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An entire function with a Baker domain and sparsely distributed singular values

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Abstract

We present an example of an entire function with a Baker domain such that the complement of the set of singular values of the function contains annuli of arbitrarily large width. This is the first known example of a function with this property.

1. *Background*

Let f be a meromorphic function which is not rational of degree one and denote by f^n, $n \in \mathbb{N}$, the nth iterate of f. The *Fatou set*, $F(f)$, is defined to be the set of points, $z \in \mathbb{C}$, such that the sequence $(f^n)_{n\in\mathbb{N}}$ is well-defined, meromorphic and forms a normal family in some neighbourhood of z. The complement, $J(f)$, of $F(f)$ is called the *Julia set* of f. An introduction to the properties of these sets can be found in, for example, [**3**] for rational functions and in [**4**] and [**7**] for transcendental meromorphic functions.

The set $F(f)$ is completely invariant so for any component U of $F(f)$ there exists, for each $n \in \mathbb{N}$, a component U_n of $F(f)$ such that $f^n(U) \subset U_n$. If $U_p = U$ for some minimal $p \in \mathbb{N}$, then we say that U is a periodic component of period p. There are five possible types of periodic components; (see [**4**, theorem 6]). In particular U is called a *Baker domain* if there exists $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$ as $n \to \infty$, for $z \in U$, where $f^p(z_0)$ is not defined. In this case, there is at least one component U_k, $1 \leqslant k \leqslant p$, with the property that $f^{np}(z) \to \infty$ as $n \to \infty$ for $z \in U_k$. If U is a Baker domain of a transcendental *entire* function f, then $f^n(z) \to \infty$ as $n \to \infty$ for $z \in U$ and, moreover, U is simply connected (see [**1**, theorem 5]).

For $p \in \mathbb{N}$, we denote by $sing(f^{-p})$ the set of finite singularities of f^{-p}; that is, the set of points $w \in \mathbb{C}$ such that some branch of f^{-p} cannot be analytically continued through w. The set $sing(f^{-1})$ consists of the critical values and finite asymptotic values of f, and we refer to these points as *singular values of* f.

In [**2**] Bargmann proved the following result.

THEOREM 1. *Let f be an entire function with an invariant Baker domain. Then there exist constants $C > 1$ and $r_0 > 0$ such that*:

$$\{z : r/C < |z| < Cr\} \bigcap sing(f^{-1}) \neq \varnothing, \quad \text{for } r \geqslant r_0. \tag{1·1}$$

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In [**8**] Rippon and Stallard generalized this result to meromorphic functions that have a finite number of poles and a p-cycle of Baker domains, with $sing(f^{-1})$ replaced by $sing(f^{-p})$.

It is natural to ask if condition (1·1) of Theorem 1 is sharp. Many known examples of functions satisfying the hypotheses of Theorem 1 have the property that $sing(f^{-1})$ meets every annulus of some uniform width. For example $f(z) = z + e^{-z} + 1$ has an invariant Baker domain with no finite asymptotic values and critical points $\{2n\pi i : n \in \mathbb{Z}\}$ so $sing(f^{-1}) = \{2n\pi i + 2 : n \in \mathbb{Z}\}$.

In this paper, we give an example of a transcendental entire function with a Baker domain such that $\mathbb{C}\backslash sing(f^{-1})$ contains a sequence of annuli whose widths tend to ∞. This is the first example shown to have this property.

We consider the function

$$f(z) = z + \frac{\sin\sqrt{z}}{\sqrt{z}} + c \tag{1·2}$$

where c is a positive constant. In [**1**] Baker proved that f has an invariant Baker domain for all c sufficiently large. He gave this example in relation to the question of whether functions of order at most $1/2$, minimal type, can have an unbounded Fatou component.

In this paper, we prove:

THEOREM 2. *For the function f defined in* (1·2) *the set $\mathbb{C}\backslash sing(f^{-1})$ contains an infinite sequence of nested annuli $\{A_n\}$ with $A_n = \{z : a_n < |z| < a_n + Kn\}$ where $\{a_n\}$ is an increasing sequence which tends to ∞ and where K is a positive constant.*

Figure 1 presents the result of a computer experiment to estimate the shape of the Julia set (black) of f with $c = 6$.

For large values of c the function f has a Baker domain that is symmetric about the real line and which contains the open interval $(0, \infty)$. For points in the plane near the real axis and far to the right, the magnitude of $\sin\sqrt{z}/\sqrt{z}$ is small and so the function behaves essentially like $z + c$. Thus, for c positive, f acts on points with large magnitude and small argument like a translation to the right.

To identify all the singular values of f, we must consider both finite asymptotic values and critical values. We begin by considering the finite asymptotic values.

2. *Finite asymptotic values of f*

First we note that the order of growth of f is $1/2$. Therefore, by the Denjoy–Carleman–Ahlfors theorem (see [**5**, theorem 5·4], for example) we see that f can have at most one finite asymptotic value. In fact f has no finite asymptotic values, and we prove this here for completeness.

THEOREM 3. *The function $f(z) = z + (\sin\sqrt{z})/\sqrt{z} + c$ has no finite asymptotic values.*

Proof. We claim that if $f(z) = z + (\sin\sqrt{z})/\sqrt{z} + c$ has a finite asymptotic value α along some path γ starting at the point 0 and tending to ∞ then the function q defined as

$$q(w) = f(w^2) \tag{2·1}$$

also has a finite asymptotic value α.

Indeed we can assume that such a path γ is simple by removing any closed curves in γ and replacing them by points (see [**6**, theorem 4·20]). Then $\mathbb{C} \setminus \gamma$ is simply connected so we

