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Entropy production at weak Gibbs measures and a generalized variational principle

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Abstract. We consider piecewise invertible systems exhibiting intermittency and establish a generalized variational principle adapted to a non-stationary process in the following sense; the supremum is attained by non-singular (not necessarily invariant) probability measures and if the system exhibits hyperbolicity, then it reduces to the usual variational principle for the pressure. Our method relies on Ruelle's program in the study of non-equilibrium statistical mechanics to analyze dissipative phenomena. We show non-positivity of entropy production at weak Gibbs measures and clarify when it indeed vanishes. We also discuss a generalized variational principle in the context of σ-finite invariant measures.

1. *Introduction*

In the study of non-equilibrium statistical mechanics, Ruelle introduced the concept of entropy production to explain irreversibility on the basis of microscopic dynamics and to give quantitative prediction for dissipative phenomena. In his program, invertible time evolution that does not preserve any smooth measure was considered [**R1, R2, GR**]. In this paper, we take the first step towards placing these approaches in a more general framework. We are concerned with dissipative phenomena observed in complex systems exhibiting *intermittency*. More specifically, we consider a non-stationary non-invertible process of which statistical laws are determined by either (bi-)non-singular probability measures or σ-finite infinite invariant measures. Let (X, d) be a complete separable metric space and let $\mathcal{F}$ be the σ-algebra of subsets of X generated by the collection of open sets. We consider piecewise invertible systems $(T, X, Q = \{X_i\}_{i\in I})$ (see the definition in §3) and (bi-)non-singular probability measures m on $(X, \mathcal{F})$ with respect to T. Let $\mathcal{L}_m$ be the transfer operator associated with m. Then we define

$$e_T(m) := \limsup_{n\to\infty} \frac{1}{n} \int_X \log \mathcal{L}_m^n 1 \circ T^n(x)\, dm(x),$$

which is called the *asymptotic averaged entropy production of* T *at* m.

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The main purpose of this article is to show the following facts.

(1) If $e_T(m)$ at m is non-zero, then there is a gap between the next two generalized entropies:

$$\overline{h}_m(T, Q) := \limsup_{n\to\infty} \frac{1}{n} H_m\Big(\bigvee_{k=0}^{n-1} T^{-k}Q\Big)$$

and

$$\hat{h}_m(T, Q) := \limsup_{n\to\infty} \frac{1}{n} \int_X I_m\Big(\bigvee_{k=0}^{n-1} T^{-k}Q \,\Big|\, \bigvee_{k=n}^{\infty} T^{-k}Q\Big)(x)\, dm(x).$$

(2) At a weak Gibbs measure m, $e_T(m)$ is non-positive. We give a sufficient condition for $e_T(m)$ being zero.

(3) For a given potential ϕ, we introduce a generalized pressure $\mathcal{GP}_T(\phi)$ in the context of non-singular probability measures. Then we establish a generalized variational principle for $\mathcal{GP}_T(\phi)$. Moreover, we clarify when it can be reduced to the usual one for the pressure $\mathcal{P}_T(\phi)$ in the framework of invariant probability measures. This allows us to obtain naturally a weaker notion of equilibrium state for ϕ. More specifically, we answer the following questions.

Question (A). When does $\mathcal{P}_T(\phi) \le \mathcal{GP}_T(\phi)$ hold?

Question (B). When does the equality $\mathcal{P}_T(\phi) = \mathcal{GP}_T(\phi)$ hold? In this case, does the usual equilibrium state for ϕ attain $\mathcal{GP}_T(\phi)$?

We should remark that $e_T(m) = 0$ if m is T-invariant. Even if m is not T-invariant, $e_T(m)$ vanishes if $\{\log \mathcal{L}_m^n 1(x)\}_{n\ge1}$ is uniformly bounded. This property fails typically in case that the potential $\phi = \log(dm/d(m\circ T))$ admits an indifferent periodic point. As we will see in §5, our results can apply to various types of intermittent systems. In particular, we have a complete answer to Questions (A) and (B) if the second moment of the stopping time over a hyperbolic region is finite. We should remark that this phenomena is observed for another type of non-hyperbolic maps (unimodal and multimodal maps) for which the usual variational principle can be established (see [**PS1, PS2, PZ**]). Finally, we can verify that as long as we restrict our attention to uniformly expanding Markov systems and potentials of summable variations there is no difference between our generalized variational principle and the usual variational principle for the pressure.

