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A note on the Witt group and the KO-theory of complex Grassmannians

by

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Abstract

For a complex Grassmannian X, there is an isomorphism

$$W^*(X) \cong KO^{2*}(X(\mathbb{C}))/KU^{2*}(X(\mathbb{C}))$$

between Balmer's Witt group and the quotient of topological K-theories.

Key Words: Grassmannian, Witt group, K-theory

Mathematics Subject Classification 2000: Primary 19G12, 55N15; Secondary 55N20

1. Introduction

Let X be a smooth variety over a field k with $1/2$. The Witt group $W(X)$ is the quotient of the Grothendieck group of vector bundles with quadratic forms over X, by the subgroup generated by bundles V with quadratic forms which admit Lagrangian subbundles E (i.e., E is its own orthogonal complement in V).

Hence when $k = \mathbb{C}$, we have a natural map (see §5 for details)

$$q : W(X) \to KO^0(X(\mathbb{C}))/KU^0(X(\mathbb{C})).$$

Here $KO^0(-)$ and $KU^0(-)$ are the usual (topological) real and complex K-theories, and $KO^0(-)/KU^0(-)$ means the cokernel of the real restriction map $r : KU^0(-) \to KO^0(-)$. One purpose of this paper is to show that the map q is an isomorphism when $X = M_{m,n}$ is the complex Grassmannian of m-planes in an $m+n$-plane. Moreover, the map q extends to an isomorphism of graded modules

$$W^*(X) \cong KO^{2*}(X(\mathbb{C}))/KU^{2*}(X(\mathbb{C}))$$

where $W^*(X)$ is Balmer's Witt group with $W^0(X) = W(X)$.

The right hand quotient of the above isomorphism is computed explicitly by Hara and Kono-Hara [Ha],[Ko-Ha] using the Atiyah-Hirzebruch spectral sequence. Here the computation of Sq^2 is most important.

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On the other hand, $W^*(X)$ has been computed by Balmer-Calmès [Ba-Ca2] in completely general form, using $m \times n$-framed *even* Young diagrams. However we can also get the results (for $k = \mathbb{C}$) using the Pardon-Gille and Balmer-Walter spectral sequences, and using the computation of Sq^2 by Kono-Hara. Another purpose of this paper is to explain the relation between the results of Balmer-Calmès and those of Kono-Hara.

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2. KO-theory

We explain the KO-theory of Grassmannians according to Hara [Ha] and Kono-Hara [Ko-Ha]. Let $M_{m,n}$ be the complex Grassmannian of m-planes in $\mathbb{C}^{m+n}$. Then there is a homeomorphism

$$M_{m,n} \cong U(m+n)/(U(m) \times U(n)).$$

By using the Serre spectral sequence induced from the fiber sequence

$$U(m+n)/(U(m) \times U(n)) \to BU(m) \times BU(n) \to BU(m+n),$$

we get quite easily for any field K that the cohomology is

$$H^*(M_{m,n};K) \cong H^*(BU(m) \times BU(n);K)/H^*(BU(m+n);K).$$

The cohomology $H^*(BU(N);K)$ is a polynomial algebra generated by Chern classes. Hence we have (§2 in [Ko-Ha])

$$H^*(M_{m,n};K) \cong K[a_1,...,a_m,b_1,...,b_n]/(c_1,...,c_{m+n}), \tag{2.1}$$

$$|a_i| = |b_i| = |c_i| = 2i$$

where a_i, b_j, c_k are Chern classes induced from maps in the above fibering, and $c_i = \sum a_{i-j} b_j$. (For other treatments, see [Fu], [La] and the arguments in §4 below.) Note that $H^*(M_{m,n};K)$ is multiplicatively generated by $a_1,...,a_m$ (or $b_1,...,b_n$).

Recall that the coefficient rings of (topological) KO^*-theory and KU^*-theory are (e.g., see §1 in [Ha])

$$KO^* \cong \mathbb{Z}[\mu,\mu^{-1},\eta,w]/(2\eta,\eta^3,w^2-4\mu,w\eta),$$

$$KU^* \cong \mathbb{Z}[\beta,\beta^{-1}]$$

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with $|\mu| = -8, |w| = -4, |\eta| = -1$ and $|\beta| = -2$. To compute $KO^*(M_{m,n})$, we consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*'} \cong H^*(M_{m,n}; KO^{*'}) \Longrightarrow KO^{*+*'}(M_{m,n}).$$

It is well known that the first differential is ((3.1) in [Ha], [Fuj])

$$d_2(\lambda \otimes x) = \lambda\eta \otimes Sq^2(\bar{x}), \quad \lambda \in KO^*$$

where $\bar{x} \in H^*(M_{m,n}; \mathbb{Z}/2)$ is the mod 2 reduction of x.

The Squaring operation is well known from the Wu formula ((3.2) in [Ha], see also Lemma 6.5 in [Ya])

$$Sq^2(a_1) = a_1^2, \ \ Sq^2(a_{2i}) = a_{2i+1} + a_{2i}a_1, \ \ Sq^2(a_{2i+1}) = a_{2i+1}a_1 \ \ for\ i \geq 1.$$

Note that $Sq^2Sq^2 = Sq^3Sq^1$ from the Adem relation. So $Sq^2Sq^2(x) = 0$ in $H^*(M_{m,n}; \mathbb{Z}/2)$ (in fact $H^{odd}(M_{m,n}; \mathbb{Z}/2) = 0$). Let us write

$$H(m,n) = H(H^*(M_{m,n}; \mathbb{Z}/2); Sq^2)$$

for the homology with the differential Sq^2. Kono and Hara compute this homology.

