2

Leptons and the electromagnetic and weak interactions

2.1 The electromagnetic interaction

The electromagnetic field is most conveniently described by a vector potential $\mathbf{A}$ and a scalar potential ϕ. For simplicity, we consider only the potential $\phi(\mathbf{r}, t)$. Using Maxwell's equations, this may be chosen to satisfy the wave-equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho(\mathbf{r}, t)}{\varepsilon_0}. \tag{2.1}$$

Here $\rho(\mathbf{r}, t)$ is the electric charge density due to the charged particles, which in atomic and nuclear physics will usually be electrons and protons, and c is the velocity of light.

In regions where $\rho = 0$, equation (2.1) has solutions in the form of propagating waves; for example, the plane wave

$$\phi(\mathbf{r}, t) = (\text{constant})\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \tag{2.2}$$

This satisfies

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{2.3}$$

provided

$$\omega^2 = c^2k^2. \tag{2.4}$$

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The wave velocity is therefore c, as we should expect. In quantum theory, unlike classical theory, the total energy and momentum of the wave are quantised, and can only be integer multiples of the basic quantum of energy and momentum given by the de Broglie relations:

$$E = \hbar\omega, \quad \mathbf{p} = \hbar\mathbf{k}. \tag{2.5}$$

Such a quantum of radiation is called a *photon*. A macroscopic wave can be considered to be an assembly of photons, and we can regard photons as particles, each carrying energy E and momentum $\mathbf{p}$.

Using (2.4) and (2.5), E and $\mathbf{p}$ are related by

$$E^2 = p^2c^2. \tag{2.6}$$

For a particle of mass m, the Einstein equation gives

$$E^2 = p^2c^2 + m^2c^4.$$

We therefore infer that the photon is a particle having zero mass.

A second important type of solution of (2.1) exists when charged particles are present. If these are moving slowly compared with the velocity of light, so that the term $\partial^2\phi/(c^2\,\partial t^2)$ can be neglected, the solution is approximately the Coulomb potential of the charge distribution. For a particle with charge density ρ_1, we can take

$$\phi(\mathbf{r}, t) \approx \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_1(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}'. \tag{2.7}$$

Another charged particle with charge density ρ_2 will have a potential energy given by

$$\begin{aligned} U_{12} &= \int \rho_2(\mathbf{r}, t)\phi(\mathbf{r}, t)\mathrm{d}^3r \\ &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_1(\mathbf{r}', t)\rho_2(\mathbf{r}, t)}{|\mathbf{r} - \mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{r}'. \end{aligned} \tag{2.8}$$

Electric potential energy is basically responsible for the binding of electrons in atoms and molecules. We shall see that, in nuclear physics, it is responsible for the instability of heavy nuclei. If magnetic effects due to the motion of the charges are included, equation (2.8) is modified to

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$$U_{12} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_1'\rho_2 + (1/c^2)\mathbf{j}_1' \cdot \mathbf{j}_2}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3\mathbf{r} \, \mathrm{d}^3\mathbf{r}', \tag{2.9}$$

where $\mathbf{j} = \rho\mathbf{v}$ is the current associated with the charge distribution which has velocity $\mathbf{v}(\mathbf{r})$. Thus this magnetic contribution to the energy is of relative order v^2/c^2.

The electromagnetic interaction also gives rise to the scattering of charged particles. For example, for two electrons approaching each other the interaction gives a mutual repulsion which leads to a transfer of momentum between the particles. The process can be represented by a diagram such as Fig. 2.1. In quantum electrodynamics, these diagrams, invented by Feynman, have a precise technical interpretation in the theory. We shall use them only to help visualise the physics involved. The scattering of the two electrons may be thought of as caused by the emission of a 'virtual' photon by one electron and its absorption by the other electron. In a virtual process the photon does not actually appear to an observer, though it appears in the mathematical formalism that describes the process.

