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Density of orbits in complex dynamics

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Abstract. In this paper we discuss whether germs of holomorphic self maps of $\mathbb{C}^n$ can have a dense orbit. We give both a positive and a negative answer depending on how the problem is interpreted more precisely.

1. *Introduction*

Let $F : \mathbb{C}_0^m \to \mathbb{C}_0^m$ be a germ of a holomorphic map with $F(0) = 0, F'(0) = \mathrm{Id}$. Cerveau [**C**] asked whether F can have a dense orbit. More precisely, does there exist a small ball $\mathbf{B}(0,\delta) = \mathbf{B}_\delta^m = \{z \in \mathbb{C}^m : \|z\| < \delta\}$ where F is defined and a sequence $S = \{p_n\}_{n=0}^\infty \subset \mathbf{B}(0,\delta)$, $F(p_n) = p_{n+1}\ \forall n$, so that $S \cap \mathbf{B}(0,\epsilon)$ is dense in some smaller ball $\mathbf{B}(0,\epsilon), 0 < \epsilon < \delta$. Our main result is that this is not possible. We prove, in fact, a more general result. Recall that a domain Ω in $\mathbb{C}^m$ is said to have a Lipschitz boundary if we can locally write, after a holomorphic rotation of coordinates, $\Omega = \{\mathrm{Im}\, z_m < r(z_1, \ldots, z_{m-1}, \mathrm{Re}\, z_m)\}$ where $|r(x) - r(y)| \leq C\|x - y\|$; the constant C is called a Lipschitz constant for r.

MAIN THEOREM 1.1. *Let $\Omega \subset\subset \mathbb{C}^m$ be a domain with a Lipschitz boundary. Suppose that $F : \Omega \to \mathbb{C}^m$ is a holomorphic map and let $V \subset \Omega$ be a non-empty open subset. Then F does not have a dense orbit in V, i.e. there does not exist a sequence $\{p_n\}_{n\geq 0} \subset \Omega$ for which $F(p_n) = p_{n+1}$ for all n and $\{p_n\} \cap V$ is dense in V.*

The condition that Ω has Lipschitz boundary is not necessary, see Remark 5.4. However, the following theorem shows that the result is false if we have no boundary conditions.

THEOREM 1.2. *There is a bounded domain U in $\mathbb{C}^m$ which is biholomorphic to the unit ball and a holomorphic map $F : U \to \mathbb{C}^m$ with a dense orbit. So there is a dense sequence $\{p_n\}_{n=0}^\infty \subset U$ with $F(p_n) = p_{n+1}$ for all n.*

The proofs depend on two results which are of independent interest.

THEOREM 1.3. *Suppose that $\Omega \subset\subset \mathbb{C}^m$ is a domain with Lipschitz boundary. Then if $p \in \partial\Omega$ there are two not mutually exclusive possibilities:*

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(i) *the point p is in the envelope of holomorphy of Ω;*
(ii) *the point p is in a local peak set for plurisubharmonic functions, i.e. there exists a neighbourhood U of p and a continuous function $\rho : U \cap \overline{\Omega} \to \mathbb{R}$ so that $\rho < 0$ on $U \cap \Omega$ and ρ is plurisubharmonic on $U \cap \Omega$. Moreover, $\rho(p) = 0$.*

THEOREM 1.4. *Let M^m be an m-dimensional Hausdorff connected complex manifold with countable topology. Then there exists a biholomorphic map $F : \mathbf{B}^m(0,1) \to U$ from the unit ball onto an open subset $U \subset M$ so that $M \setminus U$ has zero volume.*

We prove Theorem 1.4 in §2. The proof relies heavily on the previous joint work of the first author and E. L. Stout. They proved [**FS1**] that Theorem 1.4 is valid if one replaces the ball with the polydisc of the same dimension. They also proved covering theorems by balls in arbitrary complex manifolds [**FS2**].

Theorem 1.3 is proved in §3. The two main ingredients are an estimate of the local envelope of holomorphy of a domain with Lipshitz boundary together with the result by Demailly [**D**] that Lipschitz pseudoconvex domains are hyperconvex, i.e. that they allow a bounded plurisubharmonic exhaustion function. The key result about envelopes of holomorphy is the following.

THEOREM 1.5. *Let $r(z_1, \dots, z_{m-1}, \mathrm{Re}(z_m))$ be a real-valued Lipschitz function on $\mathbb{C}^{m-1} \times \mathbb{R}$ with Lipschitz constant C. Set*

$$\Omega := \{\mathrm{Im}(z_m) < r(z_1, \dots, z_{m-1}, \mathrm{Re}(z_m))\}.$$

Then the envelope of holomorphy $\tilde{\Omega}$ of Ω is schlicht and is of the form $\tilde{\Omega} := \{\mathrm{Im}(z_m) < \tilde{r}(z_1, \dots, z_{m-1}, \mathrm{Re}(z_m))\}$, where $\tilde{r} \geq r$ is a Lipschitz function with the same Lipshitz constant or $\tilde{r} \equiv \infty$.

We prove Theorem 1.2 in §4. The proof is by an explicit construction of an expanding map defined on a biholomorphic image of the polydisc, and then by using Theorem 1.4 to find a dense open subset in this domain, biholomorphic to the ball.