Theorem 2.1 *(Proposition 2 in [Ko-Ha]) Let $B(k,l)$ be the graded algebra*

$$B(k,l) = \mathbb{Z}/2[a_2^2, ..., a_{2k}^2, b_2^2, ..., b_{2l}^2)/(c_2^2, ..., c_{2k+2k}^2).$$

Then we have an isomorphism

$$H(m,n) \cong \begin{cases} B(k,l) & if\ (m,n) = (2k,2k),\ (2k+1,2l),\ (2k,2l+1) \\ B(k,l) \oplus B(k,l)\{a_{2k+1}b_{2l}\} & if\ (m,n) = (2k+1,2l+1). \end{cases}$$

They also proved the following result.

Theorem 2.2 *(Propositon 4 in [Ko-Ha]) The Atiyah-Hirzebruch spectral sequence for $KO^*(M_{m,n})$ collapses from the $E_3^{*,*'}$-term.*

For the proof of this theorem, Kono and Hara used the natural maps $U(n) \to Sp(n)$ and $Sp(n) \to U(2n)$. Let us write $N_{m,n} = Sp(m+n)/(Sp(m) \times Sp(n))$. Then the above natural maps induce maps

$$M_{m,n} \xrightarrow{q} N_{m,n} \xrightarrow{c'} M_{2m,2n}.$$

The cohomology is also computed as in the case $U(n)$,

$$H^*(N_{n,m}; \mathbb{Z}) \cong \mathbb{Z}[q_1, ..., q_m, r_1, ..., r_n]/(s_1, ..., s_{m+n}),$$

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$$|q_i| = |r_i| = |s_i| = 4i$$

where $s_i = \sum q_{i-j} r_j$ with $q^* q_i = a_i^2$ and $c'^* a_{2i} = q_i$ $(c'^*(a_{2i-1}) = 0)$.

We consider the Atiyah-Hirzebruch spectral sequence for $KO^*(N_{m,n})$. It collapses from the $E_2^{*,*'}$-term. Indeed $d_r(x) = 0$ for $x \in H^{4*}(N_{n,m};\mathbb{Z})$ and $r \geq 2$, because it can hit only the trivial group from $deg(d_r) = (r,-r+1)$. In fact $KO^{odd} = KO^{8*-1}$ and $H^*(N_{m,n};\mathbb{Z}) = H^{4*}(M_{m,n};\mathbb{Z})$. Hence we see that

$$grKO^*(N_{m,n}) \cong KO^* \otimes H^*(N_{m,n};\mathbb{Z}).$$

Proof of Theorem 2.2.: By the naturality of the spectral sequence, the maps q^*, c'^* are defined as maps of spectral sequences. The differential $d_r(a_i^2) = 0$ for all $r > 1$, because $q^* q_i = a_i^2$ and $Sq^2(a_i^2) = 0$. The fact that $c'^* a_{2i} = q_i$ implies that $a_{2i}^2 \neq 0$ in $E_\infty^{*,*'}$ and moreover each nonzero element in $B(k,l)$ is also nonzero in $E_\infty^{*,*'}$. If $d_r(a_{2k+1} b_{2l}) \neq 0$, then it is contained in $B(k,l)$ by dimensional reasons ; this is a contradiction to the preceding result. □

Let Y be a topological space (e.g., a finite dimensional CW-complex). We have the following well known (Bott) exact sequence ((1.1) in [Ha], (3.4) in [At])

$$(1) \quad ... \to KO^{*+1}(Y) \overset{\times\eta}{\to} KO^*(Y) \overset{c}{\to} KU^*(Y) \overset{r\cdot\beta^{-1}}{\to} KO^{*+2}(Y) \to ...$$

where c is the complexification map and r is the real restriction map. The map $\beta^{-1} : KU^*(Y) \to KU^{*+2}(Y)$ is an isomorphism. Let us write

$$KO^*(Y)/KU^*(Y) = KO^*(Y)/(rKU^*(Y)) = KO^*(Y)/(r\beta^{-1}KU^*(Y)).$$

Then we have

$$KO^*(Y)/KU^*(Y) \cong KO^*(Y)/(Ker\,\eta) \cong Im(\eta)(KO^*(Y))$$

(which is 2-torsion since η is), e.g., $KO^*/KU^* \cong \mathbb{Z}/2\{1,\eta\}[\mu,\mu^{-1}]$.

Recall the Atiyah-Hirzebruch spectral sequence for $KO^*(M_{m,n})$. From Theorem 2.2, we have the following isomorphism for its $E_\infty^{*,*'}$-term

$$E_\infty^{2*,odd} \cong E_\infty^{2*,8*'-1} \cong \mathbb{Z}/2\{\eta\}[\mu,\mu^{-1}] \otimes H(m,n).$$

Hereafter, in this paper, let us write

$$KO^{2*}(Y)/KU^{2*}(Y) = \oplus_{r\in\mathbb{Z}/4} KO^{2r}(Y)/KU^{2r}(Y)$$

(while $KO^*(Y)$ usually means $\oplus_{r\in\mathbb{Z}} KO^r(Y)$). Taking the degree modulo 8 in $H(m,n)$, we have a ring isomorphism :

Corollary 2.3 $KO^{2*}(M_{m,n})/KU^{2*}(M_{m,n}) \cong H(m,n)$.

