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Local Anti-Ramsey Numbers of Graphs

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A subgraph H in an edge-colouring is *properly coloured* if incident edges of H are assigned different colours, and H is *rainbow* if no two edges of H are assigned the same colour. We study properly coloured subgraphs and rainbow subgraphs forced in edge-colourings of complete graphs in which each vertex is incident to a large number of colours.

1. Introduction

In an edge-coloured host graph G, a subgraph H is *properly coloured* if no two incident edges of H receive the same colour, and *rainbow* if no two edges of H receive the same colour. Given a positive integer k, a host graph G, an edge-colouring c of G (c is not necessarily proper), then c is a *k-colouring* if c uses k colours overall, c is a *local k-colouring* if at most k colours are used at each vertex of G, and c is a *k-good* colouring if at least k different colours are used at each vertex of G.

The classical Ramsey and anti-Ramsey problems ask for the optimal *total* number of colours used on the edges of a host graph without creating a prohibited coloured subgraph. The *local* variation of these problems is concerned only with the number of colours used on the edges incident to each vertex, instead of the total number of colours used. The local Ramsey problem, for example, asks for the minimum number k such that there exists a local k-colouring of the edges of K_n with no monochromatic subgraph

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isomorphic to a given graph H: see [14, 21]. In this paper we initiate the study of a similar variation of an anti-Ramsey problem.

In anti-Ramsey problems, we are interested in rainbow subgraphs instead of monochromatic subgraphs in edge-colourings. Suppose we are given a graph H and a sufficiently large positive integer n. When we colour the edges of K_n, we can always force a rainbow copy of H to occur by using many colours. It is natural to determine the threshold on the number of colours needed to force this. The classical anti-Ramsey number of H for fixed n, denoted by $AR(n, H)$, is defined as the maximum k such that there exists a k-colouring of $E(K_n)$ that avoids rainbow copies of H. By definition, every colouring of $E(K_n)$ using more than $AR(n, H)$ colours contains a rainbow copy of H. Anti-Ramsey numbers were introduced by Erdős, Simonovits and Sós [9] in the 1970s, and have been actively studied recently (see [2, 5, 6, 7, 16, 17]). Anti-Ramsey numbers are closely related to the extremal function $ex(n, F)$. Here $ex(n, F)$ is the maximum number of edges in a graph on n vertices with no subgraph isomorphic to F. The main result in [9] is as follows:

$$AR(n, H) = \frac{n^2}{2}\left(1 - \frac{1}{\chi - 1}\right)(1 + o(1)),$$

where $\chi = \min\{\chi(H \setminus e) : e \in E(H)\}$. In particular, it shows that most Anti-Ramsey numbers are quadratic in n.

Instead of forcing rainbow copies of a given graph H, one can consider forcing properly coloured copies of H by using many colours, and study the threshold on the number of colours needed. This is thoroughly studied by Manoussakis, Spyratos, Tuza and Voigt in [19].

In this paper, we consider the local variation of the anti-Ramsey problem. Namely, we study the maximum k such that there exists a k-good edge-colouring of K_n containing no rainbow copy of a given graph H. In addition, we consider the related problem concerning properly coloured subgraphs. We give formal definitions as follows.

Given an edge-colouring c of a host graph G and a vertex v in G, define the *colour-degree* of v to be the number of different colours that are used on edges incident to v. Using this notation, a k-good edge-colouring of G is then simply an edge-colouring of G with minimum colour degree at least k. Given a positive integer n and a graph H, let $f(n, H)$ denote the maximum k such that there exists a k-good colouring of $E(K_n)$ that contains no properly coloured copy of H, and let $g(n, H)$ denote the maximum k such that there exists a k-good colouring of $E(K_n)$ containing no rainbow copy of H. By definition, every colouring of $E(K_n)$ with minimum colour degree more than $f(n, H)$ contains a properly coloured copy of H, and every colouring of $E(K_n)$ with minimum colour degree more than $g(n, H)$ contains a rainbow copy of H.

So here, we impose constraints on the number of colours locally, requiring each vertex to be incident to at least a certain number of colours, and we are interested in how large this number should be in order to force certain rainbow subgraphs or properly coloured subgraphs. Note, however, in a colouring of $E(K_n)$ with large minimum colour degree, the total number of colours used may be as small as $n = o(n^2)$ (recalling that K_n has a proper colouring using at most n colours and such a colouring has the largest possible minimum colour degree). Therefore the local anti-Ramsey problem will have a different

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flavour from the classical anti-Ramsey problem, in the sense that one cannot hope to force rainbow subgraphs merely by forcing a large overall number of colours used. Throughout the paper, we assume n is a sufficiently large positive integer, and H is a graph containing at least one edge. Also, we shall drop all ceiling and floor signs whenever they are not crucial.

Before we get into our results, we first provide an overview of the rest of the paper. In the first two sections we give general bounds on $f(n,H)$ and $g(n,H)$, respectively. Most of the results there, as expected, are closely related to Turán numbers and can be considered to be parallel to the general bounds on $AR(n,H)$ by Erdős, Simonovits and Sós in [9]. In particular, if H has chromatic number at least 3, then fairly tight bounds are obtained. There is, however, one important difference in the behaviour of $f(n,H)$ and $g(n,H)$ compared to that of $AR(n,H)$ or $ex(n,H)$. Namely, for bipartite graphs H with $e(H) > n(H)$, where $n(H), e(H)$ denote the number of vertices and number of edges in H, respectively, we have $f(n,H) = (1+o(1))n/2$ and $g(n,H) = (1+o(1))n/2$ (regardless of the structure of H). On the other hand, it is well known that for bipartite graphs H, the order of magnitude of $ex(n,H)$ (and as an indirect result, that of $AR(n,H)$, at least for certain H) depends heavily on the structure of H. Owing to the results in the first two sections, the most interesting remaining case on $f(n,H)$ and $g(n,H)$ would then be one in which H is bipartite, with each component F satisfying $e(F) \leqslant n(F)$. In other words, the case in which each component of H is a tree or is unicyclic. It is therefore natural to focus our attention on the behaviour of $f(n,H)$ and $g(n,H)$ when H is a tree or cycle. We address this issue in Section 4, which is perhaps the most intriguing section of this paper. We end with concluding remarks and questions in Section 5.

2. Properly coloured subgraphs

We start with properly coloured subgraphs in colourings with large minimum colour degree. We first derive a general upper bound on $f(n,H)$ using the chromatic number of H.

Theorem 2.1. *Let H be a graph with $\chi(H) = k \geqslant 2$. Then*

$$f(n,H) \leqslant \left[1 - \frac{1}{2(k-1)}\right] n + o(n).$$

Proof. Let a_k be any real number with $a_k > 1 - \frac{1}{2(k-1)}$. Let c be an arbitrary $a_k(n-1)$-good colouring of the edges of $G = K_n$. It suffices to show that c contains a properly coloured copy of H. Suppose the vertices of G are $v_1, \ldots, v_n$. Let $G_0 = G$, and for $i = 1, 2, \ldots, n$, define G_i iteratively as follows: G_i is obtained from G_{i-1} by deleting extra copies of each edge incident to v_i in G_{i-1} whose colour appears more than once at v_i.