Finally, we prove the Main Theorem in §5 by contradiction. A key step is to use Theorem 1.3 to show that the iterates F^n are well defined on V with image contained in $\overline{\Omega}$. The key point is to show that $F^n(V)$ is contained in $\tilde{\Omega}$ so that F^{n+1} has a holomorphic extension. This allows us to use normal families arguments.

2. *Proof of Theorem 1.4*

To prove Theorem 1.4 it suffices by Theorem 1 of [**FS1**] to assume that $M = \Delta^m$. In fact we will prove the theorem in the slightly more general case, when $M = U$, a connected open set in $\mathbb{C}^m$. This is necessary as we change U during the proof.

LEMMA 2.1. *Let U be a connected open set in $\mathbb{C}^m$. Suppose $F_j : \mathbf{B}^m_{r_j} \to U$, $j = 1, 2$, are one-to-one holomorphic maps with $F_1(\mathbf{B}^m_{r_1}) \cap F_2(\mathbf{B}^m_{r_2}) = \emptyset$. Let $0 < s < 1$. Then there exists a one-to-one holomorphic map $F_0 : \mathbf{B}^m_1 \to U$ so that $F_0(\mathbf{B}^m_1) \supset F_1(\mathbf{B}^m_{sr_1}) \cup F_2(\mathbf{B}^m_{sr_2/32})$.*

Proof. If a is a real number, we shall denote by $\mathbf{a}$ the point $(a, 0, \dots, 0)$ in $\mathbb{C}^m$. Also, for a positive number t, let K_t be the union of the closed ball $\overline{\mathbf{B}^m_1}$, the closed ball $\mathbf{4} + \overline{\mathbf{B}^m_t}$ and

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the straight line segment $[\mathbf{0}, \mathbf{4}]$. The set K_t is thus a certain compact subset of $\mathbb{C}^m$, which is polynomially convex when $t < 3$. Fix a number $\sigma \in (s, 1)$. Let $p \in F_1(b\mathbf{B}^m_{\sigma r_1})$ be the point $F_1(\sigma \mathbf{r_1})$ and let $q \in F_2(b\mathbf{B}^m_{\sigma r_2})$ be the point $F_2(-\sigma \mathbf{r_2})$. Let λ be a smooth arc in U that connects p and q and is otherwise disjoint from $F_1(\overline{\mathbf{B}^m_{\sigma r_1}}) \cup F_2(\overline{\mathbf{B}^m_{\sigma r_2}})$. Choose a smooth parametrization $f : [\mathbf{1}, \mathbf{2}] \to \lambda$ that takes $\mathbf{1}$ to the point p and $\mathbf{2}$ to q. The map $G : K_2 \to \mathbb{C}^m$ is defined by

$$G(z) = \begin{cases} F_1(\sigma r_1 z) & \text{if } z \in \overline{\mathbf{B}^m_1}, \\ F_2\left(\dfrac{\sigma r_2}{2}(z - \mathbf{4})\right) & \text{if } z \in \mathbf{4} + \overline{\mathbf{B}^m_2}, \\ f(x) & \text{if } z = \mathbf{x} \text{ with } x \in [\mathbf{1}, \mathbf{2}]. \end{cases}$$

With the proper choices of λ and f, the function G defined in this way is of class $\mathcal{C}^\infty$ on $[\mathbf{0}, \mathbf{4}]$. It is one-to-one on K_2. The function G can be approximated in the $\mathcal{C}^1$ sense on the set K_2 by polynomial maps $P : \mathbb{C}^m \to \mathbb{C}^m$ [**FS1**]. If P is such a map that approximates G well enough, then P will be one-to-one on K_2 and can be chosen to be regular on a neighbourhood of K_2. It will also satisfy the conditions

(a) $P(K_2) \subset U$,

(b) $P(K_2) \supset F_1(\mathbf{B}^m_{sr_1}) \cup F_2(\mathbf{B}^m_{sr_2})$, and

(c) $P(\mathbf{4} + \mathbf{B}^m_{1/16}) \supset F_2(\mathbf{B}^m_{sr_2/32})$.

Let $V \subset \mathbb{C}^m$ be a neighbourhood of K_2 on which P is one-to-one. Such neighbourhoods exist, because P is one-to-one on the compact set K_2 and is regular on a neighbourhood of it. By Lemma 1 of [**FS2**], there is a domain $B \subset V$ that is a neighbourhood of $K_{1/16}$ and that is biholomorphically equivalent to $\mathbf{B}^m_1$.