The paper is organized as follows. In §2, we introduce definitions of generalized entropies for non-singular transformations. In §3, we collect fundamental results for piecewise invertible sofic systems which play important roles in establishing our main results in §4. In particular, we give sufficient conditions for (bi-)non-singular probability measures satisfying the weak Gibbs property [**Y4**] and a formula of conditional probabilities that allows one to establish the Dobrushin–Lanford–Ruelle equations [**MRTMV, Y8**]. In §5, we apply our results to intermittent systems and establish a generalized variational principle in the context of σ-finite invariant measures. All proofs of the results in §§3–5 are postponed to §6.

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2. *Generalized entropies for non-singular transformations*

Let (X, d) be a complete separable metric space and let $\mathcal{F}$ be the σ-algebra of subsets of X generated by the collection of open sets. Here $(X, \mathcal{F})$ is called a *standard Borel space*. Let m be a Borel probability measure on the standard Borel space $(X, \mathcal{F})$. We call $(X, \mathcal{F}, m)$ a *standard probability space*. We consider a (bi-)non-singular transformation T of the standard probability space $(X, \mathcal{F}, m)$, (i.e. $m \circ T^{-1} \sim m$). Suppose that $Q = \{X_i\}_{i \in I}$ is a measurable disjoint countable partition of X. The *information function of* Q is defined by $I_m(Q)(x) := -\sum_{i \in I} \log m(X_i) 1_{X_i}(x)$ (where $0 \log 0 := 0$) and the *entropy of the partition* Q is defined by $H_m(Q) := \int_X I_m(Q)(x)\, dm(x)$. We use the notation

$$X_{i_1 \dots i_n} := \bigcap_{k=0}^{n-1} T^{-k} X_{i_{k+1}} \in \bigvee_{k=0}^{n-1} T^{-k} Q$$

which is called a cylinder of rank n (with respect to T).

Definition. The *generalized entropy of* T *on* $(X, \mathcal{F}, m)$ *with respect to* $Q = \{X_i\}_{i \in I}$ is defined by

$$\overline{h}_m(T, Q) := \limsup_{n \to \infty} \frac{1}{n} H_m\left(\bigvee_{k=0}^{n-1} T^{-k} Q \right)$$
$$\left(= \limsup_{n \to \infty} \int_X \left(-\frac{1}{n} \log m(X_{i_1 \dots i_n}(x)) \right) dm(x) \right),$$

where $X_{i_1 \dots i_n}(x)$ denotes the unique cylinder of rank n containing x.

When m is T-invariant, $\overline{h}_m(T, Q)$ just coincides with the entropy $h_m(T, Q)$ of T with respect to Q. We also introduce another description of the entropy of a (bi-)non-singular transformation in terms of the conditional information. In order to simplify the notation, if $\mathcal{P}$ is a sub-σ-algebra of $\mathcal{F}$ generated by elements of a partition P, then $I_m(Q \mid P)$ denotes the *conditional information of* Q *given* $\mathcal{P}$ defined by

$$I_m(Q \mid \mathcal{P})(x) = -\sum_{i \in I} \log m(X_i \mid \mathcal{P})(x) 1_{X_i}(x).$$

Definition. The *generalized conditional entropy of* T *on* $(X, \mathcal{F}, m)$ *with respect to* $Q = \{X_i\}_{i \in I}$ is defined by

$$\hat{h}_m(T, Q) := \limsup_{n \to \infty} \frac{1}{n} \int_X I_m\left(\bigvee_{k=0}^{n-1} T^{-k} Q \,\middle|\, \bigvee_{k=n}^{\infty} T^{-k} Q \right)(x)\, dm(x)$$
$$\left(= \limsup_{n \to \infty} \frac{1}{n} \int_X \sum_{k=0}^{n-1} I_m\left(T^{-k} Q \,\middle|\, \bigvee_{l=k+1}^{\infty} T^{-l} Q \right) dm(x) \right).$$

