Mathematika 56 (2010) 363–378 © University College London
doi:10.1112/S0025579310000379

CHARACTERIZATION OF ELLIPSOIDS BY MEANS OF PARALLEL TRANSLATED SECTIONS

D. LARMAN, L. MONTEJANO AND E. MORALES-AMAYA

Abstract. Let K be a convex body and let $p_0 \in \operatorname{int} K$. Suppose that in every direction we can choose continuously a section of K which is a translated copy of the corresponding parallel section of K through p_0. Our main result essentially claims that if all these pairs of sections are different almost everywhere, then K is an ellipsoid.

§1. *Introduction.* Let $K \subseteq \mathbb{R}^3$ be a convex body and let $p_0 \in \operatorname{int} K$. For a sufficiently small $\epsilon > 0$, let $K_{-\epsilon} = \{x \in K \mid \text{the } \epsilon\text{-neighborhood of } x \text{ is contained in } K\}$.

Suppose that for every plane $H(v)$ through p_0 which is orthogonal to $v \in \mathbb{S}^2$, we can choose continuously a parallel plane $H'(v)$ with the property that the sections $H(v) \cap K$ and $H'(v) \cap K$ are translated copies of each other. The purpose of this paper is to prove that if for some small $\epsilon > 0$ the collection of ϵ-reduced sections $\{H'(v) \cap K_{-\epsilon}\}_{v \in \mathbb{S}^2}$ is pairwise intersecting and the set of directions w for which $H'(w) = H(w)$ has empty interior, then K is an ellipsoid.

If, in addition, we specify that every plane $H'(v)$ passes through a fixed point q_0, the result can be considered a variation of the "false centre theorem" [1, 4] for the following reason: the orthogonal projection of K, in any direction, satisfies the hypothesis of Roger's equichordal theorem [9], and this allows us to conclude that the midpoint between p_0 and q_0 is a centre of symmetry for K; but if this is so, then every section $H(v) \cap K$ through p_0 must have a centre, and therefore p_0 is a false centre. In our case, the fact that the sections $\{H'(v)\}_{v \in \mathbb{S}^2}$ do not necessarily pass through a fixed point prevents us from working with the central symmetry of K; instead, we have to work with the shaken system of sections in the spirit of [6, 8], first introduced by Bianchi and Gruber in [2].

We will derive our main result from the following theorem.

THEOREM 1. *Let $\varphi : \mathbb{S}^2 \to \mathbb{R}$ be a continuous map with the property that $-\varphi(v) = \varphi(-v)$ if and only if $v \in \mathbb{S}^1$, and let $H_\varphi(v) = \{x \in \mathbb{R}^3 \mid \langle x, v\rangle = \varphi(v)\}$. Let $K \subseteq \mathbb{R}^3$ be a convex body and suppose that for some small $\epsilon > 0$ and every $v \in \mathbb{S}^2$ and $\mu \in \mathbb{S}^1$,*

$$H_\varphi(v) \cap H_\varphi(\mu) \cap K_{-\epsilon} \neq \emptyset,$$

where $K_{-\epsilon} = \{x \in K \mid$ the ϵ-neighborhood of x is contained in $K\}$.

Received 31 January 2009.
MSC (2000): 52A20 (primary).
The third author was supported by the SNI-CONACyT of México (no. 21120).

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

If for every $v \in \mathbb{S}^2$ *we have*

$$(H_\varphi(v) \cap K) + \eta(v) = (H_\varphi(-v) \cap K),$$

where $\eta : \mathbb{S}^2 \to \mathbb{R}^3$ *is a continuous function, then* K *is an ellipsoid.*

Essentially, this theorem claims that if in every direction we can choose continuously two sections of K which are translated copies of each other and, except for a linear curve of directions, all these pairs of parallel sections are different, then K is an ellipsoid.

§2. *Shadow and semishadow boundaries.* As in the above section, and from now on in this paper, let $\varphi : \mathbb{S}^2 \to \mathbb{R}$ be a continuous map with the property that $-\varphi(v) = \varphi(-v)$ if and only if $v \in \mathbb{S}^1$, where $\mathbb{S}^1$ is the unit circle of the xy-plane in $\mathbb{R}^3$. Let $K \subseteq \mathbb{R}^3$ be a convex body, and suppose that for some small $\epsilon > 0$ and every $v \in \mathbb{S}^2$ and $\mu \in \mathbb{S}^1$,

$$H_\varphi(v) \cap H_\varphi(\mu) \cap K_{-\epsilon} \neq \emptyset$$

and also

$$(H_\varphi(v) \cap K) + \eta(v) = (H_\varphi(-v) \cap K),$$

where $\eta : \mathbb{S}^2 \to \mathbb{R}^3$ is a continuous function.

Let L be a line through the origin. Define the *shadow cylinder* $T\partial(K, L)$ to be the union of all tangent lines of K that are parallel to L. The *shadow boundary* of K in the direction of L is defined as $S\partial(K, L) = T\partial(K, L) \cap K$. If $v \neq 0$, then $S\partial(K, v) = S\partial(K, L)$, where L is the line through the interval $[0, v]$.

If H is a plane, the section $H \cap K$ is a *semishadow boundary section of* K if there is a direction $v \in \mathbb{S}^2$ such that $H \cap \mathrm{bd}K$ is contained in the shadow boundary of K in the direction of v, that is, if $H \cap \mathrm{bd}K \subset S\partial(K, v)$. This is equivalent to $K \subset (H \cap K) + L^v$, where L^v is a line through the origin in the direction of the the unit vector v. Of course, for strictly convex bodies, a semishadow boundary section is a planar shadow boundary. Following the spirit of Blaschke's theorem as stated in [3, Theorem 16.14], it is proved in [7, Proposition 2] that if every section through a fixed point of a convex body is a semishadow boundary section, then the convex body is an ellipsoid; see also [5].

Next, we have the following technical lemma.

LEMMA 2. *Let* $K \subseteq \mathbb{R}^3$ *be a convex body. Suppose that there are two sequences of planes* $\{H_i\}$ *and* $\{\Gamma_i\}$ *with* H_i *parallel to* Γ_i *for each* $i \in \mathbb{N}$, *a sequence of unit vectors* $\{v_i\}$ *and a sequence of positive real numbers* $\{t_i\}$ *such that* $H_i \to H$, $\Gamma_i \to \Gamma$, $v_i \to v$, $t_i \to 0$ *and, for every* $i \in \mathbb{N}$, $(H_i \cap K) + t_i v_i = (\Gamma_i \cap K)$.

Then the section $H \cap K$ *is a semishadow boundary section of* K.

Proof. The convex body K is contained in the union of the cylinder $\{x + tv_i \mid x \in (H_i \cap K),\ t \in \mathbb{R}\}$ with the strip between the parallel planes H_i and Γ_i. In the limit, K is contained in the cylinder $\{x + tv \mid x \in (H \cap K),\ t \in \mathbb{R}\}$. This implies that $H \cap \mathrm{bd}K \subset S\partial(K, v)$. □

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

Note that Lemma 2 implies that for every $v \in \mathbb{S}^1$, the section $H_\varphi(\pm v) \cap K$ is a semishadow boundary section of K; that is, for every $v \in \mathbb{S}^1$, there is a line L_v through the origin such that

$$K \subset (H_\varphi(\pm v) \cap K) + L_v.$$

The purpose of this section is to prove that for every $v \in \mathbb{S}^1$, $H_\varphi(\pm v) \cap \mathrm{bd}K$ is a planar shadow boundary. We shall start by proving that there is a line ℓ parallel to the z-axis which lies in $H_\varphi(\pm v)$ for every $v \in \mathbb{S}^1$.

