$. Its stable sides are formed by two curves contained in $W^s(P)$;
- two proper rectangles are nested, disjoint or intersect each other only at their common stable sides.

On the interior of each $\omega \in \mathcal{P}_1$, the value of $r$ is constant. This value is denoted by $r(\omega)$. For each $\omega \in \mathcal{P}_n$, define its *inducing time* $\tau(\omega)$ by

$$\tau(\omega) = \sum_{i=0}^{n-1} r(\omega_i).$$

It is straightforward to see the following:

- the unstable sides of $f^{\tau(\omega)}\omega$ are formed by two curves in $\tilde{\Gamma}^u$. Its stable sides are formed by two curves contained in the stable sides of $\Theta$ (see Figure 5);
- let $k \in (0, \tau(\omega))$ be an integer. Then $f^k\omega$ intersects the interior of $\Theta$ if and only if $k = r(\omega_0) + \cdots + r(\omega_i)$ for some $i \in \{0, 1, \ldots, n-1\}$.

LEMMA 4.1. *For any $\gamma^u \in \Gamma^u$ and any proper rectangle $\omega$, $\gamma^u \cap \omega$ is a compact curve joining the stable sides of $\omega$. In addition:*

(a) $\|Df^{\tau(\omega)}|E_z^u\| \geq e^{(\lambda/3)\tau(\omega)}$ *for all* $z \in \gamma^u \cap \omega$;

(b) $\log(\|Df^{\tau(\omega)}|E_y^u\|/\|Df^{\tau(\omega)}|E_x^u\|) \leq C|f^{\tau(\omega)}x - f^{\tau(\omega)}y|$ *for all* $x, y \in \gamma^u \cap \omega$.

*Proof.* From the first property of the proper rectangles and Lemma 3.2, any curve in $\tilde{\Gamma}^u$ intersects any of the stable sides of $\omega$ exactly at one point, and this intersection is transverse. Since $\gamma^u$ is a $C^1$-limit of curves in $\tilde{\Gamma}^u$, The first assertion follows. For the rest, see [**25**, Lemma 3.5]. □

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

LEMMA 4.2. *The following holds for each* $\omega \in \mathcal{P}_n$*:*

(a) $\tau(\omega) \geq 2n$*;*

(b) *let* $\partial^u\omega$ *denote any unstable side of* $\omega$*. Then* $\text{length}(\partial^u\omega) \leq e^{-(\lambda/3)n}$*;*

(c) *if* $z \in \omega$*, then* $d_{\text{crit}}(f^n z) \geq e^{-10\tau(\omega)}$ *for every* $n \in \{0, 1, \ldots, \tau(\omega) - 1\}$.

*Proof.* (a) follows from $\min\{r(\omega) : \omega \in \mathcal{P}_1\} = 2$. (b) follows from (a) and the fact that $f^{\tau(\omega)}$ maps $\partial^u\omega$ with uniform expansion as in Lemma 4.1 to a curve in $\tilde{\Gamma}^u$ of length nearly 1. If (c) is not the case, then $f^{\tau(\omega)}z$ is still close to $Q$, which is a contradiction. □

4.2. *Points with too deep returns are negligible.* For each integer $m \geq 1$, define

$$G_m = \{z \in \Omega : d_{\text{crit}}(f^n z) \geq b^{n/10} \text{ for every } n \geq m\}.$$

Set

$$\Omega_* = \Omega \setminus \bigcup_{m=1}^{\infty} G_m.$$

This is the set of points which return very close to $\zeta_0$ quickly, infinitely many times. The next lemma states that the derivatives in the unstable directions along the orbits in $\Omega_*$ oscillate, and as a result the unstable Lyapunov exponent is undefined. Hence, we may exclude $\Omega_*$ from our consideration.

LEMMA 4.3. *If* $z \in \Omega_*$*, then* $\underline{\lambda}^u(z) \neq \bar{\lambda}^u(z)$.

*Proof.* Consider the bound/free structure in §3.2 for the forward orbit of $z$. By definition, $d_{\text{crit}}(f^n z) < b^{n/10}$ holds for infinitely many $n > 0$. For these $n$, $f^n z$ is free in the sense of §3.2. By Lemma 2.5(a) and (3), the corresponding fold period $q = q(f^n z)$ satisfies

$$q \leq -c(b)d_{\text{crit}}(f^n z) < -c(b)\frac{n}{10}\log b = \frac{n}{10}.$$

Hence, $n + q < (11/10)n$ and, by Lemma 2.5(d),

$$\|Df^q|E^u_{f^n z}\| \leq C d_{\text{crit}}(f^n z)^{1-c(b)} < Cb^{((1-c(b))/10)n} < Cb^{((1-c(b))10/11)(n+q)}.$$

Hence, we have

$$\|Df^{n+q}|E^u_z\| = \|Df^n|E^u_z\| \cdot \|Df^q|E^u_{f^n z}\| < 5^n \cdot Cb^{((1-c(b))10/11)(n+q)} < b^{n+q/2}.$$

Since this holds for infinitely many $n > 0$, we obtain $\underline{\lambda}^u(z) \leq (1/2)\log b < 0$. On the other hand, decomposing the forward orbit of $z$ into alternate bound and free segments, and then applying the expansion estimates in Lemmas 2.1 and 2.5(c) imply that $\bar{\lambda}^u(z) \geq \lambda/3 > 0$. □

COROLLARY 4.4. *For any* $\mu \in \mathcal{M}(f)$*,* $\mu(\Omega_*) = 0$.

*Proof.* From the ergodic decomposition, it suffices to consider the case where $\mu$ is ergodic. From the ergodic theorem, $\underline{\lambda}^u(z) = \bar{\lambda}^u(z)$ holds for $\mu$-almost every $z$, and therefore $\mu(\Omega_*) = 0$. □

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

4.3. *Recurrence and distortion properties of rectangles.* We elaborate more on proper rectangles intersecting $G_m$. The first two lemmas are concerned with a recurrence property to $\zeta_0$, and the third one establishes a bounded distortion.

LEMMA 4.5. *Let $\omega$ be a proper rectangle such that $\omega \cap G_m \neq \emptyset$ for some $m \geq 1$. If $\tau(\omega) > m$, then $d_{\mathrm{crit}}(f^n z) \geq b^{n/9}$ holds for all $z \in \omega$ and every $n \in \{m, m+1, \ldots, \tau(\omega) - 1\}$.*

*Proof.* Let $n \in \{m, m+1, \ldots, \tau(\omega) - 1\}$ be such that $f^n\omega \cap I(\delta) \neq \emptyset$. Choose $x_0 \in \omega \cap G_m$. The $f^{n+1}\omega$ is contained in a rectangle whose stable sides are contained in the pullbacks of two neighboring curves in $\{\tilde{\alpha}_k\}$. From the quadratic behavior near the critical points and the exponential convergence of the curves $\{\tilde{\alpha}_k\}$ to $\tilde{\alpha}_0$ with exponent close to $\log 4$, for any $z \in \omega$ we have $2d_{\mathrm{crit}}(f^n z)^2 > (1.9/4)d_{\mathrm{crit}}(f^n z_0)^2$. This yields $d_{\mathrm{crit}}(f^n z) > (1/3)d_{\mathrm{crit}}(f^n z_0) \geq (1/3)b^{n/10} \geq b^{n/9}$. □

LEMMA 4.6. *Let $\omega$ be a proper rectangle such that $\omega \cap G_m \neq \emptyset$ for some $m \geq 1$. If $\tau(\omega) > m$, then there exists $k \in [0, 1, \ldots, m]$ such that the stable sides of $f^k\omega$ are contained in long stable leaves.*

*Proof.* Let $\partial^s\omega$ denote any stable side of $\omega$ and $z$ an end point of $\partial^s\omega$. By Lemmas 3.4 and 3.5, it suffices to show that $d_{\mathrm{crit}}(f^n z) \geq b^{n/9}$ for every $n \geq m$. Since $\omega \cap G_m \neq \emptyset$, this inequality for every $n \in \{m, m+1, \ldots, \tau(\omega) - 1\}$ holds by Lemma 4.5. Since $f^{\tau(\omega)}z \in \alpha_1^- \cup \alpha_1^+$, we have $f^n z \in f^{n-\tau(\omega)}(\alpha_1^- \cup \alpha_1^+) \subset \alpha_1^+$ for every $n \geq \tau(\omega)$, which implies the desired inequality for every $n \geq \tau(\omega)$. □

LEMMA 4.7. *For every $m \geq 1$, there exists a constant $D_m > 0$ such that for any proper rectangle $\omega$ intersecting $G_m$ and $\tau(\omega) > m$,*

$$\frac{\|Df^{\tau(\omega)}|E^u_y\|}{\|Df^{\tau(\omega)}|E^u_x\|} \leq D_m \quad \text{for all } x, y \in \Omega \cap \omega.$$

*Proof.* Let $x, y \in \Omega \cap \omega$. By the last remark on long unstable leaves in §3.1, $\Omega \cap \omega \subset \mathcal{W}^u$. Take a stable side of $\omega$ and denote it by $\partial^s\omega$. Take $x' \in \partial^s\omega$ (respectively $y' \in \partial^s\omega$) such that $x$ and $x'$ (respectively $y$ and $y'$) lie on the same long unstable leaf. The chain rule gives

$$\frac{\|Df^{\tau(\omega)}|E^u_y\|}{\|Df^{\tau(\omega)}|E^u_x\|} = \frac{\|Df^{\tau(\omega)}|E^u_{x'}\|}{\|Df^{\tau(\omega)}|E^u_x\|} \cdot \frac{\|Df^{\tau(\omega)}|E^u_{y'}\|}{\|Df^{\tau(\omega)}|E^u_{x'}\|} \cdot \frac{\|Df^{\tau(\omega)}|E^u_y\|}{\|Df^{\tau(\omega)}|E^u_{y'}\|}.$$

Lemma 4.1(b) bounds the first and the third factors. For the second one, by Lemma 4.6 there exists $k \in [0, m]$ such that $f^k\partial^s\omega$ is contained in a long stable leaf. Then

$$\frac{\|Df^{\tau(\omega)}|E^u_{y'}\|}{\|Df^{\tau(\omega)}|E^u_{x'}\|} \leq \frac{\|Df^k|E^u_{y'}\|}{\|Df^k|E^u_{x'}\|} + \frac{\|Df^{\tau(\omega)-k}|E^u_{f^k y'}\|}{\|Df^{\tau(\omega)-k}|E^u_{f^k x'}\|}.$$

The first term of the right-hand side is bounded by a uniform constant which depends only on $m$ and $f$. The second one is bounded by Lemma 3.5(a). □

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

4.4. *Symbolic dynamics.* Let $\{\omega_1, \omega_2, \ldots, \omega_\ell\}$ be a finite collection of proper rectangles contained in $\Theta$. We assume that any two elements are either disjoint or intersect each other only at their stable sides. Endow $\Sigma_\ell = \{1, \ldots, \ell\}^{\mathbb{Z}}$ with the product topology of the discrete topology, and let $\sigma : \Sigma_\ell \circlearrowleft$ denote the left shift. Define a coding map $\pi : \Sigma_\ell \to \mathbb{R}^2$ by $\pi(\underline{a}) = z$, where $\underline{a} = \{a_i\}_{i\in\mathbb{Z}}$,

$$\{z\} = \left(\bigcap_{k=1}^{\infty} \omega_k^s\right) \cap \left(\bigcap_{k=1}^{\infty} \omega_k^u\right)$$

and

$$\omega_k^s = \omega_{a_0} \cap \left(\bigcap_{i=1}^{k} f^{-\tau(\omega_{a_0})} \circ \cdots \circ f^{-\tau(\omega_{a_{i-1}})} \omega_{a_i}\right)$$

and

$$\omega_k^u = \bigcap_{i=1}^{k} f^{\tau(\omega_{a_{-1}})} \circ \cdots \circ f^{\tau(\omega_{a_{-i}})} \omega_{a_{-i}}.$$

LEMMA 4.8. *The map $\pi$ is well defined, continuous, at most two-to-one and satisfies $\pi(\Sigma_\ell) \subset \Omega$. It is injective if $\omega_1, \ldots, \omega_\ell$ are contained in the interior of $\Theta$.*

*Proof.* To show that $\pi$ is well defined, it suffices to show that both $\omega_k^s$ and $\omega_k^u$ get thinner as $k$ increases, and converge to curves intersecting each other exactly at one point. However, this is not straightforward because the images of the rectangles can come close to the point $\zeta_0$ of tangency. We argue as follows.