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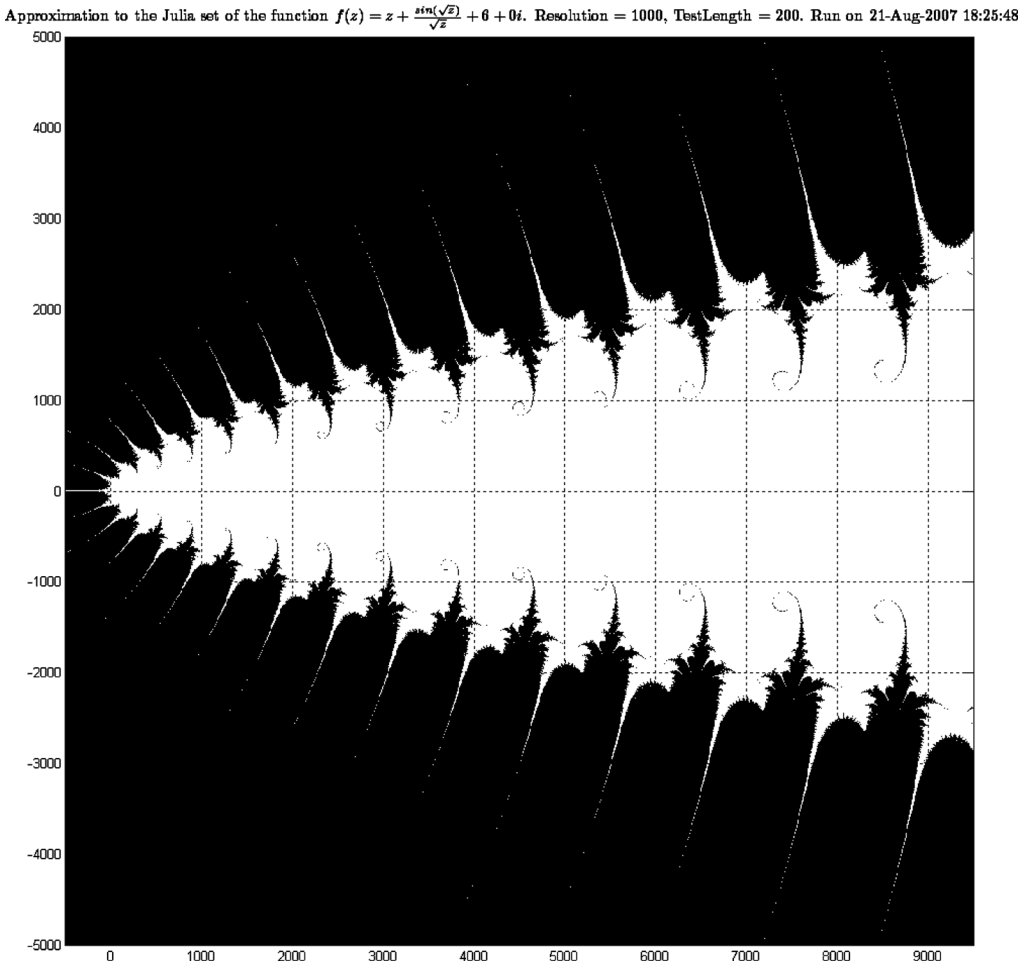


Fig. 1. Julia set of f for $c = 6$.

can define an inverse branch p^{-1} of $p(w) = w^2$ which maps $\mathbb{C} \setminus \gamma$ onto a simply connected domain bounded by a curve consisting of two pre-images of γ one of which we denote γ'. Now as $w \to \infty$ along γ', $p(w) = w^2 \to \infty$ along γ, so $f(w^2) = w^2 + (\sin w)/w + c \to \alpha$ as required. Thus, by disproving the existence of finite asymptotic values of q we can deduce that f also has none. We consider all possible types of paths tending towards infinity and show that for each type there are no finite asymptotic values of q thus proving Theorem 3.

Let γ be a curve in the w-plane tending to infinity. Then at least one of the following must occur:

(i) γ contains a sequence $w_n = u_n + iv_n$ with $|w_n|$ tending to ∞ such that, for some $d > 0$, $|v_n| \leqslant d$ for $n = 1, 2, \ldots$;
(ii) γ contains a sequence $w_n = u_n + iv_n$ with $|w_n|$ tending to ∞ such that, for some $d > 0$, $|u_n| \leqslant d$ for $n = 1, 2, \ldots$;
(iii) for each $a > 0$, $b > 0$, the path γ lies eventually outside the set $D_{a,b} = \{w : |\Re(w)| < a \text{ or } |\Im(w)| < b\}$.

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In case (i), we have

$$|q(w_n)| = \left| w_n^2 + \frac{\sin w_n}{w_n} + c \right| \geqslant |w_n|^2 - \frac{e^{v_n}}{2|w_n|} - \frac{e^{-v_n}}{2|w_n|} - c$$
$$> |w_n|^2 - e^d/|w_n| - c.$$

Thus $|q(w_n)| \to \infty$ as $n \to \infty$, so q does not have a finite asymptotic value along γ in this case.

In case (ii), we have $|v_n| \to \infty$. Without loss of generality, we assume $v_n \to \infty$, so

$$|q(w_n)| \geqslant \frac{e^{v_n}}{2|w_n|} - \frac{e^{-v_n}}{2|w_n|} - |w_n|^2 - c$$
$$\geqslant \frac{e^{v_n}}{2|w_n|} - \frac{e^{-v_n}}{2|w_n|} - d^2 - v_n^2 - c$$
$$\geqslant \frac{e^{v_n}}{2\sqrt{d^2 + v_n^2}} - \frac{1}{2} - d^2 - v_n^2 - c$$

whenever $v_n \geqslant 1$. Thus, the right-hand side certainly tends to infinity as n tends to infinity, so q does not have a finite asymptotic value along γ in this case.

It remains to consider case (iii). In this case we deduce that γ lies eventually in one quadrant. By the symmetry properties of q, that is, $q(-w) = q(w)$ and $q(\overline{w}) = \overline{q(w)}$, it is sufficient to assume that γ lies in the first quadrant. In this case, we must have $\Re(w) \to \infty$ and $\Im(w) \to \infty$ as $w \to \infty$ along γ. Also, eventually we have $\Im(w^2) > 0$. Since $\sin w = \sin u \cosh v + i \cos u \sinh v$ we deduce that $|\sin w| \to \infty$ and $\arg(\sin w) \to \infty$ as $w \to \infty$ along γ. Thus the image of γ under the sine function is a curve that tends to infinity and winds infinitely often around the point 0. Hence γ contains a sequence $w_n = u_n + i v_n$ tending to infinity (which implies that $u_n \to \infty$ and $v_n \to \infty$) such that

$$\arg(\sin w_n) = \frac{\pi}{2} \mod 2\pi$$

so $\Im(q(w_n)) = \Im(w_n^2) + \Im((\sin w_n)/w_n) \geqslant 2u_n v_n \to \infty$ as $n \to \infty$. Hence q does not have a finite asymptotic value along the path γ in this case.

Since there are no finite asymptotic values associated with any of the cases (i), (ii) or (iii), we conclude that q (and thus f too) has no finite asymptotic values so the proof of Theorem 3 is complete.

Remark 1. In Theorem 3 we introduced the function q. We will use q again in the following sections to establish certain additional properties of f.

3. *Critical points of f*

A critical value of a function is the image of a critical point. Unfortunately it is not possible to determine the critical points of f precisely. However, we can estimate their locations (and hence their distribution) with a good level of precision. We will use these estimates to obtain estimates for the locations of the critical values of the function f.

The critical points of f are the solutions of

$$f'(z) = 1 + \frac{\sqrt{z}\cos\sqrt{z} - \sin\sqrt{z}}{2z^{3/2}} = 0. \tag{3·1}$$

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To estimate their locations we make the substitution $w = \sqrt{z}$, the principal square root, to obtain the equation

$$1 + \frac{w\cos w - \sin w}{2w^3} = 0. \tag{3·2}$$

Dividing by w^2 and defining

$$h(w) = w^2 + \frac{\cos w}{2} - \frac{\sin w}{2w} = w^2 + \frac{1}{4w}(w+i)e^{iw} + \frac{1}{4w}(w-i)e^{-iw}, \tag{3·3}$$

we see that h vanishes precisely at the critical points of the function q defined in (2·1).