The lemma is proved. □

COROLLARY 2.2. *Let U be a connected open set in $\mathbb{C}^m$. Suppose $F_j : \mathbf{B}^m_{r_j} \to U$, $j = 1, 2, \ldots, N$, are finitely many one-to-one holomorphic maps with $F_i(\mathbf{B}^m_{r_i}) \cap F_j(\mathbf{B}^m_{r_j}) = \emptyset \; \forall i \neq j$. Let $0 < s < 1$. Then there exists a one-to-one holomorphic map $F_0 : \mathbf{B}^m_1 \to U$ so that $F_0(\mathbf{B}^m_1) \supset F_1(\mathbf{B}^m_{sr_1}) \cup \bigcup_{j>1} F_j(\mathbf{B}^m_{sr_j/32})$.*

Proof of the corollary. We prove the corollary by induction. By Lemma 2.1, the result holds for $N = 2$. Assume the result holds for some $N > 1$. We will show that the result holds for $N + 1$. Let $F_j : \mathbf{B}^m_{r_j} \to U$, $j = 1, 2, \ldots, N + 1$, be finitely many one-to-one holomorphic maps with $F_i(\mathbf{B}^m_{r_i}) \cap F_j(\mathbf{B}^m_{r_j}) = \emptyset$ for all $i \neq j$. Let $0 < s < 1$. Pick a σ, $s < \sigma < 1$. Define $U_1 := U \setminus \overline{F_{N+1}(\mathbf{B}^m_{\sigma r_{N+1}})}$. Then U_1 is a connected domain and $F_j : \mathbf{B}^m_{r_j} \to U_1$, $j = 1, 2, \ldots, N$, are finitely many one-to-one holomorphic maps with $F_i(\mathbf{B}^m_{r_i}) \cap F_j(\mathbf{B}^m_{r_j}) = \emptyset$ for all $i \neq j$, $i, j \leq N$. Hence the inductive hypothesis applies and we can find a holomorphic map $F_0 : \mathbf{B}^m_1 \to U_1$ so that $F_0(\mathbf{B}^m_1) \supset F_1(\mathbf{B}^m_{\sigma r_1}) \cup \bigcup_{1<j\leq N} F_j(\mathbf{B}^m_{\sigma r_j/32})$. It follows that the two maps $F_0 : \mathbf{B}^m_1 \to U$ and $F_{N+1} : \mathbf{B}^m_{\sigma r_{N+1}} \to U$ have disjoint range. Next, let λ, $s < \lambda < \sigma$, be arbitrary and pick $0 < \tau < 1$ large enough. Then $F_0(\mathbf{B}^m_\tau) \supset F_1(\mathbf{B}^m_{\lambda r_1}) \cup \bigcup_{1<j\leq N} F_j(\mathbf{B}^m_{\lambda r_j/32})$.

Next we pick μ, $0 < \mu < 1$, so large that

$$F_0(\mathbf{B}^m_{\mu\tau}) \supset F_1(\mathbf{B}^m_{sr_1}) \cup \bigcup_{1<j\leq N} F_j(\mathbf{B}^m_{sr_j/32}).$$

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We apply Lemma 2.1 to the maps $F_0 : \mathbf{B}^m_\tau \to U$ and $F_{N+1} : \mathbf{B}^m_{\sigma r_{N+1}} \to U$. Pick s', $0 < s' < 1$, so that $\mu < s'$ and $s < s'\sigma$. Then by Lemma 2.1 there exists a one-to-one holomorphic map $F : \mathbf{B}^m_1 \to U$ so that

$$\begin{aligned} F(\mathbf{B}^m_1) &\supset F_0(\mathbf{B}^m_{s'\tau}) \cup F_{N+1}(\mathbf{B}^m_{s'\sigma r_{N+1}/32}) \\ &\supset F_0(\mathbf{B}^m_{\mu\tau}) \cup F_{N+1}(\mathbf{B}^m_{sr_{N+1}/32}) \\ &\supset F_1(\mathbf{B}^m_{sr_1}) \cup \bigcup_{1<j\le N} F_j(\mathbf{B}^m_{sr_j/32}) \cup F_{N+1}(\mathbf{B}^m_{sr_{N+1}/32}). \end{aligned}$$ □

Let C denote a cube in $\mathbb{C}^m$ and let B denote a maximal concentric ball in C. We let $\rho := \rho_m := |B|/|C|$. Here $|x|$ denotes the volume of x. Obviously ρ_m is well-defined, i.e. independent of the size of C.

LEMMA 2.3. *Let V denote a bounded open set in $\mathbb{C}^m$. Then there exists a finite number of disjoint balls $B_i \subset V$ so that $|\cup B_i| > \rho|V|/2$.*

Proof. We can cover $\mathbb{C}^m$ by a grid of cubes so that the volume of the collection of cubes entirely contained in V has volume at least $|V|/2$. Then pick the largest concentric ball in each of these cubes. □

LEMMA 2.4. *Let $F : \mathbf{B}^m_1 \to U$ denote a one-to-one holomorphic map into a connected bounded open set $U \subset \mathbb{C}^m$. Let $0 < s < 1$ and $\epsilon > 0$. Then there exists a one-to-one holomorphic map $F' : \mathbf{B}^m_1 \to U$ so that $F'(\mathbf{B}^m_1) \supset F(\mathbf{B}^m_s)$ and*

$$|F'(\mathbf{B}^m_1)| \ge |F(B^m_1)| + \frac{\rho}{2 * (32)^{2m}}(|U| - |F(\mathbf{B}^m_1)|) - \epsilon.$$