Since $\ell$ is finite, $\omega_1, \omega_2, \ldots, \omega_\ell$ do not accumulate the parabola $R \cap f^{-1}\alpha_0^+$ containing the point $\zeta_0$ of tangency. By Lemma 4.6, there exists $k_0 \geq 1$ such that for each $k \geq k_0$, the stable sides of $F\omega_k^s$ are contained in long stable leaves, where $F = f^{\tau(\omega_{a_0})+\cdots+\tau(\omega_{a_{k_0}})+1}$. By the exponential decrease of the lengths of the unstable sides of this rectangle in $k$, and by Lemma 3.5(b), these long stable leaves converge as $k \to \infty$ to a single long stable leaf, denoted by $\gamma^s$. It follows that $\bigcap_{k=1}^{\infty} \omega_k^s$ is a curve contained in $F^{-1}\gamma^s$, joining the two unstable sides of $R$.

The unstable sides of $\omega_k^u$ belong to $\tilde{\Gamma}^u$. By [**24**, Lemma 2.2], the Hausdorff distance between them decreases exponentially in $k$. This implies that $\bigcap_{k=1}^{\infty} \omega_k^u \in \Gamma^u$. Hence, $(\bigcap_{k=1}^{\infty} \omega_k^s) \cap (\bigcap_{k=1}^{\infty} \omega_k^u) \neq \emptyset$ holds.

We have

$$F\left(\left(\bigcap_{k=1}^{\infty} \omega_k^s\right) \cap \left(\bigcap_{k=1}^{\infty} \omega_k^u\right)\right) \subset F\left(\bigcap_{k=1}^{\infty} \omega_k^s\right) \cap F\left(\omega_{k_0}^s \cap \bigcap_{k=1}^{\infty} \omega_k^u\right).$$

The first set of the right-hand side is a subset of $\gamma^s$ and the second is in $\Gamma^u$. Hence, the set of the left-hand side is a singleton. Since $F$ is a diffeomorphism, $(\bigcap_{k=1}^{\infty} \omega_k^s) \cap (\bigcap_{k=1}^{\infty} \omega_k^u)$ is a singleton.

Since all points outside of $R$ diverge to infinity under positive or negative iteration, we have $z \in \bigcap_{n\in\mathbb{Z}} f^n R$, and so $z \in \Omega$ from the first property of the rectangle $R$ in §2.2. In addition, the above argument shows the continuity of $\pi$. To show the injectivity, assume that $\underline{a}, \underline{b} \in \Sigma_\ell$, $\underline{a} \neq \underline{b}$ and $\pi(\underline{a}) = \pi(\underline{b})$. Then some iterate of $\pi(\underline{a})$ is contained in the stable side of two neighboring elements of $\{\omega_1, \omega_2, \ldots, \omega_\ell\}$. Hence, the orbit of $\pi(\underline{a})$ is not contained in the interior of $\Theta$, which is a contradiction. □

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

5. *Smooth ergodic theory at the first bifurcation*

In this section, using the inducing scheme in §4, we develop a smooth ergodic theory for $f$. Main results here are the following: a version of Katok's horseshoe theorem (Lemma 5.1); a Moran-like geometric construction (Lemma 5.2); an approximation by ergodic measures (Lemma 5.3).

5.1. *Approximation of ergodic measures with horseshoes.* Katok established the remarkable result that every hyperbolic measures of diffeomorphisms can be in a particular sense approximated by uniformly hyperbolic horseshoes (see [**16**, Theorem S.5.9] for the precise statement). We will need a version of this. Let $\mathcal{M}^e(f)$ denote the set of $f$-invariant ergodic Borel probability measures.

LEMMA 5.1. *Let $\mu \in \mathcal{M}^e(f)$ with $h(\mu) > 0$. For any $\varepsilon > 0$, there exist an integer $q \geq 1$ and a finite collection $\mathcal{R}$ of proper rectangles such that:*

(a) *for each $\omega \in \mathcal{R}$, $\tau(\omega) = q$;*

(b) $|(1/q) \log \#\mathcal{R} - h(\mu)| < \varepsilon$;

(c) *for any $z \in \bigcup_{\omega\in\mathcal{R}} \mathcal{W}^u \cap \omega$, $|(1/q) \sum_{i=0}^{q-1} \log J^u(f^i z) - \lambda^u(\mu)| < \varepsilon$.*

*Proof.* By [**16**, Theorem S.5.9], for any $\varepsilon \in (0, 2h(\mu))$, there exists $\nu \in \mathcal{M}^e(f)$ which is supported on a hyperbolic set and satisfies $|h(\mu) - h(\nu)| < \varepsilon/2$, $|\lambda^u(\mu) - \lambda^u(\nu)| < \varepsilon/3$. We have $\nu(\Theta) > 0$, for otherwise $\nu$ is the Dirac measure at $Q$, in contradiction to $h(\nu) > h(\mu) - |h(\mu) - h(\nu)| > \varepsilon/2 - \varepsilon/2 = 0$.

Let $\omega_{\mathrm{S}}$ (respectively $\omega_{\mathrm{R}}$) denote the connected component of $R\backslash\Theta$ at the left (respectively right) of $\zeta_0$, and define

$$\xi = \{\omega \in \mathcal{P}_1 : \nu(\omega) > 0\} \cup \{\omega_{\mathrm{S}}, \omega_{\mathrm{R}}\}.$$

Since $\nu$ is supported on a hyperbolic set, $\#\xi$ is finite. We claim that $\xi$ is a generating partition with respect to $\nu$. Indeed, by [**24**, Lemma 3.1], there is a continuous surjection $\iota$ from $\Sigma_2 = \{1, 2\}^{\mathbb{Z}}$ to $\Omega$ which gives a symbolic coding of points in $\Omega$. Since the coding is given by the maximal two rectangles intersecting only at $\zeta_0$, for any cylinder set $A$ in $\Sigma_2$, $\iota(A) \cap \bigcup\{\omega : \omega \in \xi\}$ belongs to the sigma-algebra generated by $\bigvee_{i=-\infty}^{+\infty} f^{-i}\xi$. Since cylinder sets form a base of the topology of $\Sigma_2$, the claim holds.

For $m \geq 1$, let $\Lambda_m$ denote the set of all $z \in \Theta$ for which the following holds:

- $|(1/n) \log \mu(\xi_n(z)) + h(\nu)| < \varepsilon/3$ for every $n \geq m$, where $\xi_n(z)$ denotes the element of the partition $\bigvee_{i=0}^{n-1} f^{-i}\xi$ containing $z$;
- $|(1/n) \sum_{i=0}^{n-1} \log J^u(f^i z) - \lambda^u(\nu)| \leq \varepsilon/3$ for every $n \geq m$;
- $z \in G_m$.

By the Shannon–McMillan–Breiman theorem, the ergodic theorem and Corollary 4.4, $\nu(\Lambda_m) \to \nu(\Theta) > 0$ as $m \to \infty$. Let

$$\Lambda_{m,p} = \{z \in \Lambda_m : f^q z \in \Theta \text{ for some } q \in [p, 2p]\}.$$

Denote by $\chi_\Theta$ the characteristic function of $\Theta$. Set

$$B_p = \left\{ z \in \Lambda_m : \frac{1}{p} \sum_{i=0}^{p-1} \chi_\Theta(f^i z) < \frac{5}{4}\nu(\Theta) \text{ and } \frac{1}{2p} \sum_{i=0}^{2p-1} \chi_\Theta(f^i z) > \frac{5}{8}\nu(\Theta) \right\}.$$

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

From the ergodic theorem, $\nu(B_p) \to \nu(\Lambda_m)$ as $p \to \infty$. Since $B_p \subset \Lambda_{m,p}$, it follows that $\nu(\Lambda_{m,p}) \to \nu(\Lambda_m)$ as $p \to \infty$. Choose $m \geq 1$ such that $\nu(\Lambda_m) \geq (1/2)\nu(\Theta)$, and then choose $p \geq m$ such that $\nu(\Lambda_{m,p}) \geq (1/3)\nu(\Theta)$, $-(1/p)\log(6p) + (1/p)\log \nu(\Theta) > -\varepsilon/6$ and $D_m/p < \varepsilon/3$, where $D_m$ is the constant in Lemma 4.7. For each $q \in [p, 2p]$, set

$$\Lambda_{m,p,q} = \{z \in \Lambda_{m,p} : \min\{n \in [p, 2p] : f^n z \in \Theta\} = q\}.$$

Choose $q$ such that $\nu(\Lambda_{m,p,q}) \geq (1/2p)\nu(\Lambda_{m,p})$. Define $\mathcal{R}$ to be the collection of proper rectangles intersecting $\Lambda_{m,p,q}$ with inducing time $q$. Item (a) is immediate from the construction. Note that elements of $\mathcal{R}$ are mutually disjoint, altogether cover $\Lambda_{m,p,q}$ and belong to $\bigvee_{i=0}^{q-1} f^{-i}\xi$. For each $\omega \in \mathcal{R}$, we have $\nu(\omega) \leq e^{-q(h(\nu)-\varepsilon/3)}$, and therefore

$$\#\mathcal{R} \geq \nu(\Lambda_{m,p,q})e^{q((h(\nu)-\varepsilon/3)} \geq \frac{1}{6p}\nu(\Theta)e^{q((h(\nu)-\varepsilon/3)}.$$

This yields

$$\frac{1}{q}\log \#\mathcal{R} \geq -\frac{1}{q}\log(6p) + \frac{1}{q}\log \nu(\Theta) + h(\nu) - \frac{\varepsilon}{3} > h(\nu) - \frac{\varepsilon}{2} > h(\mu) - \varepsilon.$$

Similarly, we obtain $(1/q)\log \#\mathcal{R} < h(\nu) + \varepsilon/3$, proving (b).