In particular, if m is T-invariant and satisfies $H_m(Q) < \infty$, then $\hat{h}_m(T, Q)$ coincides with the next description of $h_m(T, Q)$ in terms of conditional information,

$$H_m\left(Q \,\middle|\, \bigvee_{k=1}^{\infty} T^{-k} Q \right) = \int_X I_m\left(Q \,\middle|\, \bigvee_{k=1}^{\infty} T^{-k} Q \right)(x)\, dm(x).$$

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3. *Weak Gibbs measures for piecewise invertible systems*

We consider a measurable countable disjoint partition $Q = \{X_i\}_{i\in I}$ of X which satisfies $\mathrm{cl}(\bigcup_{i\in I} \mathrm{int}\, X_i) = X$ and $\mathrm{cl}(\mathrm{int}\, X_i) \supset X_i$ if $\mathrm{int}\, X_i \neq \emptyset$. Let T be a non-invertible transformation of X satisfying the following conditions.

(C1) $\bigcup_{i\in I} \mathrm{int}\, X_i \subset T(\bigcup_{i\in I} \mathrm{int}\, X_i)$ and $T(\bigcup_{\mathrm{int}\, X_i=\emptyset} X_i) \subset \bigcup_{\mathrm{int}\, X_i=\emptyset} X_i$;

(C2) (*local invertibility*) $T|_{X_i}$ is one-to-one for every $X_i \in Q$, and for $X_i \in Q$ with $\mathrm{int}\, X_i \neq \emptyset$, $T|_{\mathrm{int}\, X_i} : \mathrm{int}\, X_i \to T(\mathrm{int}\, X_i)$ is a homeomorphism; moreover $(T|_{\mathrm{int}\, X_i})^{-1}$ is extended to a homeomorphism $v_i : \mathrm{cl}\, T(\mathrm{int}\, X_i) \to \mathrm{cl\,int}\, X_i$;

(C3) (*T-generator condition*) $\bigvee_{n=0}^{\infty} T^{-n} Q = \epsilon$ (the partition into points).

We call (T, X, Q) a *piecewise invertible system*. We remark that for every $X_{i_1\dots i_n} := \bigcap_{k=0}^{n-1} T^{-k} X_{i_{k+1}}$ with $\mathrm{int}\, X_{i_1\dots i_n} \neq \emptyset$, $T^n|_{X_{i_1\dots i_n}} : X_{i_1\dots i_n} \to T^n X_{i_1\dots i_n}$ is one-to-one. Moreover, $T^n|_{\mathrm{int}\, X_{i_1\dots i_n}} : \mathrm{int}\, X_{i_1\dots i_n} \to T^n(\mathrm{int}\, X_{i_1\dots i_n})$ is a homeomorphism and $(T^n|_{\mathrm{int}\, X_{i_1\dots i_n}})^{-1}$ is extended to a homeomorphism $v_{i_1\dots i_n} := v_{i_1} \circ v_{i_2} \circ \cdots \circ v_{i_n} : \mathrm{cl}\, T^n(\mathrm{int}\, X_{i_1\dots i_n}) \to \mathrm{cl\,int}\, X_{i_1\dots i_n}$. For every $n \geq 1$ we define $\mathcal{A}_n := \{(i_1 \dots i_n) \in I^n \mid \mathrm{int}\, X_{i_1\dots i_n} \neq \emptyset\}$ and $\mathcal{U}^{(n)} := \{T^n(\mathrm{int}\, X_{i_1\dots i_n}) \mid \forall (i_1 \dots i_n) \in \mathcal{A}_n\}$. We use the notation $\mathcal{U} := \bigcup_{n\geq 1} \mathcal{U}^{(n)}$. Then we have the following.

LEMMA 3.1. *We have* $\bigcup_{U\in\mathcal{U}^{(1)}} U = \bigcup_{U\in\mathcal{U}} U$.

Suppose that the next condition is satisfied.

Finite range structure (FRS). The set $\mathcal{U}$ is a finite set.

