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Diophantine approximation on polynomial curves

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Abstract

In a paper from 2010, Budarina, Dickinson and Levesley studied the rational approximation properties of curves parametrised by polynomials with integral coefficients in Euclidean space of arbitrary dimension. Assuming the dimension is at least three and excluding the case of linear dependence of the polynomials together with $P(X) \equiv 1$ over the rational number field, we establish proper generalisations of their main result.

1. *Introduction*

1·1. *Definitions*

Denote $\|\alpha\|$ the distance of $\alpha \in \mathbb{R}$ to the nearest integer. For $k \geqslant 1$ an integer and a parameter $\lambda > 0$, define $\mathscr{H}_\lambda^k$ as the set of $\underline{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_k) \in \mathbb{R}^k$ for which for any $\epsilon > 0$ the estimate

$$\max_{1 \leqslant j \leqslant k} \|q\zeta_j\| \leqslant q^{-\lambda+\epsilon} \tag{1}$$

has infinitely many integral solutions q. Similarly, let $\mathscr{G}_\lambda^k$ be the set for which (1) has infinitely many integral solutions for $\epsilon = 0$. Clearly $\mathscr{G}_\lambda^k \subseteq \mathscr{H}_\lambda^k$ for all pairs $k \geqslant 1, \lambda > 0$, and the sets $\mathscr{G}_\lambda^k$ and $\mathscr{H}_\lambda^k$ diminish as λ increases.

Let $\mathscr{C}$ denote a curve in $\mathbb{R}^k$. Similar to [**4**], we predominately consider curves of the form

$$\mathscr{C} = \{(X, P_2(X), \ldots, P_k(X)) : \ X \in \mathbb{R}\}, \qquad P_j \in \mathbb{Q}[X], \tag{2}$$

where we put $P_1(X) = X$. In [**4**] the assumption $P_j \in \mathbb{Z}[X]$ was made. However, we will see soon that in both [**4**] and the present paper, the main results extend to polynomials belonging to the larger class $\mathbb{Q}[X]$. It will even be more convenient at some places, in particular in Section 2·3, to consider $\mathbb{Q}[X]$. Let d_j be the degree of P_j in (2). It will become apparent that for our purposes, without loss of generality we may assume

$$1 = d_1 \leqslant d_2 \leqslant \cdots \leqslant d_k. \tag{3}$$

We call $\underline{d} = (d_1, \ldots, d_k)$ the *type* and $\max_{1 \leqslant j \leqslant k-1}(d_{j+1} - d_j)$ the *diameter* of $\mathscr{C}$. In the special case $k = 1$ let the diameter be 0. Clearly the diameter is a non-negative integer at most $d_k - 1$. In the special case $P_j(X) = X^j$ for $1 \leqslant j \leqslant k$, we obtain the Veronese

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curve in dimension k, which we shall denote by $\mathscr{V}^k$. The curve $\mathscr{V}^k$ obviously has type $\underline{d} = (1, 2, \dots, k)$ and diameter $t = 1$.

The Hausdorff dimension of the sets $\mathscr{C} \cap \mathscr{G}_\lambda^k$ with $\mathscr{C}$ as in (2) was studied in [**4**]. In the special case $\mathscr{C} = \mathscr{V}^k$ these results were refined in [**9**]. In this paper we aim to establish results that simultaneously improve the results of [**4**] and [**9**]. In contrast to [**4**], we will mostly deal with the sets $\mathscr{C} \cap \mathscr{H}_\lambda^k$, since this will lead to a more convenient presentation of some aspects of the results. However, we point out that for the sole purpose of determining Hausdorff dimensions, the distinction between $\mathscr{C} \cap \mathscr{G}_\lambda^k$ and $\mathscr{C} \cap \mathscr{H}_\lambda^k$ will mostly not be necessary (with the only possible exception of Theorem 1·3 and $\lambda = d_k - 1$). This can be inferred from the most general forms ('zero-infinity laws') the results we use rely on. We will not explicitly carry this standard argument out and only refer to [**8**].

For $s \in \{1, 2, \dots, k\}$, define the map

$$\begin{aligned} \Pi_s : \mathbb{R}^k &\longmapsto \mathbb{R}^s, \\ (\zeta_1, \dots, \zeta_k) &\longmapsto (\zeta_1, \dots, \zeta_s). \end{aligned}$$

For a set $M \subseteq \mathbb{R}^k$ let $\Pi_s(M) = \{\Pi_s(m) : m \in M\}$. It will be of importance that Π_s are locally bi-Lipschitz continuous restricted to a curve $\mathscr{C}$ as in (2). This property guarantees that with respect to Hausdorff dimension it makes no difference whether we consider a subset of $\mathscr{C}$ in $\mathbb{R}^k$, or its image under Π_1 in $\mathbb{R}$. We remark that also bijective linear transformations of $\mathbb{R}^k$ are bi-Lipschitz continuous and hence preserve Hausdorff dimensions. Moreover, the optimal exponent in (1) is well known to be invariant under such transformations if the corresponding matrix has rational entries. We call this a *birational (linear) transformation*. This guarantees that indeed it will suffice to treat the case of $P_j \in \mathbb{Z}[X]$ in (2), otherwise we can multiply any P_j with the common denominator of its coefficients, which induces a birational transformation.

It will be convenient to define a quantity related to $\mathscr{C} \cap \mathscr{H}_\lambda^k$. For $\zeta \in \mathbb{R}$ and $\mathscr{C}$ as in (2) let $\Theta_{\mathscr{C}}(\zeta)$ be the supremum of real numbers λ such that (1) has a solution for $\underline{\zeta} = \Pi_1^{-1}(\zeta) \cap \mathscr{C}$, that is $\underline{\zeta}$ is the unique point on $\mathscr{C}$ with first coordinate $\zeta_1 = \zeta$. With this notation, for any parameter $\lambda > 0$ we have

$$\Pi_1(\mathscr{C} \cap \mathscr{H}_\lambda^k) = \{\zeta \in \mathbb{R} : \Theta_{\mathscr{C}}(\zeta) \geqslant \lambda\}. \tag{4}$$

For $\mathscr{C} = \mathscr{V}^k$ we will also write $\lambda_k(\zeta)$ for $\Theta_{\mathscr{C}}(\zeta)$. This corresponds to the quantity $\lambda_k(\zeta)$ introduced by Bugeaud and Laurent in [**6**], defined as the supremum of real numbers ν for which the estimate $\max_{1 \leqslant j \leqslant k} \|q\zeta^j\| \leqslant q^{-\nu}$ has infinitely many integer solutions q. The claimed equivalence of the definitions is evident and (4) transfers into

$$\Pi_1(\mathscr{V}^k \cap \mathscr{H}_\lambda^k) = \{\zeta \in \mathbb{R} : \lambda_k(\zeta) \geqslant \lambda\}. \tag{5}$$

The right-hand side sets have been studied for instance in [**5**]. Notice that if $k = 1$ then $\mathscr{V}^1 = \mathbb{R}$ and $\Pi_1 = \mathrm{id}$ such that (5) becomes

$$\mathscr{H}_\lambda^1 = \{\zeta \in \mathbb{R} : \lambda_1(\zeta) \geqslant \lambda\}. \tag{6}$$

Before we quote results on the sets $\mathscr{C} \cap \mathscr{G}_\lambda^k$ and $\mathscr{C} \cap \mathscr{H}_\lambda^k$ for curves $\mathscr{C}$ in Section 1·2, we remark that certain sets somehow dual to $\mathscr{C} \cap \mathscr{G}_\lambda^k$ dealing with approximation of linear forms have been intensely studied as well. The dual theory is in fact more elaborated. We refer in particular to [**1**] and [**3**] for results and also [**4**] for further references. We should also mention that sets of the type $\mathscr{M} \cap \mathscr{G}_\lambda^k$ (and their dual versions) have been studied for

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more general manifolds $\mathscr{M} \subseteq \mathbb{R}^k$. See [**7**] for example, and again [**4**] for more references. However, the theory of curves is already far from being fully understood.