Clearly, by the above procedure, G_n is properly coloured under c. At each step, at most $(1-a_k)(n-1)$ edges are deleted, so G_n has at least $\binom{n}{2} - n(n-1)(1-a_k) = (1-2+2a_k)\binom{n}{2} = (1 - \frac{1}{k-1} + \epsilon)\binom{n}{2}$ edges, where $\epsilon = a_k - 1 - \frac{1}{2(k-1)} > 0$. Since $ex(n,H) = (1-\frac{1}{k-1})\binom{n}{2} + o(n^2)$, we conclude that G_n contains a copy of H, which is properly coloured under c since G_n is also. □

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Although the upper bound given in $f(n,H)$ seems rather high, we will soon see that there is not much room for improvement in general due to the next couple of lower bounds we establish. Suppose D is a digraph and uv is an arc in D from u to v. Then u is the *tail* of uv and v is the *head* of uv.

Theorem 2.2. *If H is a graph with $e(H) > n(H)$, then*

$$f(n,H) \geqslant \lfloor n/2 \rfloor.$$

Proof. We construct a colouring of $E(K_n)$ with minimum colour degree at least $\lfloor n/2 \rfloor$ that contains no properly coloured copy of H. Let D be an orientation of $E(K_n)$ with minimum in-degree at least $\lfloor n/2 \rfloor$. Such an orientation can easily be found using a decomposition of $E(K_n)$ or $E(K_{n+1})$ into 2-factors and cyclically orienting each cycle in these 2-factors.

Let $x_1, \ldots, x_n$ denote the vertices of K_n. Define a colouring c of $E(K_n)$ by assigning colour i to all the edges with tail x_i, where $1, 2, \ldots, n$ are distinct colours. In the colouring c, each vertex has colour degree at least $\lfloor n/2 \rfloor$. Consider any copy H' of H in D, and let $D[H']$ denote the restriction of the orientation D on the edges of H'. Since $e(H') > n(H')$, there exists a vertex v with out-degree at least two in $D[H']$. By our definition of c, the out-edges at v in $D[H']$ have the same colour, preventing H' from being properly coloured. □

Note that Theorem 2.2 applies to almost all graphs. The only graphs to which it does not apply are graphs in which each component is a tree or is unicyclic. Theorem 2.2, together with Theorem 2.1, allows us to determine the asymptotics of $f(n,H)$ for almost all bipartite graphs H.

Corollary 2.3. *Let H be a bipartite graph with $e(H) > n(H)$. Then*

$$f(n,H) = (1 + o(1))n/2.$$ □

Another interesting fact that follows from Theorems 2.1 and 2.2 is as follows. Recall that a graph G is a *subdivision* of another graph H if G can be obtained from H by inserting vertices of degree two into edges of H.

Corollary 2.4. *Let H be a graph. Let ϵ be an arbitrary positive real and let $n = n(\epsilon)$ be a sufficiently large positive integer. Then every colouring of $E(K_n)$ with minimum colour degree at least $\frac{1+\epsilon}{2}n$ contains a properly coloured subdivision of H. Furthermore, if $e(H) > n(H)$ then there exists a colouring of $E(K_n)$ with minimum colour degree at least $\lfloor n/2 \rfloor$ that contains no properly coloured subdivision of H.*

Proof. For the first statement, let H' denote the graph obtained from H by subdividing each edge of H exactly once. Clearly, H' is a bipartite graph. By Theorem 2.1 $f(n,H) \leqslant n/2 + o(n)$. Thus, for sufficiently large n, every colouring of $E(K_n)$ with minimum colour degree at least $\frac{1+\epsilon}{2}n$ yields a properly coloured copy of H'.

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For the second statement, note that for any subdivision H'' of H, $e(H'') > n(H'')$ holds. Applying Theorem 2.2, there exists a colouring of $E(K_n)$ with minimum colour degree at least $\lfloor n/2 \rfloor$ that contains no properly coloured subdivision of H. □

For non-bipartite graphs H, we have the following general lower bound on H in terms of its chromatic number.

Theorem 2.5. *Let H be a graph with $\chi(H) = k \geqslant 3$. Then*

$$f(n, H) \geqslant \left(1 - \frac{1}{\lceil k/2 \rceil - 1}\right) n + \Omega(1).$$

Proof. Let $q = \lceil k/2 \rceil - 1$. Let T_n^q denote the q-partite Turán graph on n vertices (with each part of size $\lceil n/q \rceil$ or $\lfloor n/q \rfloor$). Let $V_1, \ldots, V_q$ denote the q partite sets of T. Define a colouring c of $E(K_n)$ by assigning distinct colours to $E(T_n^q)$ and then assigning a new colour 0 to all the edges in $E(K_n) - E(T_n^q)$.

It is easy to verify that each vertex has colour degree at least $n(1 - \frac{1}{\lceil k/2 \rceil - 1}) + \Omega(1)$ in c. We show that c contains no properly coloured copy of H. Let L be any properly coloured subgraph in c; we show that $\chi(L) < k$, which would imply $L \neq H$ since $\chi(H) = k$. Suppose first that L does not use any edge of colour 0; then L is a subgraph of T_n^q, and thus $\chi(L) = q = \lceil k/2 \rceil - 1 < k$. Next, suppose that L has edges of colour 0. Observe that, for each $i \in [q]$, $V(L) \cap V_i$ must induce a matching since L is properly coloured. Thus the subgraph of L induced by $V(L) \cap V_i$ has chromatic number at most 2. It follows that $\chi(L) \leqslant 2q < k$, completing the proof. □

We summarize our bounds on $f(n, H)$ for non-bipartite graphs H as follows.

Proposition 2.6. *Let H be a graph with $\chi(H) = k \geqslant 3$. Then*

$$\left[1 - \frac{1}{\lceil k/2 \rceil - 1}\right] n + \Omega(1) \leqslant f(n, H) \leqslant \left[1 - \frac{1}{2(k-1)}\right] n + o(n).$$

Further, if H is not a disjoint union of acyclic or unicyclic graphs, then $f(n, H) \geqslant \lfloor n/2 \rfloor$. □

We have seen that, for bipartite graphs H other than the ones in which each component is either a tree or is unicyclic, $f(n, H) = (1 + o(1))n/2$, and for non-bipartite graphs H other than disjoint union of acyclic or unicyclic graphs, we have lower and upper bound in the same order of magnitude. This leaves the particularly desirable task of studying $f(n, H)$ when H consists entirely of components that are acyclic or unicyclic. In particular, it is interesting to study the behaviour of $f(n, H)$ when H is a tree or cycle. We shall investigate local anti-Ramsey numbers of trees and cycles in Section 4.

