The Second Solution of Mathieu's Differential Equation. By S. GOLDSTEIN, B.A., St John's College, Isaac Newton Student.

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The Mathieu functions of period π and 2π have recently been constructed* by the help of analysis similar to that developed by Laplace, Kelvin, Darwin and Hough to find the free tides symmetrical about the axis of a rotating globe†. The purpose of this note is to show that a similar construction can be carried out for the second solution of the Mathieu equation, when one solution is periodic in π or 2π, by the help of analysis similar to that used for forced tides. The construction is effected in a form suitable for numerical computation.

The second solutions have been considered by Ince‡, and a pair of fundamental solutions of the general equation, without the restriction as to the periodicity of one solution, has been constructed by Dougall§, and asymptotic expansions and contour integral forms have been given by him; but his expressions do not lend themselves to numerical computation, while Ince's solution involves power-series in the parameter whose radius of convergence is unknown, and which can be used to yield numerical results only for a restricted range of values.

Mathieu functions are of four types, being odd or even functions of x and of $\frac{1}{2}\pi - x$. For definiteness, we shall consider only the function $ce_{2s+1}(x, q)$, even in x and odd in $\frac{1}{2}\pi - x$. Similar methods apply to the other three types. Let Mathieu's equation be taken in the form

$$y'' + (4\alpha - 16q\cos 2x)\, y = 0. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Before the second solution is constructed, it will be convenient to summarise the method of evaluating the characteristic numbers and the Fourier constants of the periodic solutions.

Let $$ce_{2s+1}(x, q) = \sum_{r=0}^{\infty} A_{2r+1} \cos(2r+1)\, x. \quad \ldots\ldots\ldots(2)$$

Then must

$$\left.\begin{aligned} &(\alpha - \tfrac{1}{4} - 2q)\, A_1 = 2qA_3, \\ &\{\alpha - (r + \tfrac{1}{2})^2\}\, A_{2r+1} = 2q\,(A_{2r-1} + A_{2r+3}) \qquad (r \geqslant 1). \end{aligned}\right\} \ldots(3)$$

* *Trans. Camb. Phil. Soc.*, Vol. XXIII, p. 303 (1927). See also Ince, *Proc. Roy. Soc. Edin.*, Vol. XLVI, p. 20 and p. 316 (1926), and Vol. XLVII, p. 294 (1927).

† Lamb, *Hydrodynamics*, fifth edition, p. 313.

‡ *Proc. Edin. Math. Soc.*, Vol. XXXIII, p. 2 (1915); *Proc. Camb. Phil. Soc.*, Vol. XXIII, p. 47 (1926).

§ *Proc. Edin. Math. Soc.*, Vol. XXXIV, p. 4 (1916); Vol. XLI, p. 26 (1923); Vol. XLIV, p. 57 (1926).

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Let $$A_{2r+3}/A_{2r+1} = v_r. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Then $$\left.\begin{aligned} &\alpha - \tfrac{1}{4} - 2q - 2qv_0 = 0, \\ &\alpha - (r+\tfrac{1}{2})^2 = 2q\,(v_r + 1/v_{r-1}) \qquad (r \geqslant 1). \end{aligned}\right\} \quad \ldots\ldots\ldots(5)$$

Thus $v_r \to 0$ or $-\infty/q$. The series (2) will converge only if $v_r \to 0$, and $v_r \to 0$ only for certain special values of α. We determine an equation whose roots are the permissible values of α as follows. From (5)

$$-v_{r-1} = \frac{2q/(r+\tfrac{1}{2})^2}{1 - \alpha/(r+\tfrac{1}{2})^2 + 2qv_r/(r+\tfrac{1}{2})^2} \qquad (r \geqslant 1), \quad \ldots(6)$$

and hence

$$v_{r-1} = \frac{-2q/(r+\tfrac{1}{2})^2}{1-\alpha/(r+\tfrac{1}{2})^2} - \frac{4q^2/(r+\tfrac{1}{2})^2(r+\tfrac{3}{2})^2}{1-\alpha/(r+\tfrac{3}{2})^2} - \frac{4q^2/(r+\tfrac{3}{2})^2(r+\tfrac{5}{2})^2}{1-\alpha/(r+\tfrac{5}{2})^2} - \cdots$$
$$\ldots \text{ to } \infty \qquad (r \geqslant 1). \quad \ldots\ldots\ldots(7)$$

v_{r-1} is thus expressed as a function of α and q. Putting $r = 1$ in (7) and substituting for v_0 in the first equation of (5), we obtain the required equation for α. The equation has an infinite number of roots, corresponding to the values $0, 1, 2, 3, \ldots$ for s. The values of A_{2r+1}/A_{2r-1} for any function ce_{2s+1} are then found by substituting the corresponding root of this equation for α in the right-hand side of (7). The actual method of computation will be found described in the papers previously cited.