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3. Balmer's Witt group.

For a smooth scheme X over a field k with $1/2 \in k$, let $W(X)$ denote the Witt group of X. Balmer defined the periodic Witt group $W^i(X) \cong W^{i+4}(X)$, $(i \in \mathbb{Z})$ with $W^0(X) = W(X)$ (for details, see [Ba]).

Balmer and Walter ((1) in [Ba-Wa]) define the Gersten-Witt complex

$$0 \to W(k(X)) \to \oplus_{x \in X^{(1)}} W(k(x)) \to \ldots \to \oplus_{x \in X^{(n)}} W(k(x)) \to 0.$$

Let $H^*(W(X))$ denote the cohomology group of the above cochain complex, with $W(k(X))$ in degree 0. Then Balmer-Walter constructed the spectral sequence (Theorem in [Ba-Wa])

$$E(BW)_2^{r,t} \cong \begin{cases} H^r(W(X)) \ (t = 0 \ mod(4)) & \Longrightarrow W^{r+t}(X). \\ 0 \ \ (t \neq 0 (mod(4)) \end{cases}$$

By the affirmative answer to the Milnor conjecture on quadratic forms by Orlov-Vishik-Voevodsky (Theorem 4.1 in [Or-Vi-Vo]), we have an isomorphism of graded rings $H^*_{et}(k(x);\mathbb{Z}/2) \cong grW(k(x))$. Using this fact, we can reinterpret the Pardon-Gille spectral sequence (Corollary 0.13 in [Pa] and [Gi]) as a spectral sequence

$$E(P)_2^{r,s} \cong H^r_{Zar}(X; H^s_{\mathbb{Z}/2}) \Longrightarrow H^r(W(X)) \cong E(BW)_2^{r,4t}$$

so that the differential d_r has degree $(1, r-1)$ for $r \geq 2$. Here $H^s_{\mathbb{Z}/2}$ is the Zariski sheaf associated to the presheaf $V \mapsto H^s_{et}(V;\mathbb{Z}/2)$ for any open subscheme V of X.

The above sheaf cohomology $H^r_{Zar}(X; H^s_{\mathbb{Z}/2})$ is related to the motivic cohomology $H^{*,*'}(X;\mathbb{Z}/2) = \oplus_{r,s \in \mathbb{Z}} H^{r,s}(X;\mathbb{Z}/2)$ (for details, see [Vo1-3]) as follows.

Let $\tau \in H^{0,1}(Spec(\mathbb{C});\mathbb{Z}/2) \cong \mathbb{Z}/2$ be a generator. (Hence $H^{*,*'}(Spec(\mathbb{C});\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau]$.) Then we get a long exact sequence from the solution of the Beilinson-Lichtenbaum conjecture (Theorem 1.3 in [To], Lemma 2.4 (5) in [Or-Vi-Vo]),

$$\ldots \to H^{m,n-1}(X;\mathbb{Z}/2) \overset{\times\tau}{\to} H^{m,n}(X;\mathbb{Z}/2)$$

$$\to H^{m-n}_{Zar}(X; H^n_{\mathbb{Z}/2}) \to H^{m+1,n-1}(X;\mathbb{Z}/2) \overset{\times\tau}{\to} \ldots$$

All elements in the above cohomology groups are 2-torsion, and we have the following additive isomorphisms :

Lemma 3.1 $E(P)_2^{m-n,n} \cong H^{m-n}_{Zar}(X; H^n_{\mathbb{Z}/2}) \cong$

$$H^{m,n}(X;\mathbb{Z}/2)/(Im(\tau)) \oplus Ker(\tau)|_{H^{m+1,n-1}(X;\mathbb{Z}/2)}.$$

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It is well known that $CH^m(X) \cong H^{2m,m}(X;\mathbb{Z})$ (Corollary 2 in [Vo2]). Since $H^{2m+1,m}(X;\mathbb{Z}) = 0$ (Corollary 2.3 in [Vo1]), from the Bockstein exact sequence, we have an isomorphism $H^{2m,m}(X;\mathbb{Z}/2) \cong H^{2m,m}(X;\mathbb{Z})/2$. In particular,

$$E(P)_2^{m,m} \cong H^m_{Zar}(X;H^m_{\mathbb{Z}/2}) \cong H^{2m,m}(X;\mathbb{Z}/2) \cong CH^m(X)/2.$$

In $CH^*(X)/2$ or more generally in $H^{*,*'}(X;\mathbb{Z}/2)$, we can define the cohomology operation Sq^i of Voevodsky (§9 in [Vo3], §3.3 in [Vo1]) (or Brosnan [Br] for $CH^*(X)/2$). This Sq^i is compatible with that in the usual (topological) mod 2 cohomology via the realization map $t_{\mathbb{C}}$ when $k \subset \mathbb{C}$ (§3.4 in [Vo1]). In fact, Voevodsky essentially used these cohomology operations to prove the Milnor conjecture [Vo1]. Moreover $Sq^2Sq^2 = \tau Sq^3Sq^1$ (Theorem 10.2 in [Vo3]). Then Totaro proved

Lemma 3.2 *(Theorem 1.1 in [To]) If $x \in E(P)_2^{m,m} \cong CH^m(X)/2$, then $d_2(x) = Sq^2(x)$.*