3. Rainbow subgraphs

In this section, we study rainbow subgraphs forced by edge-colourings with large minimum colour degree. We first prove a useful lemma which enables us to find a rainbow copy of

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a desired subgraph H in a dense properly coloured graph. Let G, H be two graphs, and let c be an edge-colouring of G. A *rainbow embedding* of H in G is an embedding σ of H in G such that $\sigma(H)$ is rainbow. Given positive integers s, t, K_t^s denotes the complete t-partite graph with s vertices in each part. Let $F[A]$ be a subgraph of F induced by a vertex set A, and let $N_F(v)$ be a neighbourhood of v in F. For convenience, we write $\sigma(A)$ for $\sigma(F[A])$ when it is clear from the context.

Lemma 3.1. *Let H be a graph with p vertices, q edges, and $\chi(H) = k \geqslant 2$. Let $s = pq$. Then every proper colouring of K_k^s contains a rainbow copy of H.*

Proof. Let c be a proper colouring of the edges of $G = K_k^s$. Let $A_1, \ldots, A_k$ denote the k partite sets of G, each of size s. Since $\chi(H) = k$, H is a k-partite graph; let $B_1, \ldots, B_k$ denote the partite sets of H. Let F be a maximal induced subgraph of H, such that there exists a rainbow embedding σ of F in G satisfying $V(\sigma(V(F) \cap B_i)) \subseteq A_i$ for all $i \in [k]$. Since the subgraph of H induced by B_1 can trivially be embedded in A_1, F is well defined. We show that $F = H$.

Suppose otherwise there is $v \in V(H) \setminus V(F)$. Without loss of generality, suppose $v \in B_k$. Let $N = N_F(v)$. Note that $N \subseteq \bigcup_{i=1}^{k-1} B_i$ and hence $V(\sigma(N)) \subseteq \bigcup_{i=1}^{k-1} A_i$. Let $D = A_k \setminus V(\sigma(F))$. Clearly, $|D| > s - p \geqslant pq - p$. If $N = \emptyset$, then we can extend σ by mapping v to any vertex in D, contradicting our choice of F. So we assume that $|N| \geqslant 1$.

For convenience, we call the colours used on edges in $\sigma(F)$ *bad colours*, and an edge of G with a bad colour a *bad edge*. Note that there are at most $q - |N|$ bad colours. Consider any vertex $u \in D$. Since c is a proper colouring, the edges u sends to $\sigma(N)$ all have different colours. If none of these edges is a bad edge, then we can extend σ by mapping v to u, contradicting our choice of F. Hence each vertex in D sends a bad edge to $\sigma(N)$. Thus there are at least $|D| > pq - p$ bad edges between D and $\sigma(N)$. On the other hand, each vertex w in $\sigma(N)$ is incident to at most $q - |N| \leqslant q - 1$ bad edges (recall that c is a proper colouring). Hence, there are at most $|N|(q-1) < p(q-1) = pq - p$ bad edges between D and $\sigma(N)$, a contradiction. □

Recall that $g(n, H)$ is the maximum k such that there exists a colouring of $E(K_n)$ with minimum colour degree at least k containing no rainbow copy of G. Note that $g(n, H) \geqslant f(n, H)$ holds trivially.

Lemma 3.2. *Let H be a graph with p vertices, q edges, and $\chi(H) = k$. Let $s = pq$. For sufficiently large n, we have $g(n, H) \leqslant f(n, K_k^s)$.*

Proof. Let c be an arbitrary edge-colouring of K_n with minimum colour degree at least $f(n, K_k^s) + 1$. It suffices to show that c contains a rainbow copy of H. By the definition of $f(n, K_k^s)$, the colouring c has a properly coloured copy B of K_k^s. By Lemma 3.1, B contains a rainbow copy of H. □

Theorem 2.1 and Lemma 3.2 together yield the following result.

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Theorem 3.3. *Let H be graph with $\chi(H) = k \geqslant 2$. Then*

$$g(n, H) \leqslant \left[1 - \frac{1}{2(k-1)}\right] n + O(n).$$

□

Since $g(n, H) \geqslant f(n, H)$ always, Theorem 2.2 together with Theorem 3.3 yield the following result.

Corollary 3.4. *Let H be a bipartite graph with $e(H) > n(H)$. Then*

$$g(n, H) = (1 + o(1))n/2.$$

□

As in the previous section, the following fact can easily be established.

Corollary 3.5. *Let H be a graph. Let ϵ be an arbitrary positive real and let $n = n(\epsilon)$ be a sufficiently large positive integer. Then every colouring of $E(K_n)$ with minimum colour degree at least $(\frac{1+\epsilon}{2})n$ contains a rainbow subdivision of H. Furthermore, if $e(H) > n(H)$ then there exists a colouring of $E(K_n)$ with minimum colour degree at least $\lfloor n/2 \rfloor$ that contains no rainbow subdivision of H.* □

Next, we give a lower bound on $g(n, H)$ for non-bipartite graphs H.

Theorem 3.6. *Let H be a graph with $\chi(H) = k \geqslant 3$. Then*

$$g(n, H) \geqslant \left(1 - \frac{1}{k-2}\right) n + \Omega(1).$$

Proof. Let T_n^{k-2} denote the $(k-2)$-partite Turán graph on n vertices. Consider a colouring c of $E(K_n)$ obtained by colouring the edges of T_n^{k-2} with distinct colours and assigning a new colour to the remaining edges. It is easy to see that c has minimum colour degree at least $(1 - \frac{1}{k-2})n + \Omega(1)$. Furthermore, if L is a rainbow subgraph under c, it is straightforward to verify that $\chi(L) \leqslant k - 1$. Hence, in particular, c contains no rainbow copy of H. □

We summarize our bounds for non-bipartite graphs H as follows.

Proposition 3.7. *Let H be a graph with $\chi(H) = k \geqslant 3$. Then*

$$\left[1 - \frac{1}{k-2}\right] n + \Omega(1) \leqslant g(n, H) \leqslant \left[1 - \frac{1}{2(k-1)}\right] n + o(n).$$

Further, if H is not a disjoint union of acyclic or unicyclic graphs, then $g(n, H) \geqslant \lfloor n/2 \rfloor$.

□

As in the previous section, we have obtained relatively satisfactory bounds on $g(n, H)$ except when H is a disjoint union of acyclic or unicyclic graphs. It is therefore particularly interesting to study $g(n, H)$ when H is a tree or a cycle. We shall leave this for Section 4.

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4. Properly coloured and rainbow trees and cycles

We first consider the case when H is a tree. This case is fairly straightforward.