Proof. If r, $0 < s < r < 1$, is large enough, then the volume of $F(\mathbf{B}^m_1 \setminus \mathbf{B}^m_r)$ is at most $\epsilon/2$. The set $V : U \setminus F(\overline{\mathbf{B}^m_r})$ is connected. By Lemma 2.3 we can find finitely many disjoint balls B_j in V with total volume at least $\rho|V|/2$. Applying Corollary 2.2 to these balls $B^m_j(p_j, r_j)$ and the ball $F(\mathbf{B}^m_r)$ we find a one-to-one holomorphic image $F'(\mathbf{B}^m_1)$ of the unit ball in U which contains $F(\mathbf{B}^m_{\sigma r})$ and $B^m_j(p_j, \sigma r_j/32)$ for any σ, $0 < \sigma < 1$, $s < \sigma r$, we want. By choosing σ large enough, we can suppose that the volume of $F'(\mathbf{B}^m_1)$ is at least

$$\begin{aligned} |F'(\mathbf{B}^m_1)| &\ge |F(\mathbf{B}^m_r)| + \frac{\rho|V|}{2 * (32)^{2m}} - \epsilon/2 \\ &\ge |F(\mathbf{B}^m_1)| + \frac{\rho|V|}{2 * (32)^{2m}} - \epsilon \\ &\ge |F(\mathbf{B}^m_1)| + \frac{\rho(|U| - |F(\mathbf{B}^m_1)|)}{2 * (32)^{2m}} - \epsilon. \end{aligned}$$

By the choice of σ, we also have that $F'(\mathbf{B}^m_1) \supset F(\mathbf{B}^m_{\sigma r}) \supset F(\mathbf{B}^m_s)$. □

COROLLARY 2.5. *Let U be a bounded connected open set in $\mathbb{C}^m$. Then there exists a sequence of one-to-one holomorphic maps $F_j : \mathbf{B}^m_1 \to \mathbb{C}^m$ and numbers $0 < s_j < 1$, $j = 1, \ldots,$ so that:*

(i) $F_{j+1}(\mathbf{B}^m_{s_{j+1}}) \supset F_j(\mathbf{B}^m_{s_j})$ *for all* $j \ge 1$;

(ii) $|F_{j+1}(\mathbf{B}^m_{s_{j+1}})| \ge |F_j(\mathbf{B}^m_{s_j})| + \frac{\rho}{3 * (32)^{2m}}(|U| - |F_j(\mathbf{B}^m_{s_j})|)$ *for all* $j \ge 1$.

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Proof. We prove the corollary by induction. Let $F_1 : \mathbf{B}_1^m \to U$ be an arbitrary ball, set $s_1 = 1/2$. Suppose we have found $F_1, \ldots, F_j$ and $s_1, \ldots, s_j$. We apply Lemma 2.4 to the map $F = F_j$ and $s = s_j + 1/2 > s_j$. Let

$$\epsilon = \frac{\rho}{12 * (32)^{2m}}(|U| - |F_j(\mathbf{B}_{s_j}^m)|).$$

Then pick F' as in Lemma 2.4 and set $F_{j+1} = F'$. Then

$$F_{j+1}(\mathbf{B}_1^m) \supset F_j(\mathbf{B}_s^m) \supset\supset F_j(\mathbf{B}_{s_j}^m)$$

and

$$\begin{aligned}
|F_{j+1}(\mathbf{B}_1^m)| &\geq |F_j(\mathbf{B}_1^m)| + \frac{\rho}{2 * (32)^{2m}}(|U| - |F_j(\mathbf{B}_1^m)|) - \epsilon \\
&= |F_j(\mathbf{B}_1^m)|\left(1 - \frac{\rho}{2 * (32)^{2m}}\right) + \frac{\rho}{2 * (32)^{2m}}|U| - \epsilon \\
&\geq |F_j(\mathbf{B}_{s_j}^m)|\left(1 - \frac{\rho}{2 * (32)^{2m}}\right) + \frac{\rho}{2 * (32)^{2m}}|U| - \epsilon \\
&= |F_j(\mathbf{B}_{s_j}^m)| + \frac{\rho}{2 * (32)^{2m}}(|U| - |F_j(\mathbf{B}_{s_j}^m)|) - \epsilon \\
&= |F_j(\mathbf{B}_{s_j}^m)| + \frac{5\rho}{(12) * (32)^{2m}}(|U| - |F_j(\mathbf{B}_{s_j}^m)|) \\
&> |F_j(\mathbf{B}_{s_j}^m)| + \frac{\rho}{3 * (32)^{2m}}(|U| - |F_j(\mathbf{B}_{s_j}^m)|).
\end{aligned}$$

Finally, we choose $0 < s_{j+1} < 1$ large enough so that

$$F_{j+1}(\mathbf{B}_{s_{j+1}}^m) \supset F_j(\mathbf{B}_{s_j}^m)$$

and

$$|F_{j+1}(\mathbf{B}_{s_{j+1}}^m)| \geq |F_j(\mathbf{B}_{s_j}^m)| + \frac{\rho}{3 * (32)^{2m}}(|U| - |F_j(\mathbf{B}_{s_j}^m)|). \qquad \square$$

PROPOSITION 2.6. *Let U be a bounded connected open set in $\mathbb{C}^m$. Then there exists a sequence of one-to-one holomorphic maps $F_j : \mathbf{B}_1^m \to \mathbb{C}^m$ and numbers $0 < s_j < 1$, $j = 1, \ldots,$ so that:*

(i) $F_{j+1}(\mathbf{B}_{s_{j+1}}^m) \supset F_j(\mathbf{B}_{s_j}^m)$ *for all* $j \geq 1$;

(ii) $|F_j(\mathbf{B}_{s_j}^m)| \to |U|$.