Now we consider the case $X = M_{m,n}$ and $k = \mathbb{C}$. (See also §6 in [Ya].) Since $M_{m,n}$ is cellular, we see that

$$H^{2*,*'}(M_{m,n};\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes CH^*(M_{m,n})/2 \cong \mathbb{Z}/2[\tau] \otimes H^{2*}(M_{m,n};\mathbb{Z}/2).$$

So we get

$$H^{2*,*'}(M_{m,n};\mathbb{Z}/2)/(Im(\tau)) \cong H^{2*}(M_{m,n};\mathbb{Z}/2), \quad Ker(\tau)|_{H^{*,*'}(M_{m,n};\mathbb{Z}/2)} = 0.$$

From Lemma 3.1, we have (if $m \neq n$,then $E(P)_2^{m,n} = 0$)

$$E(P)_2^{*,*'} \cong E(P)_2^{*,*} \cong H^{2*}(M_{m,n};\mathbb{Z}/2).$$

From the result of Totaro, we have

$$E(P)_3^{*,*'} \cong H(H^{2*}(M_{m,n};\mathbb{Z}/2);Sq^2) = H(m,n).$$

Since $deg(d_r) = (1,r-1)$ and $E(P)_3^{*,*'} = 0$ for $* \neq *'$, we get $d_r = 0$ for $r > 2$. Hence this spectral sequence collapses from the $E_3^{*,*'}$-term, i.e., $E(P)_3^{*,*'} \cong E_\infty^{*,*'}$. (Note that the above computations are valid also for any cellular variety over an algebraically closed field k.)

The Balmer-Walter spectral sequence also collapses, that is, we will prove that

$$E(P)_\infty^{*,*} \cong E(BW)_2^{*,r} \cong E(BW)_\infty^{*,r}, \quad r = 0\, mod(4).$$

For this we consider the $Sp(n)$-version of the above arguments. We consider spectral sequences for $N_{m,n}$ the quotient of algebraic groups $Sp(m+n)/Sp(m) \times Sp(n)$

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over $\mathbb{C}$. Since $N_{m,n}$ is a cellular variety, we can use the same arguments as for $M_{m,n}$. By dimensional reasons, the Pardon-Gille and Balmer-Walter spectral sequences collapse from the E_2-terms. (Note $H^*(N_{m,n};\mathbb{Z}/2) = 0$ for $* \neq 0\ mod(4)$.) That is

$$grW^*(N_{m,n}) \cong E(BW)^{*,r}_{\infty} \cong H^{2*}(N_{m,n};\mathbb{Z}/2).$$

(Note that $W^*(N_{m,n})$ is 2-torsion since $W^*(Spec(\mathbb{C})) \cong \mathbb{Z}/2$.) Now the arguments of the proof of Theorem 2.2 also work. Thus we see that the Balmer-Walter spectral sequence collapses for $M_{m,n}$.

Theorem 3.3 *We have additive isomorphisms*

$$W^*(M_{m,n}) \cong H(m,n) \cong KO^{2*}(M_{m,n})/KU^{2*}(M_{m,n}).$$

4. Young diagrams and Witt groups

In this section we recall the result of Balmer-Calmès [Ba-Ca1,2], and consider its relation with the result of Kono-Hara.

The cohomology $H^*(M_{m,n};\mathbb{Z})$ (or $CH^*(M_{m,n})$) is also computed by induction on n,m. In fact the following exact sequence

$$(4.1) \quad \ldots \to H^{*-|a_m|}(M_{m,n-1};\mathbb{Z}) \xrightarrow{g_*} H^*(M_{m,n};\mathbb{Z}) \xrightarrow{f^*} H^*(M_{m-1,n};\mathbb{Z}) \xrightarrow{\partial} \ldots$$

becomes split since $\partial = 0$, because $H^{odd}(M_{m-1,n};\mathbb{Z}) = H^{odd}(M_{m,n-1};\mathbb{Z}) = 0$ (by inductive assumption). Here g_* is the Gysin map for the embedding $M_{m,n-1} \subset M_{m,n}$. The map f^* is induced by the maps

$$M_{m-1,n} \xleftarrow{proj.} (M_{m,n} - M_{m,n-1}) \subset M_{m,n},$$

where $proj.$ is the projection of the vector bundle and hence $(proj.)^*$ is an isomorphism. (See §14 in [Fu] or Laksov [La], [Ba-Ca2]). Here note that $g_*(x) = a_m x$. Assuming the results for $H^*(M_{m-1,n};\mathbb{Z})$ and $H^*(M_{m,n-1};\mathbb{Z})$ by induction, we can compute $H^*(M_{m,n};\mathbb{Z})$.

It is well known that the cohomology of $M_{m,n}$ is stated also by using Young diagrams (see §14.6 in [Fu]). The $m \times n$-framed partition $\lambda = (\lambda_1,\ldots,\lambda_d)$ means

$$n \geq \lambda_1 \geq \ldots \geq \lambda_d \geq 1 \quad and \quad m \geq d.$$

The partition λ corresponds to a Young diagram, consisting of λ_i boxes in the i-th row from the top, lined up on the left. Then the $m \times n$-framed Young diagrams with $s = |\lambda| = \lambda_1 + \ldots + \lambda_d$ form a basis of $H^s(M_{m,n};\mathbb{Z})$, namely $H^s(M_{m,n};\mathbb{Z}) \cong \oplus_{s=|\lambda|}\mathbb{Z}$ (e.g., Proposition 14.65 in [Fu]).