Proposition 4.1. *Let H be a tree with $k \geqslant 2$ edges. Then*

$$\Delta(H) - 2 \leqslant f(n, H) \leqslant k - 1,$$
$$k - 1 \leqslant g(n, H) \leqslant 2k - 3.$$

Proof. If n is even, let c be a colouring of $E(K_n)$ in which each vertex has colour degree $\Delta(H) - 1$; such a colouring can easily be obtained from a 1-factorization of $E(K_n)$. If n is odd, let c be a colouring of $E(K_n)$ in which each vertex has colour degree either $\Delta(H) - 2$ or $\Delta(H) - 1$; such a colouring can be obtained from a 1-factorization of $E(K_{n+1})$. Clearly, in the colouring c defined above, no properly coloured copy of H can occur. Thus, $f(n, H) \geqslant \Delta(H) - 1$ when n is even and $f(n, H) \geqslant \Delta(H) - 2$ when n is odd, respectively.

On the other hand, it can easily be proved by induction on k that every colouring of $E(K_n)$ with minimum colour degree at least k contains a properly coloured copy of H. Thus $f(n, H) \leqslant k - 1$. A similar argument establishes $g(n, H) \leqslant 2k - 3$.

For a lower bound on $g(n, H)$, consider a colouring using $k - 1$ colours, in which each vertex is incident to all $k - 1$ colours; such a colouring can be obtained from a 1-factorization of K_n or 2-factorization of K_n (where n is sufficiently large). Since only $k - 1$ colours are used, no rainbow copy of H can arise. Thus $g(n, H) \geqslant k - 1$. □

It is likely that the upper bound on $g(n, H)$ in the proposition above can be improved for various classes of trees H. Next, we consider the case when H is a cycle. This case turns out to be far more tricky than we expected, and we do not know too much in this case. It is not even clear what the growth rate of $f(n, C_k)$ and $g(n, C_k)$ should be. We are, however, able to find some lower bounds on $f(n, C_k)$ and $g(n, C_k)$ and determine $f(n, C_k)$ either asymptotically or exactly for $k = 3, 4$.

First, we give some lower bounds on $f(n, C_k)$ and $g(n, C_k)$.

Proposition 4.2. *For fixed k and sufficiently large n, there exists a colouring of $E(K_n)$ with minimum colour degree at least $\lfloor (k-1)/2 \rfloor$ that contains no properly coloured cycle of length at least k. In particular, we have*

$$f(n, C_k) \geqslant \lfloor (k-1)/2 \rfloor.$$

Proof. Let $G = K_n$. Let $m = \lfloor (k-1)/2 \rfloor$. Let $A = \{a_1, \ldots, a_m\}$ be a set of m vertices in G. Let $B = V(G) - A$. We define a colouring c of $E(G)$ as follows. First colour the edges between A and B by assigning colour i to all of those incident to a_i. Then we use a new set of colours to colour the edges within A such that those edges all have different colours. Finally, we assign a new colour to all the edges within B.

It is easy to check that c has minimum colour degree at least $\lfloor (k-1)/2 \rfloor$. We show that it contains no properly coloured cycle of length at least k. Consider a properly

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coloured cycle L in c. Clearly, L cannot lie completely in B since all edges in B have the same colour. If L lies completely in A, then it has length at most $\lfloor (k-1)/2 \rfloor < k$. So we may assume that L contains vertices in both A and B. Observe that since L is properly coloured, each vertex on L must have at least one of its two neighbours on L lie in A. From this one can easily deduce that at least half of the vertices on L lie in A. Hence, in particular, L has length at most $2|A| = 2m < k$. □

Given positive integers m, n with $m \geqslant \lceil \log_2 n \rceil$, by the n smallest binary m-tuples, we mean the m-tuples that are binary representations of $1, 2, \ldots, n$ (with preceding 0s if necessary). For example, the 5 smallest binary 3-tuples are $001, 010, 011, 100, 101$.

Proposition 4.3. *For fixed k and sufficiently large n, we have*

$$g(n, C_k) \geqslant \lceil \log_2 n \rceil - 1.$$

Proof. Let $m = \lceil \log_2 n \rceil$. We define a colouring of $E(K_n)$ with minimum colour degree at least $m-1$ that contains no rainbow cycles at all. Label the vertices of K_n distinctly using the n smallest binary m-tuples. For every pair $u, v \in V(K_n)$, let $c(uv)$ be the lowest position at which the two tuples differ. It is easy to see that the colour degree of any vertex u is at least $m-1$, and that the colouring c contains no rainbow cycles. □

Next, we give a general upper bound on $g(n, C_k)$. We make an easy observation first, which appeared in several previous papers.

Lemma 4.4. *Let n, k be positive integers with $n \geqslant k$. Let c be a colouring of $E(K_n)$ that has a rainbow cycle C of length $2k-2$; then c also contains a rainbow cycle of length k.*

Proof. Let u, v denote two vertices on C at distance $k-1$. One of the u, v-portions of C avoids colour $c(uv)$ (since C is rainbow), thus completing a C_k with uv. □

Proposition 4.5. *For fixed k and sufficiently large n, we have $g(n, C_k) \leqslant n/2 + o(n)$.*

Proof. For even k, this follows from Theorem 3.3. For odd k, we have by Lemma 4.4 $g(n, C_k) \leqslant g(n, C_{2k-2}) \leqslant n/2 + o(n)$. □

Obviously, the lower and upper bounds in Propositions 4.5 and 4.3 are not satisfactory in general. They do not even give the order of magnitude for $g(n, C_k)$. However, as we shall see from the next few results, the order of magnitude of $g(n, C_k)$ most likely varies with the value of k. We shall show that $g(n, C_k)$ is sublinear in n if $k = 3$ or 4 and that it is linear in n if $k \equiv 2 \pmod 4$. For $k = 3$, by a result of Erdős and Tuza [12, Theorem 1], we have $g(n, C_3) \leqslant \lceil \log_2 n \rceil + O(1)$. By Proposition 4.3, we have $g(n, C_3) \geqslant \lceil \log_2 n \rceil - 1$.

Proposition 4.6. $f(n, C_3) = g(n, C_3) = (1 + o(1)) \log_2 n$. □

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For $k = 4$ we are able to improve the general upper bound to $g(n, C_4) = O(n^{2/3})$. In our proof below, we drop ceiling and floor signs whenever they are not crucial, and we do not attempt to optimize absolute constants.

Theorem 4.7. *Let n be a sufficiently large positive integer. Then every edge-colouring c of $K = K_n$ with minimum colour degree at least $4n^{2/3}$ contains a rainbow cycle of length 4.*

Proof. Define a colour used in c to be *sparse* if it appears on at most $n^{4/3}$ edges of K; other colours used in c will be called *dense*. Clearly we have

$$\#\text{ dense colours} \leqslant n^2/n^{4/3} = n^{2/3}. \tag{4.1}$$

Let H denote the subgraph of K consisting of edges using sparse colours. Let c_H denote c restricted to H. Let q denote the minimum colour degree of c_H. Clearly, by (4.1) we have

$$q \geqslant 4n^{2/3} - n^{2/3} = 3n^{2/3}. \tag{4.2}$$

Let S denote a largest monochromatic star in H. Suppose S is centred at x and has s leaves, each connected to x with an edge of colour 1. Note that $|U| = s \leqslant n - q$ since c_H has minimum colour degree q. Let T denote a rainbow star on $q/4$ edges centred at x, such that colour 1 is not used in T and $V(T)$ is disjoint from U. Since H has minimum colour degree q, such T clearly exists.