Regarding (c), for each $\omega \in \mathcal{R}$, choose $z_0 \in \omega \cap \Lambda_{m,p,q}$ such that $|(1/q)\sum_{i=0}^{q-1} \log J^u(f^i z_0) - \lambda^u(\nu)| < \varepsilon/3$. Using $z_0 \in G_m$ and Lemma 4.7, for all $z \in \mathcal{W}^u \cap \omega$, we have

$$\begin{aligned}\left|\frac{1}{q}\sum_{i=0}^{q-1}\log J^u(f^i z) - \lambda^u(\mu)\right| &\leq \left|\frac{1}{q}\sum_{i=0}^{q-1}\log J^u(f^i z) - \frac{1}{q}\sum_{i=0}^{q-1}\log J^u(f^i z_0)\right| \\ &\quad + \left|\frac{1}{q}\sum_{i=0}^{q-1}\log J^u(f^i z_0) - \lambda^u(\nu)\right| + |\lambda^u(\nu) - \lambda^u(\mu)| \\ &\leq \frac{\log D_m}{q} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \frac{\log D_m}{p} + \frac{2\varepsilon}{3} < \varepsilon.\end{aligned}$$

This completes the proof of Lemma 5.1. □

5.2. *Moran-like geometric construction.* In this subsection we adapt the construction of Moran-like fractals to our setting (see, e.g., [**20**, **21**, **31**] for this terminology). Let $\gamma^u(\zeta_0)$ denote the unstable side of $\Theta$ containing $\zeta_0$.

LEMMA 5.2. *Let $\beta \in I$, and let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{M}^e(f)$ such that $h(\mu_n) > 0$ and $\lambda^u(\mu_n) \to \beta$ as $n \to \infty$. There exists a closed subset $\Delta$ of $\gamma^u(\zeta_0) \cap \Omega^u(\beta)$ such that*

$$\dim_H^u(\Delta) \geq \limsup_{n\to\infty} \frac{h(\mu_n)}{\lambda^u(\mu_n)}.$$

*Proof.* An idea of a proof of this lemma is to approximate each $\mu_n$ with a horseshoe in the sense of Lemma 5.1, and then construct a set of points which wander around these horseshoes for longer and longer times as $n$ increases, in such a way that their unstable Lyapunov exponents converge to $\beta$. The proof is done in three steps. In Step 1 we construct the set $\Delta$. In Step 2 we show that $\Delta$ is indeed a subset of $\Omega^u(\beta)$. In Step 3 we estimate the Hausdorff dimension of $\Delta$ and prove the desired inequality.

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

*Step 1 (construction of $\Delta$).* Taking a subsequence if necessary, we may assume that $|\lambda^u(\mu_n)-\beta|<1/n$ for every $n$ and $h(\mu_n)/\lambda^u(\mu_n)$ converges as $n\to\infty$. In view of Lemma 5.1, for each $n$, fix an integer $q_n>0$ and a family $\mathcal{R}_n$ of proper rectangles with inducing time $q_n$ and with the following properties:

$$\left|\frac{1}{q_n}\log \#\mathcal{R}_n - h(\mu_n)\right| < \frac{1}{n}; \tag{5}$$

$$\text{for any } x\in\bigcup_{\omega\in\mathcal{R}_n}\mathcal{W}^u\cap\omega,\ \left|\frac{1}{q_n}\sum_{j=0}^{q_n-1}\log J^u(f^jx)-\lambda^u(\mu_n)\right|<\frac{1}{n}. \tag{6}$$

Let $\mathcal{R}_n(1)=\mathcal{R}_n$ and, for an integer $\kappa>1$, let

$$\mathcal{R}_n(\kappa)=\{\omega_0\cap f^{-q_n}\omega_1\cap\cdots\cap f^{-(\kappa-1)q_n}\omega_\kappa : \omega_0,\omega_1,\dots,\omega_{\kappa-1}\in\mathcal{R}_n\}.$$

Note that elements of $\mathcal{R}_n(\kappa)$ are proper rectangles with inducing time $\kappa q_n$, and $\#\mathcal{R}_n(\kappa)=(\#\mathcal{R}_n)^\kappa$. Given a sequence $\{\kappa_n\}_{n=1}^\infty$ of positive integers, for each integer $k\ge\kappa_1$, write

$$k=\kappa_1+\kappa_2+\cdots+\kappa_{N-1}+s,\quad 0\le s<\kappa_N.$$

For an $N$-tuple $(\omega_1\cdots\omega_{N-1}\omega_N)$ with $\omega_n\in\mathcal{R}_n(\kappa_n)$, $n\in\{1,2,\dots,N-1\}$ and $\omega_N\in\mathcal{R}_N(s)$, define

$$\widehat{\Delta}_{\omega_1\omega_2\cdots\omega_{N-1}\omega_N}=\omega_1\cap f^{-\kappa_1q_1}\omega_2\cap f^{-\kappa_1q_1-\kappa_2q_2}\omega_3\cap\cdots\cap f^{-\kappa_1q_1-\cdots-\kappa_{N-1}q_{N-1}}\omega_N.$$

This is a proper rectangle with inducing time $\kappa_1q_1+\cdots+\kappa_{N-1}q_{N-1}+sq_N$. Set $\Delta_{\omega_1\omega_2\cdots\omega_{N-1}\omega_N}=\gamma^u(\zeta_0)\cap\widehat{\Delta}_{\omega_1\omega_2\cdots\omega_{N-1}\omega_N}$. These sets are mutually disjoint compact curves. Define

$$\Delta=\bigcap_{k=1}^\infty\bigcup_{(\omega_1\cdots\omega_N)}\Delta_{\omega_1\cdots\omega_N}.$$

*Step 2 (proof of $\Delta\subset\Omega^u(\beta)$).* Let $z\in\Delta$. For each large integer $M\ge\kappa_1q_1$, choose a pair $(N,s)$ of integers such that $0\le s<\kappa_N$ and $0\le M-(\kappa_1q_1+\cdots+\kappa_{N-1}q_{N-1}+sq_N)<q_N$. The triangle inequality gives

$$\left|\sum_{j=0}^{M-1}\log J^u(f^jx)-M\beta\right|\le I+II+III+IV,$$

where

$$\begin{aligned}
I&:=\left|\sum_{i=0}^{\kappa_1q_1-1}\log J^u(f^iz)-\kappa_1q_1\beta\right|\\
&\le\sum_{j=0}^{\kappa_1-1}\left|\sum_{l=0}^{q_1-1}\log J^u(f^{q_1j+l}z)-q_1\beta\right|;\\
II&:=\left|\sum_{i=\kappa_1q_1}^{\kappa_1q_1+\cdots+\kappa_{N-1}q_{N-1}-1}\log J^u(f^iz)-(\kappa_2q_2+\cdots+\kappa_{N-1}q_{N-1})\beta\right|\\
&\le\sum_{n=2}^{N-1}\sum_{j=0}^{\kappa_n-1}\left|\sum_{l=0}^{q_n-1}\log J^u(f^{\kappa_1q_1+\cdots+\kappa_{n-1}q_{n-1}+jq_n+l}z)-q_n\beta\right|;
\end{aligned}$$

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

$$III := \left| \sum_{i=\kappa_1 q_1+\cdots+\kappa_{N-1}q_{N-1}}^{\kappa_1 q_1+\cdots+\kappa_{N-1}q_{N-1}+sq_N-1} \log J^u(f^i z) - sq_N\beta \right|$$
$$\leq \sum_{j=0}^{s-1} \left| \sum_{l=0}^{q_N-1} \log J^u(f^{\kappa_1 q_1+\cdots+\kappa_{N-1}q_{N-1}+jq_N+l} z) - q_N\beta \right|;$$

$$IV := \left| \sum_{i=\kappa_1 q_1+\cdots+\kappa_{N-1}q_{N-1}+sq_N}^{M-1} \log J^u(f^i z) - (M-(\kappa_1 q_1+\cdots+\kappa_{N-1}q_{N-1}+sq_N))\beta \right|$$
$$\leq \Big(\sup_{x\in\Omega} |\log J^u(x)| + \beta\Big) q_N.$$

Using (6) and $|\lambda^u(\mu_n) - \beta| < 1/n$, we have

$$\left| \sum_{l=\kappa_1 q_1+\cdots+\kappa_{n-1}q_{n-1}+jq_n}^{\kappa_1 q_1+\cdots+\kappa_{n-1}q_{n-1}+(j+1)q_n-1} \log J^u(f^l z) - q_n\beta \right|$$
$$\leq \left| \sum_{l=\kappa_1 q_1+\cdots+\kappa_{n-1}q_{n-1}+jq_n}^{\kappa_1 q_1+\cdots+\kappa_{n-1}q_{n-1}+(j+1)q_n-1} \log J^u(f^i z) - q_n\lambda^u(\mu_n) \right|$$
$$+ |q_n\lambda^u(\mu_n) - q_n\beta|$$
$$\leq \frac{2q_n}{n}$$

and similarly

$$\left| \sum_{l=0}^{q_1-1} \log J^u(f^{jq_1+l} z) - q_1\beta \right| \leq 2q_1.$$

Summing these and other reminder terms, we get

$$\left| \sum_{j=0}^{M-1} \log J^u(f^j z) - M\beta \right| \leq \sum_{n=1}^{N-1} \frac{2q_n\kappa_n}{n} + \frac{2q_N s}{N} + \Big(\sup_{x\in\Omega} |\log J^u(x)| + \beta\Big) q_N$$
$$\leq \frac{3q_{N-1}\kappa_{N-1}}{N} + \frac{2q_N s}{N}$$
$$\leq \frac{3M}{N},$$

where the second inequality holds provided the sequence $\{\kappa_n\}_{n=1}^{\infty}$ is chosen so that $\kappa_n$ is sufficiently large compared to $q_1, q_2, \ldots, q_{n+1}$ and $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$. Since $N \to \infty$ as $M \to \infty$, we obtain $\lambda^u(z) = \beta$.

*Step 3 (proof of the desired inequality).* For each compact curve $\Delta_{\omega_1\cdots\omega_N}$ at the $k$th level of $\Delta$, choose a point $x_{\omega_1\cdots\omega_N} \in \Delta_{\omega_1\cdots\omega_N} \cap \Delta$ which is not an end point of $\Delta_{\omega_1\cdots\omega_N}$. Define an atomic probability measure $\nu_k$ evenly distributed on the set $\bigcup_{(\omega_1\cdots\omega_N)} \{x_{\omega_1\cdots\omega_N}\}$. Let $\nu$ denote an accumulation point of the sequence $\{\nu_k\}_k$. Since $\Delta$ is compact, $\nu(\Delta) = 1$. For $\varepsilon > 0$ and $z \in W^u$, let $D_\varepsilon(z)$ denote the closed ball in $W^u$ of radius $\varepsilon$ about $z$. By virtue of the local property of Hausdorff dimension [**36**, Proposition 2.1], the desired inequality holds if

$$\liminf_{\varepsilon\to 0} \frac{\log \nu D_\varepsilon(z)}{\log \varepsilon} \geq \limsup_{n\to\infty} \frac{h(\mu_n)}{\lambda^u(\mu_n)} \quad \text{for every } z \in \Delta. \tag{7}$$

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

For each integer $k \geq \kappa_1$, write $k = \kappa_1 + \kappa_2 + \cdots + \kappa_{N-1} + s$, $0 \leq s < \kappa_N$ and define

$$a_k = \exp\left[-\kappa_{N-1} q_{N-1}\left(\lambda^u(\mu_{N-1}) + \frac{2}{N-1}\right) - s q_N\left(\lambda^u(\mu_N) + \frac{1}{N}\right)\right].$$

From the uniform boundedness, $\inf_n \lambda^u(\mu_n) \geq \lambda^u_m > 0$, $a_k \to 0$ as $k \to \infty$ provided $\kappa_n \to \infty$ as $n \to \infty$. In addition, we claim that $\{a_k\}$ is monotone decreasing provided $\{\kappa_n\}$ is chosen so that $\kappa_n$ is sufficiently large compared with $q_1, q_2, \ldots, q_{n+1}$ and $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$. Indeed, if $k = \kappa_1 + \kappa_2 + \cdots + \kappa_{N-1} + \kappa_N - 1$, then

$$a_k = \exp\left[-\kappa_{N-1} q_{N-1}\left(\lambda^u(\mu_{N-1}) + \frac{2}{N-1}\right) - (\kappa_{N-1} - 1) q_N\left(\lambda^u(\mu_N) + \frac{1}{N}\right)\right]$$

and

$$a_{k+1} = \exp\left(-\kappa_N q_N\left(\lambda^u(\mu_N) + \frac{2}{N}\right)\right).$$

Hence, the claim holds.

