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NEAR-INTEGRATED RANDOM COEFFICIENT AUTOREGRESSIVE TIME SERIES

ALEXANDER AUE
University of California, Davis

We determine the limiting behavior of near-integrated first-order random coefficient autoregressive RCA(1) time series. It is shown that the asymptotics of the finite-dimensional distributions crucially depends on how the critical value 1 is approached, which determines whether the process is near-stationary, has a unit root, or is mildly explosive.

In a second part, we derive the limit distribution of the serial correlation coefficient in the near stationary and the mildly explosive settings under very general conditions on the parameters. The results obtained are in accordance with those available for first-order autoregressive time series and can hence serve as an addition to existing literature in the area.

1. INTRODUCTION

Let $\{X_k\}$ be a first-order autoregressive AR(1) process, that is, the solution of the recurrence equations

$$X_k = \varphi X_{k-1} + \varepsilon_k, \qquad -\infty < k < \infty. \tag{1.1}$$

In the case that the noise $\{\varepsilon_k\}$ is an independent, identically distributed sequence, it is well known that the process is nonanticipative (future independent) and strictly stationary if $|\varphi| < 1$ and explosive if $|\varphi| > 1$ (see, e.g., Brockwell and Davis, 1991). The process $\{X_k\}$ is said to have a unit root if $\varphi = 1$. Then the process resembles the behavior of a random walk. If $\varphi \approx 1$ and $\varphi \to 1$ with increasing sample size, $\{X_k\}$ is referred to as near-integrated.

A number of papers have been devoted to examining the asymptotic structure of $\{X_k\}$ if the coefficient φ is reparameterized to

$$\varphi_n = 1 - \frac{\beta}{n^\alpha}, \qquad \alpha \in [0,1], \qquad n \geq 1, \tag{1.2}$$

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with some fixed real number β, and if model (1.1) is transformed to

$$X_k(n) = \varphi_n X_{k-1}(n) + \varepsilon_k, \qquad k = 1, \ldots, n, \tag{1.3}$$

with initial value $X_0(n) = 0$ for all n. For instance, Chan and Wei (1987, 1988) studied the near-stationary AR(1) process ($\beta > 0$) and also the mildly explosive case ($\beta < 0$) if $\alpha = 1$ and the noise $\{\varepsilon_k\}$ constitutes a sequence of martingale differences. An extensive analysis for near-integrated time series was carried out by Phillips (1988). His more general multivariate approach, which includes multiple times series in which some component series have unit roots, some are stationary, and others are explosive, is in one dimension asymptotically equivalent to the setting introduced by Chan and Wei (1987, 1988). Related work in the preceding framework is due to Ling and Li (1998), who considered unstable autoregressive moving average (ARMA) processes with generalized autoregressive conditionally heteroskedastic (GARCH) errors. In recent work, Phillips and Magdalinos (2007) established limit theorems in the general parameter case $\varphi_n \to 1$ for what they call moderate deviations from a unit root.

The asymptotic theory obtained in all these papers has consequences for testing procedures for the presence of unit roots. Usually these tests have a low power because the behavior of the test statistic under consideration does not crucially differ between the null hypothesis of a unit root and the alternatives of near-stationary or mildly explosive processes. Unit-root tests play a fundamental role in econometrics; see, among others, Nelson and Plosser (1982) and Phillips (1988) and the references therein.

Motivated by results for GARCH(1,1) sequences obtained by Berkes, Horváth, and Kokoszka (2005), we will study the finite-dimensional distributions of first-order random coefficient autoregressive RCA(1) time series, which are defined by the stochastic recurrence equations in (2.1). Originally, these processes were applied to model certain dynamical systems (see Nicholls and Quinn, 1982), but they are also used in finance (see Tong, 1990). Surveys on basic properties of this model, such as necessary and sufficient criteria for the existence of unique strictly stationary solutions and the finiteness of moments, can be found, for instance, in Aue, Horváth, and Steinebach (2006) and Nicholls and Quinn (1982).

Our results differ from the approach using model (1.3) in two directions. On one hand, the parameter φ_n is allowed to be random and to depend on the time index k, which is enabled by changing it to the corresponding quantity $\varphi_n + b_k(n)$ in (2.1) involving an independent sequence $\{b_k(n)\}$ of random variables having finite second moments. On the other hand, the parameterization (1.2) can be relaxed. The limit results of Theorems 2.3 (near-stationary case), 2.4 (unit-root), and 2.5 (mildly explosive case) are obtained assuming the asymptotic rates given in (2.6) only. The proofs of these theorems are based on an additive representation of the RCA(1) process established in Theorem 3.1.

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A great variety of the aforementioned unit-root tests are based on the least squares estimator

$$\hat{\varphi}_n = \left(\sum_{k=1}^{n} X_{k-1}^2(n) \right)^{-1} \sum_{k=1}^{n} X_{k-1}(n) X_k(n), \qquad n \geq 2,$$

of φ_n. Its limiting behavior has been examined thoroughly. However, most of the existing literature is devoted to the special case $\alpha = 1$ in (1.2). See, for instance, Chan and Wei (1987, 1988). For the general convergence $\varphi_n \to 1$, it turns out that a Gaussian limit exists under a normalization of order $\sqrt{n/|\varphi_n - 1|}$ if the AR(1) process is near-stationary. On the other hand, if $X_k(n)$ is in the explosive regime, the limit will no longer be Gaussian but a Cauchy random variable, whereas the normalization sequence now has the exponential order $e^{n(\varphi_n - 1)}/(\varphi_n - 1)$. These results can be found in Phillips and Magdalinos (2007). We will show that similar statements hold true also in the random coefficient setting provided in (2.1). Theorem 2.1 deals with mild explosion; Theorem 2.2 gives the corresponding result for near-stationarity.

Ultimately, it is worthwhile mentioning that, because all calculations can be accomplished also if $b_k(n) \equiv 0$, the results remain valid for AR(1) time series. (This applies primarily to the finite-dimensional distribution results.) So, this paper can serve as complementary work to the existing results cited previously.

The paper is organized as follows. In Section 2, we present the RCA(1) model and state and discuss the main results. Proofs are given in Section 3.

2. MAIN RESULTS

Consider the sequence of random coefficient autoregressive RCA(1) models

$$X_k(n) = [\varphi_n + b_k(n)] X_{k-1}(n) + \varepsilon_k, \qquad k = 1, \ldots, n, \tag{2.1}$$

with initial value $X_0(n) = 0$ for all n. Therein, for all n,

$$b_k(n) = \omega_n \gamma_k, \qquad k = 1, \ldots, n, \tag{2.2}$$

with $\omega_n \geq 0$ and

$$\begin{aligned} &\{(\gamma_k, \varepsilon_k)\} \quad \text{are independent, identically distributed with} \\ &\mathrm{E}(\gamma_1, \varepsilon_1) = (0,0) \quad \text{and} \quad \mathrm{E}\gamma_1^2 = 1, \qquad \mathrm{E}\varepsilon_1^2 = \sigma^2, \end{aligned} \tag{2.3}$$

where, moreover, $\{\gamma_k\}$ and $\{\varepsilon_k\}$ are uncorrelated. Note that the initial value $X_0(n) = 0$ is applied merely for convenience. The statements of all theorems to come remain true if this condition is relaxed to $X_0(n) = \mathcal{O}_P(1)$. Displays (2.2) and (2.3) yield that $\mathrm{E}[X_k(n)] = \varphi_n \mathrm{E}[X_{k-1}(n)] = 0$, by iteration. We shall assume that

$$\varphi_n \to 1 \quad \text{and} \quad \omega_n^2 \to 0 \qquad (n \to \infty), \tag{2.4}$$

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which immediately implies that $\varphi_n^2 + \omega_n^2 \to 1$. Also, by recursion, it holds for the second moment of $X_k(n)$ that $\mathrm{E}[X_k^2(n)] = \sigma^2 \sum_{j=1}^{k} (\varphi_n^2 + \omega_n^2)^{j-1} \to k\sigma^2$ as $n \to \infty$ for fixed k.

It turns out that the rate of convergence and also the limit distribution of the sequence of RCA(1) processes crucially depend on how the parameter φ_n approaches 1. Hence, we define

$$\tilde{\varphi}_n = \varphi_n - 1, \qquad n \geq 1, \tag{2.5}$$

and differ between the three cases $\tilde{\varphi}_n > 0$, $\tilde{\varphi}_n = 0$, and $\tilde{\varphi}_n < 0$ for all n. Moreover, we need the following rates:

$$n|\tilde{\varphi}_n| \to \infty \quad \text{but} \quad \sqrt{n}\tilde{\varphi}_n \to 0 \qquad (n \to \infty) \tag{2.6}$$

and

$$\sqrt{n}\omega_n \to 0 \qquad (n \to \infty). \tag{2.7}$$

The second part of condition (2.6) is a crucial assumption for the additive representation given in Theorem 3.1. It states that the nonstochastic term of the autoregressive coefficient φ_n cannot lie in a neighborhood of unity large enough to approach stationarity or the explosive region. Hence, a discussion of transition cases is impossible in the present setting.

The next part of the section is devoted to examining the serial correlation coefficient. Denote by

$$\hat{\varphi}_n = \left(\sum_{k=1}^{n} X_{k-1}^2(n) \right)^{-1} \sum_{k=1}^{n} X_{k-1}(n) X_k(n), \qquad n \geq 1, \tag{2.8}$$

the (conditional) least squares estimator for φ_n. We say that a random variable ζ is standard Cauchy if its density is given by

$$f(x) = \frac{1}{\pi(1 + x^2)}, \qquad -\infty < x < \infty.$$

We start with giving the asymptotics for the serial autocorrelation coefficient $\hat{\varphi}_n - \varphi_n$ if φ_n is larger than 1, that is, approaches 1 from the explosive region. Under a suitable normalization this limit is standard Cauchy.

THEOREM 2.1 (Mild explosion). *Let $\{X_k(n)\}$ be an RCA(1) time series satisfying model (2.1) with $\tilde{\varphi}_n > 0$ for all n. If (2.2)–(2.7) hold, then*

$$\frac{e^{n\tilde{\varphi}_n}}{2\tilde{\varphi}_n} (\hat{\varphi}_n - \varphi_n) \xrightarrow{\mathcal{D}} \zeta \qquad (n \to \infty),$$

where $\hat{\varphi}_n$ is defined in (2.8) and ζ denotes a standard Cauchy random variable.

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In contrast to the Cauchy limit for moderate deviations in the explosive direction, the near-stationary case provides a Gaussian limit theorem.

THEOREM 2.2 (Near-stationarity). *Let $\{X_k(n)\}$ be an RCA(1) time series satisfying model (2.1) with $\tilde{\varphi}_n < 0$ for all n. If (2.2)–(2.7) hold, then*

$$\sqrt{\frac{n}{2|\tilde{\varphi}_n|}}(\hat{\varphi}_n - \varphi_n) \xrightarrow{\mathcal{D}} \xi \qquad (n \to \infty),$$

where $\hat{\varphi}_n$ is defined in (2.8) and ξ denotes a standard normal random variable.

The results are in accordance with contributions existing for near-integrated first-order autoregressive time series. The most recent work has been done in Phillips and Magdalinos (2007) for an equivalent general parameter case. Previous results in the literature were mainly concerned with the special cases $\varphi_n = 1 + \beta/n^\alpha$ with $\alpha \in (0,1)$ and the parameter β determining whether the setting is mildly explosive ($\beta > 0$) or near-stationary ($\beta < 0$). Using this particular form of φ_n, one can gain deeper insight into how to bridge gaps between limit results for near-integration and unit roots on one hand and between near-integration and stationarity, respectively, explosion on the other. For a detailed discussion, we refer to Phillips and Magdalinos (2007).