Proof. Let F_j and s_j be as in Corollary 2.5. Set $A := \lim_j |F_j(\mathbf{B}_{s_j}^m)|$. From Corollary 2.5(ii) it follows that

$$|U| \geq A \geq A + \frac{\rho}{3 * (32)^{2m}}(|U| - A).$$

This is impossible if $A < |U|$. $\square$

Completion of the Proof of Theorem 1.4. The union $\bigcup_j F_j(B_{s_j}^m)$ is biholomorphic to the unit ball. This follows from [**FS1**] since the union is contained in a Kobayashi hyperbolic set. $\square$

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3. *Proof of Theorem 1.3*

We first prove Theorem 1.5.

Proof. Let c be a Lipschitz constant of ϕ. Then

$$\frac{|\phi(v,z)-\phi(v_0,z_0)|}{\|(v,z)-(v_0,z_0)\|}\le c.$$

This is equivalent to the following. Let $\Gamma = \{u = \phi(v,z)\}$ and choose a point $(u_0+iv_0,z_0)\in\Gamma$, then Γ does not intersect the cone

$$\mathcal{C}_{(z_0,u_0,v_0)} := \{(z,u+iv) : |u-u_0| > c\|(z,v)-(z_0,v_0)\|\}$$

at the point (u_0+iv_0,z_0). Moreover, let $(\alpha,\beta)\in\mathbb{C}^{m-1}\times\mathbb{R}$ be such that $\|(\alpha,\beta)\|<1/c$, then all points $(z_0,u_0+iv_0)-t(\alpha,1+i\beta)$, where $0<t$, are contained in Ω. Let

$$H_{t,\alpha,\beta}(z,u+iv) = (z,u+iv)-t(\alpha,1+i\beta).$$

For fixed t, α and β this is a holomorphic map and

$$H_{t,\alpha,\beta}(\Omega)\subset\Omega. \qquad (*)$$

We want to show that $\tilde\Omega$ also has this property $(*)$. Since Ω is bounded by a graph it follows that $\tilde\Omega$ is schlicht and also is bounded by a graph (see [**S**]). In fact, $\tilde\Omega = \bigcup_{j=0}^{\infty}\Omega_j$, $\Omega_0=\Omega$, where Ω_j consists of all the points that can be reached from Ω_{j-1} by pushing discs with boundaries in Ω_{j-1}. We make this more precise. Let H denote the Hartogs figure

$$H := \left\{p\in\overline{\Delta}_0\cup\bigcup_{0\le s\le 1}\partial\Delta_s\right\},\quad \Delta_s=\{(z,t)\in\mathbb{C}\times\mathbb{R} : t=s,\ |z|\le 1\}\subset\mathbb{C}^2.$$

Moreover, $\hat H$ denotes the convex hull of H. Let Φ denote any rank 2, one-to-one holomorphic map defined on some (not fixed) neighbourhood of $\hat H$ in $\mathbb{C}^2$ with values in $\mathbb{C}^m$. Then

$$\Omega_j := \bigcup_{\{\Phi:\Phi(H)\subset\Omega_{j-1}\}}\Phi(\hat H).$$

Now we only need to show that if Ω_{j-1} has property $(*)$ so does Ω_j.

Observe that if $\Phi(H)\subset\Omega_{j-1}$, then $H_{t,\alpha,\beta}(\Phi(H)) = (H_{t,\alpha,\beta}\circ\Phi)(H)\subset\Omega_{j-1}$. Hence, if $p\in\Omega_j$, then $H_{t,\alpha,\beta}(p)\in\Omega_j$.

Now $\partial\tilde\Omega=\tilde\Gamma$, where

$$\tilde\Gamma = \{(z,u+iv) : u=\tilde\phi(z,v)\}.$$

Obviously, by $(*)$, if $\tilde\phi(z,v)$ is infinite for some (z,v), then $\tilde\phi\equiv\infty$ and we are done. We assume hence that $\tilde\phi$ has values in $\mathbb{R}$ only.

We need to show that if $(z_0,u_0+iv_0)\in\tilde\Gamma$, then $\tilde\Gamma$ does not intersect the cone $\mathcal{C}_{(z_0,u_0,v_0)}$. We know that the cone does not meet $\tilde\Gamma$ at points $(z,u+iv)$ where $u<u_0$ since these are interior points of $\tilde\Omega$.