Since h is symmetric in the sense that $h(-w) = h(w)$ and $h(\overline{w}) = \overline{h(w)}$, we need only locate the zeros of h in $Q = \{w : \Re(w) \geqslant 0, \Im(w) \geqslant 0\}$.

The function

$$\widetilde{h}(w) = w^2 + \frac{1}{4}e^{-iw} \tag{3·4}$$

is a good approximation to h in Q since $h(w) - \widetilde{h}(w) = e^{iw}/4 + ie^{iw}/4w - ie^{-iw}/4w$ and these three terms are all small in magnitude compared to those of $\widetilde{h}(w)$ when $\Re(w)$ is positive and $\Im(w)$ is large. In particular, when $\Im(w)$ is very large, the zeros of $\widetilde{h}$ approximate those of h with considerable precision.

We start by finding the zeros of $\widetilde{h}$ and then show that all but finitely many zeros of h are close to the zeros of $\widetilde{h}$.

The zeros of $\widetilde{h}$ in Q satisfy

$$\arg\left(w^2\right) = \arg\left(-\frac{1}{4}e^{-iw}\right) \tag{3·5}$$

and

$$\left|w^2\right| = \left|-\frac{1}{4}e^{-iw}\right|. \tag{3·6}$$

For $w = u + iv \in Q$, these conditions can be expressed as

$$2\arg w = (-u + \pi) \mod 2\pi \tag{3·7}$$

and

$$|w|^2 = \frac{1}{4}e^v. \tag{3·8}$$

These equations define curves in the w-plane and we see that the zeros of $\widetilde{h}$ lie at the intersections of these curves, where both conditions are satisfied simultaneously.

Since $\arg w = \tan^{-1}(v/u)$, we can write (3·7) as

$$\frac{v}{u} = \tan\left(\frac{1}{2}(-u+\pi)\right), \text{ that is, } v = u\cot\left(\frac{1}{2}u\right). \tag{3·9}$$

Therefore we can express the family of curves defined by (3·7) using the parametrisation $\sigma(u) = u + iu\cot(u/2)$, whose image has one connected component for each interval $2n\pi < u < (2n+2)\pi$, $n \in \mathbb{N}$.

Next we consider the curve defined by (3·8). This can be written as

$$\gamma_4(v) = \sqrt{\frac{e^v}{4} - v^2} + iv.$$

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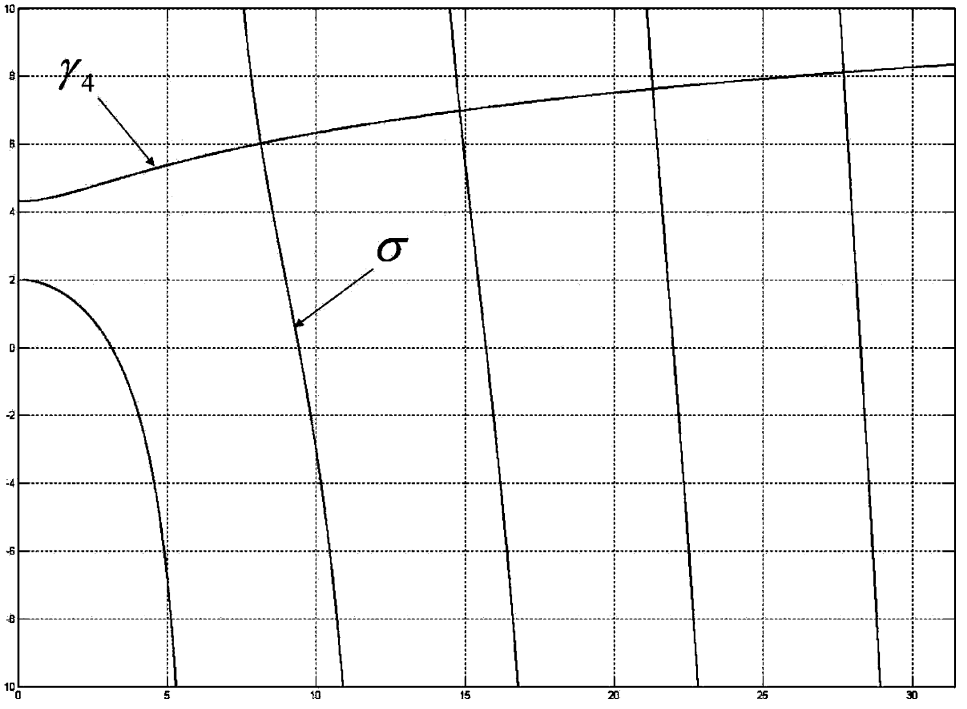


Fig. 2. The curves γ_4 and σ.

Figure 2 indicates the geometry of these two curves.

In what follows we will study the properties of $\widetilde{h}$ and h near to γ_4. To establish these properties we define a family of curves

$$\gamma_\alpha(v) = \sqrt{\frac{e^v}{\alpha} - v^2} + iv \tag{3·10}$$

parametrised by α. As the value of α varies continuously through 4, the curve γ_α sweeps out a region of the complex plane containing γ_4.

Note that if $0 < \alpha < \alpha'$, then γ_α lies to the right of $\gamma_{\alpha'}$. We proceed by describing certain other properties of the family γ_α. We prove

LEMMA 1. *Let $w_0 = u_0 + iv_0 \in Q$ with $u_0 > 0$ and $v_0 > 2$. Then there exists a unique $\alpha_0 > 0$ such that w_0 lies on the curve γ_{α_0} and $\Re(\gamma_{\alpha_0})$ is strictly increasing with v on a maximal interval of the form $[v_{\alpha_0}, \infty)$, such that $v_0 > v_{\alpha_0} \geqslant 2$. If $v_{\alpha_0} > 2$, then γ_{α_0} meets the imaginary axis (at iv_{α_0}).*

Proof. We have $u_0^2 + v_0^2 > 0$, so there exists a unique α_0 such that $\alpha_0\left(u_0^2 + v_0^2\right) = e^{v_0}$. Thus

$$u_0^2 = \frac{e^{v_0}}{\alpha_0} - v_0^2, \quad \text{so } u_0 = \sqrt{\frac{e^{v_0}}{\alpha_0} - v_0^2}. \tag{3·11}$$

Hence w_0 lies on the graph

$$u = \psi_{\alpha_0}(v) \equiv \sqrt{\frac{e^v}{\alpha_0} - v^2} \tag{3·12}$$

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and, since the inequality $e^{v_0}/\alpha_0 - v_0^2 > 0$ is strict, this graph exists for values of v that include an open neighbourhood of v_0. Now e^v/v^2 is strictly increasing for $v \geqslant 2$, so

$$\frac{e^v}{\alpha_0} - v^2 = v^2\left(\frac{e^v}{\alpha_0 v^2} - 1\right)$$

is strictly increasing on any interval where $e^v/(\alpha_0 v^2) - 1 > 0$ in $[2, \infty)$. Thus the graph $u = \psi_{\alpha_0}(v)$ exists and is strictly increasing on a maximal interval of the form $[v_{\alpha_0}, \infty)$ such that $v_0 > v_{\alpha_0} \geqslant 2$. The final statement holds because if $v_{\alpha_0} > 2$, then $\psi_{\alpha_0}(v_{\alpha_0}) = 0$.