Then (T, X, Q) provides nice (countable) symbolic dynamics similar to sofic shifts (see [**Y1, Y2**]). Therefore, we call (T, X, Q) satisfying the FRS condition a *sofic system*. If $T\ \mathrm{int}\, X_i \cap \mathrm{int}\, X_j \neq \emptyset$ implies $T\ \mathrm{int}\, X_i \supset \mathrm{int}\, X_j$, then $\mathcal{U} = \mathcal{U}^{(1)}$ and the (T, X, Q) is called a *Markov system*. In particular, if $\mathcal{U}$ consists of a single element, then (T, X, Q) is called a *Bernoulli system*. We also call $X_i \in Q$ with $T(\mathrm{int}\, X_i) = \bigcup_{U\in\mathcal{U}} U$ a *Bernoulli cylinder*. Here $\mathcal{N}_T(X)$ denotes the set of all (bi-)non-singular probability measures with respect to T and $\mathcal{M}_T(X)$ denotes the set of all T-invariant probability measures. Then $\mathcal{M}_T(X) \subset \mathcal{N}_T(X)$. We recall that a (bi-)non-singular transformation T of the standard probability space $(X, \mathcal{F}, m)$ is locally invertible (i.e. $\exists \mathcal{P} = \{Y_j\}_{j\in J}$ a disjoint partition of X such that $m(X \setminus \bigcup_{j\in J} Y_j) = 0$ and T is invertible on each $Y_j \in \mathcal{P}$) if and only if $T^{-1}\{x\}$ is countable for m-almost every $x \in X$ (cf. [**Aa**]). Since the condition (C2) implies that T is countable-to-one, if $m \in \mathcal{N}_T(X)$, then $m \circ T \leq m$ so that there exists a measurable function $\phi : X \to \mathbb{R}$ satisfying $\phi(x) = \log(dm/d(m \circ T))(x)$ (m-almost every $x \in X$). Then the transfer operator $\mathcal{L}_m : L^1(m) \to L^1(m)$ associated with m is defined as follows; for all $f \in L^1(m)$,

$$\mathcal{L}_m f(x) = \sum_{y\in T^{-1}\{x\}} \exp[\phi(y)] f(y) \quad (m\text{-almost every } x \in X).$$

We note that $m \in \mathcal{N}_T(X)$ with $m(\bigcup_{U\in\mathcal{U}} U) = 1$ satisfies $m(\bigcup_{i\in I} \mathrm{int}\, X_i) = 1$. In this case, m-almost every $x \in X$, $\mathcal{L}_m f(x) = \mathcal{L}_\phi f(x)$, where $\mathcal{L}_\phi$ is the so-called *Perron–Frobenius operator associated with* ϕ defined by

$$\mathcal{L}_\phi f(x) := \sum_{i\in I} \exp[\phi(v_i(x)] f(v_i(x)) 1_{\mathrm{cl}(T(\mathrm{int}\, X_i))}(x) \quad (\text{cf. [**Y7**]}).$$

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Since $m(\bigcup_{U\in\mathcal{U}} U)=1$ gives $m(\bigcup_{(i_1\dots i_n)\in\mathcal{A}_n} \operatorname{int} X_{i_1\dots i_n})=1$ (for all $n\geq 1$), we see that for every $n\geq 1$ and m-almost every $x\in X$

$$\mathcal{L}_m^n f(x) = \mathcal{L}_\phi{}^n f(x)$$
$$= \sum_{(i_1\dots i_n)\in\mathcal{A}_n} \exp\left[\sum_{k=0}^{n-1}\phi\circ T^k(v_{i_1\dots i_n}(x))\right] f(v_{i_1\dots i_n}(x))1_{\operatorname{cl}(T^n(\operatorname{int} X_{i_1\dots i_n}))}(x).$$

In particular, if (T, X, Q) is a Bernoulli system and $\{\exp[\phi\circ v_i]\mid X\to\mathbb{R}\}_{i\in I}$ is an equicontinuous family satisfying $\sup_{x\in X}\sum_{i\in I}\exp[\phi\circ v_i(x)]<\infty$, then $\mathcal{L}_\phi$ preserves $C(X)$.

We introduce a weaker notion of Bowen's Gibbs measure (cf. [**Y4**]).