Let V denote the set of leaves of T. Suppose $V = \{v_1, v_2, \ldots, v_p\}$, where $p = q/4$. For each $i = 1, 2, \ldots, p$, let S_i denote a rainbow star with $q/2$ edges in H centred at v_i such that (1) S_i does not use colour 1 or any colour used in T, and (2) its set of leaves L_i is disjoint from $V(T)$. Since H has minimum colour degree at least q, such stars exist. Let $W = \cup_{i=1}^{p} L_i$. Note that W is disjoint from $V(T)$ but it may overlap with U.

Let $F = \bigcup_{i=1}^{p} S_i$. Note that by definition, F does not use colour 1 or any of the colours used in T, and $e(F) = \Sigma_{i=1}^{p} |S_i| \geqslant |V|q/2 = q^2/8$. Suppose $W = \{w_1, w_2, \ldots, w_t\}$. If some w_j is incident to two edges, say $v_i w_j$ and $v_{i'} w_j$, of different colours in F, then these two edges complete a rainbow cycle of length 4 with $v_i x$ and $v_{i'} x$. Hence we may assume that, for each $j = 1, \ldots, t$, the edges in F incident to w_j all have the same colour α_j.

Call a vertex $w_j \in W$ *bad* if it is incident to at most one edge of F. Let B denote the set of bad vertices in W. Let $W^* = W - B$ and $F^* = F - B$. Note that $e(F^*) \geqslant e(F) - |B| \geqslant q^2/8 - n$. Note also that each $w_j \in W^*$ is incident to at least two edges of F (with colour α_j). We consider two cases.

Case 1: There exists a w_j that sends an edge $w_j u$ of K to some vertex $u \in U$ such that $c(w_j u) \notin \{1, \alpha_j\}$.
Suppose $v_i w_j$ and $v_{i'} w_j$ are two edges (with colour α_j) of F incident to w_j. Since T is rainbow one of $v_i x$ and $v_{i'} x$ does not use colour $c(w_j u)$. Suppose without loss of generality that $v_i x$ does not use colour $c(w_j u)$. Note by our construction that $c(w_j u) \notin \{1, \alpha_j\}$. Now $x v_i w_j u x$ is rainbow cycle of length 4.

Case 2: For each $w_j \in W^*$, all the edges of K from w_j to U use colours either 1 or α_j.
For each $w_j \in W^*$, let a_j denote the number of edges of F incident to w_j (note that they all have colour α_j), and let b_j denote the number of edges of K with colour 1 from w_j

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to U (note that w_j might lie in U). By our assumption, the other edges of K between w_j and U all have colour α_j; there are at least $|U|-1-b_j$ of them. Since V is disjoint from U, w_j is then incident to at least $a_j+|U|-1-b_j=|U|+(a_j-b_j-1)$ edges of K with colour α_j. Since we have chosen S to be a largest monochromatic star in a sparse colour, and α_j is also a sparse colour, we must have $|U|+(a_j-b_j-1)\leqslant|U|$. Hence $a_j\leqslant b_j+1$ for all $w_j\in W^*$. Now we have

$$q^2/8-n\leqslant e(F^*)=\Sigma_{w_j\in W^*}a_j\leqslant\Sigma_{w_j\in W^*}(b_j+1)\leqslant n^{4/3}+n, \tag{4.3}$$

where the last inequality follows from the fact that colour 1 is a sparse colour. By (4.2) and (4.3), we have

$$9n^{4/3}\leqslant q^2\leqslant 8\big(n^{4/3}+2n\big),$$

a contradiction. This completes our proof. □

Although we have sublinear upper bounds for C_3 and C_4, in general, however, $g(n,C_k)$ can be linear in n for certain values of k, as indicated by the next proposition.

Proposition 4.8. *Suppose $k>4$ is a positive integer with $k\equiv 2 \pmod 4$. Then*

$$g(n,C_k)\geqslant\lfloor n/4\rfloor.$$

Proof. Let $K=K_n$. Suppose the vertices of K are $x_1,\ldots,x_n$. Let V_0,V_1,V_2,V_3 be a partition of $V(K)$ into almost equal subsets, each of size $\lfloor n/4\rfloor$ or $\lceil n/4\rceil$. Let C_n^4 denote the spanning subgraph of K containing all the edges between V_i and V_{i+1}, for all $i=0,\ldots,3$ (with subscripts taken modulo 3). Next, we define an orientation $D(C_n^4)$ of C_n^4 by orienting all the edges between V_i and V_{i+1} from V_i to V_{i+1} for $i=0,1,2,3$.

We now define a colouring c of $V(K_n)$ in two steps. Step 1: we colour the edges in $D(C_n^4)$ by assigning, for each vertex x_i, colour i to all edges with tail x_i, where colours $1,\ldots,n$ are all different. Step 2: we assign a new colour 0 to all the remaining edges in K. Clearly, c has minimum colour degree at least $\lfloor n/4\rfloor$. To complete the proof it suffices to show that c contains no rainbow cycles of length 2 (mod 4). Let C be a rainbow cycle in c. We consider two cases. In each case, we show that C has length 0, 1, or 3 (mod 4).

Case 1: C does not use colour 0.
Then C lies completely in $D(C_n^4)$. Observe that C has to be a directed cycle in $D(C_n^4)$; otherwise C contains a vertex x_j of out-degree two in $D(C_n^4)$ restricted to C. These two out-edges both receive colour j by our definition of c, contradicting C being rainbow. Now, since C is a directed cycle in $D(C_n^4)$, clearly it has length 0 (mod 4).

Case 2: Colour 0 appears on an edge uv in C.
Since C is rainbow, uv is the only edge on C with colour 0. Note that since uv has colour 0, we have either u,v belong to the same V_i or $u\in V_i$ and $v\in V_j$ where $j\equiv i+2 \pmod 4$. Let $P=C-uv$. Then P lies completely in $D(C_n^4)$. Since P is rainbow, by a similar argument to that of Case 1, either P is a directed path in $D(C_n^4)$ or P contains a unique sink w. We consider two subcases.

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Subcase 2.1: P is a directed path in $D(C_n^4)$.
Then P has length either 0 (mod 4) if u, v belong to the same V_i or 2 (mod 4) if u, v belong to V_i, V_j, respectively, with $j = i + 2 \pmod 4$. So, C has length either 1 (mod 4) or 3 (mod 4).