More general limit theorems hold true also if the noise $\{\varepsilon_k\}$ is in the domain of attraction of a stable law. Corresponding results are due to Aue and Horváth (2007) for first-order autoregression. Adaptations of the proofs will also work in the present model with random coefficients.

The proof of Theorem 2.1 falls back on the representation of $X_k(n)$ given in Theorem 3.1, whereas Theorem 2.2 can be established by exploiting a result in Phillips and Magdalinos (2007). Details are given in Sections 3.5 and 3.6.

In the remainder of the section we investigate the limit behavior of the variables $X_k(n)$. Because we are actually dealing with a sequence of models, the point of view is the one of finite-dimensional distributions, and we study the asymptotic behavior of the vectors

$$[X_{\lfloor n\kappa_m \rfloor} : m = 1, \ldots, N],$$

where N is a fixed integer, $0 < \kappa_1 < \cdots < \kappa_N \leq 1$ and $\lfloor \cdot \rfloor$ the integer part.

At first, we investigate the case $\tilde{\varphi}_n > 0$ for all n. Hence, we study a sequence of nonstationary models. The asymptotics is given in the following theorem.

THEOREM 2.3 (Mildly explosive case). *Let $\{X_k(n)\}$ be an RCA(1) time series satisfying model (2.1) with $\tilde{\varphi}_n > 0$ for all n. If (2.2)–(2.7) hold, then the random variables*

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$$\left[\frac{\sqrt{2\tilde{\varphi}_n}}{e^{\kappa_m n\tilde{\varphi}_n}\sigma} X_{\lfloor n\kappa_m \rfloor} : m = 1, \ldots, N \right]$$

are asymptotically $(n \to \infty)$ *independent standard normal.*

If $\tilde{\varphi}_n = 0$ for all positive integers n, a different limiting distribution under a different normalization is obtained. Because the random part $b_k(n)$ of the coefficients dies out for $n \to \infty$ by assumption (2.4), it is not surprising to see a resemblance of the random walk here.

THEOREM 2.4 (Unit-root case). *Let* $\{X_k(n)\}$ *be an RCA(1) time series satisfying model (2.1) with* $\tilde{\varphi}_n = 0$ *for all n. If (2.2)–(2.5) and (2.7) hold, then*

$$\left[\frac{1}{\sqrt{n}\sigma} X_{\lfloor n\kappa_m \rfloor} : m = 1, \ldots, N \right] \xrightarrow{\mathcal{D}} [W(\kappa_m) : m = 1, \ldots, N] \qquad (n \to \infty),$$

where $\{W(s) : s \in [0,1]\}$ *denotes a standard Wiener process.*

Finally, we examine the stationary case, that is, $\tilde{\varphi}_n < 0$ for all n. Although the order of the oscillations, given by the corresponding normalizations, is exponential $e^{n\tilde{\varphi}_n}/\sqrt{\tilde{\varphi}_n}$ and $\sqrt{n}$ in the near-explosive and unit-root setting, respectively, fluctuations increase with the rate $1/\sqrt{|\tilde{\varphi}|}$ in the case of near-stationary models.

THEOREM 2.5 (Near-stationary case). *Let* $\{X_k(n)\}$ *be an RCA(1) time series satisfying model (2.1) with* $\tilde{\varphi}_n < 0$ *for all n. If (2.2)–(2.7) hold and if*

$$\mathrm{E}|\varepsilon_1|^{2+\delta} < \infty \quad \text{for some } \delta > 0, \tag{2.9}$$

the random variables

$$[\sqrt{2|\tilde{\varphi}_n|}\sigma^{-1} X_{\lfloor n\kappa_m \rfloor} : m = 1, \ldots, N]$$

are asymptotically $(n \to \infty)$ *independent standard normal.*

The proofs of Theorems 2.3–2.5 are based on a common asymptotic additive representation of $\{X_k(n)\}$, which is given in Theorem 3.1. Using this result as a starting point, the respective asymptotic distributions are obtained by applying and exploiting special characteristics of the individual cases to filter out the dominating part of the decomposition. Details are deferred to the next section.

We conclude the section with the remark that Theorems 2.3–2.5 hold true also if the noise sequence $\{\varepsilon_k\}$ is allowed to depend on n, that is, $\varepsilon_k = \varepsilon_k(n) = \sigma_n^2 \delta_k$, where $\{\delta_k\}$ are independent, identically distributed and centered random variables with $\mathrm{E}\delta_1^2 = 1$ and $\sigma_n^2 \to \sigma^2 > 0$ as $n \to \infty$.

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3. PROOFS

Section 3 is divided into six sections. In the first, we derive the representation of $\{X_k(n)\}$, which is the basic tool for the proofs of Theorems 2.1–2.5 in the subsequent sections.

To simplify notation, we suppress the dependence on n in what follows. Hence, we abbreviate

$$X_k = X_k(n) \quad \text{and} \quad b_k = b_k(n)$$

and

$$\varphi = \varphi_n, \qquad \tilde{\varphi} = \tilde{\varphi}_n, \quad \text{and} \quad \omega^2 = \omega_n^2.$$

Nonetheless, we are actually dealing with a triangular array of random variables. Throughout the section, we will use

$$t = \lfloor \kappa n \rfloor \quad \text{with some } 0 < \kappa \leq 1. \tag{3.1}$$

3.1. An Asymptotic Representation of RCA(1) Time Series

We shall need the following small lemma.

LEMMA 3.1. *If assumptions (2.2)–(2.7) and (3.1) are satisfied, it holds that*

$$\max_{0 \leq i < t} |\varphi - 1 + b_{t-i}| = o_P(1) \qquad (n \to \infty).$$

Proof. We can estimate

$$\max_{0 \leq i < t} |\varphi - 1 + b_{t-i}| \leq |\tilde{\varphi}| + \omega \max_{0 \leq i < t} |\gamma_{t-i}|. \tag{3.2}$$

Using Corollary 3 of Chow and Teicher (1988, p. 90), we see that

$$\max_{0 \leq i < t} |\gamma_{t-i}| = \max_{1 \leq j \leq t} |\gamma_j| = \mathcal{O}(\sqrt{t}) \quad \text{a.s.}$$

as $n \to \infty$. The claim follows from (3.2) after an application of (2.4), (2.5), and (2.7). ■

A repeated application of the defining equations (2.1) yields

$$X_t = X_0 \prod_{0 \leq i < t} (\varphi + b_{t-i}) + \sum_{0 \leq j < t} \prod_{0 \leq i < j} (\varphi + b_{t-i}) \varepsilon_{t-j}. \tag{3.3}$$

If the parameters φ and ω^2 are independent of n, display (3.3) is used as a starting point to derive necessary and sufficient criteria for the existence of a unique strictly stationary solution to the RCA(1) equations. The proof includes the application of the strong law of large numbers to the products

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$\prod_{i=0}^{j-1}(\varphi + b_{t-i})$. See Aue et al. (2006) for details. Here, however, $\varphi = \varphi_n$ and $\omega^2 = \omega_n^2$, and the notion of strict stationarity becomes obsolete. In the setting specified by (2.1), we obtain instead an additive decomposition of (3.3) by investigating the same products. Details are given next.

THEOREM 3.1. *Let $\{X_k\}$ follow model (2.1). If (2.2)–(2.7) and (3.1) are satisfied, then*

$$X_t = X_0 e^{t\tilde{\varphi}}\left(1 + \sum_{0\le j<t} b_{t-j} + R_t^{(1)}\right) + \sum_{0\le j<t} e^{j\tilde{\varphi}}\left(1 + \sum_{0\le i<j} b_{t-i} + R_{t,j}^{(2)}\right)(1 + R_{t,j}^{(3)})\varepsilon_{t-j},$$

where, as $n \to \infty$,

$$|R_t^{(1)}| = \mathcal{O}_P(t(\tilde{\varphi}^2 + \omega^2)),$$

$$\max_{0\le j<t} |R_{t,j}^{(2)}| = \mathcal{O}_P(t\omega^2),$$

$$\max_{0\le j<t} \frac{|R_{t,j}^{(2)}|}{j \log\log j} = \mathcal{O}_P(\omega^2),$$

$$\max_{0\le j<t} \frac{1}{j} |R_{t,j}^{(3)}| = \mathcal{O}_P(\tilde{\varphi}^2 + \omega^2).$$

Proof. The proof is given in three steps. Using display (3.3), we obtain first an additive representation for the product $\prod_{i=0}^{j-1}(\varphi + b_{t-i})$, which is refined in a second and third step.

(a) Define the events

$$A_n = \left\{ \max_{0\le i<t} |\tilde{\varphi} + b_{t-i}| \le \frac{1}{2} \right\}, \qquad n \ge 1.$$

Recalling that $t = \lfloor \kappa n \rfloor$, it follows from Lemma 3.1 that $\lim_{n\to\infty} P\{A_n\} = 1$. A Taylor expansion yields $|\log(1 + x) - x| \le 2x^2$ for $|x| \le \frac{1}{2}$. Hence, for all $j < t$, on A_n,

$$|S_{t,j}| = \left| \sum_{0\le i<j} \log(\varphi + b_{t-i}) - \sum_{0\le i<j} (\tilde{\varphi} + b_{t-i}) \right| \le 2 \sum_{0\le i<j} (\tilde{\varphi} + b_{t-i})^2 \le 4j\tilde{\varphi}^2 + 4\omega^2 \sum_{0\le i<j} \gamma_{t-i}^2.$$

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Using the strong law of large numbers and that, by definition, $Eb_1^2 = \omega^2$, we can estimate the sum of the γ_i^2's as

$$\max_{1\le i\le t} \frac{1}{i} |\gamma_t^2 + \cdots + \gamma_{t-i}^2| \stackrel{\mathcal{D}}{=} \max_{1\le i\le t} \frac{1}{i} |\gamma_1^2 + \cdots + \gamma_i^2| = \mathcal{O}_P(1) \qquad (n \to \infty).$$

This implies for the product of interest

$$\begin{aligned} \prod_{0\le i<j} (\varphi + b_{t-i}) &= \exp(j\tilde{\varphi}) \exp\left(\sum_{0\le i<j} b_{t-i} \right) \exp(S_{t,j}) \\ &= \exp(j\tilde{\varphi}) \exp\left(\sum_{0\le i<j} b_{t-i} \right) (1 + R_{t,j}^{(3)}) \end{aligned} \tag{3.4}$$

as $n \to \infty$, where

$$\begin{aligned} \max_{0\le j<t} \frac{1}{j} |S_{t,j}| &= \mathcal{O}_P\left(\max_{0\le j<t} \frac{1}{j} \left[4j\tilde{\varphi}^2 + 4\omega^2 \sum_{0\le i<j} \gamma_{t-i}^2 \right] \right) \\ &= \mathcal{O}_P(\tilde{\varphi}^2 + \omega^2) \qquad (n \to \infty). \end{aligned}$$

Hence, $R_{t,j}^{(3)}$ satisfies the corresponding rate given in Theorem 3.1, namely,

$$\max_{0\le j<t} \frac{1}{j} |R_{t,j}^{(3)}| = \mathcal{O}_P(\tilde{\varphi}^2 + \omega^2) \qquad (n \to \infty).$$

(b) Next, we will estimate the middle term on the right-hand side of display (3.4). Because $\{\gamma_i\}$ is a sequence of independent, identically distributed random variables with $E\gamma_1^2 = 1$, it follows from the weak convergence of partial sums to Brownian motion that

$$\max_{0\le j<t} \left| \sum_{0\le i<j} \gamma_{t-i} \right| = \mathcal{O}_P(\sqrt{t}) \qquad (n \to \infty).$$