It is straightforward to check that $v_4 \approx 4.3 > 2$, so the curve γ_4 meets the imaginary axis. From the form of the curves defined in (3·8) and (3·9), we conclude that the zeros of $\widetilde{h}$ in Q all lie on the curve γ_4 and so, for each $n \geqslant 0$, there is one zero, $\widetilde{w}_n$ say, in the set $\{w : 2n\pi < \Re(w) < 2n\pi + 2\pi\}$ with

$$\Re(\widetilde{w}_n) - 2n\pi \longrightarrow \pi \text{ as } n \longrightarrow \infty. \tag{3·13}$$

Now we fix ε such that $0 < \varepsilon < \pi/4$. It follows from (3·13) that there exists $N_0 \in \mathbb{N}$ such that

$$(2n+1)\pi - \varepsilon < \Re(\widetilde{w}_n) < (2n+1)\pi + \varepsilon, \quad \text{for all } n > N_0. \tag{3·14}$$

In the remainder of this section we will show that there exists $N \in \mathbb{N}$ such that, for all $n \geqslant N$, exactly one zero of h (a critical point of q) lies in a small neighbourhood of $\widetilde{w}_n$ and that there are only finitely many other zeros of h.

We define the following three regions.

$$E_1^b = \{w : w \in Q, \Im(w) < b\}, \quad \text{where } b > 0, \tag{3·15}$$

$$E_2^r = \{w : w \in Q, |w| < r\}, \quad \text{where } r > 0, \tag{3·16}$$

and

$$E_3 = \{w : w \in \gamma_\alpha(v), 4 - \varepsilon < \alpha < 4 + \varepsilon, v > v_\alpha\}. \tag{3·17}$$

These three regions are indicated in Figure 3.

We begin by proving:

LEMMA 2. *There exist constants v_{min}, $R_1 > 0$ such that there are no zeros of h in $D = Q \setminus (E_1^{v_{min}} \cup E_2^{R_1} \cup E_3)$.*

Proof. From Lemma 1, we see that whenever $u \geqslant 0$ and $v > 2$, there exists a unique α such that $w = u + iv \in \gamma_\alpha$.

We now define the following sets, using the function ψ_{α_0} defined in (3·12) with α_0 replaced by $4 + \varepsilon$ and $4 - \varepsilon$:

$$A = \{w = u + iv \in Q : u < \psi_{4+\varepsilon}(v), v > v_{4+\varepsilon}\}$$

and

$$B = \{w = u + iv \in Q : u > \psi_{4-\varepsilon}(v), v > v_{4-\varepsilon}\}$$

so that when $v_{min} > v_{4-\varepsilon}$ we have $D \subset A \cup B$.

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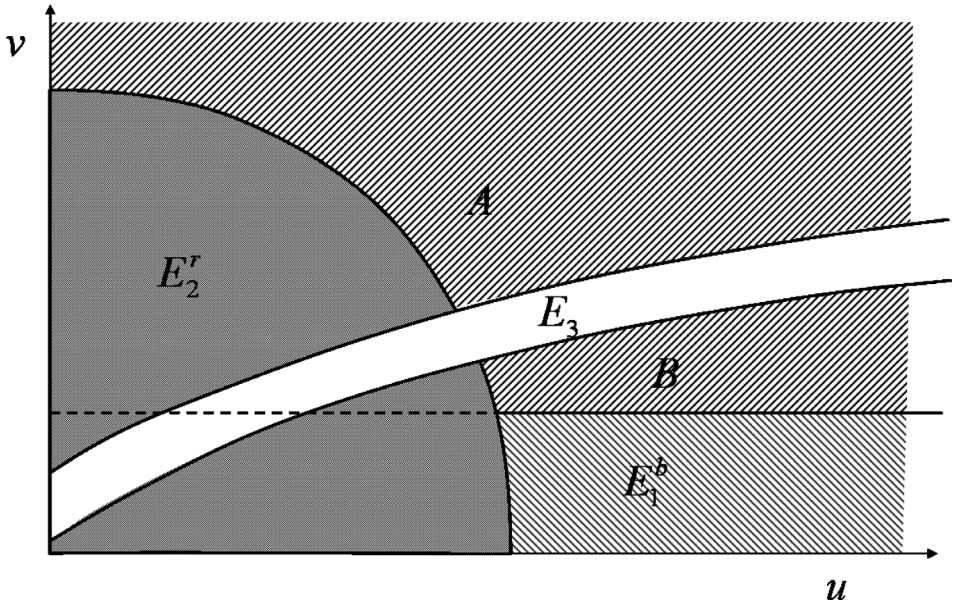


Fig. 3. The three regions E_1, E_2 and E_3.

Let $w \in A$, so $w \in \gamma_\alpha$ where $\alpha > 4 + \varepsilon$. To show that $h(w) \neq 0$ it is sufficient (from the definition of h in (3·3)) to show that

$$\frac{1}{4}\left|e^{-iw}\right| - |w|^2 > \frac{1}{4}e^{-v} + \frac{1}{4|w|}e^{-v} + \frac{1}{4|w|}e^{v}.$$

Using the substitution $|w|^2 = e^v/\alpha$ we see that this inequality is equivalent to

$$\left(\frac{1}{4} - \frac{1}{\alpha}\right)e^{v} > \frac{1}{4}e^{-v} + \frac{1}{4|w|}e^{-v} + \frac{1}{4|w|}e^{v}.$$

Now $\alpha > 4+\varepsilon$, so whenever $|w| > \max\{1, 2(4+\varepsilon)/\varepsilon\}$, we see that it is sufficient to show that

$$\frac{\varepsilon}{2(4+\varepsilon)}e^{2v} > 2.$$

Taking logs, we see that the inequality is true, and so $h(w) \neq 0$, provided that $v > (1/2)\ln(4(4+\varepsilon)/\varepsilon)$.

Now suppose $w \in B$. Then $w \in \gamma_\alpha$ where $\alpha < 4-\varepsilon$. By a similar manipulation as above we obtain the result that $h(w) \neq 0$ whenever $v > (1/2)\ln(4(4-\varepsilon)/\varepsilon)$ and $|w| > \max\{1, 2(4-\varepsilon)/\varepsilon\}$.