Alternative forms of the equation for α can be found. They will be of most use for the purpose of this note if developed in connection with the odd function se_{2n+1}, $n = 0, 1, 2, \ldots$.

Let $$se_{2n+1}(x, q) = \sum_{r=0}^{\infty} B_{2r+1} \sin(2r+1)x. \quad \ldots\ldots\ldots\ldots(8)$$

Then must

$$\left.\begin{aligned} &(\alpha - \tfrac{1}{4} + 2q)\,B_1 = 2qB_3, \\ &\{\alpha - (r+\tfrac{1}{2})^2\}\,B_{2r+1} = 2q\,(B_{2r-1} + B_{2r+3}) \qquad (r \geqslant 1). \end{aligned}\right\} \quad \ldots(9)$$

Except for the first equation, (9) is the same as (3). Hence if now we put

$$v_r = B_{2r+3}/B_{2r+1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

v_{r-1} will be the same function of α and q as before. The equation for α is therefore

$$\alpha - \tfrac{1}{4} + 2q - 2qv_0 = 0, \quad \ldots\ldots\ldots\ldots\ldots(11)$$

where v_0 is the same function of α and q as before. The values of B_{2r+1}/B_{2r-1} for any function se_{2n+1} are then found by substituting the corresponding root of (11) for α in the right-hand side of (7).

The alternative forms of the equation are found as follows. In (9) put

$$B_{2r+1}/B_{2r+3} = u_r. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

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Then
$$\left.\begin{aligned}&(\alpha - \tfrac{1}{4} + 2q)\, u_0 = 2q,\\ &\alpha - (r+\tfrac{1}{2})^2 = 2q\,(u_{r-1} + 1/u_r) \qquad (r \geqslant 1).\end{aligned}\right\} \quad \ldots\ldots(13)$$

Hence
$$\begin{aligned} u_r &= \frac{-2q}{(r+\frac{1}{2})^2 - \alpha + 2qu_{r-1}} \\ &= \frac{-2q}{(r+\frac{1}{2})^2 - \alpha} - \frac{4q^2}{(r-\frac{1}{2})^2 - \alpha} - \frac{4q^2}{(r-\frac{3}{2})^2 - \alpha} - \cdots \\ &\qquad - \frac{4q^2}{(\frac{3}{2})^2 - \alpha} - \frac{4q^2}{\frac{1}{4} - \alpha - 2q}. \end{aligned} \quad \ldots\ldots\ldots\ldots(14)$$

u_r is thus expressed as a function of α and q. In the mth equation of (9), put u_{m-1} for B_{2m-1}/B_{2m+1} and v_m for B_{2m+3}/B_{2m+1}. Then we get the equation for α in the form
$$\alpha - (m+\tfrac{1}{2})^2 - 2q\,(u_{m-1} + v_m) = 0, \quad \ldots\ldots\ldots\ldots(15)$$
where u_{m-1} is given as a function of α and q by putting $r = m-1$ in the right-hand side of (14), and v_m by putting $r = m+1$ in the right-hand side of (7).

The roots of this equation are the characteristic numbers for the functions se_{2n+1} for $n = 0, 1, 2, \ldots$. Now, by a well-known theorem, the characteristic number for ce_{2s+1} is never equal to the characteristic number for se_{2n+1} for any values of s and n and any non-zero value of q. Hence, if α be given its characteristic value for ce_{2s+1}, the left-hand side of (15) will never vanish for non-zero q, no matter what integral value m takes.

Lastly, we see from (13) that $u_r \to 0$ or $-\infty/q$. If the latter alternative holds, the series (12) converges, and this can happen only when α has its characteristic value for se_{2n+1} for some n. Hence, if we consider u_r as defined by (14), $u_r \to 0$ when α has any other value than that of a characteristic number for a function se_{2n+1}; in particular $u_r \to 0$ when α has the characteristic value for ce_{2s+1}.

If v_{r-1} and u_r had been initially defined by (7) and (14), the equations (13) and all but the first of (5) could have been deduced. On this basis, a more logical sketch could have been given of the theory, but it would have had the disadvantage of artificiality. The reader should have no difficulty in constructing the logical account for himself. In what follows, the v and u will be taken to be defined by (7) and (14), and α to have its characteristic value for ce_{2s+1}, so that the equations (13), and *all* the equations (5), are satisfied; the left-hand side of (15) never vanishes, and v_r and u_r both $\to 0$ as $r \to \infty$.