Hence, from (2.7) and (3.1),

$$\max_{0\le j<t} \left| \sum_{0\le i<j} b_{t-i} \right| = \max_{0\le j<t} \left| \omega \sum_{0\le i<j} \gamma_{t-i} \right| = \mathcal{O}_P(\sqrt{t}\omega) = o_P(1) \qquad (n \to \infty). \tag{3.5}$$

Define the events

$$B_n = \left\{ \max_{0\le j<t} \left| \sum_{0\le i<j} b_{t-i} \right| \le \frac{1}{2} \right\}, \qquad n \ge 1.$$

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By (3.5), $\lim_{n\to\infty} P\{B_n\} = 1$. Using Taylor expansion, $|\exp(x) - (1 + x)| \le (\sqrt{e}/2)x^2$ for $|x| \le \frac{1}{2}$. Thus, for all $j < t$, on B_n,

$$|R_{t,j}^{(2)}| = \left|\exp\Big(\sum_{0\le i<j} b_{t-i}\Big) - \Big(1 + \sum_{0\le i<j} b_{t-i}\Big)\right| \le \frac{\sqrt{e}}{2}\Big(\sum_{0\le i<j} b_{t-i}\Big)^2$$

with

$$\max_{0\le j<t}\Big(\sum_{0\le i<j} b_{t-i}\Big)^2 = \max_{0\le j<t} \omega^2\Big(\sum_{0\le i<j} \gamma_{t-i}\Big)^2 = \mathcal{O}_P(t\omega^2)$$

as $n \to \infty$ using (3.5). Furthermore,

$$\max_{0\le j<t} \frac{1}{j\log\log j}\Big(\sum_{0\le i<j} b_{t-i}\Big)^2 = \max_{0\le j<t} \frac{\omega^2}{j\log\log j}\Big(\sum_{0\le i<j} \gamma_{t-i}\Big)^2 = \mathcal{O}_P(\omega^2)$$

as $n \to \infty$, where the law of the iterated logarithm has been applied to the sequence $\{\gamma_i\}$ to obtain the rate $\mathcal{O}_P(\omega^2)$. Inserting these results into the right-hand side of (3.4), we arrive at

$$\prod_{0\le i<j}(\varphi + b_{t-i}) = \exp(j\tilde{\varphi})\Big(1 + \sum_{0\le i<j} b_{t-i} + R_{t,j}^{(2)}\Big)(1 + R_{t,j}^{(3)}), \tag{3.6}$$

where $R_{t,j}^{(2)}$ and $R_{t,j}^{(3)}$ satisfy the corresponding rates given in Theorem 3.1.

(c) To finally establish the additive decomposition, we study the first product in (3.3), $\prod_{i=0}^{t-1}(\varphi + b_{t-i})$. From part (a) of the proof,

$$\max_{0\le j<t}|R_{t,j}^{(3)}| = \mathcal{O}_P\Big(\max_{0\le j<t}|S_{t,j}|\Big) = \mathcal{O}_P(t(\tilde{\varphi}^2 + \omega^2)) = o_P(1) \qquad (n \to \infty).$$

Thus, we obtain from (3.6) that

$$\prod_{0\le i<t}(\varphi + b_{t-i}) = \exp(t\tilde{\varphi})\Big(1 + \sum_{0\le i<t} b_{t-i} + \mathcal{O}_P(t\omega^2)\Big)(1 + \mathcal{O}_P(t(\tilde{\varphi}^2 + \omega^2))).$$

Applying the central limit theorem to the partial sums of $\{\gamma_i\}$ and taking into account assumption (2.7) yields that

$$\Big(1 + \sum_{0\le i<t} b_{t-i} + \mathcal{O}_P(t\omega^2)\Big)(1 + \mathcal{O}_P(t(\tilde{\varphi}^2 + \omega^2)))$$
$$= 1 + \sum_{0\le i<t} b_{t-i} + \mathcal{O}_P(t(\tilde{\varphi}^2 + \omega^2))$$

as $n \to \infty$. So, we arrive at

$$\prod_{0\le i<t}(\varphi + b_{t-i}) = \exp(t\tilde{\varphi})\Big(1 + \sum_{0\le i<t} b_{t-i} + R_t^{(1)}\Big) \qquad (n \to \infty),$$

where the implicitly defined quantity $R_t^{(1)}$ satisfies the rate stated in Theorem 3.1. ■

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Theorem 3.1 is arguably the main technical contribution delivered in this paper. It modifies the techniques developed for the case in which the two major parameters φ and ω^2 are constant (as used, e.g., in Aue et al., 2006) so that they are applicable in the near-integrated setting.

3.2. Proof of Theorem 2.3

Theorems 2.3–2.5 are proved by an application of Theorem 3.1. Rewriting the decomposition derived in the latter theorem yields the statement of the following proposition.

PROPOSITION 3.1. *If the assumptions of Theorem 3.1 are satisfied, it holds that*

$$\begin{aligned} X_t &= X_0 e^{t\tilde{\varphi}} \Bigg(1 + \sum_{0\le j<t} b_{t-j} + R_t^{(1)}\Bigg) + \sum_{0\le j<t} e^{j\tilde{\varphi}} \Bigg(1 + \sum_{0\le i<j} b_{t-i} + R_{t,j}^{(2)}\Bigg) R_{t,j}^{(3)} \varepsilon_{t-j} \\ &\quad + \sum_{0\le j<t} e^{j\tilde{\varphi}} R_{t,j}^{(2)} \varepsilon_{t-j} + \sum_{0\le j<t} e^{j\tilde{\varphi}} \Bigg(1 + \sum_{0\le i<j} b_{t-i}\Bigg) \varepsilon_{t-j} \\ &= X_t^{(1)} + X_t^{(2)} + X_t^{(3)} + X_t^{(4)}. \end{aligned} \tag{3.7}$$

It turns out that the dominating part of X_t is $X_t^{(4)}$. Its limiting behavior is derived in Lemma 3.2. In Lemmas 3.3–3.5 it is proved that the terms $X_t^{(1)}$, $X_t^{(2)}$, and $X_t^{(3)}$ are negligible. Thus, Theorem 2.3 follows readily from these auxiliary results. Recall that in Section 3.2 $\tilde{\varphi} > 0$.

LEMMA 3.2. *Under the assumptions of Theorem 2.3,*

$$\sqrt{2\tilde{\varphi}} e^{-t\tilde{\varphi}} X_t^{(4)} \xrightarrow{\mathcal{D}} \sigma\xi,$$

where σ^2 is given in (2.3) and ξ denotes a standard normally distributed random variable.

Proof. From the central limit theorem, $\gamma_t + \gamma_{t-1} + \cdots + \gamma_1 = \mathcal{O}_P(\sqrt{t})$ as $n \to \infty$. Hence,

$$\sum_{0\le j<t} b_{t-j} = \omega \sum_{0\le j<t} \gamma_{t-j} = \mathcal{O}_P(\sqrt{t}\omega^2) = o_P(1) \qquad (n \to \infty), \tag{3.8}$$

by (2.7). This implies that the dominating part of $X_t^{(4)}$ is, as $n \to \infty$, the sum $\sum_{j=0}^{t-1} e^{j\tilde{\varphi}} \varepsilon_{t-j}$. Consequently, it follows from Theorem 27.2 in Billingsley (1995) that the lemma is established if the Lindeberg condition is verified for the partial sums $\sum_{j=0}^{t-1} \xi_{n,j}$, where

$$\xi_{n,j} = \sqrt{\tilde{\varphi}} e^{-(t-j)\tilde{\varphi}} \varepsilon_{t-j}.$$

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Observe first that clearly $\mathrm{E}\xi_{n,j} = 0$ and

$$\sum_{j=0}^{t-1} \mathrm{E}\xi_{n,j}^2 = 2\tilde{\varphi}\sigma^2 \sum_{j=0}^{t-1} e^{-2(t-j)\tilde{\varphi}} \to \sigma^2 \qquad (n \to \infty).$$

Lindeberg's condition then follows from the relation $0 > -(t-j)\tilde{\varphi} \to -\infty$ and the fact that $\{\varepsilon_j\}$ consists of independent, identically distributed random variables. Indeed, for any $\eta > 0$ and $n \to \infty$,

$$\begin{aligned}
&\sum_{j=0}^{t-1} \mathrm{E}[\xi_{n,j}^2 1\{|\xi_{n,j}| > \eta\}] \\
&\quad \le 2\tilde{\varphi} \sum_{j=0}^{t-1} e^{-2(t-j)\tilde{\varphi}} \mathrm{E}[\varepsilon_{t-j}^2 1\{\sqrt{2\tilde{\varphi}}|\varepsilon_{t-j}| > \eta\}] \\
&\quad = \mathrm{E}\left[\varepsilon_1^2 1\left\{|\varepsilon_1| > \frac{\eta}{\sqrt{2\tilde{\varphi}}}\right\}\right] 2\tilde{\varphi}\sigma^2 \sum_{j=0}^{t-1} e^{-2(t-j)\tilde{\varphi}} \\
&\quad \le C\mathrm{E}\left[\varepsilon_1^2 1\left\{|\varepsilon_1| > \frac{\eta}{\sqrt{2\tilde{\varphi}}}\right\}\right] \to 0,
\end{aligned}$$

because $1/\sqrt{\tilde{\varphi}} \to \infty$ and $\mathrm{E}\varepsilon_1^2 = \sigma^2 < \infty$. Here $1\{A\}$ is the indicator function of a set A and C is a positive constant. ■

LEMMA 3.3. *If the assumptions of Theorem 2.3 are satisfied, then*

$$\frac{\sqrt{\tilde{\varphi}}e^{-t\tilde{\varphi}}}{\sigma} X_t^{(1)} = o_P(1) \qquad (n \to \infty).$$

Proof. Using Theorem 3.1, (2.6), and (2.7), we obtain that

$$R_t^{(1)} = \mathcal{O}_P(t(\tilde{\varphi}^2 + \omega^2)) = o_P(1) \qquad (n \to \infty).$$

Moreover, $b_t + b_{t-1} + \cdots + b_1 = o_P(1)$ by (3.8), and clearly also

$$\frac{\sqrt{\tilde{\varphi}}e^{-t\tilde{\varphi}}}{\sigma} X_0 e^{t\tilde{\varphi}} = \frac{\sqrt{\tilde{\varphi}}X_0}{\sigma} = o_P(1) \qquad (n \to \infty)$$

by the fact that $\tilde{\varphi} \to 0$. Hence, the proof is complete. ■

LEMMA 3.4. *Under the assumptions of Theorem 2.3,*

$$\frac{\sqrt{\tilde{\varphi}}e^{-t\tilde{\varphi}}}{\sigma} X_t^{(2)} = o_P(1) \qquad (n \to \infty).$$

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Proof. Observe that, by Theorem 3.1 and (2.7),

$$\max_{0\le j<t} |R_{t,j}^{(2)}| = \mathcal{O}_P(t\omega^2) = o_P(1) \qquad (n \to \infty).$$