First we need some definitions.

Let F be a compact convex body in the plane. A line L is called a *diametral line* of F if L intersects F in an interval $[a, b]$ and there are parallel support lines L_a and L_b of F through a and b, respectively. If, in addition, every pair of parallel support lines L_a and L_b of F at a and b are such that $L_a \cap F$ and $L_b \cap F$ are both closed intervals with non-empty interior, then the line L is called an *absolute diametral line*.

So, for example, there is always a direction in which there is a diametral line parallel to that direction which is not an absolute diametral line for F. Furthermore, if $p_0 \notin F$ (but is possibly a point at ∞) and every line through p_0 that intersects the interior of F is an absolute diametral line, then F is a quadrilateral with two parallel sides and whose other pair of opposite sides are concurrent at p_0. Finally, if $p_0 \in \mathrm{bd}F$ and every line through p_0 that intersects the interior of F is an absolute diametral line, then F is a triangle with p_0 as one of its vertices.

LEMMA 3. *There is a unique chord of K that is parallel to the z-axis and of maximal length. Moreover, this chord lies in $H_\varphi(\pm v)$ for every $v \in \mathbb{S}^1$.*

Proof. Suppose H_0 is a support plane of K with the property that $(H_0 \cap K) + I \subset K$, where $I = \{(0, 0, t) \in \mathbb{R}^3 \mid 0 \le t \le \zeta\}$ and ζ is the maximal length of all chords of K that are parallel to the z-axis.

Let us first prove that for every $x_0 \in H_0 \cap K$ and every $v \in \mathbb{S}^1$, the diametral chord $x_0 + I$ is such that its endpoints x_0 and $x_0 + (0, 0, \zeta)$ lie on the shadow cylinder $T\partial(K, L_v)$. Note that by Lemma 2, $T\partial(K, L_v)$ is the boundary of the cylinder $(H_\varphi(\pm v) \cap K) + L_v$. Since $x_0 + I$ is parallel to the z-axis, there is a $\mu \in L_v$ such that $x_0 + I \subset H_\varphi(\pm v) + \mu$. Furthermore, the fact that $x_0 + I \subset K \subset (H_\varphi(\pm v) \cap K) + L_v$ implies that $(x_0 + I) - \mu \subset H_\varphi(\pm v) \cap K$; hence the endpoints of $(x_0 + I) - \mu$ lie on the boundary of $H_\varphi(\pm v) \cap K$ and, consequently, the endpoints of $x_0 + I$ lie on the boundary of the cylinder $(H_\varphi(\pm v) \cap K) + L_v$. In particular, this argument implies that for every $v \in \mathbb{S}^1$, the plane $H_\varphi(\pm v)$ contains at least one chord parallel to the z-axis that is of maximal length ζ.

Suppose that $H_0 \cap K$ is neither a single point nor a closed interval. By the argument in the previous paragraph, H_0 is not only a supporting plane of K but also a supporting plane of the cylinder $(H_\varphi(\pm v) \cap K) + L_v$, for every $v \in \mathbb{S}^1$. In fact, H_0 intersects the boundary of the cylinder $(H_\varphi(\pm v) \cap K) + L_v$ in a band, $(H_\varphi(\pm v) \cap H_0 \cap K) + L_v$, that is parallel to L_v and which contains $H_0 \cap K$; therefore, the line $H_\varphi(\pm v) \cap H_0$ is a diametral chord of $H_0 \cap K$.

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

We want to prove that for every $v \in \mathbb{S}^1$, the line $H_\varphi(\pm v) \cap H_0$ is not only a diametral line but also an absolute diametral line of $H_0 \cap K$. Let $v_0 \in \mathbb{S}^2 - \{(0, 0, \pm 1)\}$ be a vector parallel to $H_\varphi(\pm v) \cap H_0$, and let $w \in \mathbb{S}^2 - \mathbb{S}^1$ be a vector which is orthogonal to v_0 and so close to v that the planes $H_\varphi(w)$ and $H_\varphi(-w)$ both intersect the relative interior of $(H_0 \cap K)$ while the lines $H_\varphi(w) \cap H_0$ and $H_\varphi(-w) \cap H_0$ both lie in H_0 on the same side of $H_\varphi(\pm v) \cap H_0$. This is possible because, by hypothesis, for some small $\epsilon > 0$, $H_\varphi(\pm v) \cap H_\varphi(w) \cap K_{-\epsilon} \neq \emptyset$ and $H_\varphi(\pm v) \cap H_\varphi(-w) \cap K_{-\epsilon} \neq \emptyset$. Also, by hypothesis, $(H_\varphi(w) \cap K) + \eta(w) = (H_\varphi(-w) \cap K)$, and hence there are two different chords of $H_0 \cap K$ that have the same length and which are parallel and on the same side of $H_\varphi(\pm v) \cap H_0$. This implies that $H_\varphi(\pm v) \cap H_0$ is an absolute diametral chord of $H_0 \cap K$.

Parallel to each direction of H_0 there is line of the form $H_\varphi(\pm v) \cap H_0$, which is, by the above, an absolute diametral line of $H_0 \cap K$; but this is impossible if $H_0 \cap K$ is a two-dimensional compact convex subset of H_0 with non-empty interior.

Suppose now that $H_0 \cap K$ is a non-trivial closed interval, and let L_0 be a line through the origin parallel to $H_0 \cap K$. Again, the band $(H_\varphi(\pm v) \cap H_0 \cap K) + L_v$, parallel to L_v, contains $H_0 \cap K$, and hence either $L_0 = L_v$ or $H_0 \cap K \subset H_\varphi(\pm v)$. This implies that for every $v \in \mathbb{S}^1$ which is not orthogonal to L_0, we have $L_v = L_0$; in other words, for every $v \in \mathbb{S}^1$ that is not orthogonal to L_0, there is the following equivalence of cylinders:

$$K + L_v = K + L_0 = (H_\varphi(\pm v) \cap K) + L_0.$$

Therefore, for every $v \in \mathbb{S}^1$ that is not orthogonal to L_0, the intersection of the plane $H_\varphi(\pm v)$ with the cylinder $K + L_0$ is contained in K; but this is impossible unless K is unbounded. □

LEMMA 4. *For every $v \in \mathbb{S}^1$, $H_\varphi(\pm v) \cap \mathrm{bd} K$ is a planar shadow boundary of K.*

Proof. Let us think of $\mathbb{RP}^1$, the space of directions in the plane, as the quotient space of $\mathbb{S}^1$ under the equivalence relation $x \sim -x$ for every $x \in \mathbb{S}^1$. By the previous lemma, there is a line ℓ parallel to the z-axis with the property that for every $v \in \mathbb{S}^1$, $\ell \subset H_\varphi(\pm v)$ and, furthermore, $K \subset (H_\varphi(\pm v) \cap K) + L_v$.

Let H be a support plane of K with $\{q_0\} = H \cap \ell$. Note that q_0 may be considered as a point at infinity when H is parallel to the z-axis. We shall prove that there exist w_1^H, $w_2^H \in \mathbb{RP}^1$ (possibly with $w_1^H = w_2^H$) and a line L through the origin such that for every $w \in \mathbb{S}^1$ with $\pm w$ lying between w_1^H and w_2^H, we have $L_w = L$, i.e. $K \subset (H_\varphi(\pm w) \cap K) + L$; moreover, one of the following occurs.

(a) The support set $H \cap K$ is a quadrilateral (possibly degenerate) with two opposite sides parallel to L and the other two opposite sides contained in $H_\varphi(w_1^H) \cap H$ and $H_\varphi(w_2^H) \cap H$ and concurrent at q_0.