Subcase 2.2: P contains a unique sink w.
Let P_1 denote the portion of P from u to w, and let P_2 denote the portion of P from v to w. Let l, l_1, l_2 denote the length of P, P_1, P_2, respectively. If u, v belong to the same V_i, then $l_1 \equiv l_2 \pmod 4$. Thus, $l \equiv 2l_1 \pmod 4$, implying that $l \equiv 0$ or 2 (mod 4), and hence C has length 1 or 3 (mod 4). If u, v belong to V_i, V_j, respectively, with $j \equiv i + 2 \pmod 4$, then $l_1 \equiv l_2 + 2 \pmod 4$, and so $l \equiv 2l_2 + 2 \equiv 0$ or 2 (mod 4). Again, C has length 1 or 3 (mod 4). □

We now summarize our bounds on $g(n, C_k)$. The upper bound comes from Theorem 3.3.

Proposition 4.9. *Let $k \geqslant 3$ be a fixed integer and let n be a sufficiently positive large integer. We have*

$$\lceil \log_2 n \rceil - 1 \leqslant g(n, C_k) \leqslant n/2 + o(n).$$

Furthermore, $g(n, C_3) = (1 + o(1)) \log_2 n$, $g(n, C_4) = O(n^{2/3})$, and if $k \equiv 2 \pmod 4$ then $g(n, C_k) \geqslant \lfloor n/4 \rfloor$. □

In particular, we know the order of magnitude of $g(n, C_k)$, when $k = 3$ and when $k \equiv 2 \pmod 4$. We also have an upper bound on the order $n^{2/3}$ for C_4. In general, we suspect that the order of magnitude of $g(n, C_k)$ may vary with the value of k.

Let us now go back to $f(n, C_k)$. It appears that forcing properly coloured cycles is considerably easier than forcing rainbow cycles. Our speculation is that $f(n, C_k)$ may be bounded from above by a constant depending only on k (independent of n). We are able to determine the exact value of $f(n, C_4)$, which is surprisingly small.

First, let us consider the following colouring of $E(K_n)$. Let x_1, x_2 be two vertices of K_n. Let $X = \{x_1, x_2\}$ and $Y = V(K_n) - Y$. Colour the edges of K_n as follows. For $i = 1, 2$, assign colour i to all the edges between x_i and Y. Then assign colour 1 to all the edges within Y and colour 3 to all the edges within X. It is easy to see that the colouring defined above has minimum colour degree 2 and contains no properly coloured C_4. Thus $f(n, C_4) \geqslant 2$. We show next that raising the minimum colour degree to be at least 3 would ensure a properly coloured C_4 and hence $f(n, C_4) = 2$.

Theorem 4.10. *For $n \geqslant 4$, every colouring of $E(K_n)$ with minimum colour degree at least 3 contains a properly coloured C_4. In particular, we have $f(n, C_4) = 2$.*

Proof. Let c be an edge-colouring of K_n with minimum colour degree at least 3. Let G be a minimal complete subgraph of K_n such that c restricted to G has minimal colour degree at least 3. We show that G contains a properly coloured C_4. For convenience, we

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henceforth use c to denote c restricted to G. If $n(G) = 4$ then c is a proper colouring of $E(G)$ and the claim holds trivially. So we may assume that $n(G) \geqslant 5$.

Suppose G does not contain a properly coloured C_4, we derive a contradiction. By our choice of G, for each vertex u in G, c restricted to $G - u$ has minimum colour degree 2. Let $A(u)$ denote the set of vertices with colour degree 2 in c restricted to $G - u$. Create a digraph D on $V(G)$ with edge set $\bigcup_{u \in V(G)} \{uv : v \in A(u)\}$. By the definition of D and the assumption that c contains no properly coloured C_4 we can make the following observations about D.

(1) Every vertex u in D has out-degree at least 1.
(2) If $uv \in E(D)$, then v is incident to exactly three colours in G and uv is the only edge incident to v with colour $c(uv)$.
(3) Every vertex v in D has in-degree at most 2.
(4) If uv and xy are two independent edges of D then $c(uv) \neq c(xy)$.
(5) If D' is a sub-digraph of D with maximum out-degree at most 1, then the edges of D' all have different colours.

Observations (1) and (2) follow immediately from the definition of D. Observation (3) follows from (2) and the fact that $n(D) \geqslant 4$. Observation (4) holds because, if $c(uv) = c(xy) = \alpha$, then $C = uvxyu$ would be a properly coloured C_4 (noting that, by observation (2), $c(vx) \neq c(uv) = \alpha$ and $c(yu) \neq c(xy) = \alpha$). Finally, observations (2) and (4) together imply observation (5).

Observation (1) implies that D contains either a directed cycle or a double edge (joining two vertices in both directions). We consider three cases. In each case, we derive a contradiction.

Case 1: D has a directed cycle C of length at least 4.
By observations (2) and (5), the edges on C all have different colours and each vertex on C is incident to exactly three colours in G. Since G contains no properly coloured C_4, C has length at least 5. Suppose $C = u_0u_1 \cdots u_mu_0$, where $m \geqslant 4$. For each $i = 0, \ldots, m$, suppose $u_{i-1}u_i$ uses colour i (where subscripts are taken modulo m), by our discussion above, $0, 1, \ldots, m$ are all different colours, and for $i = 0, \ldots, m$, $u_{i-1}u_i$ is the only edge incident to u_i with colour i.

Since $u_0u_1u_2u_3$ is a properly coloured P_4 using colours $1, 2, 3$ in order and $c(u_0u_3) \neq a_3$, to avoid a properly coloured C_4 we must have $c(u_0u_3) = 1$. This also implies that the three colours incident to u_3 in G are $1, 3, 4$. Similarly, we have $c(u_1u_4) = 2$. Now, since $c(u_1u_3) \in \{1, 3, 4\}$ and $c(u_1u_3) \neq 1$ or 3, we have $c(u_2u_4) = 4$.

Since $u_0u_3u_1u_4$ is a properly coloured P_4 using colours $1, 4, 2$ in order, to avoid a properly coloured C_4 we must have $c(u_0u_4) \in \{1, 2\}$. If $c(u_0u_4) = 2$ then $u_0u_1u_2u_4u_0$ is a properly coloured C_4, noting that $c(u_2u_4) \neq 2$, a contradiction. So, $c(u_0u_4) = 1$, and thus C has length at least 6, otherwise we would have two edges in C having the same colour. Moreover, in this case u_4 is incident to four different colours $1, 2, 4, 5$, again a contradiction.

Case 2: D has a cycle T of length 3 and no longer cycle.
Let $T = u_1u_2u_3$ be a directed triangle in D. We first prove that there exists a set S of at most four vertices containing $V(T)$ such that no vertex in $D - S$ sends an edge into S

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in D. If no vertex outside T sends an edge into T, we let $S = V(T)$. Otherwise, let w be a vertex that sends an edge into T. Without loss of generality, suppose $wv_1 \in E(D)$, let $S = V(T) \cup \{w\}$. Suppose there exists a vertex w' outside S that sends an edge $w'x$ into S, where $x \in S$: we will derive a contradiction.