Thus, by (3.8) and Theorem 3.1, the dominating part of $X_t^{(2)}$ is $\sum_{j=0}^{t-1} e^{j\tilde{\varphi}} R_{t,j}^{(3)} \varepsilon_{t-j}$. Note that, by assumption (2.3), $R_{t,j}^{(3)}$, which has been defined in the proof of Theorem 3.1, and ε_{t-j} are uncorrelated. Hence,

$$\mathrm{E}\left[\sum_{0\le j<t} e^{j\tilde{\varphi}} R_{t,j}^{(3)} \varepsilon_{t-j}\right]^2 = \sigma^2 \sum_{0\le j<t} e^{2j\tilde{\varphi}} \mathrm{E}[R_{t,j}^{(3)}]^2$$
$$\le t^2(\tilde{\varphi}^2 + \omega^2)^2 \mathrm{E}\left[\sum_{0\le j<t} e^{j\tilde{\varphi}} \varepsilon_{t-j}\right]^2,$$

where $\mathrm{E}[R_{t,j}^{(3)}]^2$ has been estimated using the corresponding rate in Theorem 3.1. Recognizing that $\sum_{j=0}^{t-1} e^{j\tilde{\varphi}} \varepsilon_{t-j}$ is the dominating term of $X_t^{(4)}$, it follows from Lemma 3.3 and Chebyshev's inequality that, for any $\varepsilon > 0$,

$$P\{\sqrt{\tilde{\varphi}} e^{-t\tilde{\varphi}} |X_t^{(2)}| \ge \varepsilon\} \le \mathcal{O}(1) t^2 (\tilde{\varphi}^2 + \omega^2)^2 = o(1)$$

as $n \to \infty$, where (2.6) and (2.7) have been applied to obtain the rate $o(1)$. ■

LEMMA 3.5. *Under the assumptions of Theorem 2.3,*

$$\frac{\sqrt{\tilde{\varphi}} e^{-t\tilde{\varphi}}}{\sigma} X_t^{(3)} = o_P(1) \qquad (n \to \infty).$$

Proof. First note that the order of magnitude of $X_t^{(3)}$ is determined by $\sum_{j=0}^{t-1} e^{j\tilde{\varphi}} R_{t,j}^{(2)} \varepsilon_{t-j}$. Then, as in the proof of Lemma 3.4,

$$\mathrm{E}\left[\sum_{0\le j<t} e^{j\tilde{\varphi}} R_{t,j}^{(2)} \varepsilon_{t-j}\right]^2 \le t^2 \omega^4 \mathrm{E}\left[\sum_{0\le j<t} e^{j\tilde{\varphi}} \varepsilon_{t-j}\right]^2,$$

where the corresponding rate for $R_{t,j}^{(2)}$ as established in Theorem 3.1 has been applied. Consequently, Chebyshev's inequality and (2.7) yield

$$P\{\sqrt{\tilde{\varphi}} e^{-t\tilde{\varphi}} |X_t^{(2)}| \ge \varepsilon\} \le \mathcal{O}(1) t^2 \omega^4 = o(1)$$

as $n \to \infty$, completing the proof. ■

Proof of Theorem 2.3. The assertion follows from putting together the results of Lemmas 3.2–3.5. ■

3.3. Proof of Theorem 2.4

We follow the arguments of the previous section using decomposition (3.7) from Proposition 3.1. It will be shown in a series of lemmas that $X_t^{(4)}$ is determining

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the limit distribution, whereas the other terms are asymptotically negligible. The main auxiliary result is the following lemma. Recall that $t = \lfloor \kappa n \rfloor$ with some $\kappa \in (0,1]$ by (3.1) and that $\tilde{\varphi} = 0$.

LEMMA 3.6. *Under the assumptions of Theorem 2.4,*

$$\frac{1}{\sqrt{n}\sigma} X_t^{(4)} \xrightarrow{\mathcal{D}} W(\kappa) \qquad (n \to \infty),$$

where $\{W(s) : s \in [0,1]\}$ *denotes a standard Wiener process.*

Proof. Referring to (3.8), we obtain similarly that to derive the limit distribution of $1/(\sqrt{n}\sigma)X_t^{(4)}$ it suffices to study the partial sums $1/(\sqrt{n}\sigma)\sum_{0\le j<t} \varepsilon_{t-j}$. The errors $\{\varepsilon_i\}$, however, constitute an independent, identically distributed sequence of centered random variables with finite second moments. Hence, the functional central limit theorem implies that

$$\frac{1}{\sqrt{n}\sigma} \sum_{0\le j<t} \varepsilon_{t-j} = \frac{1}{\sqrt{n}\sigma} \sum_{1\le j\le t} \varepsilon_j \xrightarrow{\mathcal{D}} W(\kappa),$$

where $\{W(s) : s \in [0,1]\}$ denotes a standard Brownian motion. ■

It remains to show that all other terms in the representation (3.7) are negligible. This will be done in the following lemmas.

LEMMA 3.7. *Under the assumptions of Theorem 2.4,*

$$\frac{1}{\sqrt{n}\sigma} X_t^{(1)} = o_P(1) \qquad (n \to \infty).$$

Proof. By (2.7) and Theorem 3.1, $R_t^{(1)} = o_P(1)$. Moreover, $(\sqrt{n}\sigma)^{-1}X_0 = o_P(1)$ and, as in (3.8), $b_t + \cdots + b_1 = o_P(1)$ after an application of the central limit theorem to the partial sums of the γ_i's and the claim follows. ■

LEMMA 3.8. *Under the assumptions of Theorem 2.4,*

$$\frac{1}{\sqrt{n}\sigma} X_t^{(2)} = o_P(1) \qquad (n \to \infty).$$

Proof. The triangle inequality, Theorem 3.1, (2.7), and (3.8) imply that

$$\max_{0\le j<t} \left| \sum_{0\le i<t} b_{t-j} + R_{t,j}^{(2)} \right| \le \omega \max_{0\le j<t} \left| \sum_{0\le i<t} \gamma_{t-j} \right| + \max_{0\le j<t} |R_{t,j}^{(2)}| = o_P(1) \qquad (n \to \infty).$$

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Hence, the order of magnitude of $X_t^{(2)}$ is given by $\sum_{0\le i<t} R_{t,j}^{(3)} \varepsilon_{t-j}$. Proceeding exactly as in the proof of Lemma 3.4, we arrive at

$$P\left\{ \frac{1}{\sqrt{n}} |X_t^{(2)}| \ge \varepsilon \right\} = \mathcal{O}(1) t^2 (\tilde{\varphi}^2 + \omega^2)^2 = o(1) \qquad (n \to \infty)$$

after an application of (2.6) and (2.7). ■

LEMMA 3.9. *Under the assumptions of Theorem 2.4,*

$$\frac{1}{\sqrt{n}\sigma} X_t^{(3)} = o_P(1) \qquad (n \to \infty),$$

where $R_{t,j}^{(2)}$ satisfies the corresponding rate in Theorem 3.1.

Proof. By the same arguments developed in the proof of Lemma 3.5,

$$P\left\{ \frac{1}{\sqrt{n}} |X_t^{(3)}| \ge \varepsilon \right\} = \mathcal{O}(1) t^2 \omega^4 = o(1) \qquad (n \to \infty),$$

using (2.7). ■

Proof of Theorem 2.4. The assertion follows directly from Lemmas 3.6–3.9. ■

3.4. Proof of Theorem 2.5

The following lemma will be used throughout the proof of Theorem 2.5. It has been established in Berkes et al. (2005) within the context of near-integrated GARCH(1,1) sequences, and we state it without proof. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. We say that $a_n \sim b_n$ if $\lim_{n\to\infty} a_n b_n^{-1} = 1$. Note that in Section 3.4 always $\tilde{\varphi} < 0$.

LEMMA 3.10. *For any $\nu \ge 0$ it holds that*

$$\sum_{0\le j<t} j^\nu e^{j\tilde{\varphi}} \sim \frac{\Gamma(\nu+1)}{|\tilde{\varphi}|^{\nu+1}},$$

where $\tilde{\varphi} = \varphi - 1 < 0$ and $\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx$ denotes the gamma function.

Proof. See Berkes et al. (2005), Lemma 4.1.

We shall repeat the steps accomplished in Sections 3.2 and 3.3. First, we will derive the limiting behavior of the term $X_t^{(4)}$ and then show that all other terms of the decomposition do not contribute to the asymptotics. Fix N and define

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$$t_m = \lfloor \kappa_m n \rfloor \quad \text{with some } 0 < \kappa_m < 1, \qquad m = 1, \ldots, N, \tag{3.9}$$

where $\kappa_1 < \cdots < \kappa_N$. Our first auxiliary result is the following lemma.

LEMMA 3.11. *Let $\tilde{\varphi} < 0$. If the second part of assumption (2.6) is satisfied, then*

$$\sqrt{2|\tilde{\varphi}|}[\tau_1, \ldots, \tau_N] \xrightarrow{\mathcal{D}} \sigma[\xi_1, \ldots, \xi_N] \qquad (n \to \infty),$$

where, for $m = 1, \ldots, N$,

$$\tau_m = \sum_{0 \le i < t_m} e^{i\tilde{\varphi}} \varepsilon_{t_m - i}$$

and $\xi_1, \ldots, \xi_N$ are independent, identically distributed standard normal random variables.

Proof. The proof is given in two steps. First, the Cramér–Wold device is applied; hence, we are investigating linear combinations of $\tau_1, \ldots, \tau_N$. Then we show that the central limit theorem is satisfied by verifying Liapounov's condition in a second part.

(a) As a consequence of the Cramér–Wold device, the N-dimensional weak convergence result in Lemma 3.11 is equivalent to the one-dimensional statement

$$\sqrt{2|\tilde{\varphi}|}(\lambda_1 \tau_1 + \cdots + \lambda_N \tau_N) \xrightarrow{\mathcal{D}} \sigma(\lambda_1 \xi_1 + \cdots + \lambda_N \xi_N) \tag{3.10}$$

being true for all $\lambda_1, \ldots, \lambda_N$ (see Billingsley, 1995, Thm. 29.4). Note that the limit variable is normal with zero mean and variance $\sigma^2(\lambda_1^2 + \cdots + \lambda_N^2)$. To prove relation (3.10), choose real numbers $\lambda_1, \ldots, \lambda_N$ and observe that

$$\begin{aligned}
&\lambda_1 \tau_1 + \cdots + \lambda_N \tau_N \\
&\quad = \sum_{1 \le i \le t_1} [\lambda_1 e^{(t_1 - i)\tilde{\varphi}} + \lambda_2 e^{(t_2 - i)\tilde{\varphi}} + \cdots + \lambda_N e^{(t_N - i)\tilde{\varphi}}] \varepsilon_i \\
&\qquad + \sum_{t_1 < i \le t_2} [\lambda_2 e^{(t_2 - i)\tilde{\varphi}} + \cdots + \lambda_N e^{(t_N - i)\tilde{\varphi}}] \varepsilon_i \\
&\qquad + \cdots + \sum_{t_{N-1} < i \le t_N} \lambda_N e^{(t_N - i)\tilde{\varphi}} \varepsilon_i \\
&\quad = S_1 + \cdots + S_N.
\end{aligned} \tag{3.11}$$

In the remainder of the first part of the proof, the moment assumptions of Theorem 27.3 in Billingsley (1995) will be verified. Clearly, $S_1, \ldots, S_N$ have expectation zero. Moreover, it holds for the variance of S_1 that

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$$\mathrm{E}S_1^2 = \sigma^2 \sum_{1\le i\le t_1} [\lambda_1 e^{(t_1-i)\tilde{\varphi}} + \cdots + \lambda_N e^{(t_N-i)\tilde{\varphi}}]^2$$

$$= \sigma^2 \sum_{1\le j\le N} \lambda_j^2 \sum_{1\le i\le t_1} e^{2(t_j-i)\tilde{\varphi}} + \sigma^2 \sum_{1\le j\ne l\le N} \lambda_j \lambda_l \sum_{1\le i\le t_1} e^{(t_j+t_l-2i)\tilde{\varphi}}.$$