(b) The support set $H \cap K$ is a triangle (possibly degenerate) which has q_0 as a vertex, with the side opposite to q_0 being parallel to L, and whose two other adjacent sides are contained in $H_\varphi(w_1^H) \cap H$ and $H_\varphi(w_2^H) \cap H$.

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

(c) The support set $H \cap K$ is a closed interval (possibly degenerate) that is parallel to L with extreme points contained in $H_\varphi(w_1^H) \cap H$ and $H_\varphi(w_2^H) \cap H$.

Suppose first that the support set $H \cap K$ is neither a single point nor a closed interval. We want to prove that for every $v \in \mathbb{S}^1$ such that $H_\varphi(\pm v)$ intersects the relative interior of $H \cap K$, the line $H_\varphi(\pm v) \cap H$ is an absolute diametral line of $H \cap K$. This is so because if $H_\varphi(\pm v)$ intersects the interior of $H \cap K$, then H is not only a support plane of K but also a support plane of the cylinder $(H_\varphi(\pm v) \cap K) + L_v$. In fact, H intersects the boundary of the cylinder $(H_\varphi(\pm v) \cap K) + L_v$ in a band $(H_\varphi(\pm v) \cap H \cap K) + L_v$, parallel to L_v, that contains $H \cap K$. Furthermore, by repeating the arguments in the proof of Lemma 3, we conclude that $H_\varphi(\pm v) \cap H$ is also an absolute diametral line of $H \cap K$. Therefore, every line contained in H through q_0 that intersects the interior of the convex figure $H \cap K$ is an absolute diametral line. This implies that either $H \cap K$ is a quadrilateral with two opposite sides parallel to L_v and the other pair of opposite sides concurrent at $\ell \cap H = q_0$, or $H \cap K$ is a triangle with q_0 as one of its vertices. Suppose now that $H \cap K$ is a non-trivial closed interval. Again, by using the arguments of Lemma 3, we can prove that for every $v \in \mathbb{S}^1$ such that $H_\varphi(\pm v)$ intersects the relative interior of $H \cap K$, L_v is parallel to $H \cap K$.

Let $v_0 \in \mathbb{S}^1$ and $\Omega = \{\pm w \in \mathbb{RP}^1 \mid L_{v_0} = L_w\}$. Our next step is to prove that there are $w_1, w_2 \in \mathbb{RP}^1$ (possibly with $w_1 = w_2$) such that $\{\pm w \in \mathbb{RP}^1 \mid \pm w$ lies in the closed subarc between w_1 and w_2 that contains $\pm v_0\} = \Omega$ and, furthermore, the shadow boundary is such that $S\partial(K, L_{v_0}) = (\mathrm{bd}K) \cap \Lambda$ where $\Lambda = \bigcup_{\pm w \in \Omega} H_\varphi(\pm w)$. The fact that $\mathrm{bd}(H_\varphi(\pm w) \cap K) \subset \mathrm{bd}(K + L_{v_0})$ for every $\pm w \in \Omega$ implies, by convexity, that there are $w_1, w_2 \in \mathbb{RP}^1$ such that $\Omega = \{w \in \mathbb{RP}^1 \mid w$ lies in the closed subarc between w_1 and w_2 that contains $v_0\}$. For every $x \in \mathrm{bd}(H_\varphi(\pm v_0) \cap K)$, let $I_x = (x + L_{v_0}) \cap K$, so that $S\partial(K, L_{v_0}) = \bigcup_{x \in \mathrm{bd}(H_\varphi(\pm v_0) \cap K)} I_x$. We will prove that for every $x \in \mathrm{bd}(H_\varphi(\pm v_0) \cap K)$, the endpoints of I_x lie in $H_\varphi(w_1)$ and $H_\varphi(w_2)$. Clearly, I_x contains $(x + L_{v_0}) \cap \Lambda$. If $\{x\}$ is a support subset of $\mathrm{bd}(H_\varphi(v_0) \cap K)$, then I_x is a support subset of K and hence, by (c), $(x + L_{v_0}) \cap \Lambda = I_x$. If $\{x\}$ is not a support subset of $\mathrm{bd}(H_\varphi(v_0) \cap K)$, then x lies in the relative interior of a support subset of $\mathrm{bd}(H_\varphi(\pm v_0) \cap K)$ and hence in the relative interior of a support subset of K; therefore, by (a) and (b), $(x + L_{v_0}) \cap \Lambda = I_x$. Consequently, $S\partial(K, L_{v_0}) = \mathrm{bd}K \cap \Lambda$.

Let us now prove that $\Omega = \{\pm v_0\}$ and thus establish that $\mathrm{bd}(H_\varphi(\pm v_0) \cap K)$ is a shadow boundary for every $v_0 \in \mathbb{S}^1$. Take $\pm w_0 \in \mathbb{S}^1$ orthogonal to L_{v_0}, and let $u \in \mathbb{S}^2 - \{\pm w_0\}$ be orthogonal to L_{v_0} and sufficiently close to w_0. By hypothesis, for some small $\epsilon > 0$,

$$H_\varphi(\pm v_0) \cap H_\varphi(u) \cap K_{-\epsilon} \neq \emptyset \quad \text{and} \quad H_\varphi(\pm v_0) \cap H_\varphi(-u) \cap K_{-\epsilon} \neq \emptyset.$$

This implies that the parallel lines, $L^+ = H_\varphi(\pm v_0) \cap H_\varphi(u)$ and $L^- = H_\varphi(\pm v_0) \cap H_\varphi(-u)$ both intersect the closed interval $\ell \cap K$ on its relative

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

interior. Choose $\{x^+, y^+\} = \mathrm{bd}K \cap L^+$ and $\{x^-, y^-\} = \mathrm{bd}K \cap L^-$ in such a way that the set $\{x^+, x^-\}$ is separated from the set $\{y^+, y^-\}$ by the line ℓ in the plane $H_\varphi(\pm v_0)$. By hypothesis, $(H_\varphi(u) \cap K) + \eta(u) = (H_\varphi(-u) \cap K)$, which implies that the support subset $I_{x^*} = (x^+ + L_{v_0}) \cap K$ of $\mathrm{bd}(H_\varphi(u) \cap K)$ in the direction L_{v_0} and the support subset $I_{x^-} = (x^- + L_{v_0}) \cap K$ of $\mathrm{bd}(H_\varphi(-u) \cap K)$ in the direction L_{v_0} have the same length, because $I_{x^+} + \eta(u) = I_{x^-}$. If $w_1 \neq w_2$, this contradicts the fact that the endpoints of I_{x^+} and I_{x^-} lie in $H_\varphi(w_1)$ and $H_\varphi(w_2)$ and that this happens for every $u \in \mathbb{S}^2 - \{\pm w_0\}$ which is orthogonal to L_{v_0} and sufficiently close to w_0. Therefore $w_1 = w_2 = \pm v_0$, and hence $\mathrm{bd}(H_\varphi(\pm v_0) \cap K)$ is a shadow boundary for every $v_0 \in \mathbb{S}^1$. □

§3. *Axis of shadow boundaries.* Let K be a convex body in euclidean 3-space and let L be a line through the interior of K. We say that L is an *axis of shadow boundaries* for K if every section through L is a planar shadow boundary of K.