1·2. *Facts*

For parameters $\lambda \leqslant 1/k$, Dirichlet's box principle implies $\mathscr{H}_\lambda^k = \mathbb{R}^k$. Consequently $\mathscr{C} \bigcap \mathscr{H}_\lambda^k = \mathscr{C}$ for any curve $\mathscr{C}$, and sufficient smoothness provided we infer $\dim(\mathscr{C} \bigcap \mathscr{H}_\lambda^k) = \dim(\mathscr{C}) = 1$. The case $\lambda > 1/k$ is of interest and not well-understood so far. Our results will deal with parameters $\lambda > 1$. In this case, it is known that there exists no uniform theory applicable to all smooth curves with the regularity properties usually used in this context. On the other hand, for values λ sufficiently close to $1/k$ (in dependence of k), a general theory for sufficiently smooth curves is conjectured. This was proved for $k = 2$ and $\lambda \in (1/2, 1)$ in [**2**, **11**]. More precisely, in case of $\mathscr{C}$ parametrised by $(x, f(x))$ with a C^3-function f with the set $\{x : f''(x) = 0\}$ of dimension at most $1/2$, we have $\dim(\mathscr{C} \bigcap \mathscr{H}_\lambda^2) = (2 - \lambda)/(1 + \lambda)$. However, in dimension $k \geqslant 3$ and a generic curve $\mathscr{C}$, the sets $\mathscr{C} \bigcap \mathscr{H}_\lambda^k$ remain poorly understood for $\lambda \in (1/k, 1)$. See [**2**, section 1·4] for more information on the difference between small versus large values of λ for the behavior of the sets $\mathscr{C} \bigcap \mathscr{H}_\lambda^k$.

In the special case $k = 1$, it follows from a zero-infinity law due to Jarník [**8**] that for any $\lambda \geqslant 1$ we have

$$\dim(\mathscr{G}_\lambda^1) = \dim(\mathscr{H}_\lambda^1) = \frac{2}{1+\lambda}. \tag{7}$$

In view of the identifications (5), (6), a special case of [**5**, lemma 1] due to Bugeaud concerning the curves $\mathscr{V}^k$ turns into the following assertion.

LEMMA 1·1 (Bugeaud). *Let $k \geqslant 1$ be an integer. For any parameter $\lambda \geqslant 1/k$, we have*

$$\Pi_1(\mathscr{V}^k \bigcap \mathscr{H}_\lambda^k) \supseteq \mathscr{H}_{k\lambda+k-1}^1 = \{\zeta \in \mathbb{R} : \lambda_1(\zeta) \geqslant k\lambda + k - 1\}.$$

Thus by virtue of (7) *we conclude*

$$\dim(\mathscr{V}^k \bigcap \mathscr{H}_\lambda^k) \geqslant \frac{2}{k(1+\lambda)}.$$

This can be readily generalised for curves in (2). We additionally incorporate obvious estimates for the sake of completeness.

LEMMA 1·2. *Let $k \geqslant 1$ be an integer and $\mathscr{C}$ be a curve as in* (2) *of type $\underline{d} = (d_1, \ldots, d_k)$ as in* (3). *Then for any parameter $\lambda \geqslant 1/k$ we have*

$$\mathscr{H}_{d_k\lambda+d_k-1}^1 \subseteq \Pi_1(\mathscr{C} \bigcap \mathscr{H}_\lambda^k) \subseteq \mathscr{H}_\lambda^1. \tag{8}$$

In particular

$$\frac{2}{d_k(1+\lambda)} \leqslant \dim(\mathscr{C} \bigcap \mathscr{H}_\lambda^k) \leqslant \frac{2}{1+\lambda}. \tag{9}$$

Proof. We may restrict to $P_j \in \mathbb{Z}[X]$, see Section 1·1. The right inclusion in (8) is obvious by the definition of $\mathscr{H}_\lambda^1$. In view of (4), the left inclusion in (8) is equivalent to saying that for any $\zeta \in \mathbb{R}$ we have

$$\Theta_{\mathscr{C}}(\zeta) \geqslant \frac{\lambda_1(\zeta) - d_k + 1}{d_k}. \tag{10}$$

Let $m \geqslant 1$ be an integer. Lemma 1·1 asserts that

$$\max_{1 \leqslant j \leqslant m} \|q\zeta^j\| \leqslant q^{-\eta}$$

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has infinitely many integer solutions q for any $\eta < (\lambda_1(\zeta) - m + 1)/m$. On the other hand, observe that for any $P \in \mathbb{Z}[X]$ of degree at most m we have

$$\|qP(\zeta)\| \leqslant \tau(P) \max_{1 \leqslant j \leqslant m} \|q\zeta^j\|, \qquad 1 \leqslant j \leqslant m,$$

where $\tau(P)$ denotes the sum of the absolute values of the coefficients of P. The claim (10) follows if we let $m = d_k$ and consider the polynomials $P = P_j$ for $1 \leqslant j \leqslant k$, respectively. Similarly to Lemma 1·1, we infer the estimates (9) with (7) for the parameter λ and $d_k\lambda + d_k - 1$ respectively, since Π_1 does not affect Hausdorff dimensions for subsets of $\mathscr{C}$.

Recall that the results in [**2**] show that we cannot expect equality in the left inequality in (9) to hold for $\lambda < 1$. On the other hand, for large parameters λ, this has been established. An affirmative result based on a "zero-infinity law" due to Budarina, Dickinson and Levesley [**4**] is the following.

THEOREM 1·3 (Budarina *et al.*). *Let $k \geqslant 1$ be an integer and $\mathscr{C}$ be a curve as in* (2) *of type $\underline{d} = (d_1, \ldots, d_k)$ that satisfies* (3). *For any parameter $\lambda \geqslant \max(d_k - 1, 1)$, we have* $\dim(\mathscr{C} \cap \mathscr{G}_\lambda^k) = 2/(d_k(\lambda+1))$. *If $\lambda > \max(d_k - 1, 1)$, we have* $\dim(\mathscr{C} \cap \mathscr{H}_\lambda^k) = 2/(d_k(\lambda+1))$ *as well.*

The original version of Theorem 1·3 was formulated for $P_j \in \mathbb{Z}[X]$ and contains only the claim for the sets $\mathscr{C} \cap \mathscr{G}_\lambda^k$. However, both the transition to $\mathbb{Q}[X]$ and the equality of the dimensions of $\mathscr{C} \cap \mathscr{G}_\lambda^k$ and $\mathscr{C} \cap \mathscr{H}_\lambda^k$ for $\lambda > d_k - 1$ can be derived as remarked in Section 1·1. It might be possible to deduce the equality for $\lambda = d_k - 1$ as well with a refined argument. However, it seems not to be completely obvious and is not of much importance for us either.

In the special case $\mathscr{C} = \mathscr{V}^k$, it was shown by the author [**9**, theorem 1·6 and corollary 1·8] that the claim of Theorem 1·3 is actually valid for any parameter $\lambda > 1$. This improves Theorem 1·3 for $\mathscr{C} = \mathscr{V}^k$ in case of $k \geqslant 3$.

