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Alexander's theorem for real-compactness

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Alexander's theorem (5) states that a topological space is compact if there is a subbase, $\mathscr{A}$, for its closed sets such that every subclass of $\mathscr{A}$ with the finite intersection property has a non-empty intersection. An analysis and extension of this is given here which has applications, *inter alia*, to problems concerning real-compactness (2).

1. *Definitions.* X is a fixed non-empty set; 'σ' and 'ϕ' are to be read as 'countable' and 'finite' respectively and 'γ' may be consistently read as either 'countable' or 'finite'.

If $\mathscr{B}$ is a non-empty class of subsets of X, a subclass of $\mathscr{B}$ is called *proper* if it is not all of $\mathscr{B}$ and is non-empty; a *γ-intersection* is the intersection of a non-empty γ-subclass from $\mathscr{B}$ and a *γ-union* is similarly defined; $\mathscr{B}^\gamma = \{B \subset X : B$ contains a γ-intersection from $\mathscr{B}\}$; a *γ-filter* in $\mathscr{B}$ is a non-empty subclass $\mathscr{F}$ in $\mathscr{B}$ with $\mathscr{F}^\gamma = \mathscr{F}$; if $\mathscr{F}$ is a proper γ-filter in $\mathscr{B}$ and $(\mathscr{F} \cup \{B\})^\gamma \supset \mathscr{B}$ for all $B \in \mathscr{B} \setminus \mathscr{F}$ then $\mathscr{F}$ is called a *maximal γ-filter* in $\mathscr{B}$; a subclass $\mathscr{P}$ of $\mathscr{B}$ is *γ-prime* if $\mathscr{B} \setminus \mathscr{P}^\gamma$ is closed under γ-unions. It is easily proved that a maximal γ-filter in $\mathscr{B}$ is γ-prime: the converse is usually not true.

$L^\gamma(\mathscr{B})$ is the least class of subsets of X containing $\mathscr{B}$ and closed under γ-unions and γ-intersections. $B^\gamma(\mathscr{B})$ is the least class of subsets of X containing $\mathscr{B}$ and closed under γ-unions, γ-intersection and the operation $\langle A, B\rangle \to A' \cup B$ $(A' = X \setminus A)$. If $\mathscr{B}$ has an empty γ-intersection $B^\gamma(\mathscr{B})$ is a Boolean algebra; in any case $\{B' : B \in B^\gamma(\mathscr{B})\}$ is a Boolean ring.

Throughout this note $\mathscr{A}$ and $\mathscr{S}$ will denote non-empty classes of sets from X.

2. *The theorem. Theorem. If $\mathscr{M}$ is a maximal γ-filter in $\mathscr{S}$ and $\mathscr{B}^\gamma(\mathscr{S}) \supset L^\gamma(\mathscr{A}) \supset \mathscr{S}$ then $\mathscr{M}^\gamma \cap \mathscr{A}$ is a maximal γ-filter in $\mathscr{A}$ and has the same intersection as $\mathscr{M}$.*

Proof. (1) Let $\mathscr{E} \equiv \{E \in \mathscr{S}^\gamma : E \notin \mathscr{M}^\gamma \Rightarrow (\{E\} \cup \mathscr{M})^\gamma \supset \mathscr{S}\}$. It will be proved that $\mathscr{E} \supset B^\gamma(\mathscr{S})$:

Firstly, $\mathscr{S} \subset \mathscr{E}$, by maximality of $\mathscr{M}$.

If $\mathscr{G}$ is a γ-subset of $\mathscr{E}$ and $\bigcup \mathscr{G} \notin \mathscr{M}^\gamma$ then for each $G \in \mathscr{G}$ $G \notin \mathscr{M}^\gamma$, so given $S \in \mathscr{S}$ there is a γ-intersection I_{GS} from $\mathscr{M}$ with $G \cap I_{GS} \subset S$, whence $(\bigcup \mathscr{G}) \cap \bigcap \{I_{GS} : G \in \mathscr{G}\} \subset S$ and $\bigcup G \in \mathscr{E}$. Again, if $\bigcap G \notin \mathscr{M}^\gamma$ there is a $G \in \mathscr{G} \setminus \mathscr{M}^\gamma$, whence $(\{\bigcap \mathscr{G}\} \cup \mathscr{M})^\gamma \supset \{(G\} \cup \mathscr{M})^\gamma \supset \mathscr{S}$ and $\bigcap \mathscr{G} \in \mathscr{E}$.

