109.30 A generalisation of Sylvester's theorem with an application

1. *Theorem and Proofs*

Sylvester's theorem concerns the sum of three vectors connecting the centre of the circumcircle with the three vertices of the triangle:

Theorem 1: Let ABC be a triangle with circumcentre O and orthocentre H. Then,

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}. \tag{1}$$

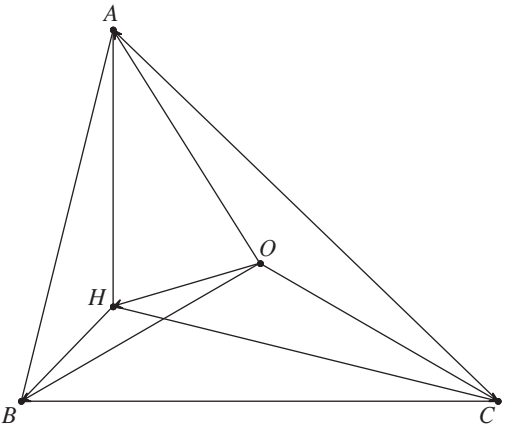


FIGURE 1: Illustration for Theorem 1

This result, as mentioned in [1], is generalised in [2]. In this Note, we present a different generalisation as follows:

Theorem 2: Let ABC be a triangle with circumcentre O and orthocentre H. Consider an arbitrary point P. Let X, Y and Z be the orthogonal projections of P onto lines OA, OB and OC, respectively. Let K be the orthocentre of triangle XYZ. Translate K by the vector $\overrightarrow{OP}$ to obtain L (see Figure 2). Then,

$$\overrightarrow{XA} + \overrightarrow{YB} + \overrightarrow{ZC} = \overrightarrow{LH}.$$

Proof of Theorem 2:

We observe that X, Y and Z lie on the circle with diameter OP, making the midpoint J of OP the circumcentre of triangle XYZ. Applying Sylvester's theorem to triangle XYZ with circumcentre J and orthocentre K, we have

$$\overrightarrow{JX} + \overrightarrow{JY} + \overrightarrow{JZ} = \overrightarrow{JK}.$$

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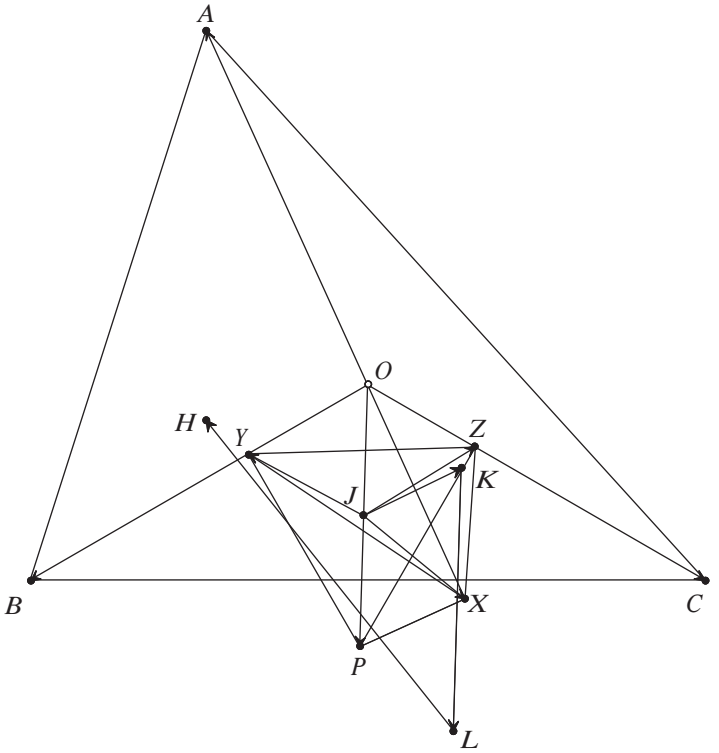


FIGURE 2: Illustration for proof of Theorem 2

Since P is an arbitrary point, inserting the origin P into (1) yields

$$\left(\overrightarrow{PX} - \overrightarrow{PJ}\right) + \left(\overrightarrow{PY} - \overrightarrow{PJ}\right) + \left(\overrightarrow{PZ} - \overrightarrow{PJ}\right) = \left(\overrightarrow{PK} - \overrightarrow{PJ}\right),$$

which is equivalent to

$$\overrightarrow{PX} + \overrightarrow{PY} + \overrightarrow{PZ} = \overrightarrow{PK} + 2\overrightarrow{PJ}. \tag{2}$$

Now, let's apply Sylvester's theorem to triangle ABC with circumcentre O and orthocentre H to obtain

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}.$$

Again, since P is an arbitrary point, inserting the origin P into (1) yields

$$\left(\overrightarrow{PA} - \overrightarrow{PO}\right) + \left(\overrightarrow{PB} - \overrightarrow{PO}\right) + \left(\overrightarrow{PC} - \overrightarrow{PO}\right) = \left(\overrightarrow{PH} - \overrightarrow{PO}\right),$$

which is equivalent to

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = \overrightarrow{PH} + 2\overrightarrow{PO}. \tag{3}$$