Let $\varepsilon \in (0, a_1)$. Choose an integer $k \geq 1$ with $a_{k+1} < \varepsilon \leq a_k$. Write $k = \kappa_1 + \kappa_2 + \cdots + \kappa_{N-1} + s$, $0 \leq s < \kappa_N$ as before. From (6), for any $x \in \Delta_{\omega_1 \cdots \omega_N}$, we have

$$\begin{aligned}
&\left|\sum_{j=0}^{\kappa_1 q_1 + \cdots + \kappa_{N-1} q_{N-1} + s q_N - 1} \log J^u(f^j x)\right| \\
&\leq \sum_{n=1}^{N-1} \kappa_n q_n\left(\lambda^u(\mu_n) + \frac{1}{n}\right) + s q_N\left(\lambda^u(\mu_N) + \frac{1}{N}\right) \\
&\leq \kappa_{N-1} q_{N-1}\left(\lambda^u(\mu_{N-1}) + \frac{2}{N-1}\right) + s q_N\left(\lambda^u(\mu_N) + \frac{1}{N}\right),
\end{aligned}$$

where the last inequality holds provided $\{\kappa_n\}$ is chosen so that $\kappa_n$ is sufficiently large compared to $q_1, q_2, \ldots, q_{n+1}$ and $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$. Since the curve $f^{\kappa_1 q_1 + \cdots + \kappa_{N-1} q_{N-1} + s q_N} \Delta_{\omega_1 \cdots \omega_N}$ belongs to $\tilde{\Gamma}^u$, the mean value theorem gives

$$\mathrm{length}(\Delta_{\omega_1 \cdots \omega_N}) \geq \frac{1}{2} \exp\left[-\kappa_{N-1} q_{N-1}\left(\lambda^u(\mu_{N-1}) + \frac{2}{N-1}\right) - s q_N\left(\lambda^u(\mu_N) + \frac{1}{N}\right)\right].$$

Hence, for any $z \in \Delta$, the number of the compact curves $\Delta_{\omega_1 \cdots \omega_N}$ at the $k$th level of $\Delta$ which intersect $D_\varepsilon(z)$ is at most

$$\frac{2\varepsilon}{\inf_{(\omega_1 \cdots \omega_N)} \mathrm{length}(\Delta_{\omega_1 \cdots \omega_N})} \leq \frac{2a_k}{\inf_{(\omega_1 \cdots \omega_N)} \mathrm{length}(\Delta_{\omega_1 \cdots \omega_N})} \leq 4.$$

By construction, each set at the $k$th level contains a finite number of sets at the $(k+1)$th level, and this number is independent of the sets at the $k$th level. Hence, for every $p \geq k$,

$$\nu_p(\Delta_{\omega_1 \cdots \omega_N}) = \frac{1}{\#\{(\omega_1 \cdots \omega_N)\}}.$$

Since $\nu$ charges no weight to the end points of $\Delta_{\omega_1 \cdots \omega_N}$,

$$\nu(\Delta_{\omega_1 \cdots \omega_N}) = \lim_{p \to \infty} \nu_p(\Delta_{\omega_1 \cdots \omega_N}) = \frac{1}{\#\{(\omega_1 \cdots \omega_N)\}}.$$

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

Using this and (5),

$$\begin{aligned}\nu D_\varepsilon(z) &\le \frac{4}{\#\{(\omega_1 \cdots \omega_N)\}} \le \frac{4}{(\#\mathcal{R}_{N-1})^{\kappa_{N-1}} \cdot (\#\mathcal{R}_N)^s} \\ &\le 4\exp\left[-\kappa_{N-1}q_{N-1}\left(h(\mu_{N-1}) - \frac{1}{N-1}\right) - sq_N\left(h(\mu_N) - \frac{1}{N}\right)\right].\end{aligned}$$

This yields

$$\begin{aligned}\frac{\log \nu D_\varepsilon(z)}{\log \varepsilon} &\ge \frac{\log \nu D_\varepsilon(z)}{\log a_k} \\ &\ge \frac{\kappa_{N-1}q_{N-1}(h(\mu_{N-1}) - 1/(N-1)) + sq_N(h(\mu_N) - 1/N)}{\kappa_{N-1}q_{N-1}(\lambda^u(\mu_{N-1}) + 2/(N-1)) + sq_N(\lambda^u(\mu_N) + 1/N)} + \frac{\log 4}{\log a_k}.\end{aligned}$$

Since $k, N \to \infty$ as $\varepsilon \to 0$, (7) follows. This completes the proof of Lemma 5.2. □

5.3. *Approximation of non-ergodic measures with ergodic ones.* The next lemma asserts that for any non-ergodic measure one can find an ergodic one with similar (positive) entropy and similar unstable Lyapunov exponent.

LEMMA 5.3. *For any $\mu \in \mathcal{M}(f)$ and $\varepsilon > 0$, there exists $\nu \in \mathcal{M}^e(f)$ such that $h(\nu) > 0$, $|h(\mu) - h(\nu)| < \varepsilon$ and $|\lambda^u(\mu) - \lambda^u(\nu)| < \varepsilon$.*

*Proof.* Considering the ergodic decomposition of $\mu$, one can find a linear combination $\mu' = a_1\mu_1 + \cdots + a_s\mu_s$ of ergodic measures such that $|h(\mu) - h(\mu')| < \varepsilon/4$ and $|\lambda^u(\mu) - \lambda^u(\mu')| < \varepsilon/4$. We claim that for each $\mu_i$, there exists $\nu_i \in \mathcal{M}^e(f)$ such that $h(\nu_i) > 0$, $|h(\mu_i) - h(\nu_i)| < \varepsilon/4$ and $|\lambda^u(\mu_i) - \lambda^u(\nu_i)| < \varepsilon/4$. Set $\nu = a_1\nu_1 + \cdots + a_s\nu_s$. Then $h(\nu) > 0$, $|h(\mu') - h(\nu)| < \varepsilon/4$ and $|\lambda^u(\mu') - \lambda^u(\nu)| < \varepsilon/4$. Hence, $|h(\mu) - h(\nu)| < \varepsilon/2$ and $|\lambda^u(\mu) - \lambda^u(\nu)| < \varepsilon/2$. The claim will be shown at the end of the proof.

Since $f \mid \Omega$ is a factor of the full shift on two symbols [**25**, Lemma 3.1], it has the specification property [**26**, Lemma 1(b)]. Hence, ergodic measures are entropy-dense [**10**]: there exists a sequence $\{\xi_n\}_n$ in $\mathcal{M}^e(f)$ such that $\xi_n \to \nu$ and $h(\xi_n) \to h(\nu)$ as $n \to \infty$. By [**24**, Lemma 4.4] and $\nu\{Q\} = 0$, we obtain $\lambda^u(\xi_n) \to \lambda^u(\nu)$.

It is left to prove the claim. We may suppose that $h(\mu_i) = 0$, for otherwise there is nothing to prove. By Katok's closing lemma [**15**, Main Lemma], there exist a periodic point $p$ and an atomic measure $\mu_{\mathcal{O}(p)}$ supported on the orbit $\mathcal{O}(p)$ of $p$ such that $|\lambda^u(\mu_i) - \lambda^u(\mu_{\mathcal{O}(p)})| < \varepsilon/8$. Since there is a transverse homoclinic point associated to $p$, from the Poincaré–Birkhoff–Smale theorem (see, e.g., [**16**, Theorem 6.5.5]), there exists a non-trivial basic set of $f$ containing $p$ and the transverse homoclinic point. The isolating neighborhood of the basic set is the saturation of a thin strip around the stable manifold of $p$. Taking a sufficiently thin strip, one can make sure that the measure of maximal entropy of $f$ restricted to the basic set, denoted by $\nu_i$, satisfies $0 < h(\nu_i) < \varepsilon/4$ and $|\lambda^u(\mu_{\mathcal{O}(p)}) - \lambda^u(\nu_i)| < \varepsilon/8$. □

*Remark 5.4.* It is possible to give a self-contained proof of Lemma 5.3 by slightly extending the argument in the proof of Lemma 5.2.

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

## 6. *Proofs of the theorems*

In this section we bring the results in the previous sections together and prove the theorems. In §6.1 we prove Theorem A. In §6.2 we complete the proof of Theorem B. In §6.3 we prove Theorem C. In §6.4 we prove Theorem D.

### 6.1. *Domain of the Lyapunov spectrum.* We now prove Theorem A.

*Proof of Theorem A.* Let $\beta \in I$. By Lemma 5.3, there exists a sequence $\{\nu_n\}_n$ in $\mathcal{M}^e(f)$ with $\lambda^u(\nu_n) \to \beta$ as $n \to \infty$. Lemma 5.2 yields $\Omega^u(\beta) \neq \emptyset$, proving the 'if' part of Theorem A.

For a proof of the 'only if' part in Theorem A, we need a couple of lemmas.

LEMMA 6.1. *If* $z \in \bigcup_{m=1}^{\infty} G_m \setminus W^s(Q)$, *then* $\bar{\lambda}^u(z) \geq \lambda^u_m$.

*Proof.* Let $z \in G_m$. Since $f^n z \in \Theta$ holds for infinitely many $n > 0$, there exists an infinite nested sequence $\omega_0 \supset \omega_1 \supset \cdots$ of proper rectangles containing $z$. From Lemma 4.8, each $\omega_n$ contains a periodic point of period $\tau(\omega_n)$, denoted by $q_n$. Since $\omega_n \cap G_m \neq \emptyset$, Lemma 4.7 gives

$$\left| \frac{1}{\tau(\omega_n)} \sum_{i=0}^{\tau(\omega_n)-1} \log J^u(f^i q_n) - \log J^u(f^i z) \right| \leq \frac{\log D_m}{\tau(\omega_n)}.$$

Since $\tau(\omega_n) \to \infty$ as $n \to \infty$, the desired inequality follows. □

The next upper semi-continuity result follows from a slight modification the proof of [**24**, Lemma 4.3] in which a convergent sequence of $f$-invariant measures was treated. For $z \in \Omega$ and an integer $n \geq 1$, write $\delta_z^n = (1/n) \sum_{i=0}^{n-1} \delta_{f^i z}$, where $\delta_{f^i z}$ denotes the Dirac measure at $f^i z$.