THEOREM 1·4 (Schleischitz). *Let $k \geqslant 1$ be an integer and $\lambda > 1$. Then we have the identity of one-dimensional sets*

$$\Pi_1(\mathscr{V}^k \cap \mathscr{H}_\lambda^k) = \mathscr{H}^1_{k\lambda+k-1}. \tag{11}$$

As a consequence

$$\dim(\mathscr{V}^k \cap \mathscr{H}_\lambda^k) = \frac{2}{k(\lambda+1)}. \tag{12}$$

In fact (12) was inferred for the dimension of $\Pi_1(\mathscr{V}^k \cap \mathscr{H}_\lambda^k)$, however the dimensions coincide by the remarks on Π_1 in Section 1·1. For any $k \geqslant 2$, the restriction $\lambda > 1$ is also necessary for equality in (11). Indeed, for $\lambda = 1$ there are counterexamples due to Bugeaud [**5**], as remarked in [**9**]. Theorem 1·3 and the quotes from [**2**] above imply that (12) is valid precisely for $\lambda \geqslant 1$ if $k = 2$, and most likely this is true for any $k \geqslant 3$ too. Hence, apart from the value $\lambda = 1$ in (12), Theorem 1·4 is supposed to be sharp.

2. *New results*

2·1. *Extension of the bound in Theorem* 1·3

In this section we refine the method used in [**9**] to show that for $k > 2$ the assertion of Theorem 1·3 holds in fact for a larger range of values λ, not only for $\mathscr{V}^k$ as in Theorem 1·4

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but much more general curves $\mathscr{C}$ as in (2). The improvement concerning the range of values λ will turn out to depend solely on the diameter t of $\mathscr{C}$. The method will be further refined in Section 2·3.

We can assume $t \geqslant 1$, since otherwise $d_k = 1$, and $\Pi_1(\mathscr{C} \cap \mathscr{H}_\lambda^k) = \mathscr{H}_\lambda^1$ and $\dim(\mathscr{C} \cap \mathscr{H}_\lambda^k) = 2/(1+\lambda)$ for $\lambda \geqslant 1$ follow from (8) and (9). We identify the latter also as the simplest case of Theorem 1·3. More generally, it is not hard to see that the constant and linear terms of the polynomials $P_j(X)$, $j \geqslant 2$, can be removed via a birational transformation without affecting the results, see Section 2·3. In particular, the linear polynomials among those P_j can be dropped. The main result of the present section is the following.

THEOREM 2·1. *Let $k \geqslant 1$ be an integer and $\mathscr{C}$ be a curve as in* (2) *of type $\underline{d} = (d_1, \ldots, d_k)$ as in* (3) *and diameter $t \geqslant 1$. Then for any parameter $\lambda > t$ we have*

$$\Pi_1(\mathscr{C} \cap \mathscr{H}_\lambda^k) = \mathscr{H}_{d_k\lambda+d_k-1}^1 = \{\zeta \in \mathbb{R} : \lambda_1(\zeta) \geqslant d_k\lambda + d_k - 1\}. \tag{13}$$

Observe that for $\mathscr{C} = \mathscr{V}^k$, Theorem 2·1 confirms (11) in Theorem 1·4. Similar to Section 1·2, we can infer a corollary on the dimensions we investigate.

COROLLARY 2·2. *Let k, $\mathscr{C}$ and λ be as in Theorem* 2·1. *Then we have*

$$\dim(\mathscr{C} \cap \mathscr{H}_\lambda^k) = \frac{2}{d_k(\lambda+1)}.$$

Proof. The right-hand side in (13) has dimension $2/(d_k(1+\lambda))$ by (7), and thus the left-hand side in (13) as well. Since the map Π_1 restricted to $\mathscr{C}$ does not affect Hausdorff dimensions, the claim follows.

Corollary 2·2 leads to an improvement of Theorem 1·3, except if either $\underline{d} = (1, 1, \ldots, 1, d_k)$ or $\underline{d} = (1, d_k, d_k, \ldots, d_k)$ for the claim on $\mathscr{C} \cap \mathscr{G}_\lambda^k$ and the exact value $\lambda = d_k - 1$. First consider $k > 2$. Then the first exceptional case is not of interest by the remarks above. The second exceptional case leads to what we will call a degenerate case in Section 2·3, and can be transformed either into the case $k \leqslant 2$ or a non-exceptional case. See Section 2·3, in particular Theorem 2·7. However, if $k = 2$, any curve is exceptional and Corollary 2·2 does not provide any new information. We illustrate the relation between Corollary 2·2 and Theorem 1·3 with an example.

Example 2·3. Consider the curve

$$\mathscr{C}_0 = \left\{\left(X, \frac{1}{6}X^3 + 5X^2, X^3 - \frac{11}{2}X + \frac{1}{3}, \frac{2}{13}X^7 - 11X^3 - 1, \frac{3}{4}X^9 + \frac{3}{8}X^5 + \frac{1}{2}\right) : X \in \mathbb{R}\right\}$$

in $\mathbb{R}^5$. Then $\mathscr{C}_0$ has type $\underline{d} = (1, 3, 3, 7, 9)$ and diameter $t = 4$. Corollary 2·2 yields $\dim(\mathscr{C}_0 \cap \mathscr{H}_\lambda^k) = 2/(9(\lambda+1))$ for $\lambda > 4$, whereas Theorem 1·3 yields (almost) the same result only for $\lambda \geqslant 8$.

The question that remains open is what happens for $k = 2$ and parameters $\lambda \in (1, t)$ and $k \geqslant 3$ and $\lambda \in (1/k, t]$. The remark below Theorem 1·4 on $\lambda = 1$ suggests that the analogue of Theorem 2·1 probably fails for any $\lambda \leqslant t$. However, one can hope that for $\lambda \in [1, t]$ the difference set $\Pi_1(\mathscr{C} \cap \mathscr{H}_\lambda^k) \setminus \mathscr{H}_{d_k\lambda+d_k-1}^1$ is always sufficiently small to preserve Hausdorff dimensions. We state this as a conjecture.

CONJECTURE 2·4. *Let $k \geqslant 1$ be an integer and $\mathscr{C}$ any curve as in* (2). *The condition $\lambda \geqslant 1$ is necessary and sufficient for equality in the left hand inequality in* (9).

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Theorem 1·3 together with the results on planar curves remarked in Section 1·2 shows that Conjecture 2·4 is true at least for $k = 2$ and curves with diameter $t = 1$ (hence only $\lambda = 1$ is of interest), in particular for $\mathscr{C} = \mathscr{V}^2$. In fact, the value $\lambda = 1$ can be included in the case $k > 2$ and $t = 1$ as well, since this case can be transformed into the case $k \leqslant 2$, see Section 2·3. However, for $t \geqslant 2$ the conjecture is very open even for $k = 2$.

2·2. *Upper bounds*

We aim to further generalize Theorem 2·1 and Corollary 2·2. Concretely, the trivial upper bound in (9) will be refined for $k, \mathscr{C}$ as in Theorem 2·1 and $\lambda \leqslant t$. Even though there is equality if $\underline{d} = (1, 1, \ldots, 1)$, for many curves $\mathscr{C}$ the method of the proof of Theorem 2·1 in Section 3 can be carried out to reduce this bound. The accuracy of the refined bounds depends heavily on the structure of the type $\underline{d}$ of $\mathscr{C}$.