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The Young diagram for the conjugate partition $\tilde{\lambda}$ of λ is obtained by interchanging rows and columns in the diagram. Let us say that a framed Young diagram λ is *even* if all its segments have even length, which are strictly inside of the frame, namely all $\lambda_i - \lambda_{i+1}$ for $1 \le i \le d-1$, and all $\tilde{\lambda}_i - \tilde{\lambda}_{i+1}$ for $1 \le i \le \tilde{d}-1$ are even.

The result by Balmer and Calmès is stated by using even Young diagrams. We can consider the generalized Witt group $W^i(X;L)$ for $i \in \mathbb{Z}/4$ and for a line bundle L on X such that the usual Witt group $W^i(X) = W^i(X;O_X)$. Let us write the total Witt group

$$W^{total}(X) = \oplus_{i\in\mathbb{Z}/4,\ L\in Pic(X)/2} W^i(X;L).$$

When $X = M_{m,n}$, we see that $Pic(X)/2 \cong \mathbb{Z}/2$, which is parametrized by $t(\lambda)$ half of the perimeter of λ.

Theorem 4.1 *(Main Theorem in [Ba-Ca2]) Let $k = \mathbb{C}$. The total Witt group $W^{total}(M_{m,n})$ has a $\mathbb{Z}/2$-basis indexed by even Young diagrams λ. The corresponding base $[\lambda]$ is in $W^{|\lambda|}(M_{m,n}, t(\lambda))$.*

Balmer and Calmès prove their theorem by inspecting the following (long) exact sequence. The Gysin and the boundary maps are defined as the maps for $W^{total}(-)$. (Gysin maps are constructed by Calmès and Hornbostel in [Ca-Ho], also see §4.1 in [Ne].) Let $g : Z \subset X$ be a regular embedding of $codim = c$, and $U = X - Z$. Morover we assume that there is a map $U \to Z$ with affine fibers. Let ω_g be the relative canonical bundle (for the definition see [Ba-Ca1]). Then we have the natural exact sequence ((11) in [Ba-Ca1])

$$... \to W^{*-c}(Z, \omega_g \otimes L|_Z) \xrightarrow{g_*} W^*(X,L) \xrightarrow{f^*} W^*(U, L|_U) \xrightarrow{\partial} ...$$

In general $\partial \neq 0$. In fact, when $X = M_{m,n}$ and $Z = M_{m,n-1}$, we have (Figures 4-6 in [Ba-Ca2])

$$g_*([\lambda]) = \begin{cases} [(1,..,\overset{m}{1}) + \lambda] \ \ if\ m-d : even \\ 0 \quad otherwise, \end{cases} \qquad f^*([\lambda]) = \begin{cases} [\lambda] \ \ if\ d < m \\ 0 \quad otherwise, \end{cases}$$

$$\partial([\lambda]) = \begin{cases} [\lambda - (1,...,\overset{d}{1})] \ \ if\ \lambda_d : odd \\ 0 \quad otherwise. \end{cases}$$

Now we explain the relation between results of Kono-Hara and Balmer-Calmès. For a Young diagram λ, we can define the Schur polynomial (e.g. see §14.5 in [Fu]) by

$$\Delta_\lambda(b) = det(b_{\lambda_i + j - i}) \in K[a_1,...,a_m, b_1,...,b_n]/(c_1,...,c_{m+n}) \quad (2.1).$$

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It is known (Lemma 14.5.1 in [Fu]) that $\Delta_\lambda(b) = \Delta_{\tilde{\lambda}}(a)$. Hence we have $\Delta_{(k)}(b) = b_k$ and $\Delta_{(\tilde{k})=(1,...,1)}(b) = a_k$. Moreover we see by the above definition of Δ_λ that,

$$\Delta_\lambda(b) = b_{\lambda_1}...b_{\lambda_d} \quad mod(F_{>\lambda}).$$

Here $F_{>\lambda}$ is the filtration of elements $b_{\lambda'_1}....b_{\lambda'_{d'}}$ with $\lambda' > \lambda$ in lexicographical order.

Using (2.1), we see that the above $\Delta_\lambda(b)$ form a basis $[\lambda]$ of $H^*(M_{m,n};\mathbb{Z})$. However we can also get this by induction using the short exact sequence (4.1) with

$$(4.2) \quad g_*([\lambda]) = [(1,...,1) + \lambda] = [(1+\lambda_1,...,1+\lambda_d,1,...,1)].$$

(Indeed, $\Delta_{g_*(\lambda)}(b) = a_m \cdot \Delta_\lambda(b)\ mod(F_{>g^*(\lambda)})$.) The induced map f^* is given by $f^*(\lambda) = \lambda$ for $d < m$, and $= 0$ for $d = m$.

Let us say that a framed Young diagram λ is *strongly even* if all its segments have even length, namely all λ_i, $\tilde{\lambda}_i$ are even. Then its Schur polynomial is written

$$\Delta_\lambda(b) = (b^2_{\lambda_1} b^2_{\lambda_3}...b^2_{\lambda_{d-1}}) \quad mod(F_{>\lambda}),$$

where $\lambda_1,...,\lambda_{d-1}$ are even.