LEMMA 6.2. *Let* $z \in \Omega$ *and* $\{n_k\}_k$, $n_k \nearrow \infty$ *be such that* $\delta_z^{n_k}$ *converges weakly to* $\mu \in \mathcal{M}(f)$. *Then*

$$\limsup_{k \to \infty} \int \log J^u d\delta_z^{n_k} \leq \lambda^u(\mu).$$

*Proof.* If $z \in W^s(Q)$, then $\delta_z^{n_k} \to \delta_Q$ and $\int \log J^u d\delta_z^{n_k} \to \lambda^u(\delta_Q)$ as $k \to \infty$, and so the desired inequality holds. Assume that $z \notin W^s(Q)$. Write $\mu = u\delta_Q + (1-u)\nu$, $0 \leq u \leq 1$, $\nu \in \mathcal{M}(f)$ and $\nu\{Q\} = 0$. Let $\varepsilon > 0$. Let $V$ be a small open set containing $Q$, $\mu(\partial V) = 0$ and $\mu(V) \leq u + \varepsilon$. Fix a partition of unity $\{\rho_0, \rho_1\}$ on $R$ such that $\operatorname{supp}(\rho_0) = \overline{\{x \in R : \rho_0(x) \neq 0\}} \subset V$ and $Q \notin \operatorname{supp}(\rho_1)$. Then

$$\lim_{k \to \infty} \frac{1}{n_k} \#\{i \in \{0, 1, \ldots, n_k - 1\} : f^i z \in V\} = \lim_{k \to \infty} \delta_z^{n_k}(V) = \mu(V) \leq u + \varepsilon.$$

Since $z \notin W^s(Q)$, the forward orbit of $z$ is a concatenation of segments in $V$ and those out of $V$. Let $l_k$ denote the number of segments in $V$ up to time $n_k$. If $0 \leq i_1 < i_2$ are integers

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

such that $f^{i_1}z \notin V$, $f^i z \in V$ for every $i \in \{i_1+1, i_1+2, \ldots, i_2-1\}$ and $f^{i_2}z \notin V$, then $\|Df^{i_2-i_1}|E^u_{f^{i_1}z}\| \le Ce^{\lambda^u(\delta_Q)(i_2-i_1)}$. Then

$$\int \rho_0 \log J^u \, d\delta_z^{n_k} = \frac{1}{n_k}\sum_{i=0}^{n_k-1}(\rho_0 \log J^u)\circ f^i(z) \le (u+2\varepsilon)\lambda^u(\delta_Q) + C\frac{l_k}{n_k}.$$

If $u<1$, then the weak convergence for the sequence $\{(\delta_z^{n_k} - u\delta_Q)/(1-u)\}_k$ of measures implies that

$$\lim_{n\to\infty}\int \rho_1 \log J^u \, d\delta_z^{n_k} = (1-u)\int \rho_1 \log J^u \, d\nu \le (1-u)\lambda^u(\nu).$$

The same inequality remains to hold for the case $u=1$. Hence, we have

$$\begin{aligned}\limsup_{k\to\infty}\int \log J^u \, d\delta_z^{n_k} &\le \limsup_{k\to\infty}\int \rho_0 \log J^u \, d\delta_z^{n_k} + \lim_{k\to\infty}\int \rho_1 \log J^u \, d\delta_z^{n_k}\\ &\le (u+2\varepsilon)\lambda^u(\delta_Q) + C\cdot\limsup_{k\to\infty}\frac{l_k}{n_k} + (1-u)\lambda^u(\nu).\end{aligned}$$

The second term can be made arbitrarily small by shrinking $V$. Then letting $\varepsilon \to 0$ yields the desired inequality. □

To finish, recall that $\hat{\Omega}^u = \{z \in \Omega^u : \underline{\lambda}^u(z) \ne \bar{\lambda}^u(z)\}$. Let $z \in \Omega^u \setminus \hat{\Omega}^u$ and suppose that $\lambda^u(z) \ne \lambda^u(\zeta_0)$. Lemma 4.3 gives $z \in \bigcup_{m=1}^{\infty} G_m$. If $z \in W^s(Q)$, then $\lambda^u(z) = \lambda^u(Q) \in I$. Otherwise, Lemma 6.1 gives $\lambda^u(z) \ge \lambda^u_m$. Since $\Omega$ is compact, there is a subsequence $\{n_k\}_k$, $n_k \nearrow \infty$ such that $\delta_z^{n_k} \to \mu \in \mathcal{M}(f)$. Lemma 6.2 gives $\lambda^u(z) \le \lambda^u(\mu) \le \lambda^u_M$. Hence, $\lambda^u(z) \in I$. This completes the proof of Theorem A. □

6.2. *Formula for the Lyapunov spectrum.* We now prove Theorem B.

*Proof of Theorem B.* Let $\beta \in I$. Lemma 5.2 implies that

$$L^u(\beta) \ge \lim_{\varepsilon\to 0}\sup\left\{\frac{h(\mu)}{\lambda^u(\mu)} : \mu \in \mathcal{M}^e(f),\ |\lambda^u(\mu)-\beta| < \varepsilon\right\}.$$

We claim that the number in the right-hand side does not change even if $\mathcal{M}^e(f)$ is replaced by $\mathcal{M}(f)$. To see this, let $\varepsilon > 0$. Let $\mu \in \mathcal{M}(f)$ be non-ergodic with $h(\mu) > 0$. By Lemma 5.3, there exists $\nu \in \mathcal{M}^e(f)$ such that $|h(\mu)-h(\nu)| < \varepsilon$ and $|\lambda^u(\mu)-\lambda^u(\nu)| < \varepsilon$. Since $h(\mu) \le \log 2$ and $\lambda^u(\mu) < \log 5$,

$$\left|\frac{h(\mu)}{\lambda^u(\mu)} - \frac{h(\nu)}{\lambda^u(\nu)}\right| < \frac{(\log 2 + \log 5)\varepsilon}{(\lambda^u_m)^2} < \frac{3\varepsilon}{(\lambda^u_m)^2}.$$

This implies the claim.

PROPOSITION 6.3. *For any $\beta \in I$ and every integer $m \ge 1$,*

$$\dim_H^u(\Omega^u(\beta)\cap G_m) \le \lim_{\varepsilon\to 0}\sup\left\{\frac{h(\mu)}{\lambda^u(\mu)} : \mu \in \mathcal{M}(f),\ |\lambda^u(\mu)-\beta| < \varepsilon\right\}.$$

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

Recall that $\Omega_* = \Omega \setminus \bigcup_{m=1}^{\infty} G_m$. By Lemma 4.3, the unstable Lyapunov exponent is undefined for points in $\Omega_*$. Hence, $\Omega^u(\beta) = \bigcup_{m=1}^{\infty} \Omega^u(\beta) \cap G_m$. From Proposition 6.3 and the countable stability of Hausdorff dimension, we obtain the reverse inequality, the upper estimate of $L^u(\beta)$.

*Proof of Proposition 6.3.* We argue in two steps. In Step 1 we reduce our consideration to a certain subset with the same Hausdorff dimension. In Step 2 we complete the proof.

*Step 1(reduction).* Recall that $\gamma^u(\zeta_0)$ is the unstable side of $\Theta$ containing $\zeta_0$. Set

$$\tilde{\Omega}^u(\beta) = \{z \in \Omega^u(\beta) \cap \gamma^u(\zeta_0) : f^n z \in \Theta \text{ for infinitely many integers } n > 0\}.$$

Since $\gamma^u(\zeta_0)$ contains a fundamental domain in $W^u$, for any $z \in \Omega^u(\beta)$ which is not the fixed point in $W^u$, there exists an integer $n$ such that $f^n z \in \gamma^u(\zeta_0)$. From the countable stability and the $f$-invariance of $\dim_H^u$, we have $\dim_H^u(\Omega^u(\beta) \cap G_m) = \dim_H^u(\Omega^u(\beta) \cap \gamma^u(\zeta_0) \cap G_m)$. Points in $\Omega^u(\beta) \cap \gamma^u(\zeta_0)$ which return to $\Theta$ under forward iteration only finitely many times form a countable subset, and so $\dim_H^u(\Omega^u(\beta) \cap \gamma^u(\zeta_0) \cap G_m) = \dim_H^u(\tilde{\Omega}^u(\beta) \cap G_m)$. Therefore,

$$\dim_H^u(\Omega^u(\beta) \cap G_m) = \dim_H^u(\tilde{\Omega}^u(\beta) \cap G_m).$$

We will estimate the right-hand side from above. For $\eta > 0$, let $D_\eta(\zeta_0)$ denote the closed ball in $W^u$ of radius $\eta$ about $\zeta_0$. For each $n \geq 1$ and $\varepsilon > 0$, consider the following finite collection of proper rectangles:

$$\left\{\omega \in \mathcal{P}_n : \omega \cap G_m \neq \emptyset,\ \omega \cap D_\eta(\zeta_0) = \emptyset,\ \inf_{z \in \omega \cap \gamma^u(\zeta_0)} \left| \frac{1}{\tau(\omega)} \sum_{i=0}^{\tau(\omega)-1} \log J^u(f^i z) - \beta \right| < \frac{\varepsilon}{2} \right\},$$

denoted by $\{\omega_{n,1}, \omega_{n,2}, \ldots, \omega_{n,\ell(n)}\}$. For each $\omega_{n,s}$, $s \in \{1, 2, \ldots, \ell(n)\}$ write $\omega_{n,s}^u = \omega_{n,s} \cap \gamma^u(\zeta_0)$. It is straightforward to see that

$$(\tilde{\Omega}^u(\beta) \cap G_m) \setminus D_\eta(\zeta_0) \subset \limsup_{n \to \infty} \bigcup_{s=1}^{\ell(n)} \omega_{n,s}^u.$$

We claim that it suffices to show that

$$\limsup_{n \to \infty} \frac{1}{n} \log \sum_{s=1}^{\ell(n)} \text{length}(\omega_{n,s}^u)^{d_\varepsilon^u} \leq 0 \quad \text{for any } \varepsilon > 0, \tag{8}$$

where

$$d_\varepsilon^u = \sup\left\{ \frac{h(\mu)}{\lambda^u(\mu)} : \mu \in \mathcal{M}(f),\ |\lambda^u(\mu) - \beta| < \varepsilon \right\}. \tag{9}$$

Indeed, then using $\text{length}(\omega_{n,s}^u) \leq e^{-(\lambda/3)n}$ from Lemma 4.2(b), for any $\rho > 0$, we have

$$\limsup_{n \to \infty} \frac{1}{n} \log \sum_{s=1}^{\ell(n)} \text{length}(\omega_{n,s}^u)^{d_\varepsilon^u + \rho} \leq -\frac{\lambda}{3}\rho.$$

It follows that $\sum_{s=1}^{\ell(n)} \text{length}(\omega_{n,s}^u)^{d_\varepsilon^u + \rho}$ has negative growth rate as $n$ increases. Therefore, the Hausdorff $(d_\varepsilon^u + \rho)$-measure of the set $(\tilde{\Omega}^u(\beta) \cap G_m) \setminus D_\eta(\zeta_0)$ is 0. Since $\rho > 0$ is arbitrary, $\dim_H^u((\tilde{\Omega}^u(\beta) \cap G_m) \setminus D_\eta(\zeta_0)) \leq d_\varepsilon^u$, and by the countable stability of $\dim_H^u$ we obtain $\dim_H^u(\tilde{\Omega}^u(\beta) \cap G_m) \leq d_\varepsilon^u$. Letting $\varepsilon \to 0$ yields the desired inequality in Proposition 6.3.