LEMMA 5. *Let K be a convex body in euclidean 3-space and let L be an axis of shadow boundaries for K. Then there are no disks in the boundary of K, and every line segment in the boundary of K lies in a plane through L. Furthermore, K is smooth at $L \cap \mathrm{bd}K$.*

Proof. Suppose that $A \subset \mathrm{bd}K$ is a support set of K. Let H be a plane such that $H \cap \mathrm{bd}K = S\partial(K, L)$ is a shadow boundary with the property that H intersects the relative interior of A; then $A \subset H$. This is so because $A \subset K \subset (H \cap K) + L$, and hence A lies in $T\partial(K, L) = (H \cap \mathrm{bd}K) + L$. Then $A \subset H$, because otherwise $K \cap T\partial(K, L) \neq H \cap \mathrm{bd}K$.

Since the relative interior of A is always intersected by a plane through L, we have that there are no disks in the boundary of K and every line segment in the boundary of K lies in a plane through L. Furthermore, by the proof of [**1**, Lemma 5], K is smooth at $L \cap \mathrm{bd}K$. □

The purpose of this section is to prove the following theorem.

THEOREM 6. *Let K be a convex body as in Theorem 1. Suppose that the z-axis is an axis of shadow boundaries. Then K is affine-equivalent to a body of revolution whose axis of revolution is the z-axis.*

Let us pick two parallel support planes, H_1 and H_2, of K at $\mathrm{bd}K \cap \{z\text{-axis}\}$ in such a way that every section through the z-axis is the shadow boundary of K in a direction v parallel to H_i. After applying an affine map, we may assume that the z-axis is orthogonal to H_1 and H_2. By [**1**, Lemma 6], every section of K orthogonal to the z-axis has a centre of symmetry in the z-axis. Moreover, all orthogonal sections to the z-axis are similar and similarly situated. In other words, if we project all these sections orthogonally in the direction of the z-axis, then all of them are homothetic with centre of homothecy being the projection of the z-axis. Note also that, for the above reason, if H is a plane through the z-axis, then the z-axis is an orthogonal axis of symmetry for the section $H \cap K$.

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

We shall prove our theorem through a series of lemmas. Throughout the rest of the proof it will be assumed, without loss of generality, that the parallel support planes at $\mathrm{bd}K \cap \{z\text{-axis}\}$ are orthogonal to the z-axis.

Let $[(0, 0, \rho_1), (0, 0, \rho_2)]$ be the orthogonal projection of $K_{-\epsilon}$ onto the z-axis. Note that $[(0, 0, \rho_1), (0, 0, \rho_2)] \subset \mathrm{int}\, K$.

LEMMA 7. *For every $v \in \mathbb{S}^2 \backslash \mathbb{S}^1$, the plane $H_\varphi(v)$ intersects the z-axis in the point $(0, 0, \kappa_v) \in [(0, 0, \rho_1), (0, 0, \rho_2)]$.*

Proof. This is so because if $\ell \subset H_\varphi(v)$ is the line through $(0, 0, \kappa_v)$ orthogonal to the z-axis, then ℓ intersects $K_{-\epsilon}$; otherwise we would have

$$\ell \cap K_{-\epsilon} = H_\varphi(v) \cap H_\varphi(\mu) \cap K_{-\epsilon} = \emptyset$$

for $\mu \in \mathbb{S}^1$ orthogonal to ℓ, which contradicts our hypothesis. □

Let $v \in \mathbb{S}^1$ and $\Sigma_v = \{w \in \mathbb{S}^2 \mid w \text{ is orthogonal to } v\}$. Denote by $\Gamma(v)$ the plane through the z-axis with the property that the section $\Gamma(v) \cap \mathrm{bd}K$ is $S\partial(K, v)$, the shadow boundary of K in the direction of v.

LEMMA 8. *For every $w \in \Sigma_v$, the translation vector $\eta(w)$ is parallel to $\Gamma(v)$, where $(H_\varphi(w) \cap K) + \eta(w) = (H_\varphi(-w) \cap K)$.*

Proof. The fact that $\Gamma(v)$ is the shadow boundary in the direction of v implies that the chord $H_\varphi(w) \cap \Gamma(v) \cap K$ is a diametral chord of the section $H_\varphi(w) \cap K$. Parallel to $H_\varphi(w) \cap \Gamma(v) \cap K$ there are no other diametral chords of $H_\varphi(w) \cap K$. Similarly, the chord $H_\varphi(-w) \cap \Gamma(v) \cap K$ is a diametral chord of the section $H_\varphi(-w) \cap K$, and parallel to it there are no other diametral chords of $H_\varphi(-w) \cap K$. Consequently,

$$(H_\varphi(w) \cap \Gamma(v) \cap K) + \eta(w) = (H_\varphi(-w) \cap \Gamma(v) \cap K),$$

and hence $\eta(w)$ is parallel to $\Gamma(v)$. □

Let

$$\beta = \min\{\text{length of a chord of } K \text{ through } (0, 0, \rho_1) \text{ or } (0, 0, \rho_2) \text{ that is orthogonal to the } z\text{-axis}\}.$$

LEMMA 9. *The midpoint of every chord of K that is parallel to v and of length less than β lies in $\Gamma(v)$.*

Proof. Let $0 < \delta < \beta$, and let $\mathrm{CH}(\delta) = \{\text{chords of } K \text{ that are parallel to } v \text{ and of length } \delta\}$. Let $\pi : \mathbb{R}^3 \to \Gamma(v)$ be the projection onto $\Gamma(v)$ along the direction v. Hence every chord of $\mathrm{CH}(\delta)$ is projected onto a point of $\Gamma(v)$ and, furthermore, $\mathrm{CH}(\delta)$ is projected onto a convex curve contained in the interior of $\Gamma(v) \cap K$, because $S\partial(K, v) = \Gamma(v) \cap \mathrm{bd}K$. Let $\alpha(\delta) = \pi(\mathrm{CH}(\delta))$ be this curve, and note that $[(0, 0, \rho_1), (0, 0, \rho_2)]$ is completely contained in the interior of $\alpha(\delta)$. So, if $\alpha(\delta) \cap \{z\text{-axis}\} = \{(0, 0, \alpha_1), (0, 0, \alpha_2)\}$, we may assume that $\alpha_1 < \rho_1 < \rho_2 < \alpha_2$.

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

Remember that $\Sigma_v = \{w \in \mathbb{S}^2 \mid w \text{ is orthogonal to } v\}$, and let Σ_v^+ and Σ_v^- be the two open subarcs of $\Sigma_v \backslash \mathbb{S}^1$. The collection of lines $\{H_\varphi(v) \cap H_\varphi(w) \mid w \in \mathrm{Cl}(\Sigma_v^+)\}$ consists of a continuous selection of a unique line in $H_\varphi(v)$ parallel to every direction. Similarly, the collection of lines $\{H_\varphi(v) \cap H_\varphi(w) \mid w \in \mathrm{Cl}(\Sigma_v^-)\}$ consists of a continuous selection of a unique line in $H_\varphi(v)$ parallel to every direction. So, for every chord $I \in \mathrm{CH}(\delta)$, except any that projects onto $(0, 0, \alpha_1)$ or $(0, 0, \alpha_2)$, there exist $w^+ \in \Sigma_v^+$ and $w^- \in \Sigma_v^-$ such that $I \subset H_\varphi(w^+)$ and $I \subset H_\varphi(w^-)$.