By observations (2) and (5), the colours used on edges $u_1u_2, u_2u_3, u_3u_1, wu_1, w'x$ are distinct: let them be $1, 2, 3, 4, 5$, respectively. Each of u_1, u_2, u_3 is incident to exactly three colours in G. Since $wu_1u_2u_3$ is a properly coloured P_4 using colours $4, 1, 2$ in order and $c(wu_3) \neq c(u_2u_3) = 2$, we must have $c(wu_3) = 4$ to avoid a properly coloured C_4.

If $x = u_1$ then we clearly also have $c(w'u_3) = 5$, in which case u_3 is incident to four colours $2, 3, 4, 5$, a contradiction. If $x = u_3$, then u_3 is incident to four colours $2, 3, 4, 5$, a contradiction. If $x = u_2$, then we switch the roles of w and w' and obtain a similar contradiction. This leaves $x = w$ as the only possibility. In this case, $c(w'u_1) = 1$ since $c(w'u_1) \in \{1, 3, 4\}$ and $c(w'u_1) \neq 3$ or 4. Now, $w'u_1u_3ww'$ is a properly coloured C_4 using colours $1, 3, 4, 5$ in order, again a contradiction.

We have shown that $V(D) - S$ sends no edge into S. Since D has minimum out-degree at least 1, so does $D - S$. Hence $D - S$ contains either a directed triangle or a double edge. We consider the two cases, assuming the colours used on u_1u_2, u_2u_3, u_3u_1 are $1, 2, 3$, respectively.

Subcase 2.1: $D - S$ contains a directed triangle $T' = \{v_1v_2v_3\}$.
By observations (2) and (5), the colours used on v_1v_2, v_2v_3, v_3v_1 are distinct, and different from colours on T, say $5, 6, 7$, respectively. Each of $u_1, u_2, u_3, v_1, v_2, v_3$ is incident to exactly three colours in G.

Now, since $u_3u_1v_3v_2$ is a properly coloured P_4 (noting that $c(u_1v_3) \notin \{3, 6\}$ by observation (2)), we must have $c(u_3v_2) \in \{3, 6\}$ to avoid a properly coloured C_4. By symmetry, we may assume that $c(u_3v_2) = 3$. Now, since $u_1u_2u_3v_2$ is a properly coloured P_4 using colours $1, 2, 3$ in order and $c(u_1v_2) \neq c(u_3u_1) = 3$, we must have $c(u_1v_2) = 1$. But then v_2 is incident to four colours $1, 3, 5, 6$, a contradiction.

Subcase 2.2: $D - S$ contains a double edge $xy \cup yx$.
Suppose the colours used on xy in G is 5. As before, colours $1, 2, 3, 5$ are all different, and each of u_1, u_2, u_3, x, y is incident to exactly three colours in G. Since $c(u_2y) \notin \{1, 5\}$, u_1u_2yx is a properly coloured P_4. This forces $c(u_1x) = 1$ (noting that $c(u_1x) \neq 5$) in order to avoid a properly coloured C_4. By similar arguments, we have $c(u_2x) = 2$ and $c(u_3x) = 3$. But then x is incident to four colours $1, 2, 3, 5$, a contradiction.

Case 3: D contains a double edge $uv \cup vu$ and no directed cycles.
If D contains another double edge $xy \cup yx$ that is independent of $uv \cup vu$, then by observation (2), one can easily see that $uvxyu$ is a properly coloured C_4. So we may assume that D has no double edge independent of D.

Let D' be a maximal subgraph of D containing uv such that the underlying graph G' of D' is a tree and all edges in D' other than uv are oriented towards $\{u, v\}$. By the definition of D', $D - V(D')$ sends no edge into $V(D')$. If $V(D) - V(D') \neq \emptyset$, then $D - V(D')$ has minimum out-degree at least 1, which then contains a double edge independent of $uv \cup vu$, a contradiction. So D' spans D. In particular, D' has at least 5 vertices.

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Note that D' contains no vertex of in-degree more than two, by observation (3). By symmetry, we may assume that there exist vertices x, y such that $xy, yu \in E(D')$. Clearly, D' contains another edge wz where $w \notin \{u, v, x, y\}$ and $z \in \{u, v, x, y\}$. Suppose the colours used on xy, yu, uv, wz are $1, 2, 3, 4$, respectively. By observations (2) and (5), colours $1, 2, 3, 4$ are all different, and each of u, v, y is incident to exactly three colours in G.

Suppose first that $z = x$ then $c(wu) = 4$ by an argument like before. By observation (2), $c(xu) \notin \{2, 3, 4\}$, otherwise $vuxyv$ forms a properly coloured C_4. This forces u to be incident to four different colours, a contradiction. Next, suppose $z = y$, then $c(wv) = 4, c(xv) = 1$. Now, by observation (2), $c(yv) \notin \{1, 3, 4\}$; this again forces v to be incident to four different colours. Finally, suppose $z = v$. In this case we have $c(xv) = 1$ and $c(yv) \notin \{1, 3, 4\}$, forcing v to be incident to four different colours. This completes our proof. □

We now summarize our bounds on $f(n, C_k)$. The general upper bounds follow from those on $g(n, C_k)$.

Proposition 4.11. *Let $k \geqslant 3$ be a fixed integer and let n be a sufficiently positive large integer. We have*

$$\lfloor (k-1)/2 \rfloor \leqslant f(n, C_k) \leqslant n/2 + o(n).$$

Furthermore, $f(n, C_3) = (1 + o(1))n/2$, and $f(n, C_4) = 2$. □

Any improvement of these bounds for $k \geqslant 5$ would be interesting. Finally, we consider a related problem by relaxing our requirement. We are now interested in the threshold on the minimum colour degree that forces properly coloured cycles of length at least k. Our bounds for this threshold are much better than for f. Let $\theta(n, k)$ denote the maximum l such that there is an l-good colouring with no properly coloured cycle of length at least k.

Using the construction from Proposition 4.2 we have $\theta(n, k) \geqslant \lfloor (k-1)/2 \rfloor$. On the other hand, we have the following result.

Theorem 4.12. *For fixed n, k, if c is a colouring of K_n with minimum colour degree at least $3k/2$, then c contains a properly coloured cycle of length at least k. Hence,*

$$\lfloor (k-1)/2 \rfloor \leqslant \theta(n, k) \leqslant 3k/2.$$

Proof. Let P be a longest properly coloured path in c. Let u, v denote the two endpoints of P. Orient P from u to v. For a vertex x on P, we use x^+ and x^- to denote its successor and predecessor, respectively. Let $a_1 = c(uu^+)$, $b_1 = c(vv^-)$. Let $a_2, \ldots, a_p$, where $p \geqslant 3k/2$, denote other colours incident to u in c. For each $i = 2, \ldots, p$, let ux_i be an edge with $c(ux_i) = a_i$. By our choice of P, $x_2, \ldots, x_p$ all lie on P. Suppose without loss of generality that $x_2, \ldots, x_p$ are at increasing distance from u on P. Define $b_1, \ldots, b_q$ and $y_2, \ldots, y_q$ similar to $a_2, \ldots, a_p$ and $x_2, \ldots, x_p$ for the other endpoint v of P, here $q \geqslant 3k/2$.