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

*Step 2 (finishing proof).* It is left to prove (8). For simplicity, write $\omega_s$ for $\omega_{n,s}$ and $\ell(n)$ for $\ell$. Up to changing labels and making $n$ larger if necessary, we may assume that

$$\tau(\omega_1) \geq \tau(\omega_s) > m \quad \text{for every } s \in \{1, 2, \ldots, \ell\}. \tag{10}$$

Let $\pi : \Sigma_\ell = \{1, 2, \ldots, \ell\}^{\mathbb{Z}} \to \bigcup_{s=1}^{\ell} \omega_s$ denote the coding map defined in §4.4 and $\sigma : \Sigma_\ell \circlearrowleft$ the left shift. The rectangles $\{\omega_1, \omega_2, \ldots, \omega_\ell\}$ can intersect each other only at their stable sides contained in $W^s(P)$, and there is at most one rectangle containing $P$ in its stable side. To avoid any minor problem arising from possible intersections at stable sides, we exclude the set

$$B = \{\underline{a} \in \Sigma_\ell : \pi\underline{a} \subset W^s(P)\backslash\{P\}\}$$

from our consideration. For any $\underline{a} = \{a_i\}_{i\in\mathbb{Z}} \in \Sigma_\ell\backslash B$, there exists a unique element of $\{\omega_1, \omega_2, \ldots, \omega_\ell\}$ containing $\pi\underline{a}$, which is $\omega_{a_0}$. Define $\Phi : \Sigma_\ell\backslash B \to \mathbb{R}$ by

$$\Phi(\underline{a}) = -d_\varepsilon^u \sum_{i=0}^{\tau(\omega_{a_0})-1} \log J^u(f^i(\pi\underline{a})).$$

Since $\pi(\Sigma_\ell) \subset \Omega\backslash\{Q\}$ and $\log J^u$ is continuous except at $Q$, $\Phi$ is continuous.

Let $\mathcal{M}(\sigma)$ denote the space of $\sigma$-invariant Borel probability measures on $\Sigma_\ell$ endowed with the topology of weak convergence.

LEMMA 6.4. *For any $\nu \in \mathcal{M}(\sigma)$, $\nu(B) = 0$.*

*Proof.* If $\nu(B) > 0$, then since $\pi(B) \subset W^s(P)\backslash\{P\}$ one can choose a set $A \subset B$ such that $\nu(A) > 0$ and $\pi(A) \cap \pi(\sigma^n A) = \emptyset$ for every $n > 0$. Since $\nu(\sigma^n A) = \nu(A)$, $\nu$ cannot be a probability, which is a contradiction. □

For each $k \geq 1$, define an atomic probability measure $\nu_k \in \mathcal{M}(\sigma)$ concentrated on the set $E_k = \{\underline{a} \in \Sigma_\ell : \sigma^k\underline{a} = \underline{a}\}$ by

$$\nu_k = \left(\sum_{\underline{b}\in E_k} \exp(S_k\Phi(\underline{b}))\right)^{-1} \sum_{\underline{a}\in E_k} \exp(S_k\Phi(\underline{a}))\delta_{\underline{a}},$$

where $S_k\Phi = \sum_{i=0}^{k-1} \Phi \circ \sigma^i$ and $\delta_{\underline{a}}$ denotes the Dirac measure at $\underline{a}$. Let $\nu_0$ denote an accumulation point of the sequence $\{\nu_k\}_k$ in $\mathcal{M}(\sigma)$. Taking a subsequence if necessary, we may assume that $\nu_k \to \nu_0$ weakly, and so $\nu_0 \in \mathcal{M}(\sigma)$. Define a Borel measure $\overline{\mu}$ on $\pi(\Sigma_\ell)$ by

$$\overline{\mu} = \sum_{s=1}^{\ell} \nu_0|_{\pi^{-1}\omega_s}.$$

By Lemma 6.4, $\overline{\mu}$ is a probability. Define a probability measure $\mu$ on $\Omega$ by

$$\mu = \left(\sum_{s=1}^{\ell} \tau(\omega_s)\overline{\mu}(\omega_s)\right)^{-1} \sum_{s=1}^{\ell} \sum_{i=0}^{\tau(\omega_s)-1} (f^i)_*(\overline{\mu}|_{\omega_s}).$$

It is straightforward to check that $\mu \in \mathcal{M}(f)$.

LEMMA 6.5. $h(\mu) - d_\varepsilon^u\lambda^u(\mu) \leq 0$.

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

*Proof.* Let $s \in \{1, 2, \ldots, \ell\}$ and $z \in \omega_s$. Choose $z_0 \in \omega_s \cap \gamma^u(\zeta_0)$ such that

$$\left|\frac{1}{\tau(\omega_s)} \sum_{i=0}^{\tau(\omega_s)-1} \log J^u(f^i z_0) - \beta\right| < \frac{\varepsilon}{2}.$$

Then we have

$$\begin{aligned}\left|\frac{1}{\tau(\omega_s)} \sum_{i=0}^{\tau(\omega_s)-1} \log J^u(f^i z) - \beta\right| &\le \left|\frac{1}{\tau(\omega_s)} \sum_{i=0}^{\tau(\omega_s)-1} \log J^u(f^i z) - \log J^u(f^i z_0)\right| \\ &\quad + \left|\frac{1}{\tau(\omega_s)} \sum_{i=0}^{\tau(\omega_s)-1} \log J^u(f^i z_0) - \beta\right| \\ &\le \frac{\log D_m}{\tau(\omega_s)} + \frac{\varepsilon}{2} \le \frac{\log D_m}{2n} + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

The upper bound of the first summand follows from Lemma 4.7. The third inequality follows from $\tau(\omega_s) \ge 2n$ from Lemma 4.2(a). The last one holds for sufficiently large $n$. Since $s \in \{1, 2, \ldots, \ell\}$ and $x \in \omega_s$ are arbitrary, $|\lambda^u(\mu) - \beta| < \varepsilon$ follows. This and the definition (9) yield the desired inequality. □

Observe that

$$\nu_k(\{\underline{a}\}) = \left(\sum_{\underline{b} \in E_k} \exp(S_k \Phi(\underline{b}))\right)^{-1} \exp(S_k \Phi(\underline{a})) \quad \text{for every } \underline{a} \in E_k. \tag{11}$$

Hence,

$$\begin{aligned}\sum_{\underline{a} \in E_k} \nu_k(\{\underline{a}\}) S_k \Phi(\underline{a}) &= \sum_{\underline{a} \in E_k} \nu_k(\{\underline{a}\}) \sum_{i=0}^{k-1} \delta_{\sigma^i \underline{a}}(\Phi) \\ &= \left(\sum_{\underline{b} \in E_k} \exp(S_k \Phi(\underline{b}))\right)^{-1} \sum_{\underline{a} \in E_k} \exp(S_k \Phi(\underline{a})) \sum_{i=0}^{k-1} \delta_{\sigma^i \underline{a}}(\Phi) \\ &= k \int \Phi \, d\nu_k\end{aligned}$$

and

$$\begin{aligned}-\sum_{\underline{a} \in E_k} \nu_k(\{\underline{a}\}) \log \nu_k(\{\underline{a}\}) + k\nu_k(\Phi) &= \sum_{\underline{a} \in E_k} \nu_k(\{\underline{a}\})(-\log \nu_k(\{\underline{a}\}) + S_k \Phi(\underline{a})) \\ &= \log \sum_{\underline{a} \in E_k} \exp(S_k \Phi(\underline{a})),\end{aligned}$$

where the last equality follows from taking logarithms of (11), rearranging and summing the result for all $\underline{a} \in E_k$. A slight modification of the argument in [**33**, p. 220] shows that for any integer $p \in \{1, 2, \ldots, k-1\}$,

$$\frac{1}{k} \log \sum_{\underline{a} \in E_k} \exp(S_k \Phi(\underline{a})) \le -\frac{1}{p} \sum_{\underline{a} \in E_p} \nu_k(\{\underline{a}\}) \log \nu_k(\{\underline{a}\}) + \nu_k(\Phi) + \frac{2p \log \# E_p}{k}. \tag{12}$$

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

LEMMA 6.6. $\int \Phi\, d\nu_k \to \int \Phi\, d\nu_0$ *as* $k \to \infty$.

*Proof.* Set $B^c = \Sigma_\ell \backslash B$. For any $\varepsilon > 0$, choose a compact set $K \subset B^c$ such that $\nu_0(B^c \backslash K) < \varepsilon$. Since the set $\Sigma_\ell \backslash K$ is open and closed, and $\nu_0(B \backslash K) = 0$ from Lemma 6.4, $\lim_{k\to\infty} \nu_k(\Sigma_\ell \backslash K) = \nu_0(\Sigma_\ell \backslash K) = \nu_0(B \backslash K) + \nu_0(B^c \backslash K) = \nu_0(B^c \backslash K) < \varepsilon$. Hence, for sufficiently large $k$,

$$\left|\int \Phi\, d\nu_k - \int \Phi\, d\nu_0\right| \le \left|\int_K \Phi\, d\nu_k - \int_K \Phi\, d\nu_0\right| + \left|\int_{\Sigma_\ell \backslash K} \Phi\, d\nu_k - \int_{\Sigma_\ell \backslash K} \Phi\, d\nu_0\right| < \varepsilon\Big(1 + \sup_{\underline{a} \in \Sigma_\ell} |\Phi(\underline{a})|\Big). \qquad \square$$

Letting $k \to \infty$ and then using Lemma 6.6,

$$\limsup_{k\to\infty} \frac{1}{k} \log \sum_{\underline{a} \in E_k} \exp(S_k \Phi(\underline{a})) \le -\frac{1}{p} \sum_{\underline{a} \in E_p} \nu_0(\{\underline{a}\}) \log \nu_0(\{\underline{a}\}) + \int \Phi\, d\nu_0.$$

Letting $p \to \infty$, we get

$$\limsup_{k\to\infty} \frac{1}{k} \log \sum_{\underline{a} \in E_k} \exp(S_k \Phi(\underline{a})) \le h(\sigma;\, \nu_0) + \int \Phi\, d\nu_0, \tag{13}$$

where $h(\sigma;\, \nu_0)$ denotes the entropy of $\nu_0 \in \mathcal{M}(\sigma)$.

To estimate the left-hand side of (13) from below, we need the next lemma. For $k \ge 1$, set $E'_k = \{\underline{a} \in E_k : a_0 = 1\}$.

LEMMA 6.7. *Let* $k \ge 1$ *and* $\underline{a} = \{a_i\}_{i\in\mathbb{Z}} \in E'_k$. *Then:*

(a) $\exp(S_0\Phi(\sigma^k \underline{a}))/\text{length}(\omega^u_{a_k})^{d^u_\varepsilon} \ge (1/2) D_m^{-d^u_\varepsilon}$;

(b) $\exp(S_{k-1}\Phi(\underline{a}))/\exp(S_{k-1}\Phi(\underline{b})) \ge D_m^{-d^u_\varepsilon}$ *for every* $\underline{b} = \{b_i\}_{i\in\mathbb{Z}} \in \Sigma_\ell$ *such that* $a_i = b_i$ *holds for every* $i \in \{0, 1, \ldots, k-1\}$.