Since $\varepsilon < \pi/4$ we have $1 < 2(4+\varepsilon)/\varepsilon$. Thus by taking $R_1 = 2(4+\varepsilon)/\varepsilon$ and $v_{min} = \max\{v_{4-\varepsilon}, (1/2)\ln(4(4+\varepsilon)/\varepsilon)\}$ we see that the proof is complete.

LEMMA 3. *There exists $u_{min} > 0$ such that there are no zeros of h in $\{w : 0 \leqslant \Im(w) \leqslant v_{min}, \Re(w) \geqslant u_{min}\}$.*

Proof. From the definition of h in (3·3) we have

$$|h(w)| \geqslant |w|^2 - \frac{1}{4}e^{-v}\left(1 + \frac{1}{|w|}\right) - \frac{1}{4}e^{v}\left(1 + \frac{1}{|w|}\right). \tag{3·18}$$

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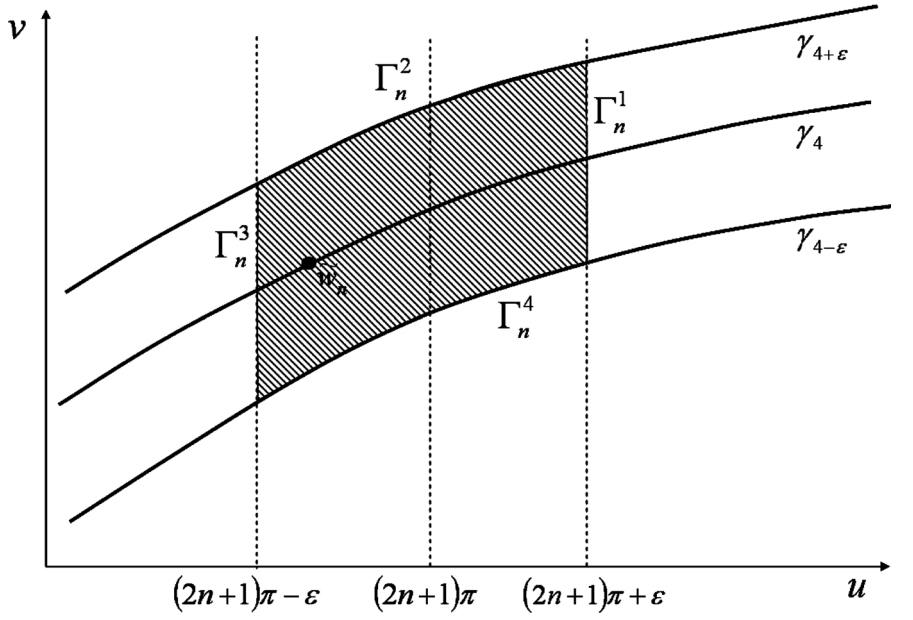


Fig. 4. Geometry of Γ_n.

Now, whenever $|w| > 1$ we have $(1 + 1/|w|) < 2$ so,

$$|h(w)| \geqslant |w|^2 - \frac{1}{2}e^{-v} - \frac{1}{2}e^{v} \geqslant |w|^2 - e^{v}, \quad \text{for } w \in Q.$$

If $0 \leqslant v \leqslant v_{min}$, we deduce that $|h(w)| \geqslant |w|^2 - e^{v_{min}}$. Thus $h(w) \neq 0$ whenever

$$|w|^2 = u^2 + v^2 > e^{v_{min}}. \tag{3·19}$$

So, it is sufficient to take $u_{min} = \max\{1, e^{v_{min}/2}\}$.

We now show that for all n sufficiently large there is exactly one zero, w_n say, of h in Q located close to each zero $\widetilde{w}_n$ of $\widetilde{h}$. We do this by using Rouché's Theorem.

To apply Rouché's Theorem near $\widetilde{w}_n$, we construct a closed curve Γ_n surrounding $\widetilde{w}_n$ on which $|\widetilde{h}(w) - h(w)| < |\widetilde{h}(w)|$ holds, so h also has exactly one zero inside Γ_n.

By (3·3) and (3·4) we see that it is sufficient to construct a curve Γ_n such that for all $w \in \Gamma_n$ we have

$$\frac{1}{4}e^{-v} + \frac{1}{4|w|}e^{-v} + \frac{1}{4|w|}e^{v} < \left| w^2 + \frac{1}{4}e^{-iw} \right|. \tag{3·20}$$

Definition 1. Let Γ_n be the closed bounded curve determined by the four unbounded curves

$$u = 2n\pi + \pi + \varepsilon, \quad \gamma_{4+\varepsilon}, \quad u = 2n\pi + \pi - \varepsilon \quad \text{and} \quad \gamma_{4-\varepsilon},$$

and let Γ_n^1, Γ_n^2, Γ_n^3 and Γ_n^4 be the parts of Γ_n lying on the above four curves respectively; see Figure 4. Note that, by (3·14), $\widetilde{h}$ has exactly one zero, $\widetilde{w}_n$, inside Γ_n whenever $n > N_0$.

First we check (3·20) on Γ_n^1.

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LEMMA 4. *There exists $M_1 \in \mathbb{N}$ such that whenever $n > M_1$, the inequality* (3·20) *holds on* Γ_n^1.

Proof. For all $w \in \Gamma_n^1$, we have $\Re w = (2n+1)\pi + \varepsilon$ and $\Im w > 0$, so $\arg(e^{-iw}/4) = \pi - \varepsilon$ and $\Im(w^2) > 0$. Thus

$$\left|w^2 + \frac{1}{4}e^{-iw}\right| \geqslant \Im\left(w^2 + \frac{1}{4}e^{-iw}\right) > \Im\left(\frac{1}{4}e^{-iw}\right) = \frac{1}{4}e^v \sin\varepsilon, \quad \text{for } w \in \Gamma_n^1.$$

Thus it is sufficient to show that if n is large enough and $w \in \Gamma_n^1$, then

$$\frac{1}{4}e^{-v} + \frac{1}{4|w|}e^{-v} + \frac{1}{4|w|}e^v < \frac{1}{4}e^v \sin\varepsilon;$$
$$\text{that is, } \left(1 + \frac{1}{|w|}\right) < \left(\sin\varepsilon - \frac{1}{|w|}\right)e^{2v}.$$

This inequality is certainly true if $|w| > \max\{1, 2/\sin\varepsilon\}$ and $4 < e^{2v}\sin\varepsilon$. We can arrange for these two conditions to be true for all $w \in \Gamma_n^1$ and all $n > M_1$ by choosing M_1 large enough, and this completes the proof.

Next we check (3·20) on Γ_n^3.

LEMMA 5. *There exists $M_3 \in \mathbb{N}$ such that whenever $n > M_3$, the inequality* (3·20) *holds on* Γ_n^3.