Let $I_0 \in \mathrm{CH}(\delta)$ be a chord of K, and let

$$\gamma(I_0) = \text{length of the chord } I_0 \text{ below } \Gamma(v) - \text{length of the chord } I_0 \text{ above } \Gamma(v).$$

By the above paragraph, let $w_0 \in \Sigma_v^+$ be such that I_0 is a chord of $H_\varphi(w_0) \cap K$ and let $I_1 = I_0 + \eta(w_0)$ be the chord of the section $H_\varphi(-w_0) \cap K$. By Lemma 8, $\eta(w_0)$ is parallel to $\Gamma(v)$, and then

$$\begin{aligned}\gamma(I_1) = {} & \text{length of the chord } I_1 \text{ below } \Gamma(v) \\ & - \text{length of the chord } I_1 \text{ above } \Gamma(v) = \gamma(I_0).\end{aligned}$$

Remember that $H_\varphi(w_0) \cap \{z\text{-axis}\} = \{(0, 0, \kappa_{w_0})\}$ and $H_\varphi(-w_0) \cap \{z\text{-axis}\} = \{(0, 0, \kappa_{-w_0})\}$. By Lemma 7 and without loss of generality, assume that $\alpha_1 < \rho_1 \le \kappa_{w_0} < \kappa_{-w_0} \le \rho_2 < \alpha_2$. Thus, the point $\pi(I_1)$ lies in the subarc of the curve $\alpha(\delta)$ between $\pi(I_0)$ and $(0, 0, \alpha_2)$, which does not contain $(0, 0, \alpha_1)$.

Now take $w_1 \in \Sigma_v^+$ such that I_1 is a chord of $H_\varphi(w_1) \cap K$, and let $I_2 = I_1 + \eta(w_1)$ be the chord of the section $H_\varphi(-w_1) \cap K$. Since Lemma 8 tells us that $\eta(w_1)$ is parallel to $\Gamma(v)$, we have that

$$\begin{aligned}\gamma(I_2) = {} & \text{length of the chord } I_2 \text{ below } \Gamma(v) \\ & - \text{length of the chord } I_2 \text{ above } \Gamma(v) = \gamma(I_1).\end{aligned}$$

Furthermore, $\kappa_{w_1} < \kappa_{-w_1}$, otherwise there would be a point $\varpi \in \Sigma_v^+$ between w_0 and w_1 with the property that $\kappa_\varpi = \kappa_{-\varpi}$, which is not possible because $H_\varphi(\varpi) \neq H_\varphi(-\varpi)$. So, the point $\pi(I_2)$ lies in the subarc of the curve $\alpha(\delta)$ between $\pi(I_1)$ and $(0, 0, \alpha_2)$, which does not contain $(0, 0, \alpha_1)$.

Define inductively, in this way, chords $I_j \in \mathrm{CH}(\delta)$ and vectors $w_j \in \Sigma_v^+$ with the following properties:

(i) each I_j is a chord of $H_\varphi(w_j) \cap K$, and $I_{j+1} = I_j + \eta(w_j)$ is the chord of the section $H_\varphi(-w_j) \cap K$;

(ii) $\gamma(I_0) = \gamma(I_1) = \cdots = \gamma(I_j) =$ length of the chord I_j below $\Gamma(v)$ − length of the chord I_j above $\Gamma(v)$;

(iii) each point $\pi(I_j)$ lies in the subarc of the curve $\alpha(\delta)$ between $\pi(I_{j-1})$ and $(0, 0, \alpha_2)$, which does not contain $(0, 0, \alpha_1)$.

By (iii), the monotone sequence $\{\pi(I_j)\}_0^\infty \subset \alpha(\delta)$ converges to a point in $\alpha(\delta)$. This implies that the sequence $\{\eta(w_j)\}_0^\infty$ converges to 0, and hence the sequence $\{w_j\}_0^\infty$ converges to a point $\mu \in \mathbb{S}^1$. Therefore, the sequence $\{\pi(I_j)\}_0^\infty$ converges to $(0, 0, \alpha_2)$. Consequently, the chords $\{I_j\}$ converge to a chord I

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

through $(0, 0, \alpha_2)$ that is parallel to v. Furthermore, the $\gamma(I_j)$ converge to $\gamma(I) = 0$, because the z-axis is an orthogonal axis of symmetry for any section of K through the z-axis. This implies that $\gamma(I_j) = 0$ and hence that the midpoint of I_0 lies in $\Gamma(v)$, as desired. This concludes the proof of the lemma. □

Proof of Theorem 6. We have that the midpoint of every chord parallel to $v \in \mathbb{S}^1$ of length less than β lies in the plane $\Gamma(v)$ through the z-axis, where $S\partial(K, v)$, the shadow boundary of K in the direction of v, is $\Gamma(v) \cap \mathrm{bd}K$. Take Δ to be the plane orthogonal to the z-axis through the point $(0, 0, \xi)$, with $\rho_2 < \xi$. Then, if $\Delta \cap K \neq \emptyset$, the diameter of the section is smaller than β and hence, by Lemma 9, for every direction $v \in \mathbb{S}^1$ the midpoints of all parallel chords of $\Delta \cap K$ in the direction v lie in a line; therefore, by Brunn's theorem, as given in [**3**, Theorem 16.13], $\Delta \cap K$ is an ellipse.

Remember that every section of K orthogonal to the z-axis has a centre of symmetry in the z-axis; moreover, all these sections are similar and similarly situated. Therefore, we conclude that there is an affine map that transforms all such sections orthogonal to the z-axis into concentric circles, as claimed. □

§4. *The theorem for bodies of revolution.* The purpose of this section is to prove the following theorem.

THEOREM 10. *Let K be a convex body as in Theorem 1. If, in addition, the z-axis is an axis of revolution, then K is an ellipsoid.*

By §§2 and 3, we may assume that the z-axis is an axis of shadow boundaries for K, that there are no disks in the boundary of K and that every line segment in the boundary of K lies in a plane through the z-axis. Furthermore, K is smooth at $\mathrm{bd}K \cap \{z\text{-axis}\}$. The following lemma is important for the proof of Theorem 10.

LEMMA 11. *Let W and V be open intervals of real numbers, and let $f(x)$ be a real continuous function defined on W and $W + V$. Suppose that for every $p \in V$, the function $g_p : W \to \mathbb{R}$ given by $g_p(x) = f(x) - f(x + p)$ is a constant map. Then $f|_W : W \to \mathbb{R}$ is a affine function, i.e. $f(w) = \alpha w + \beta$.*

Proof. Let $w_0 \in W$, and let $a > 0$ be such that $[-a + w_0, a + w_0] \subset W$. Let $\varphi(y)$ be a real continuous function defined on $[-a, a]$ and $W + [-a, a]$ by

$$\varphi(y) = f(y + w_0) - f(w_0).$$

Then $\varphi(0) = 0$ and, for every $p \in W$ and every $x \in [-a, a]$, $\varphi(x + p) - \varphi(x)$ depends only on p. In particular, $\varphi(x + p) - \varphi(x) = \varphi(0 + p) - \varphi(0)$. Therefore,

$$\varphi(x) + \varphi(p) = \varphi(p + x).$$

Let $h > 0$ and choose n such that $nh < a \leq (n + 1)h$; then $\varphi(a - h) + \varphi(p + h) = \varphi(a) + \varphi(p)$, and therefore

$$\varphi(p + h) - \varphi(p) = \varphi(a) - \varphi(a - h).$$

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

Similarly,

$$\varphi(p+h)-\varphi(p)=\varphi(a-h)-\varphi(a-2h),$$
$$\vdots$$
$$\varphi(p+h)-\varphi(p)=\varphi(a-(n-1)h)-\varphi(a-nh).$$