*Proof.* By the mean value theorem, there exists $z \in \omega^u_{a_{k-1}}$ such that

$$\|Df^{\tau(\omega_{a_{k-1}})}|E^u_z\| \cdot \text{length}(\omega^u_{a_{k-1}}) = \text{length}(f^{\tau(\omega_{a_{k-1}})}\omega^u_{a_{k-1}}).$$

Since $f^{\tau(\omega_{a_{k-1}})}\omega^u_{a_{k-1}} \in \tilde{\Gamma}^u$, $\text{length}(f^{\tau(\omega_{a_{k-1}})}\omega^u_{a_{k-1}}) < 2$. Since $\omega_{a_{k-1}}$ intersects $G_m$ and $\tau(\omega_{a_{k-1}}) > m$ by (10), Lemma 4.7 gives

$$\frac{\|Df^{\tau(\omega_{a_{k-1}})}|E^u_{\pi(\sigma^{k-1}\underline{a})}\|}{\|Df^{\tau(\omega_{a_{k-1}})}|E^u_z\|} \le D_m.$$

These estimates yield

$$\frac{\exp(S_0\Phi(\sigma^{k-1}\underline{a}))}{\text{length}(\omega^u_{a_{k-1}})^{d^u_\varepsilon}} = \left[\frac{\|Df^{\tau(\omega_{a_{k-1}})}|E^u_{\pi(\sigma^{k-1}\underline{a})}\|}{\|Df^{\tau(\omega_{a_{k-1}})}|E^u_z\|}\right]^{-d^u_\varepsilon} \text{length}(f^{\tau(\omega_{a_{k-1}})}\omega^u_{a_{k-1}})^{-d^u_\varepsilon} \ge \frac{D_m^{-d^u_\varepsilon}}{2},$$

which proves (a).

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

For each $j \in \{1, 2, \ldots, k\}$, put $t_j = \sum_{i=0}^{j-1} \tau(\omega_{a_i})$. Since $\pi\underline{a} \in \omega_{a_0}$ and $\omega_{a_0}$ intersects $G_m$, Lemma 4.5 gives

$$d_{\mathrm{crit}}(f^n(\pi\underline{a})) \geq b^{n/9} \quad \text{for every } n \in \{m, m+1, \ldots, \tau(\omega_{a_0}) - 1\}.$$

Since $f^{t_j}(\pi\underline{a}) \in \omega_{a_{j+1}}$, Lemma 4.2(c) gives

$$d_{\mathrm{crit}}(f^n(\pi\underline{a})) \geq e^{-10\tau(\omega_{a_{j+1}})} \quad \text{for every } n \in \bigcup_{j=1}^{k-1} \{t_j, t_j+1, \ldots, t_{j+1}-1\}.$$

For such $n$, we have $n \geq \tau(\omega_{a_0})$, and (10) gives $\tau(\omega_{a_0}) \geq \tau(\omega_{a_{j+1}})$. Hence, $e^{-10\tau(\omega_{a_{j+1}})} \geq e^{-10\tau(\omega_{a_0})} \geq e^{-10n} \geq b^{n/9}$ and therefore

$$d_{\mathrm{crit}}(f^n(\pi\underline{a})) \geq b^{n/9} \quad \text{for every } n \in \bigcup_{j=1}^{k-1} \{t_j, t_j+1, \ldots, t_{j+1}-1\}.$$

Since $\pi\underline{a}$ is a periodic point of $f$ of period $t_k$, the same inequality holds for every $n \geq m$. Namely, $d_{\mathrm{crit}}(f^n(\pi\underline{a})) \geq b^{n/9}$ holds for every $n \geq m$. Since $\pi\underline{a}$ and $\pi\underline{b}$ are contained in the same proper rectangle with inducing time $t_k$, the proof of Lemma 4.7 yields the same distortion bound, which implies (b). □

Let $\underline{a} \in E'_k$, $\underline{b} \in E'_{k-1}$ be such that $a_i = b_i$ for every $i \in \{0, 1, \ldots, k-1\}$. Lemma 6.7 gives

$$\frac{\exp(S_k\Phi(\underline{a}))}{\exp(S_{k-1}\Phi(\underline{b}))} = \frac{\exp(S_{k-1}\Phi(\underline{a}))}{\exp(S_{k-1}\Phi(\underline{b}))} \exp(S_0\Phi(\sigma^{k-1}\underline{a})) \geq \frac{D_m^{-2d_\varepsilon^u}}{2} \mathrm{length}(\omega_{a_k}^u)^{d_\varepsilon^u}.$$

Summing this inequality over all such $\underline{a}$, $\underline{b}$ and then using the result repeatedly gives

$$\begin{aligned}
\sum_{\underline{a}\in E_k} \exp(S_k\Phi(\underline{a})) &> \sum_{\underline{a}\in E'_k} \exp(S_k\Phi(\underline{a})) \\
&= \sum_{\underline{b}\in E'_{k-1}} \exp(S_{k-1}\Phi(\underline{b})) \sum_{\substack{\underline{a}\in E'_k \\ a_i=b_i \ 0\leq\forall i\leq k-1}} \frac{\exp(S_k\Phi(\underline{a}))}{\exp(S_{k-1}\Phi(\underline{b}))} \\
&\geq \sum_{\underline{b}\in E'_{k-1}} \exp(S_{k-1}\Phi(\underline{b})) \frac{D_m^{-2d_\varepsilon^u}}{2} \sum_{s=1}^{\ell} \mathrm{length}(\omega_s^u)^{d_\varepsilon^u} \\
&\geq \cdots \geq \sum_{\underline{b}\in E'_1} \exp(S_0\Phi(\underline{b})) \left( \frac{D_m^{-2d_\varepsilon^u}}{2} \sum_{s=1}^{\ell} \mathrm{length}(\omega_s^u)^{d_\varepsilon^u} \right)^{k-1} \\
&\geq \left( \frac{D_m^{-2d_\varepsilon^u}}{2} \sum_{s=1}^{\ell} \mathrm{length}(\omega_s^u)^{d_\varepsilon^u} \right)^{k}.
\end{aligned}$$

Hence,

$$\liminf_{k\to\infty} \frac{1}{k} \log \sum_{\underline{a}\in E_k} \exp(S_k\Phi(\underline{a})) \geq \log \sum_{s=1}^{\ell} \mathrm{length}(\omega_s^u)^{d_\varepsilon^u} - 2d_\varepsilon^u \log D_m - \log 2.$$

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

Combining this inequality with (13) and then using Lemma 6.5 yield

$$\begin{aligned}\frac{1}{n}\log\sum_{s=1}^{\ell}\text{length}(\omega_s^u)^{d_\varepsilon^u} &\le \frac{1}{n}(h(\sigma;\,\nu_0)+\nu_0(\Phi))+\frac{2}{n}d_\varepsilon^u\log D_m+\frac{\log 2}{n}\\ &= \frac{1}{n}(h(\mu)-d_\varepsilon^u\lambda^u(\mu))\sum_{s=1}^{\ell}\tau(\omega_s)\overline{\mu}(\omega_s)+\frac{2}{n}d_\varepsilon^u\log D_m+\frac{\log 2}{n}\\ &\le \frac{2}{n}d_\varepsilon^u\log D_m+\frac{\log 2}{n}.\end{aligned}$$

This implies (8) and hence finishes the proof of Lemma 6.3. □

6.3. *Properties of the Lyapunov spectrum.* We now prove Theorem C.

*Proof of Theorem C(a).* The upper semi-continuity follows from the formula in Theorem B. We derive a contradiction by assuming that $L^u$ is not lower semi-continuous at a point $\beta\in I$. Then there exist $\varepsilon>0$ and a monotone sequence $\{\beta_n\}_n$ converging to $\beta$ such that $L^u(\beta)-\varepsilon\ge L^u(\beta_n)$. If $\beta=\lambda_m^u$, then fix $\mu\in\mathcal{M}(f)$ with $\lambda^u(\mu)>\beta$. Choose a sequence $\{\mu_n\}_n$ in $\mathcal{M}(f)$ such that $h(\mu_n)/\lambda^u(\mu_n)\ge L^u(\beta)-\varepsilon/3$ and $\lambda^u(\mu_n)\to\beta$ as $n\to\infty$. Taking a subsequence if necessary, we may assume that $\beta_n\ge\lambda^u(\mu_n)$. For those sufficiently large $n$ such that $\lambda^u(\mu)\ge\beta_n$, let $t_n\in[0,1]$ be such that $\beta_n=(1-t_n)\lambda^u(\mu)+t_n\lambda^u(\mu_n)$. Then

$$\begin{aligned}L^u(\beta)-\varepsilon\ge L^u(\beta_n)&=L^u((1-t_n)\lambda^u(\mu)+t_n\lambda^u(\mu_n))\\ &\ge\frac{h((1-t_n)\mu+t_n\mu_n)}{\lambda^u((1-t_n)\mu+t_n\mu_n)}=\frac{(1-t_n)h(\mu)+t_nh(\mu_n)}{(1-t_n)\lambda^u(\mu)+t_n\lambda^u(\mu_n)}\ge L^u(\beta)-\frac{\varepsilon}{2}.\end{aligned}$$

The second inequality follows from Theorem B, and the last one follows from $t_n\to1$ and $\lambda^u(\mu_n)\ge\lambda_m^u>0$. This yields a contradiction. If $\beta=\lambda_M^u$, then we replace $\mu$ by $\nu$ with $\lambda^u(\nu)<\beta$ and proceed in the same way. The remaining case $\beta\in(\lambda_m^u,\lambda_M^u)$ is covered by the same argument. □

*Proof of Theorem C(b).* The monotonicity statement is a consequence of the next lemma.