By Lemma 3.10,

$$\sum_{1\le i\le t_1} e^{2(t_1-i)\tilde{\varphi}} = \sum_{0\le i<t_1} e^{2i\tilde{\varphi}} \sim \frac{1}{2|\tilde{\varphi}|} \qquad (n\to\infty).$$

On the other hand, for $2 \le j \le N$, Lemma 3.10 also implies that

$$\sum_{1\le i\le t_1} e^{2(t_j-i)\tilde{\varphi}} = e^{2(t_j-t_1)\tilde{\varphi}} \sum_{1\le i\le t_1} e^{2(t_1-i)\tilde{\varphi}} = o\left(\frac{1}{|\tilde{\varphi}|}\right) \qquad (n\to\infty),$$

because $(t_j - t_1)\tilde{\varphi} \to -\infty$. Similar arguments apply to the second part of $\mathrm{E}S_1^2$. In detail,

$$\sum_{1\le i\le t_1} e^{(t_j+t_l-2i)\tilde{\varphi}} = e^{(t_j-t_1)\tilde{\varphi}} e^{(t_l-t_1)\tilde{\varphi}} \sum_{1\le i\le t_1} e^{2(t_1-i)\tilde{\varphi}}$$

$$\sim \frac{1}{2|\tilde{\varphi}|} e^{(t_j-t_1)\tilde{\varphi}} e^{(t_l-t_1)\tilde{\varphi}} = o\left(\frac{1}{|\tilde{\varphi}|}\right)$$

as $n \to \infty$. Hence, putting together the previous calculations, we obtain

$$\mathrm{E}S_1^2 = \frac{\lambda_1^2\sigma^2}{2|\tilde{\varphi}|} + o\left(\frac{1}{|\tilde{\varphi}|}\right) \qquad (n\to\infty).$$

Repeating the preceding arguments for any of the remaining $\mathrm{E}S_j^2$ ($j = 2, \ldots, N$), we arrive at

$$\mathrm{E}[\lambda_1\tau_1 + \cdots + \lambda_N\tau_N]^2 = \frac{\sigma^2}{2|\tilde{\varphi}|}(\lambda_1^2 + \cdots + \lambda_N^2) + o\left(\frac{1}{|\tilde{\varphi}|}\right) \qquad (n\to\infty),$$

finishing the first part of the proof.

(b) It remains to verify Liapounov's condition. Then Theorem 27.3 in Billingsley (1995) implies that the central limit theorem is satisfied, and the assertion of Lemma 3.11 is readily proved. Observe that by a rearrangement of terms in (3.11), we can write

$$\lambda_1\tau_1 + \cdots + \lambda_N\tau_N = \mu_1\varepsilon_1 + \cdots + \mu_{t_N}\varepsilon_{t_N}$$

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with appropriately chosen coefficients μ_i $(i = 1, \ldots, t_N)$. In what follows it will be shown that

$$\frac{[|\mu_1|^{2+\delta}\mathrm{E}|\varepsilon_1|^{2+\delta} + \cdots + |\mu_{t_N}|^{2+\delta}\mathrm{E}|\varepsilon_{t_N}|^{2+\delta}]^{1/(2+\delta)}}{[\mu_1^2\mathrm{E}\varepsilon_1^2 + \cdots + \mu_{t_N}^2\mathrm{E}\varepsilon_{t_N}^2]^{1/2}} = o(1) \qquad (n \to \infty) \tag{3.12}$$

for some $\delta > 0$ (Liapounov's condition; see Billingsley, 1995, p. 312). First, we determine the precise order of magnitude of the denominator. It holds that

$$\mu_1^2\mathrm{E}\varepsilon_1^2 + \cdots + \mu_{t_N}^2\mathrm{E}\varepsilon_{t_N}^2 = \sigma^2(\mu_1^2 + \cdots + \mu_{t_N}^2) \sim \mathcal{O}\left(\frac{1}{|\tilde{\varphi}|}\right) \qquad (n \to \infty),$$

where $\sim$ is implied by the last display in part (a) of the proof. To check the numerator, select the summation range $i = 1, \ldots, t_1$ and observe that, by Jensen's inequality,

$$\begin{aligned}|\mu_i|^{2+\delta} &= |\lambda_1 e^{(t_1-i)\tilde{\varphi}} + \cdots + \lambda_N e^{(t_N-i)\tilde{\varphi}}|^{2+\delta} \\ &\le C_1(N)[|\lambda_1|^{2+\delta}e^{(t_1-i)\tilde{\varphi}(2+\delta)} + \cdots + \lambda_N|^{2+\delta}e^{(t_N-i)\tilde{\varphi}(2+\delta)}].\end{aligned}$$

Hence,

$$|\mu_1|^{2+\delta} + \cdots + |\mu_{t_1}|^{2+\delta} \sim \frac{C_1(N)|\lambda_1|^{2+\delta}}{(2+\delta)|\tilde{\varphi}|} + o\left(\frac{1}{|\tilde{\varphi}|}\right) \le \frac{C_2(N)}{|\tilde{\varphi}|} \qquad (n \to \infty).$$

Using similar calculations for the remaining summation ranges, we find that the asymptotic rate in (3.12) can be estimated by

$$\mathcal{O}\left(\left[\frac{1}{|\tilde{\varphi}|}\right]^{1/(2+\delta)-1/2}\right) = o(1) \qquad (n \to \infty).$$

This completes the proof. ■

LEMMA 3.12. *Under the assumptions of Theorem 2.5,*

$$\sqrt{2|\tilde{\varphi}|}\sigma^{-1}X_t^{(1)} = o_P(1) \qquad (n \to \infty).$$

Proof. By (2.6), (2.7), and Theorem 3.1, $R_t^{(1)} = o_P(1)$. Moreover,

$$\sqrt{2|\tilde{\varphi}|}\sigma^{-1}X_0 e^{t\tilde{\varphi}} = o_P(1) \qquad (n \to \infty),$$

because $t\tilde{\varphi} \to -\infty$ by assumption (2.6). Finally, (3.8) yields the assertion. ■

LEMMA 3.13. *Under the assumptions of Theorem 2.5,*

$$\sqrt{2|\tilde{\varphi}|}\sigma^{-1}X_t^{(2)} = o_P(1) \qquad (n \to \infty).$$

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Proof. Following the lines of the proofs of Lemmas 3.4 and 3.8, we get

$$P\{\sqrt{|\tilde{\varphi}|}|X_t^{(2)}| \geq \varepsilon\} = \mathcal{O}(1)t^2(\tilde{\varphi}^2 + \omega^2)^2 = o(1) \qquad (n \to \infty),$$

utilizing Chebyshev's inequality and the rate assumptions (2.6) and (2.7). ■

LEMMA 3.14. *Under the assumptions of Theorem 2.5,*

$$\sqrt{2|\tilde{\varphi}|}\sigma^{-1}X_t^{(3)} = o_P(1) \qquad (n \to \infty).$$

Proof. Similar to the proofs of Lemmas 3.5 and 3.9, it holds that

$$P\{\sqrt{|\tilde{\varphi}|}|X_t^{(3)}| \geq \varepsilon\} = \mathcal{O}(1)t^2\omega^4 = o(1) \qquad (n \to \infty),$$

completing the proof. ■

Proof of Theorem 2.5. It follows from Lemmas 3.11–3.14. Lemma 3.11 immediately yields the limit distribution as stated in Theorem 2.5 on noticing that, for all $m = 1, \ldots, N$,

$$\sqrt{2|\tilde{\varphi}|}X_{t_m}^{(4)} \sim \sqrt{2|\tilde{\varphi}|}\sum_{0 \leq j < t} e^{j\tilde{\varphi}}\varepsilon_{t_m - j} \qquad (n \to \infty)$$

along the lines of Lemmas 3.2 and 3.6. According to Lemmas 3.12–3.14, all other terms in the decomposition given in Proposition 3.1 are negligible. ■

3.5. Proof of Theorem 2.1

The proof is based on Proposition 3.1 and bears similarities to the methodology developed in Anderson (1959). Note, however, that we are not dealing with time indices $t = \lfloor \kappa n \rfloor$ as introduced in (3.1). Hence, we switch to using k as labeling index in decomposition (3.1), still suppressing the dependence on n. Because of the conditions imposed on φ and ω^2 in displays (2.6) and (2.7), respectively, it remains true that the asymptotic behavior of X_k is solely determined by the term $X_k^{(4)}$ and that of X_k^2 is given by $X_k^{(4)^2}$ (see Lemmas 3.18 and 3.19, which follow).

Recall the definition of the least squares estimator $\hat{\varphi}_n$ in (2.8) and note that, in Section 3.5, $\tilde{\varphi} > 0$. In a first part, we shall study the partial sums of $X_k^{(4)^2}$. Repeating the arguments developed in the preceding sections, it can be shown that

$$\sum_{k=1}^{n} X_k^{(4)^2} \sim \sum_{k=1}^{n}\left(\sum_{j=1}^{k} e^{(k-j)\tilde{\varphi}}\varepsilon_j\right)^2 = T_n.$$

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Because no new techniques are required to verify the last statement, details are omitted here to conserve space. Let $T > 0$ and denote

$$\psi_{K,L} = \sum_{j=K+1}^{L} e^{-j\tilde{\varphi}} \varepsilon_j. \tag{3.13}$$

Then we can decompose T_n as follows:

$$\begin{aligned} T_n &= \sum_{k=1}^{n} e^{2k\tilde{\varphi}} \psi_{0,n}^2 \\ &= \sum_{k=1}^{\lfloor T/\tilde{\varphi} \rfloor} e^{2k\tilde{\varphi}} \psi_{0,k}^2 + \sum_{k=\lfloor T/\tilde{\varphi} \rfloor + 1}^{n} e^{2k\tilde{\varphi}} \psi_{\lfloor T/\tilde{\varphi} \rfloor, k}^2 \\ &\quad + 2 \sum_{k=\lfloor T/\tilde{\varphi} \rfloor + 1}^{n} e^{2k\tilde{\varphi}} \psi_{\lfloor T/\tilde{\varphi} \rfloor, k} \psi_{0, \lfloor T/\tilde{\varphi} \rfloor} + \sum_{k=\lfloor T/\tilde{\varphi} \rfloor + 1}^{n} e^{2k\tilde{\varphi}} \psi_{0, \lfloor T/\tilde{\varphi} \rfloor}^2 \\ &= S_{T,1} + S_{T,2} + S_{T,3} + S_{T,4}. \end{aligned} \tag{3.14}$$

The next lemma shows that, relative to the normalization $\tilde{\varphi}^2 e^{-2n\tilde{\varphi}}$, the first three partial sums $S_{T,1}$, $S_{T,2}$, and $S_{T,3}$ are asymptotically small.

LEMMA 3.15. *Let the assumptions of Theorem 2.1 be satisfied.*

(i) For every $T > 0$ and $\varepsilon > 0$,

$$\lim_{n \to \infty} P\left\{ \frac{\tilde{\varphi}^2}{e^{2n\tilde{\varphi}}} S_{T,1} \geq \varepsilon \right\} = 0,$$

where $S_{T,1}$ is defined in (3.14).

(ii) For every $\varepsilon > 0$,

$$\lim_{T \to \infty} \limsup_{n \to \infty} P\left\{ \frac{\tilde{\varphi}^2}{e^{2n\tilde{\varphi}}} S_{T,2} \geq \varepsilon \right\} = 0,$$

where $S_{T,2}$ is defined in (3.14).

(iii) For every $\varepsilon > 0$,

$$\lim_{T \to \infty} \limsup_{n \to \infty} P\left\{ \frac{\tilde{\varphi}^2}{e^{2n\tilde{\varphi}}} S_{T,3} \geq \varepsilon \right\} = 0,$$

where $S_{T,3}$ is defined in (3.14).