Since $\mathscr{E} \subset \mathscr{S}^\gamma$ it follows that if $E \in \mathscr{E} \setminus \mathscr{M}^\gamma$ then $(\{E\} \cup \mathscr{M})^\gamma \supset \mathscr{S}^\gamma \supset \mathscr{E}$.

If $E \in \mathscr{E} \setminus \mathscr{M}^\gamma$ and $F \in \mathscr{E}$ there is a γ-intersection I_{EF} from $\mathscr{M}$ with $E \cap I_{EF} \supset F$, so $I_{EF} \subset E' \cup I_{EF} = E' \cup (E \cap I_{EF}) \subset E' \cap F$, and $E' \cup F \in \mathscr{M}^\gamma$. Consequently, if $E, F \in \mathscr{E}$ and $E' \cup F \notin \mathscr{M}^\gamma$ then $E \in \mathscr{M}^\gamma$ and $F \notin \mathscr{M}^\gamma$; so given $S \in \mathscr{S}$ there are γ-inter-

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sections I_{FS} and I from $\mathscr{M}$ with $F \cap_{FS} \subset S$ and $I \subset E$, whence $(E' \cup F) \cap (I_{FS} \cap I) \subset S$. Since S was any element of $\mathscr{S}$, $E' \cup F \in \mathscr{E}$.

(2) Let $\mathscr{F} \equiv \{F \in \mathscr{E} : F \in \mathscr{M}^\gamma \Rightarrow F$ contains a γ-intersection from $\mathscr{A} \cap \mathscr{M}^\gamma\}$. It will be shown that $\mathscr{F} \supset L^\gamma(\mathscr{A}) \supset \mathscr{S}$: Obviously $\mathscr{A} \subset \mathscr{F}$.

If $\mathscr{G}$ is a γ-subclass of $\mathscr{F}$ then clearly $\bigcap \mathscr{G} \in \mathscr{F}$; and if $\bigcup G \in \mathscr{M}^\gamma$ then the γ-primeness of $\mathscr{M}^\gamma$ shows that there is a $G \in \mathscr{G} \cap \mathscr{M}^\gamma$, so $\bigcup \mathscr{G} \supset G$, which contains a γ-intersection from $\mathscr{A} \cap \mathscr{M}^\gamma$; therefore $\bigcup \mathscr{G} \in \mathscr{F}$.

(3) The proof can now be concluded:

If $A, B \in \mathscr{A}$ and $A \notin \mathscr{M}^\gamma$ there is, by (1), a γ-intersection I_{AB} from $\mathscr{M}$ with $A \cap I_{AB} \subset B$ and so, by (2), a γ-intersection J_{AB} from $\mathscr{A} \cap \mathscr{M}^\gamma$ with $A \cap J_{AB} \subset B$. $\mathscr{M}^\gamma \cap \mathscr{A}$ cannot be all $\mathscr{A}$, for being closed under γ-unions and γ-intersections it would then contain all $\mathscr{S}$, so it is a maximal γ-filter in $\mathscr{A}$. Finally, (2) shows that $\mathscr{M}^\gamma \cap \mathscr{A}$ has the same intersection as $\mathscr{M}$.

COROLLARY 1. *$\mathscr{M}^\gamma \cap B^\gamma(\mathscr{S})$ is a maximal γ-filter in $B^\gamma(\mathscr{S})$ and has the same intersection as $\mathscr{M}$.*

Proof. Take $\mathscr{A} = B^\gamma(\mathscr{S})$.

COROLLARY 2. *If $\mathscr{N}$ is a maximal γ-filter in $L^\gamma(\mathscr{A})$ then $\mathscr{N} \cap \mathscr{A}$ is a maximal γ-filter in $\mathscr{A}$ and has the same intersection as $\mathscr{M}$.*

Proof. Take $\mathscr{S} = L^\gamma(\mathscr{A})$.

3. *Applications.* (i) First, we establish the relation of our theorem to Alexander's theorem.

Let $\mathscr{A}$ be a sub-base for the closed sets of a topology on X and take $\mathscr{S} = L^\phi(\mathscr{A})$, a base for the closed sets. Let $\mathscr{H}$ be a class of closed sets with the finite intersection property. If $\mathscr{A}$ has the finite intersection property then $\bigcap \mathscr{H} \supset \bigcap \mathscr{A}$, so if $\bigcap \mathscr{A} \neq \phi$ then $\bigcap \mathscr{H} \neq \phi$. If $\mathscr{A}$ does not have the finite intersection property then $\mathscr{S}$ contains two disjoint non-empty sets and Zorn's lemma shows that there is a maximal ϕ-filter $\mathscr{U}$ in $\mathscr{S}$ containing $\mathscr{H}^\phi \cap \mathscr{S}$ ($\bigcap (\mathscr{H}^\phi \cap \mathscr{S}) = \bigcap \mathscr{H}$). Now, by the theorem $\mathscr{U} \cap \mathscr{A}$ is a maximal ϕ-filter in $\mathscr{A}$ with the same intersection as $\mathscr{U}$. Thus if $\bigcap (\mathscr{U} \cap \mathscr{A}) \neq \phi$ then $\bigcap \mathscr{U} \neq \phi$ and so $\bigcap \mathscr{H} \neq \phi$.