Adding these n equations gives

$$n(\varphi(p+h)-\varphi(p))=\varphi(a)-\varphi(a-nh).$$

So, upon dividing by nh and writing $\eta=a-nh$ so that $0<\eta\leq h$, we have

$$\frac{\varphi(p+h)-\varphi(p)}{h}=\frac{\varphi(a)-\varphi(\eta)}{a-\eta}.$$

Therefore, taking the limit as $h\to 0+$ yields

$$\lim_{h\to 0+}\left(\frac{\varphi(p+h)-\varphi(p)}{h}\right)=\frac{\varphi(a)}{a}.$$

Similarly, if $h<0$, we take $(n+1)h\leq -a<nh$ and obtain

$$\varphi(p+h)-\varphi(p)=\varphi(0)-\varphi(0-h),$$
$$\vdots$$
$$\varphi(p+h)-\varphi(p)=\varphi(0-(n-1)h)-\varphi(0-nh).$$

Adding gives

$$n(\varphi(p+h)-\varphi(p))=\varphi(0)-\varphi(0-h)=\varphi(0)-\varphi(0-nh).$$

Then, dividing by nh and writing $\lambda=a+nh$, we get

$$\frac{\varphi(p+h)-\varphi(p)}{h}=\frac{\varphi(a-\lambda)}{a-\lambda}.$$

Therefore, taking the limit as $h\to 0-$ yields

$$\lim_{h\to 0-}\left(\frac{\varphi(p+h)-\varphi(p)}{h}\right)=\frac{\varphi(a)}{a}.$$

Consequently, for every $p\in W$,

$$\lim_{h\to 0}\left(\frac{\varphi(p+h)-\varphi(p)}{h}\right)=\frac{\varphi(a)}{a}.$$

Thus, for every $p\in W$, $\varphi'(p)$ exists and does not depend on p. This implies that for every $p\in W$, $\varphi(p)=\alpha p$; and, since for every $p\in W$ and every $x\in[-a,a]$ we have $\varphi(x)+\varphi(p)=\varphi(p+x)$, for every $x\in[-a,a]$ we have $\varphi(x)=\alpha x$.

This implies that for every $y\in[-a+w_0,a+w_0]$, we have $\alpha y-\alpha w_0=\alpha(y-w_0)=\varphi(y-w_0)=f(y)-f(0)$. Thus $f(y)=\alpha y+\beta$. Finally, the fact that for every $w\in W$ the map f is an affine map in a small neighborhood of w in W implies that f is an affine function in W. □

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

Proof of Theorem 10. Let us assume first that K is a body of revolution with the property that every point $A \in \partial K$ which is not in the axis of revolution has a unique support plane. We choose coordinates so that K is the set of points (x, y, z) satisfying

$$y^2 + z^2 \leq r(x)^2, \qquad -\rho \leq x \leq \rho,$$

where $r(x)$ is a continuous concave function on the interval $[-\rho, \rho]$ which has $r(-\rho) = r(\rho) = 0$ and is smooth in $(-\rho, \rho)$.

For every point $(x, y, 0) \in K$, let us denote by $I(x, y)$ the length of the vertical chord of K through the point $(x, y, 0)$; that is,

$$I(x, y) = 2\sqrt{r(x)^2 - y^2}.$$

Let L be a line of slope $m \neq 0$ in the plane $z = 0$, and let Γ be a vertical plane through L. Suppose that there is a vertical plane Γ' which is parallel to Γ and has the property that

$$(\Gamma \cap K) + (p, q, 0) = (\Gamma' \cap K).$$

Let us fix, for the time being, a line L of slope m and let $U = \{x \in \mathbb{R} \mid (x, (x + b)m, 0) \in L \cap \operatorname{int} K\}$, where b is such that L cuts the x-axis at the point $(-b, 0, 0)$. For every $x \in U$, define

$$\Psi(x) = \left(\frac{I(x, (x + b)m, 0)}{2}\right)^2.$$

First, notice that

$$\Psi(x) = r(x)^2 - (x + b)^2 m^2;$$

but since $I(x, (x + b)m, 0) = I(x + p, (x + b)m + q, 0)$, we have

$$\Psi(x) = r(x + p)^2 - ((x + b)m + q)^2.$$

So, upon differentiating the first expression for Ψ we get

$$\frac{d\Psi}{dx}(x) = 2r(x)\frac{dr}{dx}(x) - 2(x + b)m^2,$$

but differentiating the second expression gives

$$\frac{d\Psi}{dx}(x) = 2r(x + p)\frac{dr}{dx}(x + p) - 2((x + b)m + q)m.$$

Therefore, we have the following equality in U:

$$2r(x)\frac{dr}{dx}(x) - 2r(x + p)\frac{dr}{dx}(x + p) = 2qm.$$

Note that for $m \neq 0$, $p = 0$ implies $q = 0$.

For every $x \in U$, let

$$\lambda(x) = 2r(x)\frac{dr}{dx}(x)$$

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

and note that in this open interval, $\lambda(x) - \lambda(x + p)$ is a constant function because q and m do not depend on x but depend only on the line L, which is fixed. In order to prove that λ is an affine function using Lemma 11, we need to show that for every p' in a small neighborhood around p, the map $\lambda(x) - \lambda(x + p')$ depends only on p' and not on x.

If we allow the line $L(m)$ to change with the slope m, then clearly the translation vector $(p(m), q(m), 0)$ depends on m. Therefore, for every $x \in U$ there is a small neighborhood W of x in U and a small open interval V such that if $y \in W$ and $m \in V$, then the point $(y, (y + b(m))m, 0)$ is in int K and so, by the above arguments,

$$\lambda(y) - \lambda(y + p(m)) = 2q(m)m.$$

Thus, to conclude our argument using Lemma 11 to show that λ is an affine function in a small neighborhood of x, we need to prove that for every $x \in (-\rho, \rho)$ there is an slope m such that

$$x \in U_m = \{y \in R \mid (y, (y + b)m, 0) \in L(m) \cap \text{int } K\}$$

and $p(m)$ is contained in the interior of the domain of p, where $L(m)$ and $p(m)$ are defined as in the above paragraphs. Note that $L(m)$ tends to the x-axis while the slope m tends to zero. So, given $x \in (-\rho, \rho)$, there is $\delta > 0$ such that if $m < \delta$, then $x \in U_m = \{y \in R \mid (y, (y + b)m, 0) \in L(m) \cap \text{int } K\}$. On the other hand, by hypothesis, there is a slope m_0 with the property that $(p(m_0), q(m_0)) \neq (0, 0)$, and therefore the function p is not zero. This, together with the fact that $p(0) = 0$, implies that we can always choose a sufficiently small slope m with the property that $p(m)$ is contained in the interior of the domain of p.

We have proven that for every $x \in (-\rho, \rho)$, there is a small neighborhood in which

$$\lambda(x) = \frac{dr^2}{dx}(x) = \alpha x + \beta.$$

This implies that in the same neighborhood, the function $r(x)^2$ is quadratic; hence $r(x)^2$ is globally quadratic and, since $r(-\rho) = r(\rho) = 0$, we have

$$r(x)^2 = -\eta(x - \rho)(x + \rho) = -\eta x^2 + \eta\rho^2 \quad \text{for } \eta > 0.$$

Therefore, K is the set of points (x, y, z) with

$$\frac{y^2}{\eta\rho^2} + \frac{z^2}{\eta\rho^2} + \frac{x^2}{\rho^2} \leq 1,$$

and this concludes the first part of the proof.