Hereafter, we only consider the case $W^*(X;O_X) = W^*(X)$ but not $W^{total}(X)$. Hence we assume that *half* of the *perimeter* is always zero, i.e., $t(\lambda) = 0$. Then we easily see that when m or n is even, all even Young diagrams are strongly even. On the other hand $B(k,l)$ is additively generated by elements $b^2_{2i_1}...b^2_{2i_s}$ for $i_1 \leq ... \leq i_s$ from Theorem 2.1 by Kono-Hara. Hence if m or n is even, we have an isomorphism from $W^*(M_{m,n})$ to $B(k,l) = H(m,n)$ defined by $[\lambda] \mapsto \Delta_\lambda(b)$.

Next we consider the case $(m,n) = (2k+1,2l+1)$. Each even $m \times n$-framed diagram λ with $t(\lambda) = 0$ is easily seen to be strongly even $[\lambda^{se}]$ or $[\Gamma\lambda^{se}]$ which is defined as

$$(4.3) \quad [(2l+1,1,...,\overset{m}{1}) + (0,\lambda^{se})] = [(2l+1,\lambda^{se}_1+1,....,\lambda^{se}_{d^{se}}+1,1...,\overset{m}{1})].$$

Note that $\mu = (2l+1,1,..,1)$ is even but not strongly even, and $|\mu| = odd$, $t(\mu) = 0$. We also know that the $[\lambda^{se}]$ form the $\mathbb{Z}/2$-basis of $B(k,l)$ in Theorem 2.1. Note that

$$\Delta_\mu(b) = b_{2l+1}a_{2k} = a_{2k+1}b_{2l} \quad mod(F_{>\mu}).$$

Hence the $[\Gamma\lambda^{se}]$ form a basis of $B(k,l)\{a_{2k+1}b_{2l}\}$. Therefore we see that the even $m \times n$-framed Young diagrams form a basis of

$$B(k,l) \oplus B(k,l)\{a_{2k+1}b_{2l}\} = H(m,n).$$

Thus we can explain the relation between Kono-Hara and Balmer-Calmès

Theorem 4.2 *The map defined by* $[\lambda] \mapsto \Delta_\lambda(b)$ *gives an additive isomorphism* $W^*(M_{m,n}) \cong H(m,n)$.

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5. The map from Witt groups to K-theory

Let Y be a topological space. Let us write by $W(Y)^{top}$ the topological version of the Witt group. Namely, $W(Y)^{top}$ is the quotient of the Grothendieck group of complex vector bundles with quadratic forms over Y, by the subgroup generated by complex vector bundles which admit Lagrangian subbundles E (i.e., $E = E^{\perp}$ for the bilinear form defined from the quadratic form). Hence for the realization map $t_{\mathbb{C}}$ (whenever $k \subset \mathbb{C}$), we see that

$$t_{\mathbb{C}} : W(X) \to W(X(\mathbb{C}))^{top}.$$

Let V be an n-dimensional complex vector bundle on Y with a quadratic form q. Such a complex bundle V is classified by a homotopy class of maps $Y \to BO_n(q)$. Here $O_n(q)$ is the subgroup of $U(n)$ generated by elements which preserve q, and $BO_n(q)$ is its classifying space.

Over $\mathbb{C}$, any q is isomorphic to q_I where

$$q_I(x) = x_1^2 + ... + x_n^2$$

is the standard quadratic form. So $O(q)$ is isomorphic to $O(q_I)$. Note that $A \in O_n(q_I) \subset U(n)$ is represented by a matrix A such that $\bar{A}^t A = A^t(Id.)A = Id.$, where A^t is the transpose of A. Hence $O(q_I)$ is the real (usual) orthogonal group $O(n)$. Thus for each V, we have a map $Y \to BO(n)$, which of course induces an element in $KO^0(Y)$.

Let E be an m-dimensional Lagrangian subbundle of a complex bundle V of dimension $2m$, i.e., $E = E^{\perp}$ for a nonsingular symmetric bilinear form $b(x,y)$. Take a basis $\{e_1,...,e_m\}$ of E and a basis $\{e_1,...,e_{2m}\}$ of V such that $b(e_i,e_{m+j}) = \delta_{i,j}$. Then the linear form is written $b(x,y) = \sum x_i y_{m+i} + x_{m+i} y_m$ for $x = \sum x_j e_j$ and $y = \sum y_j e_j$. Hence the induced quadratic form is the split form

$$q_S(x) = \frac{1}{2} b(x,x) = x_1 x_{m+1} + ... + x_m x_{2m}.$$

Let us write by $KO(q_I)^0(Y)$ (resp. $KO(q_S)^0(Y)$) the K-theory defined by the quadratic form q_I (resp. q_S). Then we have the following diagram

$$\begin{array}{ccccc} KU^0(Y) & \xrightarrow{H} & KO(q_S)^0(Y) & \longrightarrow & W^{top}(Y) \to 0 \\ {\scriptstyle =}\downarrow & & {\scriptstyle \cong}\downarrow & & \downarrow \\ KU^0(Y) & \xrightarrow{r} & KO^0(Y) = K(q_I)^0(Y) & \longrightarrow & KO^0(Y)/KU^0(Y) \to 0 \end{array}$$

where $H(L) = L \oplus \bar{L}$ with the quadratic form q_S for $L \in KU^0(Y)$.

The following fact may be well known to experts, but we give here an elementary proof.