Proof. Let $w \in \Gamma_n^3$. We consider the term on the right-hand side of (3·20) and we derive a lower bound expressed in terms of an exponential only. First note that, for all $w \in \Gamma_n^3$ we have $\Re w = (2n+1)\pi - \varepsilon$, so

$$\left|w^2 + \frac{1}{4}e^{-iw}\right| = \left|-\frac{1}{4}e^{-iw} - w^2\right| \geqslant \Im\left(-\frac{1}{4}e^{-iw} - w^2\right)$$
$$= \frac{1}{4}e^v \sin\varepsilon - \left|w^2\right| \sin(\arg(w^2)).$$

We deduce from the definition of $\gamma_{4+\varepsilon}$ (equation (3·10)) that $\max\{\arg(w^2) : w \in \Gamma_n^3\}$ tends to zero from above as $n \to \infty$. Thus there exists $N_1 \in \mathbb{N}$ such that $\arg(w^2) < \varepsilon/2$ whenever $w \in \Gamma_n^3$ for all $n \geqslant N_1$. So, whenever $n \geqslant N_1$, we have

$$\left|w^2 + \frac{1}{4}e^{-iw}\right| \geqslant \frac{1}{4}e^v \sin\varepsilon - |w|^2 \sin\frac{\varepsilon}{2}.$$

Now whenever $0 < \varepsilon < \pi/4$, we have $\cos(\varepsilon/2) > 9/10$ so $\sin(\varepsilon) \geqslant (9/5)\sin(\varepsilon/2)$, and hence

$$\left|w^2 + \frac{1}{4}e^{-iw}\right| \geqslant \left(\frac{e^v}{4} - \frac{5|w|^2}{9}\right)\sin\varepsilon.$$

Next, since $w \in \gamma_\alpha$ for some $\alpha \geqslant 4 - \varepsilon$, we have

$$|w|^2 \leqslant \frac{e^v}{4-\varepsilon},$$

so

$$\left|w^2 + \frac{1}{4}e^{-iw}\right| \geqslant \left(\frac{1}{4} - \frac{5}{9(4-\varepsilon)}\right)e^v \sin\varepsilon.$$

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Since $0 < \varepsilon < 1$ we deduce that

$$\left| w^2 + \frac{1}{4}e^{-iw} \right| \geqslant \left(\frac{1}{4} - \frac{5}{27} \right) e^v \sin\varepsilon > \frac{1}{20} e^v \sin\varepsilon, \quad \text{for } w \in \Gamma_n^3.$$

Using the same re-arrangements as in Lemma 4 we deduce that it is sufficient to show that

$$\left(1 + \frac{1}{|w|} \right) < \left(\frac{\sin\varepsilon}{5} - \frac{1}{|w|} \right) e^{2v}, \quad \text{for all } w \in \Gamma_n^3.$$

This inequality is certainly true if $|w| > \max\{1, 10/\sin\varepsilon\}$ and $20 < e^{2v}\sin\varepsilon$. We can arrange for these two conditions to be true for all $w \in \Gamma_n^1$ and all $n > N_2$ by choosing N_2 large enough. Then, taking $M_3 \equiv \max\{N_1, N_2\}$, the proof is complete.

Finally we check (3·20) on $\gamma_{4+\varepsilon}$ and $\gamma_{4-\varepsilon}$.

LEMMA 6. *There exists $R_2 > 0$ such that the inequality* (3·20) *holds for all w in $\gamma_{4-\varepsilon} \cup \gamma_{4+\varepsilon}$ such that $|w| > R_2$.*

Proof. On $\gamma_{4+\varepsilon}$ and $\gamma_{4-\varepsilon}$ we have $|w|^2 = e^v/(4+\varepsilon)$ and $|w|^2 = e^v/(4-\varepsilon)$ respectively. Substituting for $|w|^2$, we can estimate the right-hand side of (3·20) by

$$\left| w^2 + \frac{1}{4}e^{-iw} \right| \geqslant \frac{\varepsilon}{4(4+\varepsilon)} e^v, \quad \text{for } w \in \gamma_{4+\varepsilon},$$

and

$$\left| w^2 + \frac{1}{4}e^{-iw} \right| \geqslant \frac{\varepsilon}{4(4-\varepsilon)} e^v, \quad \text{for } w \in \gamma_{4-\varepsilon}.$$

Thus, when $|w| > \max\{1, 2(4+\varepsilon)/\varepsilon\}$, inequality (3·20) holds as long as

$$2 < \frac{\varepsilon}{2(4+\varepsilon)} e^{2v} \text{ on } \gamma_{4+\varepsilon} \quad \text{and} \quad 2 < \frac{\varepsilon}{2(4-\varepsilon)} e^{2v} \text{ on } \gamma_{4-\varepsilon},$$

that is,

$$v > \frac{1}{2}\ln(4(4+\varepsilon)) - \frac{1}{2}\ln\varepsilon \equiv c_1 \text{ on } \gamma_{4+\varepsilon} \text{ and } v > \frac{1}{2}\ln(4(4-\varepsilon)) - \frac{1}{2}\ln\varepsilon \equiv c_2 \text{ on } \gamma_{4-\varepsilon}.$$

From the form of the curves γ_α it is clear that there exists some $R_2 > \max\{1, 2(4+\varepsilon)/\varepsilon\}$ such that $v > \max\{c_1, c_2\}$ for $w \in \gamma_{4-\varepsilon} \cup \gamma_{4+\varepsilon}$ whenever $|w| > R_2$. Thus the proof is complete.

It follows from Lemmas 4, 5 and 6 that there is exactly one zero of h in Γ_n for each $n > \max\{N_0, M_1, R_2/2\pi, M_3\}$.

To show that there are no other zeros of h near γ_4 we note that equation (3·20) holds on the closed curve defined by Γ_n^1, Γ_{n+1}^3, $\gamma_{4-\varepsilon}$ and $\gamma_{4+\varepsilon}$ for all $n > \max\{N_0, M_1, R_2/2\pi, M_3\}$. Since there are no zeros of $\widetilde{h}$ in the interior of this closed curve, by Rouché's Theorem, there are no zeros of h there either. Thus, we have proved

THEOREM 4. *There exists $N \in \mathbb{N}$ such that Γ_n contains exactly one critical point w_n of q for all $n \geqslant N$. There are only finitely many other critical points of q.*

4. *Distribution of the Critical Values of f*

In this section we complete the proof of Theorem 2 by estimating the positions of the critical values of f. Since the non-zero critical values of the function q as defined in (2·1) are equal to the critical values of f, it is sufficient to work with q.

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In Theorem 4 we only address the critical points of q in Q. For each critical point w in Q there are three additional corresponding critical points contained in the other three quadrants which are given by $-w$, $\overline{w}$ and $-\overline{w}$. The corresponding critical values are $q(-w)=q(w)$ and $q(\pm\overline{w})=\overline{q(w)}$. Therefore, to prove Theorem 2 it is sufficient to obtain estimates of the images of the critical points lying in Q.