THEOREM 2·5. *Let $k, \mathscr{C}$ be as in Theorem 2·1. For a parameter $\tau \geqslant 1/k$, let $r = r(\tau)$ be the smallest index such that $d_{r+1} - d_r > \tau$, and $r = k$ if there is no such index (that is if $\tau \geqslant t$). Then for any parameter $\lambda > \tau$, we have*

$$\mathscr{H}^{\lambda}_{d_k\lambda + d_k - 1} \subseteq \Pi_1(\mathscr{C} \cap \mathscr{H}^k_{\lambda}) \subseteq \mathscr{H}^{\lambda}_{d_r\lambda + d_r - 1} \tag{14}$$

and hence

$$\frac{2}{d_k(1+\lambda)} \leqslant \dim(\mathscr{C} \cap \mathscr{H}^k_{\lambda}) \leqslant \frac{2}{d_r(1+\lambda)}. \tag{15}$$

The claim of the theorem is of interest for $\tau \geqslant 1$ only. We may put $\lambda = \tau$ if $\tau \notin \mathbb{Z}$. Theorem 2·5 generalizes Theorem 2·1 in a non-trivial way for a parameter $\lambda > \tau$ if and only if $d_r > 1$ for $r = r(\tau)$. Consequently, one checks that the theorem provides new information at least for some parameters λ, if and only if $d_2 - d_1 = d_2 - 1 < t$. Roughly speaking, Theorem 2·5 provides good bounds if large gaps between d_j and d_{j+1} appear for large j only. We give the following an example.

Example 2·6. Consider the curves

$$\mathscr{C}_a = \{(X, X^3, X^6, X^{10}, X^{15}) : X \in \mathbb{R}\} \subseteq \mathbb{R}^5$$
$$\mathscr{C}_b = \{(X, X, X^5, X^6, X^7, X^{11}) : X \in \mathbb{R}\} \subseteq \mathbb{R}^6.$$

For $\mathscr{C}_a$ and $\tau \geqslant t = 5$, there is no index r as in the theorem and hence $d_r = d_5 = 15$. Hence $\dim(\mathscr{C}_a \cap \mathscr{H}^6_{\lambda}) = 2/(15(1+\lambda))$ for $\lambda > 5$. For $\tau \in [4, 5)$, we have $d_r = d_4 = 10$. Thus by Theorem 2·5 we infer

$$\frac{2}{15(1+\lambda)} \leqslant \dim(\mathscr{C}_a \cap \mathscr{H}^6_{\lambda}) \leqslant \frac{2}{10(1+\lambda)}, \qquad \lambda \in (4, 5].$$

For $\lambda = 5$ we used that $\mathscr{H}^k_{\lambda}$ diminish as λ increases and the continuous dependency of the right-hand side from λ. Similarly

$$\frac{2}{15(1+\lambda)} \leqslant \dim(\mathscr{C}_a \cap \mathscr{H}^6_{\lambda}) \leqslant \frac{2}{6(1+\lambda)}, \qquad \lambda \in (3, 4],$$
$$\frac{2}{15(1+\lambda)} \leqslant \dim(\mathscr{C}_a \cap \mathscr{H}^6_{\lambda}) \leqslant \frac{2}{3(1+\lambda)}, \qquad \lambda \in (2, 3],$$
$$\frac{2}{15(1+\lambda)} \leqslant \dim(\mathscr{C}_a \cap \mathscr{H}^6_{\lambda}) \leqslant \frac{2}{1+\lambda}, \qquad \lambda \in (1/5, 2].$$

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Thus an improvement to the trivial upper bound is made for $\lambda > 2$. For $\mathscr{C}_b$ on the other hand, we readily check that any $\tau < 4$ yields $d_r = d_2 = 1$, and hence

$$\frac{2}{11(1+\lambda)} \leqslant \dim(\mathscr{C}_b \cap \mathscr{H}_\lambda^6) \leqslant \frac{2}{1+\lambda}, \qquad \lambda \in (1/6, 4],$$

which we recognize as the trivial bounds from Lemma 1·2. Theorem 2·1 implies

$$\dim(\mathscr{C}_b \cap \mathscr{H}_\lambda^6) = \frac{2}{11(1+\lambda)}, \qquad \lambda \in (4, \infty].$$

2·3. *Normalisation of curves*

For some curves, the results in Section 2·1 and Section 2·2 can be improved by a suitable transformation. In Section 1·1 we noticed that the optimal parameter in (1) is invariant under a birational linear transformation, and any such map preserves Hausdorff dimensions. Notice also that we may assume the constant coefficients of the polynomials $P_j(X)$ to vanish without affecting Corollary 2·2 (this is obvious if they are integers, otherwise multiply the common denominators, subtract the constant coefficients and divide again). For fixed $k \geqslant 2$, consider all curves as in (2) labeled as in (3), but possibly with $P_{m+1} \equiv P_{m+2} \equiv \cdots \equiv P_k \equiv 0$ for some $m < k$. We define an equivalence relation by $\mathscr{C} \sim \widetilde{\mathscr{C}}$ if after possibly canceling constant coefficients in the involved $P_j(X)$, $\widetilde{P}_j(X)$, there exists a suitable transformation that maps $\mathscr{C}$ on $\widetilde{\mathscr{C}}$. We call curves in the same class *birational equivalent*. The highest appearing degree d_k for birational equivalent curves coincides, since linear combinations of polynomials obviously cannot extend the maximum of their degrees and $\sim$ is an equivalence relation. On the other hand, types and diameters do not necessarily coincide. By the above observations, for given curves $\widetilde{\mathscr{C}} \sim \mathscr{C}$, Theorem 2·1 and Corollary 2·2 apply to both with the bounds inherited from either curve (with the $P_\cdot \equiv 0$ omitted in the definition of the diameter, see below). Thus for given $\mathscr{C}$ one aims to find $\widetilde{\mathscr{C}} \sim \mathscr{C}$ with smallest possible diameter. Call a curve $\mathscr{C}$ as in (2) *normalised*, if for some $m \leqslant k$ we have

$$1 = d_1 < d_2 < \ldots < d_m, \qquad P_{m+1}(X) \equiv \cdots \equiv P_k(X) \equiv 0. \tag{16}$$

In case of $m < k$, call $(d_1, \ldots, d_m)$ the type and $\max_{1 \leqslant j \leqslant m-1}(d_{j+1} - d_j)$ the diameter. We refer to $\widetilde{\mathscr{C}}$ as a *normalisation of* $\mathscr{C}$ if $\widetilde{\mathscr{C}}$ is normalised and $\widetilde{\mathscr{C}} \sim \mathscr{C}$. Normalisations of any $\mathscr{C}$ in (2) can be recursively constructed, similar to the algorithmic solution of a system of linear equations. First cancel the constant coefficients of all polynomials. Then start with the highest degree h that is not unique. Pick one fixed polynomial P_e among those (let $e = 1$ if $h = 1$) and subtract suitable multiples of P_e of the other polynomials of degree h such that the leading coefficients vanish. This process must become stationary and, after possibly relabeling, will lead to a normalisation. Moreover, it is not hard to see that the types of normalisations of a fixed curve $\mathscr{C}$ coincide, and the diameter is minimised for any normalisation within the class of $\mathscr{C}$. Thus normalisations are optimal for our purposes. We remark that for $\mathscr{C}$ as in (2), we can find a normalisation where all linear coefficients of $P_j(X)$ for $j \geqslant 2$ vanish as well, since we can subtract a suitable rational multiple of $P_1(X) = X$ from any $P_j(X)$. We call a curve in (2) *degenerate* if its normalisations contain at least one identically-vanishing polynomial, that is $m < k$ in (16), and otherwise *non-degenerate*. A curve is degenerate if and only if the $P_j(X)$ together with $P(X) \equiv 1$ are $\mathbb{Q}$-linearly dependent. For degenerate curves, normalisation reduces the problem to lower dimension. By definition, a curve in (2) is non-degenerate if and only if the type of its normalisations satisfies $1 = d_1 < d_2 < \cdots < d_k$. Since the maximum degree is invariant under birational

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transformations, the relation $d_k < k$ implies $\mathscr{C}$ is degenerate. Moreover, a normalisation of a non-degenerate curve $\mathscr{C}$ of type $\underline{d} = (d_1, \dots, d_k)$ has diameter at most $d_k - k + 1 \geqslant 1$. Hence the results of Section 2·1 yield that this value is a uniform lower bound on the parameter for non-degenerate curves.