2.2 The weak interaction

There are three *weak interaction* fields associated with the W^+, W^- and Z particles. Each one, like the electromagnetic field, is described by a vector and a scalar potential. However, the bosons associated with the weak

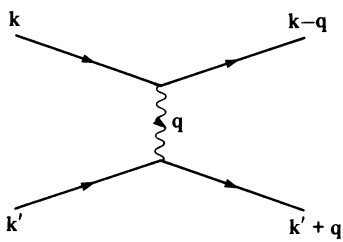


Fig. 2.1 The scattering of two electrons of momenta $\hbar\mathbf{k}$, $\hbar\mathbf{k}'$ by the exchange of a virtual photon carrying momentum $\hbar\mathbf{q}$. Time runs from left to right in these diagrams. (In principle, the exchange of a Z particle (§2.2) also contributes to electron–electron scattering, but the very short range and weakness of the weak interaction makes this contribution almost completely negligible: the electrons are in any case kept apart by the Coulomb repulsion induced by the photon exchange.)

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fields all have mass, and the W^- and W^+ bosons are electrically charged. The Z boson is neutral, and most similar to the photon, but it has a mass

$$M_Z = (91.187 \pm 0.007)\ \text{GeV}/c^2 \sim 100 \text{ proton masses},$$

which is very large by nuclear physics standards.

The interactions between leptons and the electromagnetic and weak fields were combined into a unified 'electro-weak' theory by Weinberg and by Salam. The existence of the Z and $W^{\pm}$ bosons was predicted by the theory, and the theory together with experimental data from neutrino–nuclear scattering also suggested values for their masses. These predictions were confirmed by experiments at CERN in 1983.

The wave equation satisfied by the scalar potential ϕ_Z associated with the Z boson is a generalisation of (2.1) and includes a term involving M_Z:

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\frac{M_Z c}{\hbar}\right)^2\right]\phi_Z(\mathbf{r}, t) = -\frac{\rho_Z(\mathbf{r}, t)}{\varepsilon_0}, \tag{2.10}$$

where ρ_Z is the neutral weak-charge density. There is a close, but not exact, analogy between weak-charge density and electric-charge density, and particles carry weak charge somewhat as they carry electric charge. In the case of a nucleus, ρ_Z will extend over the nuclear dimensions.

In free space where $\rho_Z = 0$ there exist plane wave solutions of (2.10),

$$\phi_Z(\mathbf{r}, t) = (\text{constant})\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)},$$

but now to satisfy the wave equation we require

$$\omega^2 = c^2k^2 + c^2(M_Z c/\hbar)^2,$$

and with the de Broglie relations (2.5) for the field quanta we obtain the Einstein energy–momentum relation for the Z boson:

$$E^2 = p^2c^2 + M_Z^2c^4.$$

The static solution of (2.10) which corresponds to a point unit weak charge at the origin is

$$\phi_Z(r) = \frac{1}{4\pi\varepsilon_0}\frac{e^{-\kappa r}}{r}, \quad \text{writing } \kappa = \frac{M_Z c}{\hbar}. \tag{2.11}$$

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At points away from the origin where $\nabla^2\phi_Z - \kappa^2\phi_Z = 0$, this satisfies equation (2.10), as may be easily checked by substitution, using the formula $\nabla^2\phi_Z = (1/r)\mathrm{d}^2(r\phi_Z)/\mathrm{d}r^2$. Close to the origin the solution (2.11) behaves like the corresponding Coulomb potential $1/(4\pi\varepsilon_0 r)$ of a unit point electric charge, and hence has the correct point source behaviour. The generalisation of (2.11) to a distribution of weak charge gives the quasi-static solution (cf. (2.7))

$$\phi_Z(\mathbf{r}, t) \approx \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_Z(\mathbf{r}', t)\,\mathrm{e}^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}'. \tag{2.12}$$