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Assume that $(z_1, u_1 + iv_1) \in \tilde{\Gamma} \cap \mathcal{C}_{(z_0,u_0,v_0)}$ and $u_1 > u_0$. We want to show that this implies that $(z_0, u_0 + iv_0) \in \tilde{\Omega}$ which contradicts the fact that this is a boundary point. But the fact that $(z_1, u_1 + iv_1) \in \partial\tilde{\Omega}$ implies that the set

$$\{(z, u + iv) = (z_1, u_1 + iv_1) - t(1, \alpha, \beta) : \|(\alpha, \beta)\| < 1/c, 0 < t < 1\}$$

is contained in $\tilde{\Omega}$ and a simple computation shows that $(z_0, u_0 + iv_0)$ is an interior point of this set. □

Proof of Theorem 1.3. Pick a point $p \in \partial\Omega$. If p is an interior point of the envelope of holomorphy then we are done. So suppose not. Since Ω has Lipschitz boundary, we can rotate and translate coordinates so that $p = 0$ and Ω is locally given by $\Omega = \{(z, u + iv) : u < \phi(z, v), \|(z, v)\| < \rho\}$ for some Lipshitz function ϕ. Let $A > \phi(0)$ and pick $L > 0$ large enough so that $A - L\|(z, v)\| < \phi(z, v)$ if $\rho/2 < \|(z, v)\| < \rho$. We globalize Ω by setting

$$\Omega' = \{(z, u + iv) \in \mathbb{C}^m : u - L\|(z, v)\| \text{ if } \|(z, v)\| > \rho/2,$$
$$u < \min\{A - L\|(z, v)\|, \phi(z, v)\} \text{ if } \|(z, v)\| \leq \rho/2\}.$$

Letting $\tilde{\Omega}'$ be the envelope of holomorphy of Ω', $\tilde{\Omega}' = \{u < \tilde{\phi}(z, v)\}$, we get $\tilde{\phi}(0) = \phi(0) = 0, \tilde{\phi} \geq \phi$, near 0. By a theorem of Demailly [**D**] it follows that there is a locally defined continuous function ρ on the set $\{u \leq \tilde{\phi}(z, v)\}$ which is plurisubharmonic on $\{u < \phi(z, v)\} \subset \{u < \tilde{\phi}(z, v)\}$ and with $\rho(0) = 0$. □

4. *Proof of Theorem 1.2*

We prove the theorem in steps.

Step 1. Let R' be the closed rectangle in $\mathbb{C}$ with corners

$$(0, 0), \quad (2\sqrt{2}\pi, 0), \quad (0, 4\pi), \quad (2\sqrt{2}\pi, 4\pi).$$

This is covered by two closed rectangles R'_1, R'_2, where R'_1 has corners

$$(0, 0), \quad (2\sqrt{2}\pi, 0), \quad (0, 2\pi), \quad (2\sqrt{2}\pi, 2\pi)$$

and R'_2 has corners

$$(0, 2\pi), \quad (2\sqrt{2}\pi, 2\pi), \quad (0, 4\pi), \quad (2\sqrt{2}\pi, 4\pi).$$

Let $F_1 : R'_1 \to R'$, $F_1(z) = \sqrt{2}iz + 2\sqrt{2}\pi$ and $F_2 : R'_2 \to R'$, $F_2(z) = \sqrt{2}iz + 4\sqrt{2}\pi$. Then the F_j are homeomorphisms. Next let R, R_j denote the interiors of the same rectangles. Let $\Omega := R_1 \cup R_2$ and set $F : \Omega \to R$, $F(z) = F_1(z)$, $z \in R_1$, $F(z) = F_2(z)$, $z \in R_2$. Let p be any point in Ω and let $U \subset \Omega$ be some neighbourhood of p. Then there exists a connected open subset $V \subset U$ and an integer $N \geq 1$ so that $F^j(V) \subset \Omega$ for each $1 \leq j \leq N$, $F^N(V) = R_1$ or R_2 and hence $\Omega \subset F^{N+1}(V)$. From this observation it is easy to inductively find a dense orbit $\{p_n\}$ in Ω. Moreover, we can choose $\{p_n\}$ to be contained in any open subset of full measure or just dense.

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Step 2. Let D denote a disc,

$$D := \{w \in \mathbb{C} : 1 < |w| < e^{2\sqrt{2}\pi},\ w \notin \mathbb{R}^+ \setminus 0\}.$$

Then the maps $\tau_j : R_j \to D,\ \tau_j(z) = e^z$, are biholomorphisms, $j = 1, 2$. Suppose that $z_j \in R_j,\ z_2 = z_1 + 2\pi i$. Then $\tau_1(z_1) = \tau_2(z_2)$. Notice also that

$$\begin{aligned} F_2(z_2) &= \sqrt{2}i z_2 + 4\sqrt{2}\pi \\ &= \sqrt{2}i z_1 - 2\sqrt{2}\pi + 4\sqrt{2}\pi \\ &= F_1(z_1). \end{aligned}$$

Hence, we can define $\Phi : D \to \mathbb{C}$ by $\Phi(w) = e^{F_j(z_j)},\ j = 1$ or $2,\ w = e^{z_j}, z_j \in R_j$. Then Φ is a well defined holomorphic map. Let $\{p_n\}_{n=0}^{\infty}$ be a dense orbit of F in Ω. Define $q_n := e^{p_n} \in D$. Then $\Phi(q_n) = e^{F_j(p_n)}$ if $p_n \in R_j$. Hence $\Phi(q_n) = e^{F(p_n)} = e^{p_{n+1}} = q_{n+1}$. Hence $\{q_n\} \subset D$ and is a dense orbit for Φ.

Step 3. Let $D^m := D \times \cdots \times D$, m times and define $\Psi : D^m \to \mathbb{C}^m$ by $\Psi(z_1, \ldots, z_m) = (\Phi(z_1), \ldots, \Phi(z_m))$. Then we can find a dense sequence $\{p_n\} \subset D^m$ contained in any given dense open subset.