Now consider the edges ux_k and vy_k. If $c(x_k{}^-, x_k) \neq a_k$ then $P[u, x_k] \cup ux_k$ is properly coloured cycle of length at least k. So we may assume that $c(x_k{}^-, x_k) = a_k$, and hence

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$c(x_k, x_k^+) \neq a_k$. Similarly, we may assume that $c(y_k, y_k^+) = b_k$ and $c(y_k^-, y_k) \neq b_k$. We consider two cases.

Case 1: x_k lies between y_k and v in P.
In this case, $P[u, y_k] \cup y_k v \cup P[x_k, v] \cup u x_k$ is a properly coloured cycle of length at least k.

Case 2: x_k lies between u and y_k in P.
Let $m = \lfloor k/2 \rfloor$. Since $C' = P[x_m, y_m] \cup x_m y_m$ is cycle of length at least k, we may assume that either $c(x_m, y_m) = c(x_m, x_m^+)$ or $c(x_m, y_m) = c(y_m^-, y_m)$, otherwise C' is properly coloured. By symmetry we may assume that $c(x_m, y_m) = c(x_m, x_m^+)$. Then $c(x_m, y_m) \neq c(x_m^-, x_m)$. If $c(x_m, y_m) \neq c(y_m^-, y_m)$ then $P[u, x_m] \cup x_m y_m \cup P[x_k, y_m] \cup u x_k$ is a properly coloured cycle of length at least k. Otherwise, $P[u, x_m] \cup x_m y_m \cup P[y_m, v] \cup v y_k \cup P[x_k, y_k] \cup u x_k$ is a properly coloured cycle of length at least k. □

5. Concluding remarks

It is clear that much more work needs to be done regarding $f(n, C_k)$ and $g(n, C_k)$. For $g(n, C_k)$, we see that the order of magnitude varies with the value of k. We know that $g(n, C_3) = (1 + o(1)) \log_2 n$, $g(n, C_4) = O(n^{2/3})$, while $g(n, C_k) \geqslant \lfloor n/4 \rfloor$ when $k \equiv 2 \pmod 4$. Besides those results, little is known.

The behaviour of $f(n, C_k)$ might be more predictable. Although there is a big gap between our general lower and upper bounds on $f(n, C_k)$, we suspect that for $k \geqslant 4$ $f(n, C_k)$ is bounded by a constant depending only on k. Formally, we raise the following question.

Question 5.1. *Is it true that for all positive integers $k \geqslant 4$ there exists a constant λ_k depending only on k such that $f(n, C_k) \leqslant \lambda_k$?*

We know that this is true for $k = 4$ as $f(n, C_4) = 2$. In a related problem, Alon and Gutin [3] showed that there exists a universal constant α a little smaller than $1 - \frac{1}{\sqrt{2}}$ such that, for all sufficiently large n, every edge-colouring of K_n in which no colour appears more than αn times at any vertex contains a properly coloured C_k for every k between 3 and n. For more general results on properly coloured and rainbow subgraphs in edge-colourings of K_n in which no colour appears more than a given number of times at any vertex: see [4].

References

[1] Ahlswede, R., Cai, N. and Zhang, Z. (1992) Rich colorings with local constraints. *J. Combin. Inform. System Sci.* **17** 203–216.
[2] Alon, N. (1983) On a conjecture of Erdős, Simonovits and Sós concerning anti-Ramsey theorems. *J. Graph Theory* **7** 91–94.
[3] Alon, N. and Gutin, G. (1997) Properly colored Hamiltonian cycles in edge-colored complete graphs. *Random Struct. Alg.* **11** 179–186.
[4] Alon, N., Jiang, T., Miller, Z. and Pritikin, D. Properly colored subgraphs and rainbow subgraphs in edge-colorings of complete graphs with local constraints. Submitted.

https://doi.org/10.1017/S0963548303005868 Published online by Cambridge University Press

[5] Alon, N., Lefmann, H. and Rödl, V. (1992) On an anti-Ramsey type result. In *Sets, graphs and numbers* (Budapest, 1991), Vol. 60 of *Colloq. Math. Soc. János Bolyai*, North-Holland, Amsterdam, p. 9–22.
[6] Axenovich, M. and Jiang, T. Anti-Ramsey numbers for small complete bipartite graphs. To appear in *Ars Combinatoria*.
[7] Axenovich, M., Jiang, T. and Kündgen, A. Bipartite anti-Ramsey numbers of cycles and path covers in bipartite graphs. To appear in *J. Graph Theory*.
[8] Bollobás, B. (1978) *Extremal Graph Theory*, Academic Press, London/New York.
[9] Erdős, P., Simonovits, M. and Sós, V. T. (1975) Anti-Ramsey theorems. In *Infinite and finite sets* (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, North-Holland, Amsterdam, Vol. 10 of *Colloq. Math. Soc. Janos Bolyai*, pp. 633–643.
[10] Erdős, P. and Simonovits, M. (1966) A limit theorem in graph theory. *Studia Sci. Math. Hungar.* **1** 51–57.
[11] Erdős, P. and Stone, A. H. (1946) On the structure of linear graphs. *Bull. Amer. Math. Soc.* **52** 1087–1091.
[12] Erdős, P. and Tuza, Zs. (1993) Rainbow subgraphs in edge-colorings of complete graphs. *Ann. Discrete Math.* **55** 81–88.
[13] Graham, R., Rothschild, B. and Spencer, J. (1990) *Ramsey Theory*, 2nd edn, Wiley-Interscience Series in Discrete Mathematics and Optimization.
[14] Gyárfás, A., Lehel, J., Schelp, R. and Tuza, Zs. (1987) Ramsey numbers for local colorings. *Graphs Combin.* **3** 267–277.
[15] Jiang, T. (2002) Edge-colorings with no large polychromatic stars. *Graphs Combin.* **18**.
[16] Jiang, T. and West, D. B. On the Erdős–Simonovits–Sós conjecture about the anti-Ramsey number of a cycle. Submitted.
[17] Jiang, T. and West, D. B. Edge-colorings of complete graphs that avoid polychromatic trees. To appear in *Discrete Math.*
[18] Lefmann, H., Rödl, V. and Wysocka, B. (1996) Multicolored subsets in colored hypergraphs. *J. Combin. Theory Ser. A* **74** 209–248.
[19] Manoussakis, Y., Spyratos, M., Tuza, Zs. and Voigt, M. (1996) Minimal colorings for properly colored subgraphs. *Graphs Combin.* **12** 345–360.
[20] Simonovits, M. and Sós, V. T. (1984) On restricted colorings of K_n. *Combinatorica* **4** 101–110.
[21] Truszczyński, M. and Tuza, Zs. (1987) Linear upper bounds for local Ramsey numbers. *Graphs Combin.* **3** 67–73.

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