Proof. (i) The proof is based on an application of Markov's inequality. Note that

$$\mathrm{E}[S_{T,1}] = \mathrm{E}\left[\sum_{k=1}^{\lfloor T/\tilde{\varphi} \rfloor} e^{2k\tilde{\varphi}} \psi_{0,k}^2 \right] \leq \frac{\sigma^2 T}{\tilde{\varphi}} \sum_{k=1}^{\lfloor T/\tilde{\varphi} \rfloor} e^{2k\tilde{\varphi}} \leq \frac{\sigma^2 T^2}{\tilde{\varphi}^2} e^{2T}$$

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for any fixed $T > 0$. Now Markov's inequality yields

$$P\left\{\frac{\tilde{\varphi}^2}{e^{2n\tilde{\varphi}}} S_{T,1} \geq \varepsilon\right\} \leq \frac{\tilde{\varphi}^2}{\varepsilon e^{2n\tilde{\varphi}}} \mathrm{E} S_{T,1}$$

for any $\varepsilon > 0$. The quantity on the right-hand side of the latter inequality becomes arbitrarily small as $n \to \infty$; hence part (i) is proved.

(ii) Note that $S(x) = \varepsilon_0 + \cdots + \varepsilon_{\lfloor x \rfloor}$ is a right-continuous function. Set $M = \lfloor T/\tilde{\varphi} \rfloor$. Then using integration by parts for $\psi_{M,k}^2$ gives the estimate

$$S_{T,2} \leq 2 \sum_{k=M+1}^{n} e^{2k\tilde{\varphi}} \left(e^{-2k\tilde{\varphi}} S^2(k) + e^{-2M} S^2(M) + \left(\tilde{\varphi} \int_M^k S(x) e^{-x\tilde{\varphi}}\, dx \right)^2 \right). \tag{3.15}$$

We shall only investigate the third term in what follows. Adapting the methods used here will work also for the remaining two terms on the right-hand side of inequality (3.15). First, observe that, by a change of variables and the definition of M,

$$\frac{\tilde{\varphi}^2}{e^{2n\tilde{\varphi}}} \sum_{k=M+1}^{n} e^{2k\tilde{\varphi}} \left(\tilde{\varphi} \int_M^k |S(x)| e^{-x\tilde{\varphi}}\, dx \right)^2$$
$$\leq \frac{\tilde{\varphi}^2}{e^{2n\tilde{\varphi}}} \sum_{k=M+1}^{n} e^{2k\tilde{\varphi}} \left(\int_{T-1}^{\infty} |S(x/\tilde{\varphi})| e^{-x}\, dx \right)^2.$$

By assumption (2.3), the innovation sequence $\{\varepsilon_k\}$ consists of independent, identically distributed random variables with finite second moments, so that $\sqrt{\tilde{\varphi}} S(x/\tilde{\varphi})$ converges weakly to $\sigma W(x)$ on $[0,\infty)$, where $\{W(s) : s \geq 0\}$ denotes standard Brownian motion. Thus, Donsker's theorem yields that

$$\tilde{\varphi} \left(\int_{T-1}^{\infty} |S(x/\tilde{\varphi})| e^{-x}\, dx \right)^2 \xrightarrow{\mathcal{D}} \left(\int_{T-1}^{\infty} |W(x)| e^{-x}\, dx \right)^2 \qquad (n \to \infty).$$

Moreover, for any $T > 0$,

$$\frac{\tilde{\varphi}}{e^{2n\tilde{\varphi}}} \sum_{k=M+1}^{n} e^{2k\tilde{\varphi}} = \frac{\tilde{\varphi}}{e^{2\tilde{\varphi}} - 1} \frac{e^{2(n+1)\tilde{\varphi}} - e^{2(M+1)\tilde{\varphi}}}{e^{2n\tilde{\varphi}}} \to \frac{1}{2} \qquad (n \to \infty)$$

by Taylor expansion for $e^{2\tilde{\varphi}}$ in the first denominator in the middle expression of the preceding display. The second ratio clearly converges to 1. Hence, for all $\varepsilon \geq 0$,

$$\lim_{T\to\infty} \limsup_{n\to\infty} P\left\{ \frac{\tilde{\varphi}^2}{e^{2n\tilde{\varphi}}} \left(\tilde{\varphi} \int_M^k S(x) e^{-x\tilde{\varphi}}\, dx \right)^2 \geq \varepsilon \right\} = 0,$$

which proves the assertion for the first part in (3.15).

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(iii) Assertion (iii) of Lemma 3.15 can be obtained using arguments similar to those applied in the first two parts of this proof. Details are hence omitted. ■

Lemma 3.15 implies that it suffices to derive the limit distribution of the remaining term $S_{T,4}$. Before turning our attention to its asymptotics, on recalling the definition of $\hat{\varphi}_n$ in (2.8), the second part of the proof deals with the partial sums of $X_{k-1}^{(4)}\varepsilon_k$. Let $T > 0$. Then

$$\begin{aligned}\sum_{k=1}^{n} X_{k-1}^{(4)}\varepsilon_k &\sim \sum_{k=1}^{\lfloor T/\tilde{\varphi}\rfloor} X_{k-1}^{(4)}\varepsilon_k + \sum_{k=\lfloor T/\tilde{\varphi}\rfloor+1}^{n} e^{(k-1)\tilde{\varphi}}\varepsilon_k \sum_{j=\lfloor T/\tilde{\varphi}\rfloor+1}^{k} e^{-j\tilde{\varphi}}\varepsilon_j \\ &\quad + \sum_{k=\lfloor T/\tilde{\varphi}\rfloor+1}^{n} e^{(k-1)\tilde{\varphi}}\varepsilon_k \sum_{j=1}^{\lfloor T/\tilde{\varphi}\rfloor} e^{-j\tilde{\varphi}}\varepsilon_j \\ &= R_{T,1} + R_{T,2} + R_{T,3}. \end{aligned} \tag{3.16}$$

It will be shown next that, relative to the normalization $\tilde{\varphi}e^{-n\tilde{\varphi}}$, $R_{T,1}$ and $R_{T,2}$ do not contribute to the limit distribution.

LEMMA 3.16. *Let the assumptions of Theorem 2.1 be satisfied.*

(i) For every $T > 0$ and $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left\{\frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}}\,|R_{T,1}| \geq \varepsilon\right\} = 0,$$

where $R_{T,1}$ is defined in (3.16).

(ii) For every $\varepsilon > 0$,

$$\lim_{T\to\infty} \limsup_{n\to\infty} P\left\{\frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}}\,|R_{T,2}| \geq \varepsilon\right\} = 0,$$

where $R_{T,2}$ is defined in (3.16).

Proof. (i) The proof is based on an application of Chebyshev's inequality. Because $\mathrm{E}R_{T,i} = 0$, for all $i = 1,2,3$, we have that $\operatorname{Var} R_{T,i} = \mathrm{E}R_{T,i}^2$, and it is enough to give estimates for the second moments. Let C denote a universal but varying constant. Observe that it suffices to study $\sum_{j=0}^{k-2} e^{j\tilde{\varphi}}\varepsilon_{k-j}\varepsilon_k$ instead of $X_{k-1}^{(4)}\varepsilon_k$ because of arguments established and repeatedly applied in the discourse. Hence, the order of magnitude of $\mathrm{E}[R_{T,1}^2]$ is given by

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$$\mathrm{E}\left[\sum_{k=1}^{\lfloor T/\tilde{\varphi}\rfloor}\sum_{j=0}^{k-1} e^{2j\tilde{\varphi}}\varepsilon_{k-j}\varepsilon_k\right]^2 \le C\,\frac{\sigma^4 T}{\tilde{\varphi}}\sum_{k=1}^{\lfloor T/\tilde{\varphi}\rfloor} e^{2k\tilde{\varphi}} \le C\,\frac{\sigma^4 T^2}{\tilde{\varphi}^2}\,e^{2T},$$

where the first inequality sign is obtained using the independence of the variables ε_k and $\sum_{j=1}^{k-2} e^{j\tilde{\varphi}}\varepsilon_{k-j}$. On recognizing that, by Chebyshev's inequality,

$$P\left\{\frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}}\,|R_{T,1}| \ge \varepsilon\right\} \le \frac{\tilde{\varphi}^2}{\varepsilon^2 e^{2n\tilde{\varphi}}}\,\mathrm{E}R_{T,1}^2,$$

the probability on the left-hand side gets arbitrarily small for any fixed $T > 0$ if $n \to \infty$. Consequently, part (i) is readily proved.

(ii) Using that $\psi_{M,k}^2 - \psi_{M,k-1}^2 = (\psi_{M,k} + \psi_{M,k-1})(\psi_{M,k} - \psi_{M,k-1})$, it can be shown by elementary algebra that

$$R_{T,2} = \sum_{k=M+1}^{n} e^{(k-1)\tilde{\varphi}}\varepsilon_k\psi_{M,k} = \frac{1}{2e^{\tilde{\varphi}}}\left(\sum_{k=M+1}^{n} e^{2k\tilde{\varphi}}(\psi_{M,k}^2 - \psi_{M,k-1}^2) + \sum_{k=M+1}^{n}\varepsilon_k^2\right). \tag{3.17}$$

Therein, the second term on the right-hand side is negligible, because

$$\frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}}\sum_{k=M+1}^{n}\varepsilon_k^2 \le \frac{n\tilde{\varphi}}{e^{n\tilde{\varphi}}}\,\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k^2 = o(1) \quad \text{a.s.} \qquad (n \to \infty),$$

because, on account of $n\tilde{\varphi} \to \infty$ by (2.6), $n\tilde{\varphi}e^{-n\tilde{\varphi}} \to 0$ and, applying the strong law of large numbers to $\{\varepsilon_k^2\}$, $(1/n)\sum_{k=1}^{n}\varepsilon_k^2 \to \sigma^2$ with probability one.

With Abel's summation formula, the first term in (3.17) can be decomposed into

$$\sum_{k=M+1}^{n} e^{2k\tilde{\varphi}}(\psi_{M,k}^2 - \psi_{M,k-1}^2)$$

$$= e^{2(M+1)\tilde{\varphi}}\psi_{M,n}^2 + \sum_{k=M+1}^{n-1}(e^{2k\tilde{\varphi}} - e^{2(k+1)\tilde{\varphi}})(\psi_{M,k}^2 - \psi_{M,n}^2). \tag{3.18}$$

Note that $e^{2(M+1)\tilde{\varphi}} \to e^{2T}$, and so it is bounded for any fixed $T > 0$. As in the estimations following display (3.15), it holds that $\tilde{\varphi}\psi_{M,n}^2 = \mathcal{O}_P(1)$ as $n \to \infty$. Consequently the first term on the right-hand side of equation (3.18) vanishes asymptotically if multiplied with the normalization $\tilde{\varphi}e^{-n\tilde{\varphi}}$.

