Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Beth Woollacott, WAV.1.05, Wavy Top Building, Loughborough University, Loughborough LE11 3TU.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is not necessary to submit solutions to both. Entries should arrive by 20th May 2022 and solutions will be published in the July 2022 edition.

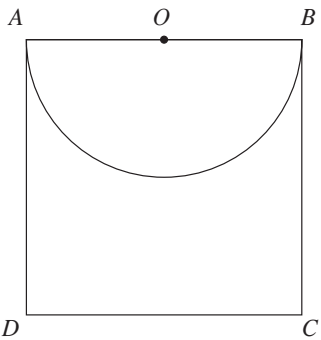
The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the MA website

https://www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others****.*

Problem 2022.1 **(Geoffrey Strickland)**

Given a square *ABCD* with a semi-circle, diameter *AB* and centre *O*, inscribed inside it, show how to construct the golden points on each side of the square using only a straight edge and a pencil.

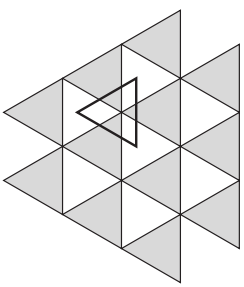


Golden points are points which divide a side internally in the Golden Ratio. There are thus two such points on each side.

Problem 2022.2 **(Paul Stephenson)**

A regular triangular tiling is two-coloured so that adjacent triangles have opposite colours. The grey triangles point left. A free triangle of the same size, also pointing left, may be translated anywhere on the tiling.

What is the smallest fraction of the free triangle which can be grey?



Note from the Editor

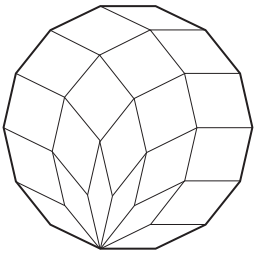
See *www.m-a.org.uk/the-mathematical-gazette* for the SPC questions.

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Solutions to 2021.5 and 2021.6

With thanks to Ewan Collar (BKHS Bablake, Coventry, UK) and Christopher Mattocks (Bishops Wordsworth's Grammar School, Salisbury, UK) for their excellent solutions to both problems.

Problem 2021.5 **(Paul Stephenson)**



The figure shows that a regular polygon with an even number of sides, $2n$, can be dissected into equilateral rhombuses.

Find a simple expression for the total number of rhombus sides, which holds regardless of the arrangement of the rhombuses within the polygon.

Solution (by Paul Stephenson)

Firstly, imagine that each edge comprises two line segments, of which one belongs to the rhombus on one side, the other to the rhombus on the other side. Therefore, at the perimeter, there are $2n$ surplus lines belonging to the exterior.

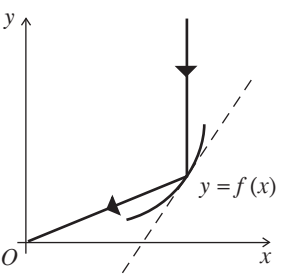
Inspecting the rhombus packing, there are T_{n-1} rhombuses, where $T_k = \frac{1}{2}k(k+1)$. Since each rhombus accounts for 4 line segments, the total number of line segments $= \frac{1}{2} \times 4n(n-1) + 2n = 2n^2$.

Since there are twice as many line segments as edges, there are n^2 edges.

Problem 2021.6 **(Mark Emson)**

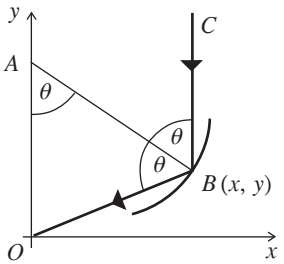
A light ray parallel to the y-axis hits a curve $y = f(x)$ and subsequently passes through the origin O, as shown.

Given that the incident angle is equal to the reflection angle at the tangent, show that $y = f(x)$ is a parabola ($x \neq 0$).



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Solution



Let $B(x, y)$ be a point on $y = f(x)$.

Let the normal to the curve at B meet the y-axis at A and let C be as shown.

$\angle ABC = \angle ABO$ (angle of incidence equals angle of reflection)
$\angle ABC = \angle BAO$ (alternate angles)
$\therefore \angle ABO = \angle BAO$ so that ΔOAB is isosceles.
$\therefore OA = OB = \sqrt{x^2 + y^2}$ (by Pythagoras).

The normal to the curve at B has gradient $\dfrac{\sqrt{x^2 + y^2} - y}{0 - x}$.

Hence the tangent to the curve at B has gradient

$$\frac{x}{\sqrt{x^2 + y^2} - y} = \frac{x\left(\sqrt{x^2 + y^2} + y\right)}{x^2 + y^2 - y^2} = \frac{\sqrt{x^2 + y^2} + y}{x}.$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{x^2 + y^2} + y}{x}$$ which is of homogeneous type.

Let $y = vx$ to get $v + x\dfrac{dv}{dx} = \dfrac{x\sqrt{1 + v^2} + vx}{x} = \sqrt{1 + v^2} + v.$

$$\therefore x\frac{dv}{dx} = \sqrt{1 + v^2} \text{ so that } \int \frac{dx}{x} = \int \frac{dv}{\sqrt{1 + v^2}}.$$

$\therefore \ln x = \ln\left(v + \sqrt{1 + v^2}\right) + \ln k$ where k is a constant.

$$\therefore x = k\left(\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right) \text{ or } cx^2 = y + \sqrt{x^2 + y^2} \quad \left(c = \tfrac{1}{k}\right).$$

$\therefore y - cx^2 = -\sqrt{x^2 + y^2}.$
$\therefore y^2 - 2cx^2y + c^2x^4 = x^2 + y^2.$
$\therefore 2cx^2y = c^2x^4 - x^2.$

$$\therefore y = \frac{cx^2}{2} - \frac{1}{2c}$$, the equation of a parabola.

Prizewinners

The first prize of £25 is awarded to Ewan Collar. The second prize of £20 is awarded to Christopher Mattocks.

Beth Woollacott

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