The exponential factor in the integral effectively vanishes for $|\mathbf{r}-\mathbf{r}'|$ greater than a few times $\kappa^{-1} = \hbar/M_Z c$ and

$$\hbar/M_Z c \approx 2 \times 10^{-3}\ \text{fm}.$$

This is a very small distance compared with the size of the nucleus. Hence in the integral in (2.12) the factor ρ_Z is slowly varying over the range of the exponential and may be taken outside the integral (which is then elementary):

$$\begin{aligned}\phi_Z(\mathbf{r}, t) &\simeq \frac{1}{4\pi\varepsilon_0}\rho_Z(\mathbf{r}, t)\int \frac{\mathrm{e}^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3\mathbf{r}' \\ &= \frac{1}{4\pi\varepsilon_0}\rho_Z(\mathbf{r}, t)\int_0^\infty \frac{\mathrm{e}^{-\kappa R}}{R}4\pi R^2\mathrm{d}R \\ &= \frac{1}{\varepsilon_0}\left(\frac{\hbar}{M_Z c}\right)^2\rho_Z(\mathbf{r}, t).\end{aligned} \tag{2.13}$$

The potential energy between two particles associated with the scalar field ϕ_Z is, by analogy with (2.8),

$$\begin{aligned}U_{12}^Z &= \int \rho_{Z2}(\mathbf{r}, t)\phi_{Z1}(\mathbf{r}, t)\mathrm{d}^3\mathbf{r} \\ &= \frac{1}{\varepsilon_0}\left(\frac{\hbar}{M_Z c}\right)^2\int \rho_{Z1}(\mathbf{r}, t)\rho_{Z2}(\mathbf{r}, t)\mathrm{d}^3\mathbf{r},\end{aligned}$$

and there is also a contribution from the vector part of the field, analogous to the magnetic contribution in (2.9), of the form

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$$\frac{1}{\varepsilon_0 c^2}\left(\frac{\hbar}{M_Z c}\right)^2 \int \mathbf{j}_{Z1}(\mathbf{r}, t) \cdot \mathbf{j}_{Z2}(\mathbf{r}, t) \mathrm{d}^3\mathbf{r},$$

where $\mathbf{j}_Z$ is the weak-current density.

The physical consequences of these expressions are quite different from the physical consequences of the electromagnetic interaction. U_{12}^Z is very much suppressed by the large mass factor in the denominator, and it is this which largely accounts for the 'weakness' of the weak interaction. Also the interaction at low energies appears as a 'contact interaction', effectively having zero range.

The electrically charged W^+ and W^- boson fields give rise to the most important weak interactions, and in particular to β-decay. They obey equations similar to those of the Z field, but the masses of the associated particles are somewhat smaller;

$$M_{W^+} = M_{W^-} = (80.41 \pm 0.10)\ \mathrm{GeV}/c^2.$$

2.3 Mean life and half life

Not all particles are stable: some, for example the $W^\pm$ and Z bosons, have only a transient existence. Suppose that an unstable particle exists at some instant $t = 0$; its *mean life* is the mean time it exists in isolation, before it undergoes radioactive decay. If we denote by $P(t)$ the probability that the particle survives for a time t, and make the assumption that the particle has a *constant* probability $1/\tau$ per unit time of decaying, then

$$P(t + \mathrm{d}t) = P(t)(1 - \mathrm{d}t/\tau),$$

since $(1 - \mathrm{d}t/\tau)$ is the probability it survives the time interval $\mathrm{d}t$. Hence

$$\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{1}{\tau},$$

and integrating,

$$P(t) = P(0)\mathrm{e}^{-t/\tau}.$$

Since $P(0) = 1$ we have

$$P(t) = \mathrm{e}^{-t/\tau}. \tag{2.14}$$

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Equation (2.14) is the familiar exponential-decay law for unstable particles. It is well verified experimentally.