We now turn our attention to the second solution corresponding to ce_{2s+1}. This is known to be of the form
$$in_{2s+1}(x, q) = Fx\, ce_{2s+1}(x, q) + el_{2s+1}(x, q)^*, \quad \ldots\ldots(16)$$

* *Trans. Camb. Phil. Soc.*, Vol. XXIII, p. 330.

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where F is independent of x, and el_{2s+1} is a function of x periodic in 2π, odd in x and even in $\frac{1}{2}\pi - x$. The value of F is to be fixed by convention, and we shall consider it later.

Let $$el_{2s+1}(x, q) = \sum_{r=0}^{\infty} b_{2r+1} \sin(2r+1)x. \quad \ldots\ldots\ldots(17)$$

By substituting from (16), (17) and (2) in (1), rearranging, and equating to zero the coefficients of $\sin x$, $\sin(2r+1)x$, we obtain, after division by 4,

$$\left.\begin{aligned} &(\alpha - \tfrac{1}{4} + 2q)\, b_1 - 2qb_3 = \tfrac{1}{2}FA_1, \\ &\{\alpha - (r+\tfrac{1}{2})^2\}\, b_{2r+1} - 2q\,(b_{2r-1} + b_{2r+3}) = F(r+\tfrac{1}{2})\, A_{2r+1}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(18)$$

It is assumed that α and the A are known; the problem is that of evaluating the b. If an arbitrary value were assigned to b_1, and b_3, b_5, ... found in turn, the series (17) would not converge. The following procedure is therefore adopted.

In (18) put all the A equal to zero except one, say A_{2m+1}, and solve for the b as explained below. If the value thus found for b_{2r+1} be $b_{2r+1}^{(2m+1)}$, then

$$b_{2r+1} = \sum_{m=0}^{\infty} b_{2r+1}^{(2m+1)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(19)$$

provided that the infinite series is absolutely convergent, and the value thus found for b_{2r+1} makes $\sum_{r=0}^{\infty} (2r+1)^2 b_{2r+1}$ absolutely convergent.

To find $b_{2r+1}^{(2m+1)}$, we have, for $m \neq 0$, $r \neq m$,

$$\left.\begin{aligned} &(\alpha - \tfrac{1}{4} + 2q)\, b_1^{(2m+1)} - 2qb_3^{(2m+1)} = 0, \\ &\{\alpha - (r+\tfrac{1}{2})^2\}\, b_{2r+1}^{(2m+1)} - 2q\,(b_{2r-1}^{(2m+1)} + b_{2r+3}^{(2m+1)}) = 0; \end{aligned}\right\} \quad \ldots(20)$$

while

$$\{\alpha - (m+\tfrac{1}{2})^2\}\, b_{2m+1}^{(2m+1)} - 2q\,(b_{2m-1}^{(2m+1)} + b_{2m+3}^{(2m+1)}) = F(m+\tfrac{1}{2})\, A_{2m+1}. \quad \ldots\ldots\ldots(21)$$

Except for the mth equation, (20) is the same as (9) with $b_{2r+1}^{(2m+1)}$ in place of B_{2r+1}. Hence

$$\left.\begin{aligned} b_{2r+1}^{(2m+1)} / b_{2r+3}^{(2m+1)} &= u_r \text{ for } r < m, \\ b_{2r+3}^{(2m+1)} / b_{2r+1}^{(2m+1)} &= v_r \text{ for } r \geqslant m; \end{aligned}\right\} \quad \ldots\ldots\ldots(22)$$

and from (21)

$$\{\alpha - (m+\tfrac{1}{2})^2 - 2q\,(u_{m-1} + v_m)\}\, b_{2m+1}^{(2m+1)} = F(m+\tfrac{1}{2})\, A_{2m+1}. \quad \ldots\ldots\ldots(23)$$

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The multiplier of $b_{2m+1}^{(2m+1)}$ never vanishes for $q \neq 0$. Thus we find $b_{2m+1}^{(2m+1)}$ and then

$$\left.\begin{aligned} b_{2r+1}^{(2m+1)} &= u_{m-1}u_{m-2}\dots u_r \quad b_{2m+1}^{(2m+1)} \text{ for } r < m, \\ b_{2r+1}^{(2m+1)} &= v_m \quad v_{m+1}\dots v_{r-1}\, b_{2m+1}^{(2m+1)} \text{ for } r > m. \end{aligned}\right\} \dots\dots(24)$$