Step 4. To finish the proof of Theorem 1.2 we use Theorem 1.4 to find an open subset of D^m of full measure and biholomorphic to the unit ball. □

5. *Proof of the Main Theorem*

To prove the theorem, we assume to the contrary that there are such V and $\{p_n\}$. We can assume V is connected and $(0) \in V$. We let $\tilde{\Omega}$ denote the (possibly multisheeted) envelope of holomorphy of Ω and $\Omega^* \subset \tilde{\Omega}$ consists of $\Omega \cup \{q \in \partial\Omega : q \in \tilde{\Omega}\}$. The map F extends holomorphically to a map $\tilde{F} : \tilde{\Omega} \to \mathbb{C}^m$. We denote by F^* the restriction of $\tilde{F}$ to Ω^*. So F^* is a continuous extension of F to Ω^* and in turn F^* extends holomorphically. We define $\Omega_k := \{z = z_0 \in \Omega, z_1 := F^*(z) \in \Omega^*, z_2 := F^*(z_1) \in \Omega^*, \ldots, z_k := F^*(z_{k-1}) \in \Omega^*\}$. Note that in general the Ω_k are not open sets but the maps $(\tilde{F})^{k+1}$ are well defined holomorphic maps in small neighbourhoods of the Ω_k.

LEMMA 5.1. *We have that $V \subset \Omega_k$ for all k.*

Proof of the Lemma. We show first that $F(V) \subset \Omega_1$. If $p_k \in V$, then $F(p_k) = p_{k+1} \in \Omega$. Since $\{p_n\} \cap V$ is dense in V, it follows by continuity that $F(V) \subset \overline{\Omega}$. Suppose there is a point $q = F(p) \in \partial\Omega,\ p \in V$. We will show that $q \in \Omega^*$. If $q \notin \Omega^*$, we can find, by Theorem 1.3, an open neighbourhood $U(q)$ and a continuous function $\rho : U \cap \overline{\Omega} \to (-\infty, 0]$ which is plurisubharmonic on $U \cap \Omega$, $\rho < 0$ on $U \cap \Omega$ and $\rho(q) = 0$. Since Ω has Lipschitz boundary, we can find an outward vector $\mathbf{n}$ in a possibly smaller neighbourhood of q. Let $\Phi_\epsilon,\ \epsilon > 0$, denote the inward translation by ϵ in the direction of $\mathbf{n}$. Then there exists a fixed neighbourhood $q \in W \subset V$ so that $\rho \circ \Phi_\epsilon \circ F$ is plurisubharmonic on W for all small enough ϵ and, moreover, $\rho \circ \Phi_\epsilon \circ F \to \rho \circ F$ uniformly as $\epsilon \searrow 0$. It follows that $\rho \circ F$ is plurisubharmonic on W and since this function takes its maximum at q it must be constant on W. This is a contradiction since $\rho \circ F < 0$ at each point of $\{p_n\} \cap U$. This shows that $V \subset \Omega_1$.

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Suppose we have shown that $V \subset \Omega_k$. Then F^{k+1} extends holomorphically to V. The same argument as above shows that the image of the extension is contained in $\overline{\Omega}$. Furthermore, applying Theorem 1.3 again in the same way as above, we obtain that the image is actually contained in Ω^*. This shows that $V \subset \Omega_{k+1}$, completing the induction. □

Proof of the Theorem. Since $(0) \in V$, we can choose sequences $m(k) < \ell(k)$ so that $p_{m(k)}, p_{\ell(k)} \to 0$. We set $n(k) = \ell(k) - m(k)$. We can assume that $n(k) \nearrow \infty$. By Lemma 5.1, the map $F^{n(k)}$ has a holomorphic extension to V and $F^{n(k)}(p_{m(k)}) = p_{\ell(k)}$. Restricting to a thinner subsequence if necessary, we can assume that $F^{n(k)} \to G : V \to \mathbb{C}^m$, $G(0) = 0$.

For any positive integer τ the sequence $F^{\tau n(k)} = (F^{n(k)})^\tau$ converges in a small enough neighbourhood to G^τ and by a normal families argument, this convergence extends to all of V and a suitable extension of G^τ to V. The image of V is still contained in $\overline{\Omega}$.

We obtain the following.

LEMMA 5.2. *Suppose that $F^{n_k} : V \to \overline{\Omega}$ is any sequence of iterates converging to some map $G : V \to \Omega$, $G(0) = 0$, then all iterates G^τ can also be analytically continued from 0 to a map from V to $\overline{\Omega}$. Moreover, if G^{τ_s} is any convergent subsequence to a map $H : V \to \mathbb{C}^m$, then there is a subsequence $F^{m(s)}$ converging to H on V. Moreover, $H(0) = 0$, $H(V) \subset \overline{\Omega}$.*

Using this lemma we can refine the properties of a limit G of some subsequence F^{n_k}.