For the second term in (3.18), using $\psi_{M,k}^2 - \psi_{M,n}^2 = (\psi_{M,k} + \psi_{M,n})(\psi_{M,k} - \psi_{M,n})$, we obtain that

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$$\sum_{k=M+1}^{n-1}(e^{2k\tilde{\varphi}}-e^{2(k+1)\tilde{\varphi}})(\psi_{M,k}^2-\psi_{M,n}^2)$$

$$=\sum_{k=M+1}^{n-1}(e^{2k\tilde{\varphi}}-e^{2(k+1)\tilde{\varphi}})\psi_{k,n}^2-2\sum_{j=M+1}^{n}$$

$$\times e^{-j\tilde{\varphi}}\varepsilon_j\sum_{k=M+1}^{n-1}(e^{2k\tilde{\varphi}}-e^{2(k+1)\tilde{\varphi}})\psi_{k,n}$$

$$=C_{T,1}-C_{T,2}. \tag{3.19}$$

At first, we will prove that, for all $\varepsilon>0$,

$$\lim_{T\to\infty}\limsup_{n\to\infty}P\left\{\frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}}\,|C_{T,1}|\geq\varepsilon\right\}=0. \tag{3.20}$$

To this end, note that using preceding estimates it holds that

$$\psi_{k,n}^2\leq 8\left(e^{-2n\tilde{\varphi}}S^2(n)+e^{-2(k+1)\tilde{\varphi}}S^2(k)+\left(\tilde{\varphi}\int_{k+1}^{n}S(x)e^{-x\tilde{\varphi}}\,dx\right)^2\right).$$

We have to estimate all three terms on the right-hand side of the latter inequality. To start with, the strong law of large numbers applied to $S(x)$ and assumption (2.6) yield that, as $n\to\infty$,

$$\frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}}\sum_{k=M+1}^{n-1}(e^{2\tilde{\varphi}}-1)e^{2(k-n)\tilde{\varphi}}S^2(n)\leq 4\,\frac{(n\tilde{\varphi})^2}{e^{n\tilde{\varphi}}}\,\frac{e^{2n\tilde{\varphi}}-e^{2(M+1)\tilde{\varphi}}}{e^{2n\tilde{\varphi}}}\,\frac{1}{n^2}\,S^2(n)$$

$$=\mathcal{O}_P\left(\frac{n\tilde{\varphi}}{e^{n\tilde{\varphi}}}\right)=o_P(1),$$

where we have also used that $e^{2\tilde{\varphi}}-1=2\tilde{\varphi}+o(2\tilde{\varphi})$. By similar arguments,

$$\frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}}\sum_{k=M+1}^{n-1}(e^{2\tilde{\varphi}}-1)e^{2k\tilde{\varphi}}e^{-2(k+1)\tilde{\varphi}}S^2(k)\leq 4\,\frac{(n\tilde{\varphi})^2}{e^{n\tilde{\varphi}}}\,\frac{1}{n}\sum_{M+1}^{n-1}\frac{1}{n}\,S^2(k)$$

$$=o_P(1)\qquad(n\to\infty),$$

because, letting $\{W(s):s\in[0,1]\}$ denote a standard Brownian motion,

$$\frac{1}{n}\sum_{k=M+1}^{n-1}\frac{1}{n\sigma^2}\,S^2(k)\leq\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{\sqrt{n}\,\sigma}\,S(k)\right)^2\xrightarrow{\mathcal{D}}\int_0^1 W^2(s)\,ds\qquad(n\to\infty).$$

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Finally,

$$\begin{aligned}
&\frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}} \sum_{k=M+1}^{n-1} (e^{2\tilde{\varphi}} - 1) e^{2k\tilde{\varphi}} \left(\tilde{\varphi} \int_{k+1}^{n} S(x) e^{-x\tilde{\varphi}}\, dx \right)^2 \\
&\quad \le 4 \frac{\tilde{\varphi}^5}{e^{n\tilde{\varphi}}} \sum_{k=M+1}^{n-1} \left(\int_{k+1}^{n} |S(x)| e^{(k+1-x)\tilde{\varphi}}\, dx \right)^2 \\
&\quad \le 4 \frac{n\tilde{\varphi}^5}{e^{n\tilde{\varphi}}} \left(\int_{0}^{n} |S(x)|\, dx \right)^2 \\
&\quad \le 4 \frac{(n\tilde{\varphi})^5}{e^{n\tilde{\varphi}}} \left(\int_{0}^{1} \frac{1}{n} S(nx)\, dx \right)^2 \\
&\quad = \mathcal{O}_P\left(\frac{(n\tilde{\varphi})^5}{e^{n\tilde{\varphi}}} \right) = o_P(1)
\end{aligned}$$

as $n \to \infty$, completing the proof of (3.20). The assertion of part (ii) of Lemma 3.16 will now follow, once it is established that, for any $\varepsilon > 0$,

$$\lim_{T\to\infty} \limsup_{n\to\infty} P\left\{ \frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}} |C_{T,2}| \ge \varepsilon \right\} = 0. \tag{3.21}$$

To do so, observe that

$$\lim_{T\to\infty} \limsup_{n\to\infty} P\left\{ \frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}} |\psi_{M,n}| \ge \varepsilon \right\} = 0,$$

which in turn implies that (3.21) is proved if, as $n \to \infty$,

$$\frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}} \sum_{k=M+1}^{n-1} (e^{2k\tilde{\varphi}} - e^{2(k+1)\tilde{\varphi}}) \psi_{k,n} = \mathcal{O}_P(1).$$

Toward this end, straightforward algebra gives that

$$\sum_{k=M+1}^{n-1} (e^{2k\tilde{\varphi}} - e^{2(k+1)\tilde{\varphi}}) \psi_{k,n} = e^{2(M+1)\tilde{\varphi}} \tilde{\varphi} \sum_{j=M+2} e^{-j\tilde{\varphi}} \varepsilon_j - \sum_{j=M+2}^{n} e^{j\tilde{\varphi}} \varepsilon_j.$$

Therein, for any fixed $T > 0$, the first term is clearly of order $\mathcal{O}_P(1)$, reasoning as before. For the second term, it holds that

$$\frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}} \left| \sum_{j=M+2}^{n} e^{j\tilde{\varphi}} \varepsilon_j \right| \stackrel{\mathcal{D}}{=} \tilde{\varphi} \left| \sum_{i=0}^{n-M-2} e^{-i\tilde{\varphi}} \varepsilon_i \right| = \mathcal{O}_P(1) \qquad (n \to \infty),$$

which proves (3.21) and part (ii) of Lemma 3.16. ■

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So, it suffices to investigate the term $R_{T,3}$ to obtain the limit distribution. Checking the remaining summation ranges in the corresponding double sum reveals that $R_{T,3}$ is the product of two independent factors, whose limits can be calculated separately.

LEMMA 3.17. *Let the assumptions of Theorem 2.1 be satisfied. Then*

$$\left(\frac{4\tilde{\varphi}^2}{\sigma^2 e^{2n\tilde{\varphi}}} S_{T,4}, \frac{2\tilde{\varphi}}{\sigma^2 e^{n\tilde{\varphi}}} R_{T,3} \right) \xrightarrow{\mathcal{D}} (N_1^2, N_1 N_2) \qquad (n \to \infty)$$

for any fixed $T > 0$, where N_1 and N_2 denote independent zero mean normal random variables with $\mathrm{E}N_1^2 = 1 - e^{-2T}$ and $\mathrm{E}N_2^2 = 1$.

Proof. The proof is given in two steps, each of them investigating one of the components of the vector of interest. The joint behavior is obtained on combining both results.

(a) Observe that, to determine the limit distribution of $S_{T,4}$, we have to investigate the quantity on the right-hand side of the following equation array. It holds that, as $n \to \infty$,

$$S_{T,4} = \sum_{k=\lfloor T/\tilde{\varphi} \rfloor}^{n} e^{2k\tilde{\varphi}} \left(\sum_{j=0}^{\lfloor T/\tilde{\varphi} \rfloor} e^{-j\tilde{\varphi}} \varepsilon_j \right)^2 = \frac{e^{2n\tilde{\varphi}}}{2\tilde{\varphi}^2} \left(\sqrt{\tilde{\varphi}} \sum_{j=0}^{\lfloor T/\tilde{\varphi} \rfloor} e^{-j\tilde{\varphi}} \varepsilon_j \right)^2 (1 + o(1)).$$

Let $S(x) = \varepsilon_0 + \cdots + \varepsilon_{\lfloor x \rfloor}$ and note that this is a right-continuous function. Set $M = \lfloor T/\tilde{\varphi} \rfloor$. Using the proof of Lemma 3.2, it can be established that

$$U_{T,1} = \sqrt{\tilde{\varphi}} \sum_{j=0}^{M} e^{-j\tilde{\varphi}} \varepsilon_j \xrightarrow{\mathcal{D}} \sigma \int_0^T e^{-x} \, dW(x), \tag{3.22}$$

which has the same distribution as $(\sigma/\sqrt{2})N_1$, where N_1 denotes a zero mean normal random variable with variance $1 - e^{-2T}$. This completes the proof for the first coordinate.

(b) It holds for the second component that

$$\frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}} R_{T,3} = U_{T,1} U_{T,2},$$

where

$$U_{T,2} = \frac{\sqrt{\tilde{\varphi}}}{e^{n\tilde{\varphi}}} \sum_{k=M+1}^{n} e^{(k-1)\tilde{\varphi}} \varepsilon_k.$$

The second sum, $U_{T,2}$, contains noise terms starting with index $k = M + 1$, whereas these terms stop at $j = M$ in the first sum, $U_{T,1}$. Hence, $U_{T,1}$ and $U_{T,2}$ are independent, yielding the independence of N_1 and N_2. The limit of $U_{T,1}$ is already established in (3.22). So, it remains to determine the limit distribution of

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$$\frac{\sqrt{\tilde{\varphi}}}{e^{n\tilde{\varphi}}}\sum_{k=M+1}^{n} e^{(k-1)\tilde{\varphi}}\varepsilon_k = \sqrt{\tilde{\varphi}}\sum_{k=M+1}^{n} e^{-(n-k+1)\tilde{\varphi}}\varepsilon_k \overset{\mathcal{D}}{=} \sqrt{\tilde{\varphi}}\sum_{j=0}^{n-M} e^{-j\tilde{\varphi}}\varepsilon_j + o_P(1).$$

Let $\{\widetilde{W}: t \geq 0\}$ be a standard Wiener process independent of $\{W(t): t \in [0,T]\}$. Then, arguing as in part (a) of the proof,

$$\frac{\sqrt{\tilde{\varphi}}}{e^{n\tilde{\varphi}}}\sum_{k=M+1}^{n} e^{(k-1)\tilde{\varphi}}\varepsilon_k \xrightarrow{\mathcal{D}} \sigma\int_0^{\infty}\exp(-x)\,d\widetilde{W}(x) \qquad (n\to\infty),$$

where the limit is a centered normal random variable with variance $\frac{1}{2}$. On combining the results of parts (a) and (b) the proof of the lemma is complete. ■

It is enough to show that the squares of the remaining terms in (3.1) are negligible.

LEMMA 3.18. *Let the assumptions of Theorem 2.1 be satisfied. Then*

$$\frac{\tilde{\varphi}^2}{e^{2n\tilde{\varphi}}}\sum_{k=1}^{n} X_k^{(i)^2}(n) = o_P(1) \qquad (n\to\infty)$$

for $i = 1,2,3$.

Proof. Let $i = 1$. Then, using the definition in (3.1), it holds that

$$\begin{aligned}\frac{\tilde{\varphi}^2}{e^{2n\tilde{\varphi}}}\sum_{k=1}^{n} X_k^{(1)^2} &= \frac{\tilde{\varphi}^2}{e^{2n\tilde{\varphi}}}\sum_{k=1}^{n} X_0^2 e^{2k\tilde{\varphi}}\left(1+\sum_{j=0}^{k-1} b_{k-j} + R_k^{(1)}\right)^2\\ &\leq \frac{8\tilde{\varphi}^2}{e^{2n\tilde{\varphi}}}\sum_{k=1}^{n} X_0^2 e^{2k\tilde{\varphi}}\left(1+\left(\sum_{j=0}^{k-1} b_{k-j}\right)^2 + R_k^{(1)^2}\right)\\ &= \mathcal{O}_P(n\tilde{\varphi}^2) + \mathcal{O}(n\tilde{\varphi}^2 n\omega^2) + \mathcal{O}(n\tilde{\varphi}^2 n(\tilde{\varphi}^2+\omega^2))\\ &= o_P(1)\end{aligned}$$

as $n\to\infty$, where the respective orders of magnitude have been obtained by applying the corresponding asymptotic rates (2.6) and (2.7) and utilizing the central limit theorem for $b_k = \omega\gamma_k$.