We proceed by obtaining an asymptotic formula for the positions of the points $q(w_n)$ where $w_n=u_n+iv_n$ is as defined in Theorem 4. From the definition of q, we have

$$\begin{aligned} q(w_n) &= w_n^2+\frac{1}{2iw_n}\left(e^{iw_n}-e^{-iw_n}\right)+c \\ &= w_n^2+\frac{1}{2i}\frac{1}{|w_n|^2}\overline{w}_n\left(e^{-v_n}e^{iu_n}-e^{v_n}e^{-iu_n}\right)+c \\ &= w_n^2-\frac{\overline{w}_n}{2i}\frac{e^{v_n}}{|w_n|^2}e^{-iu_n}+\frac{\overline{w}_n}{2i}\frac{e^{-v_n}}{|w_n|^2}e^{iu_n}+c. \end{aligned}$$

Since $v_n\to\infty$ as $n\to\infty$, we have

$$q(w_n)=w_n^2-\frac{\overline{w}_n}{2i}\frac{e^{v_n}}{|w_n|^2}e^{-iu_n}+O(1)\text{ as }n\longrightarrow\infty.$$

Since w_n lies inside Γ_n, we can write

$$q(w_n)=w_n^2-\frac{\overline{w}_n}{2i}\alpha_n e^{-iu_n}+O(1)=w_n^2-\frac{\overline{w}_n}{2i}\alpha_n e^{-i\theta_n}+O(1)\text{ as }n\longrightarrow\infty, \tag{4·1}$$

where $|\alpha_n-4|<\varepsilon$ and $u_n=(2n+1)\pi+\theta_n$ for some θ_n satisfying $|\theta_n|<\varepsilon$.

Now we can estimate the sizes of a family of annuli centred at the origin which contain no critical values of q. We do this by estimating the difference between the modulii of adjacent critical values as n gets large.

We begin with the following two results.

LEMMA 7. *For each $n\geqslant N$, let $w_n=u_n+iv_n$ be as defined in Theorem 4 and let $\widehat{v}_n$ be such that $\widehat{w}_n=u_n+i\widehat{v}_n\in\gamma_4$. Then, for all sufficiently large n, $|v_n-\widehat{v}_n|<\varepsilon$.*

Proof. Let $w=u+iv$ be any point inside Γ_n and let $\widehat{w}=u+i\widehat{v}\in\gamma_4$ as indicated in Figure 5. We show that if n is sufficiently large, then the point $w^+=u+i(\widehat{v}+\varepsilon)$ lies above the curve $\gamma_{4+\varepsilon}$ and the point $w^-=u+i(\widehat{v}-\varepsilon)$ lies below the curve $\gamma_{4-\varepsilon}$.

Since $\widehat{v}+\varepsilon>2$, it follows from Lemma 1 that there exists a unique α such that $w^+\in\gamma_\alpha$. It is sufficient to show that $\alpha>4+\varepsilon$.

Since w^+ lies in γ_α we can write

$$\begin{aligned} \alpha &= \frac{e^{\widehat{v}+\varepsilon}}{|w^+|^2}=\frac{e^{\widehat{v}}e^{\varepsilon}}{u^2+\widehat{v}^2+2\widehat{v}\varepsilon+\varepsilon^2} \\ &= \left(\frac{e^{\widehat{v}}}{u^2+\widehat{v}^2}\right)\left(\frac{e^{\varepsilon}\left(u^2+\widehat{v}^2\right)}{u^2+\widehat{v}^2+2\widehat{v}\varepsilon+\varepsilon^2}\right)=4\left(\frac{e^{\varepsilon}}{1+\varepsilon\left(\dfrac{2\widehat{v}+\varepsilon}{u^2+\widehat{v}^2}\right)}\right). \end{aligned}$$

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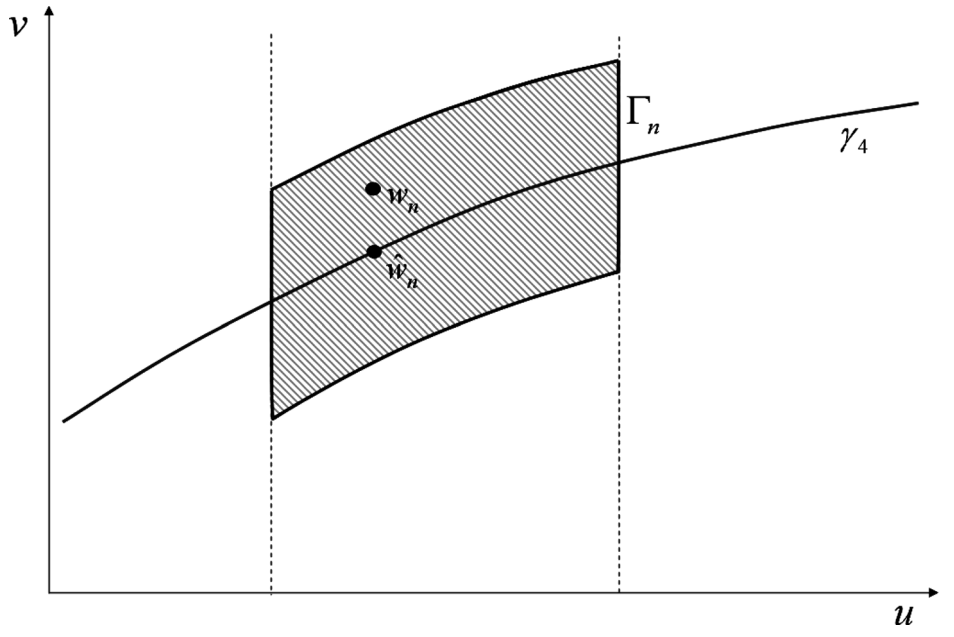


Fig. 5. The points w_n and $\hat{w}_n$ from Lemma 7.

So it is sufficient to show that

$$\frac{e^{\varepsilon}}{1+\varepsilon\left(\dfrac{2\widehat{v}+\varepsilon}{u^2+\widehat{v}^2}\right)} > 1+\frac{\varepsilon}{4}.$$

Now $e^{\varepsilon} > 1+\varepsilon$ and, for large n, we have $(2\widehat{v}+\varepsilon)/(u^2+\widehat{v}^2) < 1/2$ so

$$\frac{e^{\varepsilon}}{1+\varepsilon\left(\dfrac{2\widehat{v}+\varepsilon}{u^2+\widehat{v}^2}\right)} > \frac{1+\varepsilon}{1+\frac{\varepsilon}{2}} = 1+\frac{\varepsilon}{2+\varepsilon} > 1+\frac{\varepsilon}{4},$$

as required. Note that in the last inequality we have used the fact that $\varepsilon < \pi/4$. A similar argument shows that w^- lies on γ_α for some α satisfying $\alpha < 4-\varepsilon$.