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Lemma 5.1 *Let Y be a topological space. Then the above diagram is commutative, and hence $W(Y)^{top} \cong KO^0(Y)/KU^0(Y)$. Therefore we have a natural map (for smooth schemes X over $k \subset \mathbb{C}$)*

$$q : W(X) \to KO^0(X(\mathbb{C}))/KU^0(X(\mathbb{C})).$$

Proof: Let V be a complex $2m$-dimensional bundle with q_S, and E be an m-dimensional Lagrangian subbundle of V. Let $g = g_{ij}|U_i \cap U_j$ be a transition function for the vector bundles V and E, for open sets $U_i \subset Y$. Let $A = A_{ij}$ be the corresponding matrix in $O_{2m}(q_S)$. Namely, $\bar{A}^t A = I_{2m}$ and $A^t I' A = I'$ where $I' = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$. Take $K \in U(m)$ such that $K^t K = I'$, e.g., $K = \frac{1}{\sqrt{2}} \begin{pmatrix} I_m & I_m \\ iI_m & -iI_m \end{pmatrix}$. Then we see that $A' = KAK^{-1} \in O_{2m}(q_I)$, indeed

$$(KAK^{-1})^t (KAK^{-1}) = K^{-1t} A^t I' A K^{-1} = K^{-1t} I' K^{-1} = I.$$

Let $L \in U(m)$ be the matrix representing a transition function on E. Then the matrix representing the transition function for V is written as

$$A = \begin{pmatrix} L & M \\ 0 & N \end{pmatrix} \in U(2m) \qquad with \;\; A^t I' A = I'.$$

The fact that $A \in U(2m)$ implies $M = 0$, and the fact that $A^t I' A = I'$ implies $N = \bar{L}$. So $A = \begin{pmatrix} L & 0 \\ 0 & \bar{L} \end{pmatrix}$. This means that

$$W^{top}(Y) \cong KO(q_S)^0(Y)/H(KU^0(Y)).$$

Moreover, we can compute

$$A' = K \begin{pmatrix} L & 0 \\ 0 & \bar{L} \end{pmatrix} K^{-1} = \frac{1}{2} \begin{pmatrix} L + \bar{L} & -iL + i\bar{L} \\ iL - i\bar{L} & L + \bar{L} \end{pmatrix}.$$

On the other hand, for $L \in U(m)$, its real restriction matrix is given by

$$L \mapsto \begin{pmatrix} Re(L) & iIm(L) \\ -iIm(L) & Re(L) \end{pmatrix} \in O(2m)$$

where $Re(L)$ (resp. $Im(L)$) is the real part (resp. imaginary part) of L. Thus we see the commutativity of the left hand side of the above diagram. □

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In the $\mathbb{A}^1$-homotopy category, Hornbostel proved that $W^*(-)$ is represented as a $\mathbb{P}^1$-spectrum and also proved (Theorem 5.7 in [Ho]) that

$$W^{-i}(X) \cong W((\mathbb{P}^1)^{\wedge i} \wedge X).$$

In topology, $\mathbb{P}^1 \cong S^2$ and we have

$$KO^{-2i}(Y) \cong KO^0((S^1)^{\wedge 2i} \wedge Y) \cong KO^0((S^2)^{\wedge i} \wedge Y).$$

Therefore we can define the natural map

$$q^* : W^*(X) \to KO^{2*}(X(\mathbb{C}))/KU^{2*}(X(\mathbb{C})) \quad for\ all * \in \mathbb{Z}/4.$$

To simplify notation, let us write $KO/U^*(Y) = KO^{2*}(Y)/KU^{2*}(Y)$. Of course, the map q^* is an isomorphism when $X = pt.$, in fact an odd dimensional $\mathbb{C}$-vector space generates

$$W^*(\mathbb{C}) = W^0(\mathbb{C}) \cong \mathbb{Z}/2 \cong KO/U^0(pt.) = KO/U^*(pt.).$$

We will see that the map q^* induces an isomorphism also for $X = M_{m,n}$ (and $k = \mathbb{C}$).

First we consider the cases $(m,n) = (2k,2l)$. For this case we still know

$$W^*(X) = W^0(X), \quad KO/U^*(X(\mathbb{C})) = KO/U^0(X(\mathbb{C})).$$

Let $\tilde{g} : M_{m,n-2} \subset M_{m,n-1} \subset M_{m,n}$ be the embedding and $\tilde{f}^* : W^*(M_{m-2,n}) \to W^*(M_{m,n})$ be the map induced by the maps

$$M_{m-2,n} \overset{proj.}{\leftarrow} (M_{m-1,n} - M_{m-1,n-1}) \subset M_{m-1,n} \overset{proj.}{\leftarrow} (M_{m,n} - M_{m,n-1}) \subset M_{m,n}.$$

Then $\tilde{g}_*([\lambda]) = [(2,...,2) + \lambda]$ in $W^{total}(M_{m,n})$ by using (4.2) twice. If half of the perimeter $t([\lambda]) = 0$, then $t([(2,...,2) + \lambda]) = 0$. Hence the Gysin map is defined as the map $\tilde{g}_* : W^0(X) \to W^0(X)$.