THEOREM 2·7. *Let $k \geqslant 1$ be an integer and $\mathscr{C}$ be a non-degenerate curve as in* (2) *of type $\underline{d} = (d_1, \dots, d_k)$ as in* (3). *Then the claims of Theorem* 2·1 *and Corollary* 2·2 *hold for any parameter $\lambda > d_k - k + 1$.*

Note that the bound in Theorem 2·7 is better than the one in Theorem 1·3 for $k \geqslant 3$.

Example 2·8. Consider the curves:

$$\mathscr{C}_1 = \{(X, 4X^3 + 12X^2 + 5X - 7, 3X^4 + 6X^2 - 10X + 33) : X \in \mathbb{R}\} \subseteq \mathbb{R}^3,$$
$$\mathscr{C}_2 = \left\{\left(X, \frac{2}{7}X^8 + \frac{5}{2}X^3, \frac{1}{3}X^8 + X^4 + \frac{2}{5}, \frac{5}{4}X^8 + 3\right) : X \in \mathbb{R}\right\} \subseteq \mathbb{R}^4,$$
$$\mathscr{C}_3 = \left\{\left(X, X^2, X^3, X^3 + X^2\right) : X \in \mathbb{R}\right\} \subseteq \mathbb{R}^4.$$

Obviously $\mathscr{C}_1$ is non-degenerate and normalised, such that we cannot improve the bound $\lambda > t_1 = 2$ inferred from Theorem 2·1 and Corollary 2·2. Define the matrices

$$R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{8}{35} \\ 0 & 0 & 1 & -\frac{6}{7} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix}.$$

The matrix R_2 induces a normalisation of $\mathscr{C}_2$ given by

$$\widetilde{\mathscr{C}}_2 = \left\{\left(X, \frac{5}{2}X^3 - \frac{24}{35}, X^4 - \frac{76}{35}, \frac{5}{4}X^8 + 3\right) : X \in \mathbb{R}\right\}.$$

Thus $\mathscr{C}_2$ is non-degenerate. Furthermore $\widetilde{\underline{d}}_2 = (1, 3, 4, 8) \neq (1, 8, 8, 8) = \underline{d}_2$, and the diameter $\widetilde{t}_2 = 4$ of $\widetilde{\mathscr{C}}_2$ is smaller than the diameter $t_2 = 7$ of $\mathscr{C}_2$, where the latter also coincides with the bound from Theorem 1·3. Hence Theorem 2·1 and Corollary 2·2 hold for $\mathscr{C}_2$ and $\lambda > 4$. Finally, the curve $\mathscr{C}_3$ is degenerate since a normalisation via R_3 is given by

$$\widetilde{\mathscr{C}}_3 = \{(X, X^2, X^3, 0) : X \in \mathbb{R}\}$$

with vanishing $P_4(X) \equiv 0$. Theorem 2·1 and Corollary 2·2 apply for $\lambda > \widetilde{t}_3 = t_3 = 1$.

3. *Preparatory results*

We recall [**9**, lemma 2·1].

LEMMA 3·1 (Schleischitz). *Let $\zeta \in \mathbb{R}$. Suppose that for a positive integer x we have the estimate*

$$\|\zeta x\| < \frac{1}{2}x^{-1}. \tag{17}$$

Then there exist positive integers x_0, y_0, M_0 such that $x = M_0 x_0$, $(x_0, y_0) = 1$ and

$$|\zeta x_0 - y_0| = \|\zeta x_0\| = \min_{1 \leqslant v \leqslant x} \|\zeta v\|. \tag{18}$$

Moreover, we have the identity

$$\|\zeta x\| = M_0 \|\zeta x_0\|. \tag{19}$$

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The integers x_0, y_0, M_0 are uniquely determined by the fact that y_0/x_0 is the convergent (in lowest terms) of the continued fraction expansion of ζ with the largest denominator not exceeding x, and $M_0 = x/x_0$.

A possible proof is based on elementary facts on continued fractions. The most technical ingredient in the proofs of Theorem 2·1 and Theorem 2·5 is the following Lemma 3·3, a refinement of [**9**, lemma 2·3]. It restricts to $P_j \in \mathbb{Z}[X]$. Preceding the lemma, we recall some basic facts from elementary number theory that we will implicitly apply in its proof, in form of a proposition. Especially the last claim will be crucial.

PROPOSITION 3·2. *Let A, s, l and $B = B_1, \ldots, B_s$ be positive integers. Then $\|A/B\| \geqslant 1/B$ unless $B|A$. If $(A, B) = 1$, then $(A, B^l) = 1$. Moreover, $(A^l, B^l) = (A, B)^l$. Furthermore $(A, \prod B_i)|\prod(A, B_i)$ and a sufficient condition for equality is that the B_i are pairwise coprime. Finally, if for a prime number p we denote by $\nu_p(.)$ the multiplicity of p in ., then $\nu_p(A + B) \geqslant \min(\nu_p(A), \nu_p(B))$ and $\nu_p(A) \neq \nu_p(B)$ is sufficient for equality.*

To avoid heavy notation in the formulation of Lemma 3·3, we prepone some definitions. For $\mathscr{C}$ as in (2) with polynomials $P_j \in \mathbb{Z}[X]$ of degrees d_j labeled as in (3), write

$$P_j(X) = c_{0,j} + c_{1,j}X + \cdots + c_{d_j,j}X^{d_j}, \qquad c_{.,j} \in \mathbb{Z}, \quad 1 \leqslant j \leqslant k. \tag{20}$$

Moreover, for $\zeta \in \mathbb{R}$ we define

$$\Delta = \Delta(\mathscr{C}) := \prod_{1 \leqslant j \leqslant k} |c_{d_j,j}|, \quad D = D(\mathscr{C}) := \Delta^{d_k}, \quad \Sigma(\mathscr{C}, \zeta) := \max_{1 \leqslant j \leqslant k} \max_{|z-\zeta| \leqslant 1/2} |P_j'(z)|. \tag{21}$$

Furthermore, for x_0 an integer variable that will appear in the lemma and Δ in (21), let

$$x_1 := \frac{x_0}{(x_0, \Delta)}, \tag{22}$$

where (., .) denotes the greatest common divisor.

LEMMA 3·3. *Let $\mathscr{C}$ be a curve as in (2) with $P_j \in \mathbb{Z}[X]$ as in (20), of type $\underline{d} = (d_1, \ldots, d_k)$ labelled as in (3) and diameter $t \geqslant 1$. Further let $\zeta \in \mathbb{R}$ be arbitrary. For an integer x denote by y the closest integer to ζx and write $y/x = y_0/x_0$ for integers $(x_0, y_0) = 1$.*

There exists a constant $C = C(\mathscr{C}, \zeta) > 0$ such that for any integer $x > 0$ the estimate

$$\max_{1 \leqslant j \leqslant k} \|P_j(\zeta)x\| < C \cdot x^{-t} \tag{23}$$

implies $x_1^{d_k}$ divides x, where x_1 is defined via (21), (22) for x_0 as above. A suitable choice for C is given by

$$C = C_0 := \frac{1}{2D \cdot \Sigma(\mathscr{C}, \zeta)},$$

with $D = D(\mathscr{C})$ and $\Sigma(\mathscr{C}, \zeta)$ from (21).