Definition. A Borel probability measure m on $(X,\mathcal{F})$ is called a *weak Gibbs measure for ϕ with a constant P* if there exists a sequence of positive numbers $\{K_n\}_{n\geq 1}$ with $\lim_{n\to\infty}(1/n)\log K_n=0$ such that for all $n\geq 1$, for all $X_{i_1\dots i_n}\in\bigvee_{k=0}^{n-1}T^{-k}Q$ with $m(X_{i_1\dots i_n})>0$ and for m-almost every $x\in X_{i_1\dots i_n}$

$$K_n^{-1}\leq\frac{m(X_{i_1\dots i_n})}{\exp[\sum_{k=0}^{n-1}\phi T^k(x)+nP]}\leq K_n.$$

PROPOSITION 3.1. *Let m be a T-invariant weak Gibbs measure for ϕ with 0. If either $H_m(Q)<\infty$ or $\int_X\phi\,dm>-\infty$ is satisfied, then $h_m(T,Q)+\int_X\phi\,dm=h_m(T)+\int_X\phi\,dm=0$.*

We should remark that if $H_m(Q)<\infty$, then $h_m(T,Q)=h_m(T)$ because of (C3). Next we introduce another notion of Gibbs measures which is more closely related to the Gibbsian states in statistical mechanics.

Definition. A Borel probability measure m on $(X,\mathcal{F})$ is called a (WG-1) measure for ϕ if for all $n\geq 1$, for all $X_{i_1\dots i_n}\bigvee_{k=0}^{n-1}T^{-k}Q$ with $m(X_{i_1\dots i_n})>0$, and for m-almost every $x\in T^{-n}T^nX_{i_1\dots i_n}$,

$$m\left(X_{i_1\dots i_n}\,\middle|\,T^{-n}\left(\bigvee_{k=0}^{\infty}T^{-k}Q\right)\right)(x)=\left(\frac{\mathcal{L}_\phi{}^n 1_{X_{i_1\dots i_n}}}{\mathcal{L}_\phi{}^n 1}\right)\circ T^n(x).$$

Indeed, the (WG-1) property allows one to establish the Dobrushin–Lanford–Ruelle (DLR) equations (see [**Y8**] and [**MRTMV**] for more details).

Definition. We say that $\phi: X\to\mathbb{R}$ is a potential of *weak bounded variation* (WBV) if there exists a sequence of positive numbers $\{C_n\}_{n\geq 1}$ satisfying $\lim_{n\to\infty}(1/n)\log C_n=0$ and for all $n\geq 1$,

$$\sup_{X_{i_1\dots i_n}\in\bigvee_{k=0}^{n-1}T^{-k}Q}\frac{\sup_{x\in X_{i_1\dots i_n}}\exp[\sum_{k=0}^{n-1}\phi(T^kx)]}{\inf_{x\in X_{i_1\dots i_n}}\exp[\sum_{k=0}^{n-1}\phi(T^kx)]}\leq C_n.$$

Remark (A). If $\operatorname{Var}_n(T,\phi):=\sup_{Y\in\bigvee_{k=0}^{n-1}T^{-k}Q}\sup_{x,y\in Y}|\phi(x)-\phi(y)|\to 0$ as $n\to\infty$, then ϕ satisfies the WBV property. If $\{\phi\circ v_i\mid \operatorname{cl}(T\operatorname{int} X_i)\to\mathbb{R}\}_{i\in I}$ is a family of partially defined equicontinuous functions and $\sigma(n):=\sup_{Y\in\bigvee_{k=0}^{n-1}T^{-k}Q}\operatorname{diam} Y\to 0$ as $n\to\infty$, then $\operatorname{Var}_n(T,\phi)\to 0 (n\to\infty)$.

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THEOREM 3.1. *Let $(X, \mathcal{F})$ be a standard Borel space and let $(T, X, Q = \{X_i\}_{i\in I})$ be a piecewise invertible system. Suppose that $m \in \mathcal{N}_T(X)$ admits a function $\phi : X \to \mathbb{R}$ satisfying $\phi = \log(dm/d(m \circ T))$. Then we have the following:*

(i) *m is a (WG-1)-measure for ϕ;*

(ii) *if ϕ satisfies the WBV property and $\inf_{U\in\mathcal{U}} m(U) > 0$, then m is a weak Gibbs measure for ϕ with 0.*

We refer the reader to [**Y3, Y7**] which contains sufficient conditions for ϕ possessing $m \in \mathcal{N}_T(X)$ with $\phi = \log(dm/d(m \circ T))$.

4. *The asymptotic averaged entropy production and a generalized variational principle*

Now we come to discuss properties of the asymptotic averaged entropy production of T associated with (bi-)non-singular probability measures.