The probability that the particle decays between times t, $t + \mathrm{d}t$ is clearly $P(t) \times (\mathrm{d}t/\tau)$, so that the mean life is

$$\int_0^\infty tP(t)(\mathrm{d}t/\tau) = \int_0^\infty t\mathrm{e}^{-t/\tau}\mathrm{d}t/\tau = \tau.$$

The 'half life' $T_{\frac{1}{2}}$ is the time at which there is a 50% probability that the particle has decayed, i.e.

$$P(T_{\frac{1}{2}}) = \mathrm{e}^{-T_{\frac{1}{2}}/\tau} = \tfrac{1}{2}.$$

Hence

$$T_{\frac{1}{2}} = \tau \ln 2 = 0.693\tau.$$

In this book we have preferred to quote mean lives rather than half lives. We refer to $(1/\tau)$ as the *decay rate*.

2.4 Leptons

Leptons are spin $\frac{1}{2}$ fermions which interact through the electromagnetic and weak interactions, but not through the strong interaction. The known leptons are listed in Table 2.1.

The electrically charged leptons all have magnetic moments of magnitude $\approx e\hbar/2$ (mass) anti-aligned with their spins.

Of these charged leptons, only the familiar electron is stable. Electrons are structureless particles that are described by the Dirac relativistic wave-equation. This equation explains the spin and magnetic moment of the electron, and has the remarkable feature that it predicts the existence of anti-particles: these are particles of the same mass and spin, but of opposite charge and magnetic moment to the particle. The anti-particle of the electron is called the *positron*. Positrons were identified experimentally by Anderson in 1932 soon after their theoretical prediction.

Since leptons do not interact with the strong interaction field, electrons and positrons interact principally through the electromagnetic field. A positron will eventually annihilate with an electron, usually to produce

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Table 2.1. *Known leptons*

	Mass (MeV/c^2)	Mean life (s)	Charge
Electron e^-	0.5110	∞	$-e$
Electron neutrino ν_e	$< 15 \times 10^{-6}$	∞?	0
Muon μ^-	105.658	2.197×10^{-6}	$-e$
Muon neutrino ν_μ	< 0.17	∞?	0
Tau τ^-	1777	290×10^{-15}	$-e$
Tau neutrino ν_τ	< 18.2	∞?	0

two or three photons, so that all the lepton energy appears as electromagnetic radiation. We write these processes as

$$e^- + e^+ \rightarrow 2\gamma$$
$$e^- + e^+ \rightarrow 3\gamma.$$

Annihilation with the production of a single photon is not allowed, by energy and momentum conservation (Problem 2.4).

The converse processes of *pair-production* by photons are also possible, and pair-production from a single photon is possible provided another (charged) particle is present to take up momentum. Quantum electrodynamics, based on the Dirac and Maxwell equations, describes all processes involving electrons, positrons and photons to a high degree of accuracy.

It is a curious fact that nature provides us also with the electrically charged *muon* μ^- and *tau* τ^- and their anti-particles the μ^+ and τ^+. Apart from their greater masses and finite lifetimes, muons and taus seem to be just copies of the electron, and like the electron they are accurately described by Dirac equations. We shall see that the μ^- can be used as a probe of nuclear charge density, but otherwise neither the muons nor the taus play any significant role in nuclear physics.

The remaining leptons are the *neutrinos* ν and their corresponding anti-neutrinos denoted by $\bar{\nu}$. The experimental evidence (Table 2.1) suggests that the mass of a neutrino is certainly very small compared with the mass of its charged lepton partner. If the mass of a neutrino were zero, it would, like the photon, travel with the velocity of light.

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It is exceedingly difficult and expensive to carry out experiments with neutrinos, but there is compelling experimental evidence that the electron, muon and tau have different neutrinos, ν_e, ν_μ, ν_τ associated with them.

2.5 The instability of the heavy leptons: muon decay

The W^+ and W^- bosons lead to processes called β-decay, which neither photons nor Z bosons can induce. In this chapter we illustrate this with the example of the β-decay of the muon; in the next chapter we shall describe β-decay processes involving hadrons.