For $m = 0$, we have $b_{2r+3}^{(1)}/b_{2r+1}^{(1)} = v_r$, ...(25)

$$(\alpha - \tfrac{1}{4} + 2q - 2qv_0)\, b_1^{(1)} = \tfrac{1}{2}FA_1. \quad \dots\dots(26)$$

Since, from (5), $\alpha - \tfrac{1}{4} - 2q - 2qv_0 = 0$, ...(27)

(26) gives $b_1^{(1)} = FA_1/8q$; .....(28)

and then, since $b_{2r+3}^{(1)}/b_{2r+1}^{(1)} = v_r = A_{2r+3}/A_{2r+1}$, ...(29)

it follows that $b_{2r+1}^{(1)} = FA_{2r+1}/8q$. .....(30)

This completes the construction. We now consider the convergence of the series in (19), and the justification of the operations used in obtaining (18).

In (19) we need consider only the terms for which $m > r$. Then

$$\frac{b_{2r+1}^{(2m+3)}}{b_{2r+1}^{(2m+1)}} = u_m \frac{b_{2m+3}^{(2m+3)}}{b_{2m+1}^{(2m+1)}}. \quad \dots\dots(31)$$

From (13), since $u_{r-1} \to 0$ as $r \to \infty$,

$$u_r \sim -2q/(r+\tfrac{1}{2})^2. \quad \dots\dots(32)$$

Similarly, from (5), $v_{r-1} \sim -2q/(r+\tfrac{1}{2})^2$ .....(33)

as $r \to \infty$.

From (23)

$$b_{2m+3}^{(2m+3)}/b_{2m+1}^{(2m+1)} \sim A_{2m+3}/A_{2m+1} = v_m \sim -2q\,(m+\tfrac{3}{2})^2 \dots(34)$$

as $m \to \infty$. Hence

$$b_{2r+1}^{(2m+3)}/b_{2r+1}^{(2m+1)} \sim 4q^2/(m+\tfrac{1}{2})^2(m+\tfrac{3}{2})^2 \quad \dots\dots(35)$$

as $m \to \infty$, and the series in (19) converges absolutely.

Now consider fully b_{2r+1}. Since $u_r \to 0$ as $r \to \infty$, it follows from the first equation in (14) that for sufficiently large r,

$$|u_r| < 2|q|/r^2, \quad \dots\dots(36)$$

and therefore $|u_n| < 2|q|/r^2$ for any $n > r$. .....(37)

Similarly, from (23), for $m > r$, and r sufficiently large,

$$|b_{2m+1}^{(2m+1)}| < (m+\tfrac{1}{2})|FA_{2m+1}|/m^2 < |FA_{2r+1}|/r, \dots\dots(38)$$

since $|A_{2m+1}| < |A_{2r+1}|$ under the conditions stated.

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Hence, from the first equation of (24),

$$\left| \sum_{m=r}^{\infty} b_{2r+1}^{(2m+1)} \right| < \frac{|FA_{2r+1}|}{r} \left\{ 1 + \frac{2|q|}{r^2} + \frac{4|q|^2}{r^4} + \ldots \right\} = \frac{r|FA_{2r+1}|}{r^2 - 2|q|}. \quad \ldots\ldots\ldots\ldots(39)$$

For $m < r$, from (23) and (24),

$$b_{2r+1}^{(2m+1)} = -v_m v_{m+1} \ldots v_{r-1} \frac{F(m+\frac{1}{2}) A_{2m+1}}{(m+\frac{1}{2})^2 - \alpha - 2q(v_m + u_{m-1})}$$

$$= -\frac{F(m+\frac{1}{2}) A_{2r+1}}{(m+\frac{1}{2})^2 - \alpha - 2q(v_m + u_{m-1})}. \quad \ldots\ldots\ldots\ldots(40)$$

The denominator of the expression on the right never vanishes. Hence the multiplier of $-FA_{2r+1}$ in the last expression is bounded. Let Λ be its upper bound. Then

$$\left| \sum_{m=0}^{r-1} b_{2r+1}^{(2m+1)} \right| \leqslant |FA_{2r+1}| r\Lambda. \quad \ldots\ldots\ldots\ldots(41)$$

Hence, for sufficiently large r,

$$|b_{2r+1}| < r|FA_{2r+1}| \{\Lambda + 1/(r^2 - 2|q|)\}. \quad \ldots\ldots\ldots(42)$$

Since $$|A_{2r+1}/A_{2r-1}| \sim 2|q|/(r+\tfrac{1}{2})^2$$

as $r \to \infty$, $$\sum_{r=0}^{\infty} (2r+1)^2 b_{2r+1}$$

converges absolutely, and the operations carried out in obtaining (6) are justified.