LEMMA 6.8. *For all $\beta_0,\beta_1\in I$ with $\beta_0<\beta_1$ and all $0\le t\le1$,*

$$\min\{L^u(\beta_0),\,L^u(\beta_1)\}\le L^u(t\beta_0+(1-t)\beta_1).$$

*Proof.* From Theorem B, for any $\varepsilon>0$, there exist $\mu_0,\,\mu_1\in\mathcal{M}(f)$ such that $L^u(\beta_0)-\varepsilon<h(\mu_0)/\lambda^u(\mu_0)$, $L^u(\beta_1)-\varepsilon<h(\mu_1)/\lambda^u(\mu_1)$ and $|\lambda^u(\mu_0)-\beta_0|<\varepsilon$, $|\lambda^u(\mu_1)-\beta_1|<\varepsilon$. Then

$$\min\{L^u(\beta_0),\,L^u(\beta_1)\}<\varepsilon+\min\left\{\frac{h(\mu_0)}{\lambda^u(\mu_0)},\,\frac{h(\mu_1)}{\lambda^u(\mu_1)}\right\}.$$

Set $\nu=t\mu_0+(1-t)\mu_1$. It is easy to see that the minimum of the right-hand side is $\le h(\nu)/\lambda^u(\nu)$, which is $\le L^u(t\beta_0+(1-t)\beta_1)$. Letting $\varepsilon\to0$ yields the desired inequality. □

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

*Proof of Theorem C(c).* If $\beta \in (\lambda^u_m, \lambda^u_M)$, then by Lemma 5.3 it is possible to choose $\mu_1,\ \mu_2 \in \mathcal{M}^e(f)$ with positive entropy and satisfying $\lambda^u(\mu_1) < \beta < \lambda^u(\mu_2)$. Choose $t \in (0, 1)$ such that $t\lambda^u(\mu_1) + (1-t)\lambda^u(\mu_2) = \beta$. By Lemma 5.3 again, there exists a sequence $\{\nu_n\}_n$ in $\mathcal{M}^e(f)$ with $\lim_{n\to\infty} h(\nu_n) > 0$ and $\lambda^u(\nu_n) \to \beta$ as $n \to \infty$. Lemma 5.2 yields $L^u(\beta) > 0$. □

*Proof of Theorem C(d).* The 'if' part follows from Theorem B. To show the 'only if' part, let $\beta \in I$ be such that $L^u(\beta) = t^u$. Theorem B allows us to choose a sequence $\{\mu_n\}_n$ in $\mathcal{M}(f)$ such that $h(\mu_n)/\lambda^u(\mu_n) \to t^u$ and $\lambda^u(\mu_n) \to \beta$ as $n \to \infty$. Choosing a subsequence if necessary, we may assume that $\mu_n \to \mu \in \mathcal{M}(f)$ weakly. Write $\mu = u\delta_Q + (1-u)\nu$, $0 \le u \le 1$, $\nu\{Q\} = 0$. Since $h(\delta_Q) = 0$, the upper semi-continuity of entropy [**24**, Corollary 3.2] implies that $u \ne 1$ and $\limsup_{n\to\infty} h(\mu_n) \le h(\mu) = (1-u)h(\nu)$. On the other hand, [**24**, Lemma 4.3] gives $\liminf_{n\to\infty} \lambda^u(\mu_n) \ge (1-u)\lambda^u(\nu)$. If $u \ne 0$, then this inequality would be strict, and so

$$\frac{h(\nu)}{\lambda^u(\nu)} = \frac{(1-u)h(\nu)}{(1-u)\lambda^u(\nu)} > \frac{\limsup_{n\to\infty} h(\mu_n)}{\liminf_{n\to\infty} \lambda^u(\mu_n)} \ge \lim_{n\to\infty} \frac{h(\mu_n)}{\lambda^u(\mu_n)} = t^u,$$

which yields $P(t^u) > 0$, which is a contradiction. Hence, $u = 0$. By [**24**, Lemma 4.4], $\lambda^u(\mu_n) \to \lambda^u(\mu)$, and so $h(\mu_n) \to t^u\lambda^u(\mu)$ and $h(\mu) - t^u\lambda^u(\mu) \ge 0$. The uniqueness of the equilibrium measure for the potential $-t^u \log J^u$ [**25**, Theorem A] yields $\mu = \mu_{t^u}$, and therefore $\beta = \lambda^u(\mu_{t^u})$. □

6.4. *Hausdorff dimension of the set of irregular points.* We now prove Theorem D.

*Proof of Theorem D.* Let $\varepsilon > 0$. In view of Theorem B, choose $\mu,\ \nu \in \mathcal{M}^e(f)$ with $\lambda^u(\mu) > \lambda^u(\nu)$ and $h(\mu)/\lambda^u(\mu) \ge t^u - \varepsilon$, $h(\nu)/\lambda^u(\nu) \ge t^u - \varepsilon$. Choose sequences $\{\mu_n\}_n$, $\{\nu_n\}_n$ in $\mathcal{M}^e(f)$ with $\lambda^u(\mu_n) \to \lambda^u(\mu)$ and $\lambda^u(\nu_n) \to \lambda^u(\nu)$ as $n \to \infty$. Define $\xi_n \in \mathcal{M}^e(f)$ by

$$\xi_n = \begin{cases} \mu_n & \text{for } n \text{ odd,} \\ \nu_n & \text{for } n \text{ even.} \end{cases}$$

A slight modification of the proof of Lemma 5.2 applied to the sequence $\{\xi_n\}_n$ yields a subset $\Delta$ of $\Omega^u$ such that $\bar{\lambda}^u(z) = \lambda^u(\mu)$ and $\underline{\lambda}^u(z) = \lambda^u(\nu)$ for all $z \in \Delta$, and

$$\dim^u_H(\Delta) \ge \min\left\{\frac{h(\mu)}{\lambda^u(\mu)}, \frac{h(\nu)}{\lambda^u(\nu)}\right\}.$$

Hence, $\Delta \subset \hat{\Omega}^u$ and $\dim^u_H(\hat{\Omega}^u) \ge \dim^u_H(\Delta) \ge t^u - \varepsilon$. Letting $\varepsilon \to 0$, we obtain Theorem D. □

*Acknowledgements.* I thank the anonymous referees for their careful reading of the manuscript and giving useful comments. This research is partially supported by the Grant-in-Aid for Young Scientists (A) of the JSPS, Grant No. 15H05435, the Grant-in-Aid for Scientific Research (B) of the JSPS, Grant No. 16KT0021, and the JSPS Core-to-Core Program 'Foundation of a Global Research Cooperative Center in Mathematics Focused on Number Theory and Geometry'.

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

## REFERENCES


[**1**] L. Barreira and G. Iommi. Multifractal analysis and phase transitions for hyperbolic and parabolic horseshoes. *Israel J. Math.* **181** (2011), 347–479.
[**2**] L. Barreira and J. Schmeling. Sets of 'non-typical' points have full Hausdorff dimension and full topological entropy. *Israel J. Math.* **116** (2000), 29–70.
[**3**] E. Bedford and J. Smillie. Real polynomial diffeomorphisms with maximal entropy: II. Small Jacobian. *Ergod. Th. & Dynam. Sys.* **26** (2006), 1259–1283.
[**4**] M. Benedicks and L. Carleson. The dynamics of the Hénon map. *Ann. of Math. (2)* **133** (1991), 73–169.
[**5**] M. Benedicks and M. Viana. Solution of the basin problem for Hénon-like attractors. *Invent. Math.* **143** (2001), 375–434.
[**6**] Y. Cao, S. Luzzatto and I. Rios. The boundary of hyperbolicity for Hénon-like families. *Ergod. Th. & Dynam. Sys.* **28** (2008), 1049–1080.
[**7**] Y. M. Chung. Birkhoff spectra for one-dimensional maps with some hyperbolicity. *Stoch. Dyn.* **10** (2010), 53–75.
[**8**] Y. M. Chung and H. Takahasi. Multifractal formalism for Benedicks–Carleson quadratic maps. *Ergod. Th. & Dynam. Sys.* **34** (2014), 1116–1141.
[**9**] R. Devaney and Z. Nitecki. Shift automorphisms in the Hénon mapping. *Comm. Math. Phys.* **67** (1979), 137–146.
[**10**] A. Eizenberg, Y. Kifer and B. Weiss. Large deviations for $\mathbb{Z}^d$-actions. *Comm. Math. Phys.* **164** (1994), 433–454.
[**11**] K. Gelfert, F. Przytycki and M. Rams. On the Lyapunov spectrum for rational maps. *Math. Ann.* **348** (2010), 965–1004.
[**12**] K. Gelfert and M. Rams. The Lyapunov spectrum of some parabolic systems. *Ergod. Th. & Dynam. Sys.* **29** (2009), 919–940.
[**13**] G. Iommi and M. Todd. Dimension theory for multimodal maps. *Ann. Henri Poincaré* **12** (2011), 591–620.
[**14**] A. Johansson, T. Jordan, A. Öberg and M. Pollicott. Multifractal analysis of non-uniformly hyperbolic systems. *Israel J. Math.* **177** (2010), 125–144.
[**15**] A. Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.* **51** (1980), 137–173.
[**16**] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1995.
[**17**] L. Mora and M. Viana. Abundance of strange attractors. *Acta Math.* **171** (1993), 1–71.
[**18**] K. Nakaishi. Multifractal formalism for some parabolic maps. *Ergod. Th. & Dynam. Sys.* **20** (2000), 843–857.
[**19**] L. Olsen. Multifractal analysis of divergence points of deformed measure theoretical Birkhoff averages. *J. Math. Pures Appl.* **82** (2003), 1591–1649.
[**20**] Y. Pesin. *Dimension Theory in Dynamical Systems*. University of Chicago Press, Chicago, 1997.
[**21**] Y. Pesin and H. Weiss. A multifracatal analysis of equilibrium measures for conformal expanding maps and Moran-like geometric constructions. *J. Stat. Phys.* **86** (1997), 233–275.
[**22**] F. Przytycki and J. Rivera-Letelier. Nice inducing schemes and the thermodynamics of rational maps. *Comm. Math. Phys.* **70** (2011), 661–707.
[**23**] D. Ruelle. An inequality for the entropy of differentiable maps. *Bol. Soc. Brasil. Mat.* **9** (1978), 83–87.
[**24**] S. Senti and H. Takahasi. Equilibrium measures for the Hénon map at the first bifurcation. *Nonlinearity* **26** (2013), 1719–1741.
[**25**] S. Senti and H. Takahasi. Equilibrium measures for the Hénon map at the first bifurcation: uniqueness and geometric/statistical properties. *Ergod. Th. & Dynam. Sys.* **36** (2016), 215–255.
[**26**] K. Sigmund. On dynamical systems with the specification property. *Trans. Amer. Math. Soc.* **190** (1974), 285–299.
[**27**] H. Takahasi. Abundance of nonuniform hyperbolicity in bifurcations of surface endomorphisms. *Tokyo J. Math.* **34** (2011), 53–113.
[**28**] H. Takahasi. Prevalent dynamics at the first bifurcation of Hénon-like families. *Comm. Math. Phys.* **312** (2012), 37–85.
[**29**] H. Takahasi. Equilibrium measures at temperature zero for Hénon-like maps at the first bifurcation. *SIAM J. Appl. Dyn. Syst.* **15** (2016), 106–124.
[**30**] H. Takahasi. Prevalence of non-uniform hyperbolicity at the first bifurcation of Hénon-like families. *Preprint.* Available at http://arxiv.org/abs/1308.4199.
[**31**] F. Takens and E. Verbitskiy. On the variational principle for the topological entropy of certain non-compact sets. *Ergod. Th. & Dynam. Sys.* **23** (2003), 317–348.


https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press

[**32**] M. Urbański and C. Wolf. Ergodic theory of parabolic horseshoes. *Comm. Math. Phys.* **281** (2008), 711–751.
[**33**] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts in Mathematics, 79)*. Springer, New York, 1982.
[**34**] Q. D. Wang and L.-S. Young. Strange attractors with one direction of instability. *Comm. Math. Phys.* **218** (2001), 1–97.
[**35**] H. Weiss. The Lyapunov spectrum for conformal expanding maps and Axiom A surface diffeomorphisms. *J. Stat. Phys.* **95** (1999), 615–632.
[**36**] L.-S. Young. Dimension, entropy and Lyapunov exponents. *Ergod. Th. & Dynam. Sys.* **2** (1982), 109–124.

https://doi.org/10.1017/etds.2016.65 Published online by Cambridge University Press