For $i = 2$, the order of magnitude of $X_k^{(2)}$ is determined by $\sum_{j=0}^{k-1} e^{j\tilde{\varphi}} R_{k,j}^{(3)}\varepsilon_{k-j}$. Using, in addition, the independence of $R_{k,j}^{(3)}$ and ε_{k-j} we can estimate as in the proof of Lemma 3.4

$$\mathrm{E}\left[\sum_{k=1}^{n}\left(\sum_{j=0}^{k-1} e^{j\tilde{\varphi}} R_{k,j}^{(3)}\varepsilon_{k-j}\right)^2\right] \leq Cn^2(\tilde{\varphi}^2+\omega^2)^2\mathrm{E}\left[\sum_{k=1}^{n}\left(\sum_{j=0}^{k-1} e^{j\tilde{\varphi}}\varepsilon_{k-j}\right)^2\right] \qquad (n\to\infty),$$

so that Markov's inequality gives the assertion.

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Finally, similar arguments work also in the case $i = 3$. The proof of the lemma is complete. ■

LEMMA 3.19. *Let the assumptions of Theorem 2.1 be satisfied. Then*

$$\frac{\tilde{\varphi}}{e^{n\tilde{\varphi}}} \sum_{k=1}^{n} X_{k-1}^{(i)} \varepsilon_k = o_P(1) \qquad (n \to \infty)$$

for $i = 1, 2, 3$.

Proof. Similar arguments as in Lemma 3.18 lead to the conclusions utilizing suitable parts of the proofs of Lemmas 3.3–3.5. ■

Proof of Theorem 2.1. Collecting the results of Lemmas 3.15–3.19, we arrive at

$$\lim_{T\to\infty} \limsup_{n\to\infty} P\left\{ \left| \frac{e^{n\tilde{\varphi}_n}}{2\tilde{\varphi}_n} (\hat{\varphi}_n - \varphi_n) - \frac{\sigma^2 e^{2n\tilde{\varphi}}}{4\tilde{\varphi}^2} \frac{1}{S_{T,4}} \frac{2\tilde{\varphi}}{\sigma^2 e^{n\tilde{\varphi}}} R_{T,3} \right| > \varepsilon \right\} = 0$$

for all $\varepsilon > 0$. Lemma 3.17 implies that, for any $T > 0$,

$$\frac{\sigma^2 e^{2n\tilde{\varphi}}}{4\tilde{\varphi}^2} \frac{1}{S_{T,4}} \frac{2\tilde{\varphi}}{\sigma^2 e^{n\tilde{\varphi}}} R_{T,3} \xrightarrow{\mathcal{D}} \frac{N_1 N_2}{N_1^2} \qquad (n \to \infty).$$

Observe that

$$(N_1, N_2) \stackrel{\mathcal{D}}{=} (\sqrt{1 - e^{-2T}} M_1, M_2),$$

where M_1 and M_2 are independent standard normal random variables. Consequently, we get

$$\frac{N_1 N_2}{N_1^2} \stackrel{\mathcal{D}}{=} \frac{1}{\sqrt{1 - e^{-2T}}} \frac{M_2}{M_1},$$

which converges to a standard Cauchy random variable as $T \to \infty$. ■

3.6. Proof of Theorem 2.2

We follow the pattern provided in Section 3.5.

LEMMA 3.20. *Let the assumptions of Theorem 2.2 be satisfied. Then, as* $n \to \infty$,

$$\frac{2|\tilde{\varphi}|}{\sigma^2 n} \sum_{t=1}^{n} X_{t-1}^{(4)^2} \xrightarrow{P} 1 \quad \textit{and} \quad \sqrt{\frac{2|\tilde{\varphi}|}{\sigma^4 n}} \sum_{t=1}^{n} X_{t-1}^{(4)} \varepsilon_t \xrightarrow{\mathcal{D}} \xi,$$

where ξ *denotes a standard normal random variable.*

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Proof. An application of Markov's inequality yields

$$P\left\{\left|\frac{2|\tilde{\varphi}|}{\sigma^2 n}\sum_{t=1}^{n} X_{t-1}^{(4)^2} - 1\right| \geq \varepsilon\right\} \leq \frac{1}{\varepsilon}\,\mathrm{E}\left|\frac{2|\tilde{\varphi}|}{\sigma^2 n}\sum_{t=1}^{n} X_{t-1}^{(4)^2} - 1\right|.$$

It is therefore enough to consider the expectation on the right-hand side, for which we can write, utilizing the dominating part of $X_t^{(4)}$ for the estimation, that

$$\mathrm{E}\left[\frac{2|\tilde{\varphi}|}{\sigma^2 n}\sum_{t=1}^{n} X_{t-1}^{(4)^2}\right] \sim \frac{2|\tilde{\varphi}|}{\sigma^2 n}\sum_{t=1}^{n}\sum_{j=0}^{t-1} e^{2j\tilde{\varphi}}\sigma^2 = \frac{2|\tilde{\varphi}|}{1-e^{2\tilde{\varphi}}}\,\frac{1}{n}\sum_{k=1}^{n}(1-e^{2k\tilde{\varphi}}).$$

Note that, by Taylor expansion of $e^{2\tilde{\varphi}}$, the first term $2|\tilde{\varphi}|/(1-e^{2\tilde{\varphi}})$ converges to 1, whereas the second term can be estimated by

$$\frac{1}{n}\sum_{k=1}^{n}(1-e^{2k\tilde{\varphi}}) = 1 - \frac{1}{n}\sum_{k=1}^{n} e^{2k\tilde{\varphi}} \to 1 \qquad (n\to\infty),$$

because $1/n\sum_{k=1}^{n} e^{2k\tilde{\varphi}} \leq e^{2n\tilde{\varphi}} \to 0$ because of $n\tilde{\varphi}\to -\infty$. So, the first claim follows readily. The second statement of Lemma 3.20 can be found as Theorem 3.2(b) in Phillips and Magdalinos (2007). ■

LEMMA 3.21. *Let the assumptions of Theorem 2.2 be satisfied. Then*

$$\frac{\tilde{\varphi}^2}{e^{2n\tilde{\varphi}}}\sum_{k=1}^{n} X_k^{(i)^2} = o_P(1) \qquad (n\to\infty)$$

for $i = 1,2,3$.

Proof. Let $i = 1$. Then, as $n\to\infty$,

$$\begin{aligned}\frac{2|\tilde{\varphi}|}{n}\sum_{k=1}^{n} X_t^{(1)^2} &= \frac{2|\tilde{\varphi}|}{n}\sum_{k=1}^{n} X_0^2 e^{2k\tilde{\varphi}}\left(1+\sum_{j=0}^{k-1} b_{k-j} + R_k^{(1)}\right)^2\\ &\leq \frac{16|\tilde{\varphi}|}{n}\sum_{k=1}^{n} X_0^2 e^{2k\tilde{\varphi}}\left(1+\left(\sum_{j=0}^{k-1} b_{k-j}\right)^2 + R_k^{(1)^2}\right)\\ &= \mathcal{O}_P(|\tilde{\varphi}|) + \mathcal{O}_P(|\tilde{\varphi}|n\omega^2) + \mathcal{O}_P(|\tilde{\varphi}|n(\tilde{\varphi}^2+\omega^2))\\ &= o_P(1).\end{aligned}$$

Observe that $e^{2k\tilde{\varphi}} \leq 1$, because $\tilde{\varphi} < 0$. In addition, assumptions (2.6) and (2.7) and the central limit theorem for $b_k = \omega\gamma_k$ have been applied to obtain the asymptotic negligibility.

The assertions for the cases $i = 2$ and $i = 3$ follow from adapting the proofs of the corresponding parts in Lemma 3.18. ■

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LEMMA 3.22. *Let the assumptions of Theorem 2.2 be satisfied. Then*

$$\frac{2|\tilde{\varphi}|}{n}\sum_{t=1}^{n} X_{t-1}^{(i)}\varepsilon_t = o_P(1) \qquad (n\to\infty)$$

for $i = 1,2,3$.

Proof. Similar arguments as in Lemma 3.19 lead to the conclusions. ■

Proof of Theorem 2.2. Collecting the results of the lemmas in Sections 3.6 leads to the convergence in distribution result

$$\sqrt{\frac{n}{2|\tilde{\varphi}_n|}}(\hat{\varphi}_n - \varphi_n) = \left[\frac{2|\tilde{\varphi}|}{\sigma^2 n}\sum_{t=1}^{n} X_{t-1}^{(4)^2}\right]^{-1}\sqrt{\frac{2|\tilde{\varphi}|}{\sigma^4 n}}\sum_{t=1}^{n} X_{t-1}^{(4)}\varepsilon_t + o_P(1) \xrightarrow{\mathcal{D}} \zeta$$

as $n\to\infty$, where ζ denotes a standard normal random variable. ■

REFERENCES

Anderson, T.W. (1959) On asymptotic distributions of estimates of parameters of stochastic difference equations. *Annals of Mathematical Statistics* 30, 676–687.

Aue, A. & L. Horváth (2007) A limit theorem for mildly explosive autoregression with stable errors. *Econometric Theory* 23, 201–220.

Aue, A., L. Horváth, & J. Steinebach (2006) Estimation in random coefficient autoregressive models. *Journal of Time Series Analysis* 27, 61–76.

Berkes, I., L. Horváth, & P. Kokoszka (2005) Near-integrated GARCH sequences. *Annals of Applied Probability* 15, 890–913.

Billingsley, P. (1995) *Probability and Measure*, 3rd ed. Wiley.

Brockwell, P.J. & R.A. Davis (1991) *Time Series: Theory and Methods*, 2nd ed. Springer-Verlag.

Chan, N.H. & C.Z. Wei (1987) Asymptotic inference for nearly nonstationary AR(1) processes. *Annals of Statistics* 15, 1050–1063.

Chan, N.H. & C.Z. Wei (1988) Limiting distributions of least squares estimates of unstable autoregressive processes. *Annals of Statistics* 16, 367–401.

Chow, Y.S. & H. Teicher (1988) *Probability Theory: Independence, Interchangeability, Martingales*, 2nd ed. Springer-Verlag.

Ling, S. & W.K. Li (1998) Limiting distributions of maximum likelihood estimators for unstable autoregressive moving-average time series with general autoregressive heteroscedastic errors. *Annals of Statistics* 26, 84–125.

Nelson, C.R. & C. Plosser (1982) Trends and random walks in macroeconomic time series: Some evidence and implications. *Journal of Monetary Economics* 10, 139–162.

Nicholls, D.F. & B.G. Quinn (1982) *Random Coefficient Autoregressive Models: An Introduction*. Springer-Verlag.

Phillips, P.C.B. (1988) Regression theory for near-integrated time series. *Econometrica* 56, 1021–1043.

Phillips, P.C.B. & T. Magdalinos (2007) Limit theory for moderate deviations from a unit root. *Journal of Econometrics* 136, 115–130.

Tong, H. (1990) *Non-Linear Time Series: A Dynamical System Approach*. Clarendon.

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