Since $w \in \Gamma_n$ was chosen arbitrarily, and since w_n lies inside Γ_n, we see that the proof is complete.

LEMMA 8. *For each $n \geqslant N$, let $w_n = u_n + i v_n$ be as defined in Theorem 4. Then $|v_n| = o(|u_n|)$ as $n \to \infty$.*

Proof. Let $w = u + iv \in E_3$. Then there exists some α satisfying $|\alpha - 4| < \varepsilon$ such that $w \in \gamma_\alpha$. Thus

$$\frac{|v|}{|u|} = \left|\frac{v}{\sqrt{e^v/\alpha - v^2}}\right| = \left|\frac{1}{\sqrt{(e^v/(\alpha v^2) - 1)}}\right| \to 0 \text{ as } v \to \infty. \tag{4·2}$$

Since $w_n \in E_3$ for each $n \geqslant N$, and $v_n \to \infty$ as $n \to \infty$, the result follows.

We are now in a position to estimate the difference between the modulii of adjacent critical values. Using (4·1) we can write

$$|q(w_{n+1})| \geqslant |w_{n+1}|^2 - \frac{\alpha_{n+1}}{2}|\overline{w}_{n+1}| + O(1) \text{ as } n \longrightarrow \infty$$

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and

$$|q(w_n)| \leqslant |w_n|^2 + \frac{\alpha_n}{2}|\overline{w}_n| + O(1) \text{ as } n \longrightarrow \infty.$$

Combining these estimates we have

$$|q(w_{n+1})| - |q(w_n)| \geqslant |w_{n+1}|^2 - |w_n|^2 - \frac{\alpha_{n+1}}{2}|w_{n+1}| - \frac{\alpha_n}{2}|w_n| + O(1) \text{ as } n \longrightarrow \infty. \tag{4·3}$$

Now, since $\alpha_n \leqslant 4 + \varepsilon$ for all $n \geqslant N$, we deduce from (4·3) that

$$\begin{aligned} |q(w_{n+1})| - |q(w_n)| &\geqslant |w_{n+1}|^2 - |w_n|^2 - \frac{4+\varepsilon}{2}(|w_{n+1}| + |w_n|) + O(1) \\ &\geqslant |w_{n+1}|^2 - |w_n|^2 - \left(2 + \frac{\pi}{8}\right)(|w_{n+1}| + |w_n|) + O(1) \text{ as } n \longrightarrow \infty, \end{aligned} \tag{4·4}$$

where we have used the fact that $\varepsilon < \pi/4$.

Now we consider the expression $|w_{n+1}|^2 - |w_n|^2$ in (4·4) above. Multiplying out, we have

$$\begin{aligned} |w_{n+1}|^2 - |w_n|^2 &= u_{n+1}^2 + v_{n+1}^2 - u_n^2 - v_n^2 \\ &= \left(u_{n+1}^2 - u_n^2\right) + \left(v_{n+1}^2 - v_n^2\right) \\ &= (u_{n+1} - u_n)(u_{n+1} + u_n) + (v_{n+1} - v_n)(v_{n+1} + v_n). \end{aligned}$$

Since w_n is in the interior of Γ_n, we have $2n\pi + 3\pi/4 < u_n < 2n\pi + 5\pi/4$, so

$$\begin{aligned} |w_{n+1}|^2 - |w_n|^2 &\geqslant \frac{3\pi}{2}\left((2n+2)\pi + \frac{3\pi}{4} + 2n\pi + \frac{3\pi}{4}\right) + (v_{n+1} - v_n)(v_{n+1} + v_n) \\ &= \left(6n\pi^2 + \frac{21\pi^2}{4}\right) + (v_{n+1} - v_n)(v_{n+1} + v_n). \end{aligned}$$

Now by Lemma 7, since $\widehat{v}_{n+1} > \widehat{v}_n$, we have $v_{n+1} - v_n \geqslant -2\varepsilon \geqslant -\pi/2$, so

$$|w_{n+1}|^2 - |w_n|^2 \geqslant \left(6n\pi^2 + \frac{21\pi^2}{4}\right) - \frac{\pi}{2}(v_{n+1} + v_n).$$

Since $v_n = o(|u_n|)$ as $n \to \infty$ by Lemma 8, and since $u_n, v_n > 0$ for all n, we deduce that for all sufficiently large n we have $v_n < 2u_n/\pi^3$, so

$$\frac{\pi}{2}(v_{n+1} + v_n) < \frac{\pi}{2}\frac{2}{\pi^3}(u_{n+1} + u_n) < \frac{4n}{\pi} + \frac{9}{2\pi},$$

where we have used the facts that $\varepsilon < \pi/4$ and $u_n < 2n\pi + 5\pi/4$. So, for large n,

$$|w_{n+1}|^2 - |w_n|^2 \geqslant 6n\pi^2 + \frac{21\pi^2}{4} - \frac{4n}{\pi} - \frac{9}{2\pi}.$$

Substituting this estimate for $|w_{n+1}|^2 - |w_n|^2$ into (4·4) we have

$$|q(w_{n+1})| - |q(w_n)| \geqslant 6n\pi^2 - \frac{4n}{\pi} - \left(2 + \frac{\pi}{8}\right)(|w_{n+1}| + |w_n|) + O(1) \text{ as } n \longrightarrow \infty. \tag{4·5}$$

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Now we consider the $|w_{n+1}| + |w_n|$ term. Since $|w_n| \leqslant u_n + v_n$ and since $v_n = o(u_n)$ by Lemma 8, we can write $|w_n| \leqslant u_n(1 + o(1))$ as $n \to \infty$. Thus

$$\begin{aligned}\left(2 + \frac{\pi}{8}\right)(|w_{n+1}| + |w_n|) &\leqslant \left(2 + \frac{\pi}{8}\right)(u_{n+1} + u_n)(1 + o(1)) \\ &\leqslant \left(2 + \frac{\pi}{8}\right)\left(4n\pi + \frac{9\pi}{2}\right)(1 + o(1)) \\ &= n(8\pi + \pi^2/2) + o(n) \text{ as } n \longrightarrow \infty.\end{aligned}$$

Substituting this estimate into (4·5), we obtain

$$\begin{aligned}|q(w_{n+1})| - |q(w_n)| &\geqslant 6n\pi^2 - \frac{4n}{\pi} - 8n\pi - \frac{n\pi^2}{2} - o(n) \\ &= n\left(\frac{11\pi^2}{2} - \frac{4}{\pi} - 8\pi\right) - o(n) \\ &\geqslant Kn \text{ as } n \longrightarrow \infty\end{aligned}$$

for some positive constant K. The proof of Theorem 2 can now be completed by taking $a_n = |q(w_n)|$ for all n sufficiently large.

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