As a numerical example, I have calculated the b for el_1 for $q = 1$. The characteristic value of α for ce_1 for $q = 1$, the v and the A had been calculated previously. The previous calculation was slightly extended, and, to eight places of decimals, it was found that

$$\alpha = -0{\cdot}10898590.$$

The method of calculating the v is to assume that for a sufficiently large value of r, $v_r = 0$, and then calculate v_{r-1}, v_{r-2}, ... v_0 in turn from (6). In the calculation given below, v_8 was put equal to zero. The ratios of the A are thus found, and their evaluation is completed so that*

$$\int_{-\pi}^{\pi} [ce(x, q)]^2 dx = \pi,$$

and $A_1 \to +1$ as $q \to 0$.

The u are computed from the first equation of (13) and the first equation of (14). Then (30), (23) and (24) give the $b_{2r+1}^{(2m+1)}$ in terms of F. $F = 2$ was taken for convenience.

* *Trans. Camb. Phil. Soc.*, Vol. XXIII, p. 303.

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The A are given below to eight places of decimals.

r	0	1	2	3	4
A_{2r+1}	+·6264 1793	−·7388 5554	+·2450 5696	−·0403 0135	+·0039 8494

r	5	6	7	8
A_{2r+1}	−·0002 6334	+·0000 1245	−·0000 0044	+·0000 0001

The values of the v and the u are suppressed, but the $b_{2r+1}^{(2m+1)}$ are shown in the table at the end so as to exhibit the convergence.

It should be borne in mind that once the value of α is known, very little work indeed is needed for each entry in this table, and the whole calculation is neither very long nor very difficult.

To six places of decimals, with $F=2$, the values of the b_{2r+1} are found to be as under.

r	0	1	2	3
b_{2r+1}	+·934639	+·453669	−·361456	+·082935

r	4	5	6	7
b_{2r+1}	−·009983	+·000756	−·000040	+·000002

Finally let us consider what is the best convention to follow in fixing the value of F. When q is zero, F is zero, and therefore F should not be given a value independent of q*. There are several possibilities. One is to take in and the x-derivative of jn equal to 1 for $x=0$. Another is to take the first Fourier coefficient of el or fl equal to 1. At this stage it is best to leave the matter undecided, until more experience has been gained in computing and using the functions.

* This point was overlooked in *Trans. Camb. Phil. Soc.*, Vol. XXIII, p. 330, where F was taken as 1.

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The values of $b_{2r+1}^{(2m+1)}$ for el_1 for $q=1$, $F=2$.

m	$r=0$	$r=1$	$r=2$	$r=3$
0	+ ·1566 0448	− ·1847 1388	+ ·0612 6424	− ·0100 7534
1	+ ·6536 0617	+ ·5362 8847	− ·1778 7134	+ ·0292 5220
2	+ ·1198 3396	+ ·0983 2461	− ·2358 0715	+ ·0387 8015
3	+ ·0045 3744	+ ·0037 2302	− ·0089 2874	+ ·0246 6584
4	+ ·0000 3704	+ ·0000 4680	− ·0001 1224	+ ·0003 1006
5	+ ·0000 0031	+ ·0000 0025	− ·0000 0060	+ ·0000 0166
6	+ ·0000 0000	+ ·0000 0000	− ·0000 0000	+ ·0000 0000
7	—	—	—	—
8	—	—	—	—

m	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
0	+ ·0009 9624	− ·0000 6584	+ ·0000 0311	− ·0000 0011	+ ·0000 0000
1	− ·0028 9242	+ ·0001 9114	− ·0000 0904	+ ·0000 0032	− ·0000 0001
2	− ·0038 3453	+ ·0002 5340	− ·0000 1198	+ ·0000 0043	− ·0000 0001
3	− ·0024 3892	+ ·0001 6117	− ·0000 0762	+ ·0000 0027	− ·0000 0001
4	− ·0018 0377	+ ·0001 1920	− ·0000 0564	+ ·0000 0020	− ·0000 0001
5	− ·0000 0963	+ ·0000 9635	− ·0000 0456	+ ·0000 0016	− ·0000 0000
6	− ·0000 0003	+ ·0000 0025	− ·0000 0384	+ ·0000 0014	—
7	− ·0000 0000	+ ·0000 0000	− ·0000 0001	+ ·0000 0012	—
8	—	—	− ·0000 0000	+ ·0000 0000	—

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