The muon decays to a muon neutrino, together with an electron and an electron anti-neutrino:

$$\mu^- \rightarrow \nu_\mu + e^- + \bar{\nu}_e.$$

The W fields play the mediating role in this decay through the two virtual processes illustrated in Fig. 2.2. Again, in a virtual process actual W bosons do not appear to an observer.

The W bosons can in principle produce any charged lepton and its anti-neutrino or an anti-lepton and its neutrino, but energy must be conserved overall. Hence in the case of muon decay the charged lepton must be an electron. A tau decay can produce a muon or an electron (and indeed it is sufficiently massive to decay alternatively to hadrons).

It is of fundamental significance that electric charge is conserved at every stage of a decay. It is also believed to be true of all interactions that

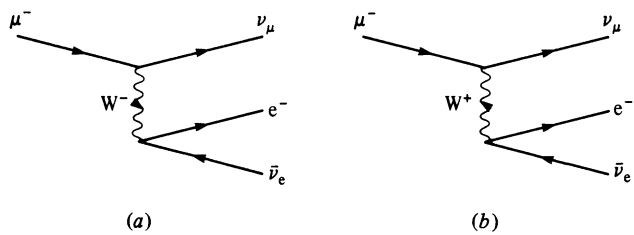


Fig. 2.2 The decay $\mu^- \rightarrow \nu_\mu + e^- + \bar{\nu}_e$. In (*a*) the muon changes to its neutrino and a 'virtual' W^- boson, which then decays to the electron and the electron anti-neutrino. In (*b*) a 'virtual' W^+ is created from the vacuum with the electron and the electron anti-neutrino. The W^+ then transforms the muon into a muon neutrino. In these diagrams, the direction of the arrows on the fermion lines follows the direction of fermion number. (The arrows on anti-particle lines then run backwards in time.)

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to a high degree of approximation a single lepton can only change to another of the same type, and a lepton and an anti-lepton of the same type can only be created or destroyed together. There is thus a conservation law, the '*conservation of lepton number*' (anti-leptons being counted negatively), for each separate type of lepton. The experimental evidence for a possible breakdown of this law will be discussed in Chapter 13.

2.6 Parity violation in muon decay

It is observed experimentally that in the decay of the negative muon, the electron momentum $\mathbf{p}_e$ is strongly biased to be in the direction opposite to that of the muon spin $\mathbf{s}_\mu$. To explain the implication of this observation for parity violation, we must first point out that there are two types of vector.

Under the reflection in the origin (Fig. 1.1), the position vector $\mathbf{r}$ of a particle and its momentum $\mathbf{p}$ transform:

$$\mathbf{r} \to \mathbf{r}' = -\mathbf{r} \quad \text{and} \quad \mathbf{p} = m\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \to \mathbf{p}' = m\frac{-\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -\mathbf{p}, \tag{2.15}$$

$\mathbf{r}$ and $\mathbf{p}$ are both *true vectors*.

The angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ has many of the attributes of a vector, but under reflection

$$\mathbf{L} \to \mathbf{L}' = (-\mathbf{r}) \times (-\mathbf{p}) = +\mathbf{L}.$$

Thus $\mathbf{L}$ does not have the reflection property (2.14) of the true vectors $\mathbf{r}$ and $\mathbf{p}$. It is called an *axial vector* or *pseudo-vector*. The intrinsic angular momentum $\mathbf{s}$ of a particle is likewise an axial vector.

Returning to muon decay, in the reflected coordinate system, $\mathbf{p}_e \to -\mathbf{p}_e$, $\mathbf{s}_\mu \to +\mathbf{s}_\mu$, so that the momentum would be said to be biased in the same direction as the muon spin! It appears that the equations of the theory are only valid in the original right-handed frame, and would have to be rewritten to hold in the left-handed reflected frame. Thus the laws are not invariant under reflection and hence parity is not conserved in muon decay. More generally, parity is not conserved in any process involving the weak interaction fields.