Let $\{\lambda_j\}_{j=1}^m$ denote the eigenvalues of $G'(0)$. Since $G^\tau(V) \subset \overline{\Omega}$ for all τ, it follows that all eigenvalues have modulus less than or equal to 1. We can assume that $1 \geq |\lambda_1| \geq \cdots \geq |\lambda_m|$. Using Lemma 5.2 again, we can assume that $\lambda_i = 1$, $i \leq j$, $\lambda_i = 0$, $i > j$ for some $0 \leq j \leq m$. If we put the matrix $G'(0)$ in Jordan normal form, we see that for the blocks with eigenvalue 1 there can be no off-diagonal term because these would grow under iteration violating Schwarz's lemma. Also, if we look at the off-diagonal terms for blocks with eigenvalue zero, these vanish after iteration, so hence we can assume that there are no off-diagonal terms.

Case (i): $\lambda_1 = \cdots = \lambda_m = 0$. Then there exist $0 < s < r$ and k sufficiently large so that $F^{n_k}(\mathbf{B}(0, r)) \subset \mathbf{B}(0, s) \subset \mathbf{B}(0, r) \subset V$ and $\|(F^{n_k})'(a)(v)\| \leq c\|v\|$ for some $c < 1$ and for all $a \in \mathbf{B}(0, r)$. Hence F^{n_k} is a contraction and hence has an attracting fixed point. Therefore F has an attracting periodic orbit. This is impossible since there is a dense orbit.

Case (ii): $\lambda_1 = \cdots = \lambda_m = 1$. In this case $G'(0) = \mathrm{Id}$. If G has non-zero higher-order terms in the Taylor expansion around the origin, the iterates of G cannot be a normal family on V, so $G(z) \equiv z$ on V.

Let $\hat{V} := V \cup \bigcup_{n \geq 1} \tilde{F}^n(V)$. Then $\hat{V}$ is open and $\tilde{F}(\hat{V}) \subset \hat{V}$.

LEMMA 5.3. *The set $\hat{V}$ has finitely many connected components $U_1, \ldots, U_r = U_1$ with $\tilde{F}(U_i) \subset U_{i+1}$. In fact the map $\tilde{F} : U_i \to U_{i+1}$ is a biholomorphism and $\tilde{F}^{n_k} \to \mathrm{Id}$ on each U_i.*

Proof of the lemma. Clearly $\tilde{F}^m$ must be one-to-one on V for each iterate $\tilde{F}^m$. Hence the Jacobians of each iterate are everywhere non-zero and each iterate is an open map.

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Hence $\hat{V}$ is an open set. Moreover, $\tilde{F}(\hat{V}) \subset \hat{V}$. In particular, each connected component is mapped into another connected component by $\tilde{F}$. Let U_1 be the connected component containing V and define inductively connected components U_i by $\tilde{F}(U_i) \subset U_{i+1}$. Since $\tilde{F}^{n_k} \to \mathrm{Id}$ on V it follows that for a minimal r, $U_r = U_1$. Hence there are only finitely many connected components. The condition that $\tilde{F}^{n_k} \to \mathrm{Id}$ on V implies that $F : \hat{V} \to \hat{V}$ is onto. Hence each map $\tilde{F} : U_i \to U_{i+1}$ is onto. By the identity theorem it follows that $\tilde{F}^{n_k} \to \mathrm{Id}$ on each U_i. Hence $\tilde{F} : U_i \to U_{i+1}$ is one-to-one and hence a biholomorphism. □

This implies that $\tilde{F}^r : U_1 \to U_1$ is an automorphism with a dense orbit of a bounded domain. The automorphism group is a Lie group and the closure of the set of iterates of $\tilde{F}^r$ is a commutative closed subgroup, isomorphic to a product of a torus and Euclidean space and a finite commutative group. Since there is a sequence of iterates of $\tilde{F}^r$ converging to the identity, the Euclidean component must vanish, but then the orbit of any point is nowhere dense, a contradiction.

Case (iii): $\lambda_1 = \cdots = \lambda_j = 1, \lambda_{j+1} = \cdots = \lambda_m = 0$ *for some* $1 \leq j < m$. Then G is of the form

$$G(z_1, \ldots, z_m) = (z_1 + f_1, \ldots, z_j + f_j, f_{j+1}, \ldots, f_m), \quad f_i = \mathcal{O}(\|z\|^2). \qquad (**)$$

Fix a small neighbourhood W, $0 \in W \subset\subset U_1$. Let $U \subset\subset W$ and suppose that $\tilde{F}^r(U) \subset W$. Then we have the functional equation

$$\tilde{F}^r \circ G(z) = G \circ \tilde{F}^r(z), \quad z \in W.$$

Let $z_\delta := (0, \ldots, 0, \delta)$ for small enough $\delta \neq 0$. Using the density of orbits and equicontinuity there exists for any $\epsilon > 0$ a neighbourhood $U(0) \subset W$ and an integer $r_\epsilon > 1$ so that $\|\tilde{F}^{r_\epsilon}(0) - z_\delta\| \leq \epsilon$ and $\tilde{F}^r(U) \subset W$. Since $G(0) = 0$, $\tilde{F}^{r_\epsilon}(0) = G(\tilde{F}^{r_\epsilon}(0))$. This is clearly impossible by $(**)$. This completes the proof. □

Remark 5.4. We can relax the Lipschitz condition of the Main Theorem if we add more hypotheses on F. Let Ω be any domain with $\Omega = \mathrm{int}(\overline{\Omega})$ and assume in addition that the map F has discrete fibers. Then F has no dense orbit.

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