Proof. Suppose (23) holds for some x and $C = C_0$. Denote by y the closest integer to ζx and let y_0/x_0 be the fraction y/x in lowest terms.

Since $P_1(X) = X$, assumption (23) for $j = 1$ leads to

$$\left|\frac{y_0}{x_0} - \zeta\right| = \left|\frac{y}{x} - \zeta\right| < C_0 x^{-t-1}.$$

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We have $\Sigma(\mathscr{C}, \zeta) \in [1, \infty)$ since $P_1'(X) \equiv 1$ and polynomials are bounded on compact sets. Hence $C_0 \leqslant 1/(2D) \leqslant 1/2$, and we infer $|y_0/x_0 - \zeta| \leqslant 1/2$. Thus the Mean Value Theorem of differentiation yields for $1 \leqslant j \leqslant k$ the estimate

$$\left| P_j\left(\frac{y_0}{x_0}\right) - P_j(\zeta) \right| \leqslant \frac{1}{2}\Sigma(\mathscr{C}, \zeta)\left|\frac{y_0}{x_0} - \zeta\right| < \frac{1}{2}\Sigma(\mathscr{C}, \zeta)C_0 x^{-t-1} = \frac{1}{4D}x^{-t-1}. \tag{24}$$

Suppose $x_1^{d_k} \nmid x$. Let u be the smallest index such that $x_1^{d_u} \nmid x$, which exists since by assumption $u = k$ is such an index. Notice $u \geqslant 2$, since $d_1 = 1$ and $x_1 | x_0$ and $x_0 | x$ by definition and thus $x_1^{d_1} | x$. Observe $d_u - d_{u-1} \leqslant t$ by definition of the diameter. Write

$$P_u(y_0/x_0) = \frac{c_{0,u}x_0^{d_u} + c_{1,u}x_0^{d_u-1}y_0 + \cdots + c_{d_u,u}y_0^{d_u}}{x_0^{d_u}} =: \frac{S_u(x_0, y_0)}{x_0^{d_u}},$$

where $S_u \in \mathbb{Z}[X, Y]$ is a fixed polynomial independent from x_0, y_0. We want a lower estimate for $\|P_u(y_0/x_0)x\|$. Assume we have already proved

$$x_0^{d_u} \nmid (x \cdot S_u(x_0, y_0)). \tag{25}$$

Then since $x_1^{d_{u-1}} | x$ by definition of u and since $x_0/x_1 \leqslant \Delta$, we have

$$\left\| x P_u\left(\frac{y_0}{x_0}\right)\right\| = \left\| \frac{x S_u(x_0, y_0)}{x_0^{d_u}} \right\| \geqslant \left\| \frac{x}{x_0^{d_u}} \right\| \geqslant \frac{1}{\Delta^{d_u} x_1^{d_u - d_{u-1}}} \geqslant \frac{1}{D}x_1^{-t} \geqslant \frac{1}{D}x_0^{-t}. \tag{26}$$

On the other hand, the estimate (24) for $j = u$ implies

$$\left| x\left(P_u(\zeta) - P_u\left(\frac{y_0}{x_0}\right)\right)\right| < \frac{1}{4D}x^{-t} \leqslant \frac{1}{4D} \cdot x_0^{-t}. \tag{27}$$

The combination of (26) and (27) and triangular inequality imply

$$\max_{1 \leqslant j \leqslant k} \|P_j(\zeta)x\| \geqslant \|P_u(\zeta)x\| > \frac{3}{4D}x_0^{-t} \geqslant \frac{3}{4D}x^{-t}. \tag{28}$$

Since $C_0 \leqslant 1/(2D) < 3/(4D)$, the estimate (28) contradicts (23). Thus the assumption was false and we must have $x_1^{d_k} | x$.

It remains to be shown that (25) holds. Write $x_0 = Q_1Q_2Q_3$ with pairwise coprime Q_j uniquely defined in the following way. Let Q_1 consist of those common prime factors of Δ and x_0 (with the multiplicity they appear in x_0) that are contained strictly more often in x_0 than in Δ. Let Q_2 contain the remaining common prime factors of Δ and x_0 (again with the multiplicity they appear in x_0). Finally, let Q_3 consist of the remaining prime factors of x_0, such that $(Q_3, \Delta) = 1$. It follows from the form of S_u and $(x_0, y_0) = 1$ that the integers $S_u(x_0, y_0)$ and x_0 contain only common primes that divide $c_{d_u,u}$ and thus Δ. Consequently $(Q_3, S_u(x_0, y_0)) = 1$ and hence also $(Q_3^{d_u}, S_u(x_0, y_0)) = 1$. The primes in Q_1 can appear in $S_u(x_0, y_0)$ at most with the multiplicity they appear in $c_{d_u,u}$ and thus in Δ. Thus $(Q_1^{d_u}, S_u(x_0, y_0))|\Delta$, and in particular $(Q_1^{d_u}, S_u(x_0, y_0))|\Delta^{d_u}$. Since all prime factors in Q_2 appear at most as often as in Δ, we have $Q_2|\Delta$. Hence in particular $(Q_2^{d_u}, S_u(x_0, y_0))|\Delta^{d_u}$.

Since $(Q_1, Q_2) = 1$ and $x_0^{d_u} = Q_1^{d_u}Q_2^{d_u}Q_3^{d_u}$, from the derived properties we infer

$$(x_0^{d_u}, S_u(x_0, y_0))|\Delta^{d_u}. \tag{29}$$

Assume (25) is false, that is $x_0^{d_u} | (xS_u(x_0, y_0))$. Then the remaining factors of $x_0^{d_u}$ must be

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contained in x. In other words, (29) would imply $(x_0^{d_u}/(x_0^{d_u}, \Delta^{d_u}))|x$. But

$$\frac{x_0^{d_u}}{(x_0^{d_u}, \Delta^{d_u})} = \left(\frac{x_0}{(x_0, \Delta)}\right)^{d_u} = x_1^{d_u}$$

and hence $x_1^{d_u}|x$. However, this contradicts the choice of u. Thus the assumption is disproved and (25) must hold. This completes the proof.

Remark 3·4. The constant C_0 can be improved if we restrict to large x in (23). Since the fractions y_0/x_0 as in the lemma converge to ζ as $x_0 \to \infty$, indeed the claim can be verified with $\Sigma(\mathscr{C}, \zeta)$ altered to $\max_{1\leqslant j\leqslant k} |P_j'(\zeta)| + \epsilon$ for any $\epsilon > 0$ and $x \geqslant \hat{x}(\epsilon)$.

Remark 3·5. Suppose some x satisfies (23), and define x_0, x_1 via x as in the lemma. In general, it is not clear whether the analogue of (23) also holds for its divisor $x' := x_1^{d_k}$. This cannot be the case for large x_0 that has a prime factor not contained in x_1 (this prime must divide Δ). Indeed, in this case $\|x'(y_0/x_0)\| \geqslant 1/x_0$, but since $x_0|x$ and due to (23) also

$$|x'(\zeta - y_0/x_0)| \leqslant |x(\zeta - y_0/x_0)| < C_0 x^{-1} \leqslant \frac{1}{2} x_0^{-1}.$$

Since $x_0/x_1 \leqslant \Delta$ and $d_k \geqslant 2$, for large x_0 clearly $x' > x_0$. Triangular inequality gives a contradiction, as in the lemma. However, $x^* := x_0^{d_k}$ is a suitable choice for (23), which can be shown very similar to the proof of Lemma 1·1 if we anticipate the claim of Theorem 2·1. Thus $D = 1$, i.e. all polynomials are monic, is a sufficient criterion for (23) to hold for x', since then $x' = x^*$. See also Corollary 3·7.