Suppose now that there is a corner point $A = (a, b, c) \in \partial K$ which is not in the axis of revolution of K. Then every point on the boundary of K with coordinates (x, y, z) where $x = a$ is also a corner point of K. By the first part of

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

the proof (and using the same notation), there is a slope m such that $a \in U_m$ and $p(m)$ is contained in the interior of the domain of p. First, note that

$$(\Gamma(m) \cap K) + (p(m), q(m), 0) = (\Gamma'(m) \cap K).$$

Therefore, if $a \in U_m$, then there is a point $A' = (a, b', 0) \in \partial K \cap \Gamma(m)$ which is a corner point of the section $K \cap \Gamma(m)$. This implies that $(a + p(m), b' + q(m), 0)$ is a corner point of the section $\Gamma'(m) \cap K)$ and hence that $(a + p(m), b' + q(m), 0)$ is a corner point of K; therefore, every point on the boundary of K with coordinates (x, y, z) where $x = a + p(m)$ is a corner point of K. Finally, the fact that $p(m)$ is contained in the interior of the domain of p implies that the interior of the corner points of K is non-empty, which is a contradiction. This concludes the proof of Theorem 10. □

Theorem 1 follows from Lemmas 3 and 4 and Theorems 6 and 10.

Remark 1. There is an analogue of Theorem 1 which claims, essentially, that if in every direction we can choose continuously two sections of K which are homothetic copies of each other and, except for a linear curve of directions, all these pairs of parallel sections are different, then K is an ellipsoid. The proof of this result, although technically more complicated, uses the same ideas as those developed in this paper.

§5. *The main theorem.* Our main result is the following.

THEOREM 12. *Let $K \subseteq \mathbb{R}^3$ be a convex body and let $p_0 \in \operatorname{int} K$. Suppose that for every plane $H(v)$ through p_0 which is orthogonal to $v \in \mathbb{S}^2$, we can choose continuously a parallel plane $H'(v)$ such that the sections $H(v) \cap K$ and $H'(v) \cap K$ are translated copies of each other. Suppose also that for some small $\epsilon > 0$, the collection of ϵ-reduced sections $\{H'(v) \cap K_{-\epsilon}\}_{v \in \mathbb{S}^2}$ is pairwise intersecting and the set Φ of directions w for which $H'(w) = H(w)$ has empty interior. Then K is an ellipsoid.*

For the proof of Theorem 12, we need two technical lemmas and a definition.

If F is a compact convex body in the plane, then a line L is called a *diametral line* of F if L intersects F in an interval $[a, a']$ and there are parallel support lines L_a and $L_{a'}$ of F through a and a', respectively. If, in addition, a lies in the relative interior of the non-trivial closed interval $L_a \cap F$ and a' lies in the relative interior of the non-trivial closed interval $L_{a'} \cap F$, then the line L is called an *absolute internal diametral line in the direction* L_a. If F is a compact closed interval in the plane that is contained in the line L_a, then a line L is called an *absolute internal diametral line of F in the direction* L_a if L intersects L_a transversely in some point a of the relative interior of F.

LEMMA 13. *Let K be a convex body as in Theorem 12. Let $[a, b]$ be a diametral chord of K through p_0, let H_a and H_b be parallel support planes of K through a and b, respectively, and let $H(w)$ be a plane through $[a, b]$. If $H'(w) \neq H(w)$ and either $H(w)$ or $H'(w)$ intersects the relative interior of $H_a \cap K$, then the line $H_a \cap H(w)$ is an absolute internal diametral line of*

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

$H_a \cap K$ in the direction L_a, the line $H_b \cap H(w)$ is an absolute internal diametral line of $H_b \cap K$ in the direction L_b, and L_a is parallel to L_b.

Proof. Let $A = H_a \cap K$ and $B = H_b \cap K$. Let

$$I_a = H(w) \cap H_a \cap K, \qquad I'_a = H'(w) \cap H_a \cap K,$$
$$I_b = H(w) \cap H_b \cap K, \qquad I'_b = H'(w) \cap H_a \cap K.$$

Let $L \subset H_a$ be a line parallel to $H(w)$ between $H(w)$ and $H'(w)$ and sufficiently close to I_a, and let $H(w_1)$ be the plane through p_0 and L. By hypothesis,

$$(H(w_1) \cap K) + t_{w_1} = H'(w_1) \cap K.$$

Suppose first that $H'(w)$ intersects the relative interior of $H_a \cap K$. Let

$$L' = H'(w_1) \cap H_a$$

and suppose that L is so close to I_a that $L' \cap K \neq \emptyset$. Consequently, we have that $(L \cap K) + t_{w_1} = L' \cap K$, because L is a support line of $H(w_1) \cap K$ and the corresponding support line for $H'(w_1) \cap K$ is L'. So the vector t_{w_1} is parallel to the plane H_a. Let $\ell \subset H(w_1)$ be the support line of $H(w_1) \cap K$ parallel to L, and let $\ell' \subset H'(w_1)$ be the support line of $H'(w_1) \cap K$ parallel to L'. Therefore

$$(\ell \cap K) + t_{w_1} = (\ell' \cap K).$$

First, note that the plane $H(w)$ separates the line ℓ from the line L and that the plane $H'(w)$ separates the line ℓ' from the line L', because the collection of ϵ-reduced sections $\{H'(v) \cap K_{-\epsilon}\}_{v \in \mathbb{S}^2}$ is pairwise intersecting. Hence the line ℓ' lies between $H(w)$ and $H'(w)$. This implies that $\ell' \subset H_b$ lies between I_b and $I_{b'}$. Therefore the length of $\ell' \cap K$ is greater than or equal to the length of I_b and $I_{b'}$. Since t_{w_1} is parallel to the plane H_b, $\ell \subset H_b$ and the length of $\ell \cap K$ is equal to the length of $\ell' \cap K$. All of this implies that the line $H_b \cap H(w)$ is an absolute internal diametral line of $H_b \cap K$ in the direction parallel to the vector t_{w_1}.

Similarly, if $H(w)$ intersects the relative interior of $H_a \cap K$, we can prove that the line $H_b \cap H(w)$ is an absolute internal diametral line of $H_b \cap K$ in the direction parallel to t_{w_1}. Furthermore, by interchanging the roles of a and b, we can show that $H_a \cap H(w)$ is an absolute internal diametral line of $H_a \cap K$ in the direction parallel to the vector t_{w_1}.

The case where $H(w)$ intersects the relative interior of $H_a \cap K$ is analogous. This finishes the proof of the lemma. □

LEMMA 14. *Let K be a convex body as in Theorem 12. Let $[a, b]$ be a diametral chord of K through p_0, let H_a and H_b be parallel support planes of K through a and b, respectively, and suppose that $H_a \cap K$ is a closed interval. Then there is a line L through p_0 and $H_a \cap K$ such that every plane through L is a semishadow boundary section.*

Proof. If $H_a \cap K$ is a point, then every plane $H(w)$ through $[a, b]$ is such that $H'(w) = H(w)$. Since the set Φ of directions $w \in \mathbb{S}^2$ for which $H'(w) = H(w)$ has empty interior, $H(w)$ is a semishadow boundary section by Lemma 2.

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

Let $H_a \cap K = [c, d]$ be a non-trivial interval, and let $\Lambda \subset H_b$ be the line through b parallel to $[c, d]$. Let L be a line through p_0 transverse to $[c, d]$ and $\Lambda \cap K$, with the property that L contains an extreme point of either $[c, d]$ or $\Lambda \cap K$. Note that L is still a diametral line of K.