LEMMA 4.1. *Let $(X, \mathcal{F})$ be a standard Borel space and let (T, X, Q) be a piecewise invertible system. If $m \in \mathcal{N}_T(X)$, then for all $n \geq 1$ and for m-almost every $x \in X$*

$$I_m\left(\bigvee_{k=0}^{n-1} T^{-k}Q \,\middle|\, T^{-n}\epsilon\right)(x) + \sum_{k=0}^{n-1} \log \frac{dm}{d(m\circ T)} \circ T^k(x) = \log \mathcal{L}_m^n 1 \circ T^n(x).$$

As we have already remarked in §1, our main interest is in the case that the WBV sequence $\{C_n\}_{n\geq 1}$ for $\phi := \log(dm/d(m \circ T))$ diverges as $n \to \infty$.

Definition. A periodic point x_0 with $T^q x_0 = x_0$ is called *indifferent with respect to m* if $\sum_{k=0}^{q-1} \phi \circ T^k(x_0) = 0$ for $\phi : X \to \mathbb{R}$ with $\exp[\phi] = dm/d(m \circ T)$.

Neither summability of variations of ϕ nor the uniformly bounded distortion property for $\exp \phi$ holds under the existence of an indifferent periodic point with respect to m (see [**Y5**, Lemma 6.1]). The next result insists that $e_T(m)$ at a weak Gibbs measure m is non-positive and it indeed becomes zero under certain conditions.

THEOREM 4.1. *Let $(X, \mathcal{F})$ be a standard Borel space and let (T, X, Q) be a piecewise invertible sofic system. Suppose that $m \in \mathcal{N}_T(X)$ with $m(\bigcup_{U\in\mathcal{U}} U) = 1$ admits a potential ϕ of WBV satisfying $\exp[\phi] = dm/d(m \circ T)$. Then m is a weak Gibbs measure for ϕ (with 0) and $e_T(m) \leq 0$. Assume further that either of the following conditions is satisfied:*

(i) *(T, X, Q) is a Bernoulli system;*

(ii) *m possesses an indifferent periodic point x_0 with period q which is contained in a Bernoulli cylinder of rank q.*

Then we have

$$e_T(m) = \lim_{n\to\infty} \frac{1}{n} \int_X \log \mathcal{L}_m^n 1 \circ T^n(x)\, dm(x) = \hat{h}_m(T, Q) - \overline{h}_m(T, Q) = 0.$$

LEMMA 4.2. *Under the assumptions in the Theorem 4.1,*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \log \mathcal{L}_m^n 1 \right\|_{L^\infty(m)} = 0$$

and the decay rate is subexponential.

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Definition. For a given potential $\phi : X \to \mathbb{R}$ we define

$$\mathcal{N}_T(X, \phi) := \Big\{ \mu \in \mathcal{N}_T(X) | I_\mu(T^{-k}Q|T^{-(k+1)}\epsilon) + \phi \circ T^k, \\ I_\mu(T^{-k}Q|T^{-(k+1)}\epsilon) + \log \frac{d\mu}{d(\mu T)} \circ T^k \in L^1(\mu)\ (\forall k \geq 0) \Big\}.$$

For every $\mu \in \mathcal{N}_T(X, \phi)$, we use the notation

$$\mathcal{F}(\phi, \mu) := \hat{h}_\mu(T, Q) + \liminf_{n\to\infty} \frac{\int_X \sum_{k=0}^{n-1} \phi T^k(x)}{n} d\mu(x) \\ - e_T(\mu) - \limsup_{n\to\infty} \frac{1}{n} \min\{A_n, B_n\},$$

where

$$A_n := \int_X \sum_{k=0}^{n-1} \left(\mathcal{L}_\mu^k 1(x) \frac{\exp[\phi]}{dm/d(\mu T)}(x) - 1 \right) d\mu(x)$$

and

$$B_n := \int_X (\mathcal{L}_\phi{}^n 1(x) - 1)\, d\mu(x).$$

THEOREM 4.2. *Let $(X, \mathcal{F})$ be a standard Borel space and let $(T, X, Q = \{X_i\}_{i\in I})$ be a piecewise invertible sofic system. If $m \in \mathcal{N}_T(X)$ with $m(\bigcup_{U\in\mathcal{U}} U) = 1$ admits a function ϕ satisfying $\exp[\phi] = dm/d(m \circ T)$ and $\sup_{x\in X} \mathcal{L}_\phi 1(x) < \infty$, then the next two properties hold.*