Remark 3·6. The proof is less technical if we assume that all P_j are monic, since then $S_u(x_0, y_0)$ is simply coprime with x_0. In this context, notice that one could replace the product by the lowest common multiple in the definition of Δ.

The following corollary is inferred basically as the last part of the proof of [**9**, lemma 3·1], so we omit the proof.

COROLLARY 3·7. *Keep the notation and assumptions from Lemma* 3·3. *Then* $P_j(y_0/x_0)$ *is a convergent of the continued fraction expansion of* $P_j(\zeta)$ *for* $1 \leqslant j \leqslant k$. *Furthermore, if* (23) *holds for some pair* $(x, C) = (Nx_0^{d_k}, C)$ *with an integer* $N \geqslant 1$ *and* $C \leqslant C_0$, *then*

$$\max_{1\leqslant j\leqslant k} \|P_j(\zeta)x\| = N \cdot \max_{1\leqslant j\leqslant k} \|P_j(\zeta)x_0^{d_k}\|. \tag{30}$$

In particular, (23) *holds for any pair* $(\tilde{x}, C) = (Mx_0^{d_k}, C)$ *with* $1 \leqslant M \leqslant N$ *as well, and the minimum of the left hand sides among those* $\tilde{x}$ *is obtained for* $\tilde{x} = x_0^{d_k}$.

Observe that (23) might be true for some proper divisor of $x_0^{d_k}$, since Lemma 3·3 only asserts $x_1^{d_k}|x$. See also Remark 3·5. However, Corollary 3·7 and its omitted proof in fact show that the rational approximation vectors of good approximations as in (23) have j-th coordinate $P_j(y_0/x_0)$, which means they are elements of the curve $\mathscr{C}$. Compare this to [**4**, lemma 1].

4. *Proof of Theorem* 2·1 *and Theorem* 2·5

We prove Theorem 2·1 using Lemma 3·1 and Lemma 3·3. The proof is very similar to the proof of [**9**, theorem 1·6] with Lemma 3·1 and [**9**, lemma 2·3], with the value 1 replaced by t throughout.

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Proof of Theorem 2·1 We may assume $P_j \in \mathbb{Z}[X]$ without any loss of generality, see Section 1·1. Since Lemma 1·2 applies to our situation, it remains to be shown that for any $\lambda > t$ we have

$$\Pi_1(\mathscr{C} \bigcap \mathscr{H}^k_\lambda) \subseteq \mathscr{H}^A_{d_k\lambda+d_k-1}.$$

In view of (4), this is equivalent to the claim that provided that $\Theta_{\mathscr{C}}(\zeta) > t$ holds for some $\zeta \in \mathbb{R}$, we have

$$\Theta_{\mathscr{C}}(\zeta) \leqslant \frac{\lambda_1(\zeta) - d_k + 1}{d_k}. \tag{31}$$

The definition of the quantity $\Theta_{\mathscr{C}}(\zeta)$ implies that for any fixed $t < T < \Theta_{\mathscr{C}}(\zeta)$, the inequality

$$\max_{1\leqslant j\leqslant k} \|P_j(\zeta)x\| \leqslant x^{-T} \tag{32}$$

has arbitrarily large integer solutions x. One checks that for any $\nu > 0$ and sufficiently large $x > \hat{x}(\nu, T) := \nu^{1/(1-T)}$ we have $x^{-T} < \nu x^{-1}$. Choosing $\nu \leqslant C_0$ with $C_0 \leqslant 1/2$ from Lemma 3·3, condition (32) and $T > t \geqslant 1$ ensure we may apply both Lemma 3·1 and Lemma 3·3 for $x \geqslant \hat{x}$, with coinciding pairs x_0, y_0 such that y_0/x_0 is the reduced fraction y/x. Further let M_0 be as in Lemma 3·1. Since $(x_0/x_1)|\Delta$, by Lemma 3·3 we have $M_0 \geqslant x_1^{d_k} x_0^{-1} \geqslant x_0^{d_k-1}/D$. Note the factor $1/D$ depends on $\mathscr{C}$ only. Moreover, define T_0 and W_0 respectively implicitly by $x_0^{-T_0} = |\zeta x_0 - y_0|$ and $x_0^{W_0} = D$ respectively, i.e.

$$T_0 = -\frac{\log|\zeta x_0 - y_0|}{\log x_0}, \qquad W_0 := \frac{\log D}{\log x_0}.$$

Since $P_1(\zeta) = \zeta$, the derived properties yield

$$T \leqslant -\frac{\log\|\zeta x\|}{\log x} = -\frac{\log(M_0|\zeta x_0 - y_0|)}{\log(M_0 x_0)} \leqslant \frac{T_0 - (d_k - 1 - W_0)}{d_k - W_0} = \frac{T_0 - d_k + 1 + W_0}{d_k - W_0}.$$

Note that since D is fixed, W_0 tends to 0 as x_0 tends to infinity. Since we may choose T arbitrarily close to $\Theta_{\mathscr{C}}(\zeta)$, the definition of T_0 implies (31).

The proof shows that Theorem 2·1 can be refined, similar to [**9**, corollary 3·1].

COROLLARY 4·1. *Let k and $\mathscr{C}$ be as in Theorem* 2·1 *and D defined in* (21). *For any fixed $T > t$, there exists $\hat{x} = \hat{x}(T, \mathscr{C}, \zeta)$, such that the estimate*

$$\max_{1\leqslant j\leqslant k} \|P_j(\zeta)x\| \leqslant x^{-T}$$

for an integer $x \geqslant \hat{x}$ implies the existence of x_0, y_0, M_0 as in Lemma 3·1 *with the properties*

$$x \geqslant x_0^{d_k}/D, \qquad M_0 \geqslant x_0^{d_k-1}/D, \qquad |\zeta x_0 - y_0| \leqslant x_0^{-d_k T - d_k + 1}. \tag{33}$$

Similarly, if for $C_0 = C_0(k, \zeta)$ from Lemma 3·3 *the inequality*

$$\max_{1\leqslant j\leqslant k} \|P_j(\zeta)x\| < C_0 \cdot x^{-t}$$

has an integer solution $x > 0$, then (33) *holds with $T = t$.*

We enclose the proof of Theorem 2·5.

Proof of Theorem 2·5. The left inclusion in (14) is due to (8) again. For the right inclusion, we first prove the following modification (in fact, generalisation) of Lemma 3·3: let

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C_0, x_0, x_1 as in (21), (22) or Lemma 3·3, and $\tau, r = r(\tau)$ as in Theorem 2·5. Then for any integer $x > 0$ the estimate

$$\max_{1 \leqslant j \leqslant k} \|P_j(\zeta)x\| < C_0 \cdot x^{-\tau} \tag{34}$$

implies $x_1^{d_r} | x$.

Proceed as in the proof of Lemma 3·3. Assume the claim is false and define u in the same way. Observe that if $r \leqslant u-1$, then $x_1^{d_r} | x_1^{d_{u-1}}$, but on the other hand $x_1^{d_{u-1}} | x$ by definition of u. Hence indeed $x_1^{d_r} | x$. Otherwise, if $r \geqslant u$, then apply Lemma 3·3 to the integer r and $\widetilde{\mathscr{C}} := \Pi_r(\mathscr{C})$, with Π_r defined as in Section 1·1. Since by assumption τ is greater or equal than the diameter of $\widetilde{\mathscr{C}}$, again $x_1^{d_r} | x$. The remainder of the proof of (14) is established precisely as the proof of Theorem 2·1 with t replaced by τ and d_k replaced by d_r and if one prefers $\widetilde{D} := \Delta^{d_r} \leqslant \Delta^{d_k} = D$ instead of D. The upper bound in (15) follows similarly to Corollary 2·2 and we recognise the lower bound as the one in (9).