(ii) If X is given a completely regular Hausdorff topology and $\mathscr{Z}$, $\mathscr{P}$ are respectively the classes of zero sets and of cozero sets in X then

$$L^\sigma(\mathscr{Z}) = B^\sigma(\mathscr{Z}) = L^\sigma(\mathscr{P}) = B^\sigma(\mathscr{P}).$$

(Here use is made of (3), p. 27.) Furthermore, every ϕ-prime σ-filter in $\mathscr{P}$ or $\mathscr{Z}$ is a maximal σ-filter in $\mathscr{P}$ or $\mathscr{Z}$ respectively.

Taking $\mathscr{S} = \mathscr{Z}$, $\mathscr{A} = \mathscr{P}$ we see that the following are equivalent:

(a) X is real-compact.

(b) Every maximal σ-filter in $B^\sigma(\mathscr{Z})$ has non-empty intersection (4).

This list of equivalent statements can be further extended.

(d) Take $\mathscr{S} = \mathscr{Z}$ and $\mathscr{A} = B(\mathscr{Z})$ and let $\mathscr{M}$ be a maximal ϕ-filter in $\mathscr{Z}$ such that $\mathscr{P} \cap \mathscr{M}^\phi$ is a σ-filter in $\mathscr{P}$. $\mathscr{M}^\phi \cap B^\phi(\mathscr{Z})$ is a maximal ϕ-filter in $B^\phi(\mathscr{Z})$ and $\mathscr{P} \cap \mathscr{M}^\phi = \mathscr{M}^\phi \cap B^\phi(Z) \cap \mathscr{P}$ is then a ϕ-prime σ-filter in $\mathscr{P}$; so it is a maximal σ-filter

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in $\mathscr{P}$ and has the same intersection as $\mathscr{M}$. Thus if X is real-compact $\bigcap \mathscr{M} \neq \phi$. The converse result clearly holds.

(*e*) Similarly, if $\mathscr{V}$ is a maximal ϕ-filter in $\mathscr{P}$ and $\mathscr{V}^{\phi} \cap Z$ is a σ-filter in Z then this latter is a maximal σ-filter in $\mathscr{Z}$ which has non-empty intersection when X is real-compact. Again the converse is evident.

Frolík (1) has called X *almost real-compact* if whenever $\mathscr{U}$ is a maximal ϕ-filter in the open sets of X such that the class $\overline{\mathscr{U}}$ of closures of its sets has no empty σ-intersection then $\bigcap \overline{\mathscr{U}} \neq \phi$. His result that real-compactness implies almost real-compactness follows by taking $\mathscr{S}$ as the open sets $\mathscr{A} = B^{\phi}(\mathscr{S})$ and arguing as in the proof of (*d*).

As a final application:

If $B \in B^{\sigma}(Z)$ and $x \in B$ then $\mathscr{U} = \{Z \in \mathscr{Z} : x \in Z\}$ is a maximal σ-filter in $\mathscr{Z}$. $B^{\sigma}(\mathscr{Z}) \cap \mathscr{U}$ is a maximal σ-filter in $B^{\sigma}(\mathscr{Z})$ contained in $\mathscr{V} \equiv \{A \subset B^{\sigma}(\mathscr{Z}) : x \in \mathscr{A}\}$. So $\mathscr{V} = \mathscr{U}$ and there is a $Z \in \mathscr{Z}$ with $x \in Z \subset B$. Thus every set in $B^{\sigma}(\mathscr{Z})$ is an intersection of cozero sets. Now if X is real compact then every cozero set is real-compact (2) and since intersections of real-compact subsets are real-compact (2) each intersection of sets from $B^{\sigma}(\mathscr{Z})$ is real-compact. This class of real-compact sets is closed under σ-unions and intersections and it contains all closed sets.

The last result has been announced (for sets of $B^{\sigma}(\mathscr{Z})$) in (6). It can be proved without appeal to the results cited.

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