For every $w \in \mathbb{S}^2$ orthogonal to L, let $(H(w) \cap K) + t(w) = H'(w) \cap K$. If $t(w) \neq 0$, we may assume that $c \in H(w)$ and $d \in H'(w)$; otherwise, by Lemma 13, we get a contradiction to the way we chose L. So either $t(w) = 0$, or $|t(w)|$ has the length of $[c, d]$. The continuity of $t(w)$ implies that $t(w) = 0$ for every $w \in \mathbb{S}^2$ orthogonal to L. This concludes the proof of this lemma because, by Lemma 2, every plane through L is a semishadow boundary section. □

Proof of Theorem 12. Let $[a, b]$ be the chord of K through p_0 of maximal length, and assume without loss of generality that $[a, b] \subset \{z\text{-axis}\}$. Let H_a and H_b be parallel support planes of K through a and b, respectively. We claim that $\mathbb{S}^1$ is contained in Φ, i.e. for every plane $H(w)$ through $[a, b]$ we have $H'(w) = H(w)$.

Assume first that $H_a \cap K$ is a convex figure with non-empty interior. Suppose that either a lies in the relative interior of $H_a \cap K$ or a is not a corner point of the boundary of $H_a \cap K$; then $\mathbb{S}^1$ is contained in Φ, because otherwise, by Lemma 13, $[a, b]$ would not be the chord of K through p_0 of maximal length.

Suppose that a is a corner point of the boundary of $H_a \cap K$, and for every $w \in \mathbb{S}^1$ let

$$(H(w) \cap K) + t(w) = H'(w) \cap K.$$

If the line $H(w) \cap H_a$ intersects the interior of $H_a \cap K$, then $t(w) = 0$, otherwise, by Lemma 13, $[a, b]$ would not be the chord of K through p_0 of maximal length. If the line $H(w) \cap H_a$ is a support line of $H_a \cap K$, then either $t(w) = 0$ or $|t(w)|$ is greater than or equal to the width of $H_a \cap K$, otherwise $H'(w)$ would intersect the interior of $H_a \cap K$ and therefore, by Lemma 13, $[a, b]$ would not be the chord of K through p_0 of maximal length. The continuity of $t(w)$ implies that $t(w) = 0$ for every $w \in \mathbb{S}^1$.

Assume now that $H_a \cap K = [c, d]$ is a non-trivial closed interval. Let

$$(H(w) \cap K) + t(w) = H'(w) \cap K.$$

By the arguments of the above paragraphs, we have $a = c$ and $d \in H'(w)$. So either $t(w) = 0$, or $|t(w)|$ has the length of $[c, d]$. The continuity of $t(w)$ implies that $t(w) = 0$. This proves that $\mathbb{S}^1 \subset \Phi$.

We want to prove now that $\Phi = \mathbb{S}^1$. Suppose there is $w_0 \in \Phi \backslash \mathbb{S}^1$, that is, $H'(w_0) = H(w_0)$. Since the set Φ of directions $w \in \mathbb{S}^2$ for which $H'(w) = H(w)$ has empty interior, by Lemma 2 we have that $H(w_0)$ is a semishadow boundary section of K and hence that there exists a line L_{w_0} through the origin such that $K \subset (H(w_0) \cap K) + L_{w_0}$. We claim that every line $\Upsilon \subset H(w_0)$ through p_0 is a diametral chord of K because Υ is the intersection of two semishadow boundary sections.

Choose x_0 to be a regular point on the boundary of $H(w_0) \cap K$ such that $x_0 + L_{w_0}$ does not intersect the z-axis. If Υ is the line through p_0 such that $\Upsilon \cap K = [x_0, y_0]$, then, by the above, Υ is a diametral chord of K and $\ell + L_{w_0}$ is the support plane of K through x_0, where $\ell \subset H(w_0)$ is the support line

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press

of $H(w_0) \cap K$ such that $\{x_0\} = \ell \cap K$. Let $I = (\ell + L_{w_0}) \cap K \subset x_0 + L_{w_0}$. By Lemma 14, there is a line L through p_0 which is transverse to $x_0 + L_{w_0}$ and has the property that every plane through L is a semishadow boundary of K. Since $x_0 + L_{w_0}$ does not intersect the z-axis, L is not the z-axis. Therefore, every line which is not in the plane generated by the z-axis and L is the intersection of two semishadow boundary sections and is thus a diametral line of K. This implies that K is centrally symmetric with centre p_0, which is impossible because $\Phi \neq \mathbb{S}^2$. This proves that $\Phi = \mathbb{S}^1$.

Suppose now that p_0 is the origin and, by hypothesis, let $\mu : \mathbb{S}^2 \to \mathbb{R}^3$ be the continuous function such that $\mu(v) = \mu(-v)$ for every $v \in \mathbb{S}^2$ and

$$(H(\pm v) \cap K) + \mu(\pm v) = (H'(v) \cap K).$$

Define $\varphi : \mathbb{S}^2 \to \mathbb{R}$ to be the continuous map such that $\varphi(v) = 0$ and $\varphi(-v) = \mu(\pm v)$ for every v in the closed upper hemisphere of $\mathbb{S}^2$. Hence, $\varphi : \mathbb{S}^2 \to \mathbb{R}$ is a continuous map with the property that $-\varphi(v) = \varphi(-v)$ if and only if $v \in \mathbb{S}^1$, and if $H_\varphi(v) = \{x \in \mathbb{R}^3 \mid \langle x, v\rangle = \varphi(v)\}$, then for every $v \in \mathbb{S}^2$ we have

$$(H_\varphi(v) \cap K) + \eta(v) = (H_\varphi(-v) \cap K),$$

where $-\eta(v) = \eta(-v)$ and $\eta(v) = \mu(\pm v)$ for every v in the closed upper hemisphere of $\mathbb{S}^2$. Furthermore, for some small $\epsilon > 0$ and every $v \in \mathbb{S}^2$ and $\mu \in \mathbb{S}^1$, we have $H_\varphi(v) \cap H_\varphi(\mu) \cap K_{-\epsilon} \neq \emptyset$. Therefore, by Theorem 1, K is an ellipsoid. □

Acknowledgements. We would like to thank the referee of this paper, whose important observations have allowed us to greatly improve this work.

References

1. P. W. Aitchison, C. M. Petty and C. A. Rogers, A convex body with a false centre is an ellipsoid. *Mathematika* **18** (1971), 50–59.
2. G. Bianchi and P. Gruber, Characterization of ellipsoids. *Arch. Math.* **49** (1987), 344–350.
3. H. Busemann, *The Geometry of Geodesics*, Academic Press (New York, 1955).
4. D. G. Larman, A note on the false centre problem. *Mathematika* **21** (1974), 216–217.
5. A. Marchaud, Un theoreme sur les corps convexes. *Ann. Sci. École Norm. Sup.* **76** (1959), 283–304.
6. L. Montejano, Two applications of topology to convex geometry. *Proc. Steklov Inst. Math.* **247** (2004), 164–167.
7. L. Montejano and E. Morales, Variations of classic characterizations of ellipsoids and a short proof of the false centre theorem. *Mathematika* **54** (2007), 37–42.
8. L. Montejano and E. Morales, Shaken false centre theorem I. *Mathematika* **54** (2007), 43–48.
9. C. A. Rogers, An equichordal problem. *Geom. Dedicata* **10** (1981), 73–78.

D. Larman,
Department of Mathematics,
University College London,
Gower Street, London WC1E 6BT,
U.K.
E-mail: d.larman@math.ucl.ac.uk

L. Montejano,
Instituto de Matemáticas,
Universidad Nacional Autonóma de México,
Mexico
E-mail: luis@matem.unam.mx

E. Morales-Amaya,
Instituto de Matemáticas,
Universidad Nacional Autonóma de México,
Mexico
E-mail: efren@cimat.mx

https://doi.org/10.1112/S0025579310000379 Published online by Cambridge University Press