5. *Sets of accurately prescribed approximation*

Let $\mathscr{C}$ be as in (2) of type $\underline{d} = (d_1, \ldots, d_k)$ as in (3) and diameter $t \geqslant 1$, and $\lambda \in (t, \infty]$ be arbitrary. Theorem 2·1 or Corollary 2·2 imply that the set of $\zeta \in \mathbb{R}$ with $\Theta_{\mathscr{C}}(\zeta) = \lambda$ is non empty. By virtue of (4), this can be translated into the corresponding set of points on the curve $\underline{\zeta} \in \mathscr{C}$. Proceeding as in [**10**], we can apply Lemma 3·3, Corollary 3·7 and Corollary 4·1 to obtain $\underline{\zeta} \in \mathscr{C}$ with much sharper prescribed approximation properties.

Consider any function $\Psi : \mathbb{R} \mapsto \mathbb{R}$ of fast decay to 0. Define $\mathscr{K}_{\mathscr{C}}(\Psi)$ the set of points on the curve that is approximable to degree Ψ, that is

$$\mathscr{K}_{\mathscr{C}}(\Psi) = \left\{ \underline{\zeta} \in \mathscr{C} \colon \max_{1 \leqslant j \leqslant k} \|q\zeta_j\| \leqslant \Psi(q) \text{ for infinitely many integers } q \right\}.$$

Notice that for $\Psi(X) = X^{-\lambda}$ the set $\mathscr{K}_{\mathscr{C}}(\Psi)$ equals $\mathscr{C} \bigcap \mathscr{G}_\lambda^k$, and is contained in $\mathscr{C} \bigcap \mathscr{H}_\lambda^k$ but in general not in $\mathscr{C} \bigcap \mathscr{H}_{\lambda+\epsilon}^k$ for any $\epsilon > 0$. For $\mathscr{C}$ in (2) write $P_j(X) = Q_j(X)/K_j$ with $Q_j \in \mathbb{Z}[X]$ with coprime coefficients and K_j the corresponding integer. Let $K := \prod_{1 \leqslant j \leqslant k} K_j$ and define D as in (21) for the polynomials Q_j. The final theorem shows that for fixed $c < D^{-1}K^{-1}$, some $\underline{\zeta} \in \mathscr{C}$ are approximable to degree Ψ but not to degree $c\Psi$. In particular, if the polynomials have integral coefficients and are monic, we can take c arbitrarily close to 1.

THEOREM 5·1. *Let $k \geqslant 1$ an integer and $\mathscr{C}$ be a curve as in* (2) *of type $\underline{d} = (d_1, \ldots, d_k)$ labelled as in* (3) *with diameter $t \geqslant 1$. Define D, K as above. Let $\Psi : \mathbb{R} \mapsto \mathbb{R}$ be a decreasing function with $\Psi(x) = o(x^{-t})$ as $x \to \infty$. Moreover, let $I \subseteq \mathbb{R}$ be a non-empty interval. Then for any $c < D^{-1}K^{-1}$, the set*

$$(\mathscr{K}_{\mathscr{C}}(\Psi) \setminus \mathscr{K}_{\mathscr{C}}(c\Psi)) \bigcap I$$

is uncountable.

We only sketch the proof, as it is very similar to the proof of the second claim of [**10**, theorem 1·4]. First restrict to monic integral coefficients, i.e. $K = D = 1$. Consequently $x_0 = x_1$ and by Remark 3·5 and Corollary 3·7 the optimal choices of x, in the sense that $\max_{1 \leqslant j \leqslant k} \|x P_j(\zeta)\|$ is small, are of the form $x = x_0^{d_k}$. Hence one can proceed as in [**10**, theorem 1·4], with Lemma 3·3, Corollary 3·7 and Corollary 4·1 instead of [**9**, lemma 2·3] and [**9**, corollary 3·1], and $L_k(\zeta)$ replaced by $M_k(\mathscr{C}, \zeta) := \max_{1 \leqslant j \leqslant k} |P_j'(\zeta)|$ in view of

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Remark 3·4. Note that $P_1'(X) \equiv 1$, so clearly M_k vanishes for no value ζ. In the general case $D \geqslant 2, K \geqslant 2$, consider x of the form $KDx_0^{d_k}$. The factor $1/D$ arises in view of Remark 3·5. The knowledge of the exact value $|\zeta - y/x| = |\zeta - y_0/x_0|$ enables us to predict $\max_{1\leqslant j\leqslant k} \|xQ_j(\zeta)\|$ only up to a multiplicative factor $1/(D+\epsilon)$, as $x_1^{d_k} \leqslant x \leqslant x_0^{d_k}$ and $x_0^{d_k}/x_1^{d_k} \leqslant D$. Finally, the transition to rational coefficients gives another factor $1/K$ by a similar argument. The decay assumption on Ψ is sufficient due to the bounds in Lemma 3·3, Corollary 3·7 and Corollary 4·1 (and in fact can be slightly relaxed, as in [**10**, theorem 1·4]).

REFERENCES

[**1**] R. C. BAKER. Dirichlet's theorem on Diophantine approximation. *Math. Proc. Camb. Phil. Soc.* **83** (1978), 37–59.

[**2**] V. BERESNEVICH, D. DICKINSON and S. L. VELANI. Diophantine approximation on planar curves and the distribution of rational points. with an appendix by R.C. Vaughan: 'Sums of two squares near perfect squares'. *Ann. of Math.* **166** (2007), 367–426.

[**3**] V. I. BERNIK. An application of Hausdorff dimension in the theory of Diophantine approximation. *Acta Arith.* **42** (1983), 219–253 (in Russian).

[**4**] N. BUDARINA, D. DICKINSON and J. LEVESLEY. Simultaneous Diophantine approximation on polynomial curves. *Mathematika* **56** (2010), 77–85.

[**5**] Y. BUGEAUD. On simultaneous rational approximation to a real numbers and its integral powers. *Ann. Inst. Fourier (Grenoble)* (6) **60** (2010), 2165–2182.

[**6**] Y. BUGEAUD and M. LAURENT. Exponents in Diophantine approximation. *Diophantine Geometry Proceedings, Scuola Normale Superiore Pisa*, Ser. CRM, **4** (2007), 101–121.

[**7**] M. M. DODSON, B. P. RYNNE and J. A. G. VICKERS. Metric Diophantine approximation and Hausdorff dimension on manifolds. *Math. Proc. Camb. Phil. Soc.* **130** (2001), 515–522.

[**8**] V. JARNÍK. Über die simultanen Diophantische Approximationen. *Math. Z.* **33** (1931), 505–543.

[**9**] J. SCHLEISCHITZ. On the spectrum of Diophantine approximation constants. *Mathematika* **62** (2016), no. 1, 79–100.

[**10**] J. SCHLEISCHITZ. Generalizations of a result of Jarnik on simultaneous approximation. To appear in *Mosc. J. Comb. Number Theory* arXiv: 1410.6697.

[**11**] R. C. VAUGHAN and S. L. VELANI. Diophantine approximation on planar curves: the convergence theory. *Invent. Math.* **166** (2006), 103–124.

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