The normal bundle ν_1 for $M_{m,n-1} \subset M_{m,n}$ is the dual of the universal bundle, and its first Chern class $c_1(\nu_1) = -b_1$. Hence the first Chern class $c_1(\nu_2) = 0\ mod(2)$ for $\nu_2 : M_{m,n-2} \subset M_{m,n}$ and $dim(\nu_2) = 0\ mod(4)$. So the second Stiefel-Whitney class $w_2(\nu_2) = 0$. Therefore ν_2 has a spin structure, and hence KO-orientable. This means that we have a Thom isomorphism for (topological) KO-theory and a Gysin map (Theorem 12.3 in [At-Bo-Sh]). (Note that each complex vector bundle is always KU-orientable.) In fact, we can define

$$\tilde{g}_* : KO^0(M_{m,n-2}) \overset{Thom\ iso.}{\to} KO^0(Th(\nu_2)) \overset{p^*}{\to} KO^0(M_{m,n})$$

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where $Th(\nu_2)$ is the Thom space and $p : M_{m,n} \to Th(\nu_2)$ is the quotient map. Note here that $\tilde{g}_*(x) = a_m^2 x$ in $KO/U^*(M_{m,n})$.

Moreover we have the following commutative diagram. Indeed, the Gysin maps in Witt groups are also defined by using the Thom spaces and the fact $W^{*-dim(L)}(X;det(L)) \cong \tilde{W}^*(Th(L))$ (see §4.1 in [Ne]).

$$\begin{array}{ccccccc} 0 \to W^0(M_{m,n-2}) & \xrightarrow{\tilde{g}_*} & W^0(M_{m,n}) & \xrightarrow{\tilde{f}^*} & W^0(M_{m-2,n}) \to 0 \\ \downarrow q_1 & & \downarrow q_2 & & \downarrow q_3 \\ 0 \to KO/U^0(M_{m,n-2}) & \xrightarrow{\tilde{g}_*} & KO/U^0(M_{m,n}) & \xrightarrow{\tilde{f}^*} & KO/U^0(M_{m-2,n}) \to 0. \end{array}$$

The exactness of rows follow from Theorem 2.1, 4.1, 4.2 and Corollary 2.3. (There is a bijection $H^*(M_{k,l};\mathbb{Z}/2) \to W^{2*}(M_{m,n})$ defined by $[\lambda] \mapsto [\lambda']$ where $[\lambda]$ and $[\lambda']$ are similar of the one to two, and there is a ring isomorphism $H^*(M_{k,l};\mathbb{Z}/2) \to KO/U^{2*}(M_{m,n})$ defined by $a_i \mapsto a_{2i}^2$.) Assuming that q_1 and q_3 are isomorphisms (in Theorem 4.2), we see that q_2 is also an isomorphism (in Theorem 4.2), by the five lemma.

The case m or n even follows from the above result by the naturality of the map g^* for the embedding $g : M_{m,n-1} \subset M_{m,n}$.

The case $(m,n) = (2k+1, 2l+1)$ is proved as follows. For this case, we still know

$$W^*(X) = W^0(X) \oplus W^{2k+2l+1}(X),$$

$$KO/U^*(X(\mathbb{C})) = KO/U^0(X(\mathbb{C})) \oplus KO/U^{2k+2l+1}(X(\mathbb{C})).$$

Let $g' : M_{m-1,n-1} \to M_{m,n}$ be the embedding. Then we see $g'_*([\lambda]) = [\Gamma\lambda]$ by (4.3) (using (4.2) and its conjugate). Let ν_2 be the normal bundle for g'. Then $c_1(\nu_2) = -a_1 - b_1 = 0 \ mod(2)$ but $dim(\nu_2) = (2k+2l+1) \neq 0 \ mod(4)$. Let $i = -(2k+2l+1) \ mod(4)$ and e be a trivial complex line bundle on $M_{m-1,n-1}$. The Gysin map for KO-theories is defined for this case by

$$g'_* : KO^0(M_{m-1,n-1}) \overset{Thom\ iso.}{\to} KO^0(Th(\nu_2 \oplus e^{\oplus i})) \overset{p^*}{\to} KO^0(M_{m,n} \wedge (\mathbb{P}^1)^{\wedge i}).$$

Therefore we have the following commutative diagram

$$\begin{array}{ccc} W^0(M_{2k,2l}) & \xrightarrow[\cong]{g'_*} & W^{2k+2l+1}(M_{2k+1,2l+1}) \\ \downarrow q_1 & & q_2 \downarrow \\ KO/U^0(M_{2k,2l}) & \xrightarrow[\cong]{g'_*} & KO/U^{2k+2l+1}(M_{2k+1,2l+1}). \end{array}$$

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We see that q_2 is an isomorphism in this case.

Similarly we consider the map $f'^* : W^*(M_{2k+1,2l+1}) \to W^*(M_{2k,2l})$ induced by the maps

$$M_{m-1,n-1} \overset{proj.}{\leftarrow} (M_{m-1,n} - M_{m-2,n}) \subset M_{m-1,n} \overset{proj.}{\leftarrow} (M_{m,n} - M_{m,n-1}) \subset M_{m,n}.$$

By Theorem 4.1 (or Theorem 3.3) and Corollary 2.3, we have an isomorphism $f'^* : W^0(M_{2k+1,2k+1}) \cong W^0(M_{2k,2k})$.

This also proves that q_2 is an isomorphism for $W^0(M_{m,n}) \to KO/U^0(M_{m,n})$. Thus we can show

Theorem 5.2 *The map $q^* : W^*(M_{m,n}) \to KO^{2*}(M_{m,n})/KU^{2*}(M_{m,n})$ induces an isomorphism.*

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