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Given the assumption that L is the image of K through translation by vector $\overrightarrow{OP}$, we have

$$\overrightarrow{KL} = \overrightarrow{OP},$$

or

$$\overrightarrow{PO} = \overrightarrow{PK} - \overrightarrow{PL}. \tag{4}$$

Now, we can simplify the vector expressions as follows:

$$\begin{aligned}
\overrightarrow{XA} + \overrightarrow{YB} + \overrightarrow{ZC} &= \left(\overrightarrow{PA} - \overrightarrow{PX}\right) + \left(\overrightarrow{PB} - \overrightarrow{PY}\right) + \left(\overrightarrow{PC} - \overrightarrow{PZ}\right) \\
&= \left(\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}\right) - \left(\overrightarrow{PX} + \overrightarrow{PY} + \overrightarrow{PZ}\right) \\
&= \left(\overrightarrow{PH} + 2\overrightarrow{PO}\right) - \left(\overrightarrow{PK} + 2\overrightarrow{PJ}\right) \text{(using (2) and (3))} \\
&= \left(\overrightarrow{PH} - \overrightarrow{PK}\right) + 2\left(\overrightarrow{PO} - \overrightarrow{PJ}\right) \\
&= \overrightarrow{KH} + 2\overrightarrow{JO} \text{ (since } J \text{ is the midpoint of } PO) \\
&= \overrightarrow{KH} + \overrightarrow{PO} \text{ (because } J \text{ is the midpoint of } PO) \\
&= \overrightarrow{KH} + \left(\overrightarrow{PK} - \overrightarrow{PL}\right) \text{(using equation (4))} \\
&= \left(\overrightarrow{PK} + \overrightarrow{KH}\right) - \overrightarrow{PL} \\
&= \overrightarrow{PH} - \overrightarrow{PL} \\
&= \overrightarrow{LH}.
\end{aligned}$$

This completes the proof of Theorem 2.

Remark

If P coincides with O, then X, Y, Z and L also coincide with O, leading to Theorem 1.

We get a consequence of Theorem 2 as follows:

Consequence 1

Under the same assumptions as in Theorem 2, let G represent the centroid of triangle XYZ. Then, we can express this relationship as

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{LH}.$$

Proof: Given that G is the centroid of triangle XYZ, we can state that

$$3\overrightarrow{OG} = \overrightarrow{OX} + \overrightarrow{OY} + \overrightarrow{OZ}.$$

Referring to the result of Theorem 2, we obtain

$$\overrightarrow{XA} + \overrightarrow{YB} + \overrightarrow{ZC} = \overrightarrow{LH}$$

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and so

$$(\overrightarrow{OA} - \overrightarrow{OX}) + (\overrightarrow{OB} - \overrightarrow{OY}) + (\overrightarrow{OC} - \overrightarrow{OZ}) = \overrightarrow{LH}.$$

Combining these two results, we arrive at:

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} - 3\overrightarrow{OG} = \overrightarrow{LH}.$$

Which can be rewritten as:

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{LH}.$$

This concludes the proof.

Sylvester's theorem can be expressed in the equivalent form

$$\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\overrightarrow{HO}.$$

We can extend this result as follows:

Theorem 3:

Let ABC be a triangle with circumcentre O and orthocentre H. Take an arbitrary point P, and let X, Y and Z be the orthogonal projections of P onto the lines HA, HB and HC, respectively. Define K as the orthocentre of triangle XYZ, and L as the midpoint of PK. Then,

$$\overrightarrow{XA} + \overrightarrow{YB} + \overrightarrow{ZC} = 2\overrightarrow{LO}.$$

Proof: Observe that X, Y and Z lie on the circle with diameter OP, making the midpoint J of OP the circumcentre of triangle XYZ. Thus, for any point P, equation (2) remains valid for triangle XYZ with circumcentre J and orthocentre K, and (3) is also applicable to triangle ABC with circumcentre O and orthocentre H.

With these considerations, we can manipulate the vector expressions as follows:

$$\begin{aligned}
\overrightarrow{XA} + \overrightarrow{YB} + \overrightarrow{ZC} &= \left(\overrightarrow{PA} - \overrightarrow{PX}\right) + \left(\overrightarrow{PB} - \overrightarrow{PY}\right) + \left(\overrightarrow{PC} - \overrightarrow{PZ}\right) \\
&= \left(\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}\right) - \left(\overrightarrow{PX} + \overrightarrow{PY} + \overrightarrow{PZ}\right) \\
&= \left(\overrightarrow{PH} + 2\overrightarrow{PO}\right) - \left(\overrightarrow{PK} + 2\overrightarrow{PJ}\right) \text{ (using (2) and (3))} \\
&= \left(\overrightarrow{PH} - \overrightarrow{PK}\right) + 2\left(\overrightarrow{PO} - \overrightarrow{PJ}\right) \\
&= \left(2\overrightarrow{PJ} - 2\overrightarrow{PL}\right) + 2\overrightarrow{JO} \\
&\qquad\qquad (J \text{ and } L \text{ are the midpoints of } PH \text{ and } OK) \\
&= 2\overrightarrow{LJ} + 2\overrightarrow{JO} \\
&= 2\overrightarrow{LO}.
\end{aligned}$$

This concludes the proof of Theorem 3.

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In a similar manner to the consequence of Theorem 2, we have the following consequence of Theorem 3, along with a similar proof.

Consequence 2

Under the same assumptions as in Theorem 3, let G represent the centroid of triangle XYZ. Then, we can express this relationship as

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = 2\overrightarrow{LO}.$$

2. *An application*

Using Theorem 2 and its Consequence, we have the following application:

Theorem 4

Under the same assumptions as in *Theorem* 2, let G represent the centroid of triangle XYZ. Then, we have the following equation:

$$LH^2 = 3\left(GA^2 + GB^2 + GC^2\right) - \left(a^2 + b^2 + c^2\right). \tag{5}$$

We leave this as an exercise for the reader with the remark that it can be resolved by the Generalised Parallel Axis Theorem (GPAT). A statement and proof of this can be found in [3].

Remark: If point P coincides with O, then the points X, Y, Z, L and G also coincide with O, and the result (5) coincides with the celebrated result

$$OH^2 = 9R^2 - \left(a^2 + b^2 + c^2\right)$$

which is due to Euler.

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