The inequivalence of right-handedness and left-handedness is most extreme in the case of neutrinos. Neutrinos produced in a weak interaction process are always 'left-handed', with their spin anti-parallel to their

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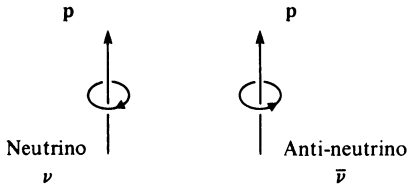


Fig. 2.3 The relation between momentum **p** and spin for a neutrino ν and an anti-neutrino $\bar{\nu}$.

direction of motion, and anti-neutrinos are always 'right-handed' (Fig. 2.3). There is no evidence that right-handed neutrinos (or left-handed anti-neutrinos) exist at all.

The breakdown of parity conservation may be expressed slightly differently. The reflection in the origin $\mathbf{r} \to \mathbf{r}' = -\mathbf{r}$ is easily seen to be equivalent to mirror reflection in a plane, followed by a rotation through π about an axis perpendicular to that plane (e.g. the xy-plane and the z-axis, cf. Problem 1.2). There is no evidence that the laws of physics break down under rotations, so the breakdown is in the mirror reflection: the assumption that the mirror image of a physical process is also a possible physical process is wrong, in so far as the weak interaction is involved.

Problems

2.1 Plane wave solutions of the relativistic wave-equation for a free particle of mass m are of the form

$$\psi(\mathbf{r}, t) = (\text{constant})\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

where

$$\omega^2 = c^2k^2 + (m^2c^4/\hbar^2).$$

Show that the group velocity of a wave-packet representing a particle of total energy $E = \hbar\omega$ is the same as the velocity of a relativistic classical particle having the same total energy.

2.2 The weak charge density of an electron bound in an atom has a similar magnitude to the electric charge density and has, similarly, a probability distribution over the atomic dimensions of the electron's wave-function. Show that the ratio of the weak interaction energy to the electrostatic interaction energy between two electrons bound in an atom is of order of magnitude $4\pi(\hbar/(a_0M_Zc))^2 \sim 10^{-15}$, where a_0 is the Bohr radius. (Compare this result with Problem 1.1.)

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2.3 An electron-positron pair bound by their Coulomb attraction is called *positronium*. Show that when positronium decays from rest to two photons, the photons have equal energy.

2.4 Use energy and momentum conservation to show that pair annihilation with the emission of a single photon, $e^+ + e^- \rightarrow \gamma$, is impossible in free space.

2.5 Show that a muon in free space with a kinetic energy of 1 MeV will travel a mean distance of about 90 m before it decays.

2.6 An electron and a μ^+ bound by the Coulomb attraction is called *muonium*.Which of the following decays can occur?

(*a*) $(\mu^+ e^-) \rightarrow \gamma + \gamma$

(*b*) $(\mu^+ e^-) \rightarrow \nu_e + \bar{\nu}_\mu$

(*c*) $(\mu^+ e^-) \rightarrow e^+ + e^- + \nu_e + \bar{\nu}_\mu$

2.7 The masses of the electron and neutrinos from a muon decay are negligible compared with the muon mass. Show that if the muon decays from rest and the kinetic energy released is divided equally between the final leptons then the angle between the paths of any two of them is approximately 120°.

2.8 Starting from the Coulomb law and the Biot–Savart law, show that the electric field **E** is a true vector field, but that the magnetic field **B** is an axial vector field.

2.9 Show that a muon is more tightly bound in the lowest state of a ^{3}H–muon atom than in the lowest state of a deuteron–muon atom, by about 48 eV. (Note that in the expression $m(e^2/4\pi\varepsilon_0)^2/2\hbar^2$ for the binding energy of a hydrogen-like system, m is the *reduced* mass.) Take the mass of the deuteron to be 1876 MeV and the mass of the triton ^{3}H to be 2809 MeV.

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