(i) *If ϕ satisfies the WBV property and m possesses an indifferent periodic point with period q in a Bernoulli cylinder of rank q, then*

$$e_T(m) = \hat{h}_m(T, Q) + \liminf_{n\to\infty} \int_X \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ T^k(x)\, dm(x) = 0.$$

(ii) *For every $\mu \in \mathcal{N}_T(X, \phi)$*

$$\limsup_{n\to\infty} \frac{1}{n} \int_X \left(I_\mu\left(\bigvee_{k=0}^{n-1} T^{-k}Q \,\middle|\, T^{-n}\epsilon \right)(x) + \sum_{k=0}^{n-1} \phi \circ T^k(x) \right) d\mu(x) \\ \leq e_T(\mu) + \limsup_{n\to\infty} \frac{1}{n} \min\{A_n, B_n\} < \infty.$$

Definition. We define a *generalized pressure $\mathcal{GP}_T(\phi)$ of ϕ* by

$$\mathcal{GP}_T(\phi) := \sup\{\mathcal{F}(\phi, \mu) \mid \mu \in \mathcal{N}_T(X, \phi)\}.$$

We say that $\mu \in \mathcal{N}_T(X, \phi)$ is a *weak equilibrium state for ϕ* if $\mathcal{F}(\phi, \mu) = \mathcal{GP}_T(\phi)$. Here $\mathcal{WE}_T(\phi)$ denotes the set of all weak equilibrium states for ϕ.

THEOREM 4.3. (A generalized variational principle) *Let ϕ be a potential of WBV with $\sup_{x\in X} \mathcal{L}_\phi 1(x) < \infty$. Assume that there exists $m \in \mathcal{N}_T(X)$ with $m(\bigcup_{U\in\mathcal{U}} U) = 1$ satisfying $\exp[\phi] = dm/d(m \circ T)$ and possesses an indifferent periodic point x_0 with $T^q x_0 = x_0$ in a Bernoulli cylinder of rank q. Then $\mathcal{GP}_T(\phi) = 0$ and we have the following.*

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(i) *Every $\mu = hm$ with $h/(hT) \equiv 1$ (m-almost everywhere) satisfies $\mu \in \mathcal{WE}_T(\phi)$.*
(ii) *If the indifferent periodic point x_0 satisfies $(1/q)\sum_{k=0}^{q-1} \exp[\phi(T^k x_0)] = 1$, then $(1/q)\sum_{k=0}^{q-1} \delta_{T^k x_0} \in \mathcal{WE}_T(\phi)$.*

Remark (B). If m is not T-invariant, then $\mu = hm$ with $h/(hT) \equiv 1$ is also not T-invariant. If m is a weak Gibbs measure for ϕ, then the supremum is attained by every weak Gibbs measure for the common potential ϕ.

Since m itself and $(1/q)\sum_{k=0}^{q-1} \delta_{T^k x_0}$ are weak equilibrium states for ϕ, we have the following result.

COROLLARY 4.1. *Here $\mathcal{WE}_T(\phi)$ consists of more than two elements.*

We observe further details for (ii) in Theorem 4.2 in the case when $\mu \le m$ with $d\mu/dm = h$. First, we note that

$$A_n := \int_X \sum_{k=0}^{n-1} \left(\frac{hT}{h}(T^k x) - 1 \right) d\mu(x)$$

and

$$B_n := \int_X \left(\frac{hT^n}{h}(x) - 1 \right) d\mu(x).$$

Then the following are obtained.

(a) In the case when $\mu \in \mathcal{M}_T(X)$, we see that

$$A_n = n\left(\int_X hT(x)\, dm(x) - 1 \right), \quad B_n = \int_X hT^n(x)\, dm(x) - 1$$

and

$$\mathcal{F}(\phi, \mu) = \int_X \log \frac{hT}{h}(x)\, d\mu(x) - \min\left\{ \int_X hT(x)\, dm(x) - 1, \limsup_{n\to\infty} \int_X \frac{hT^n(x)}{n}\, dm(x) \right\}.$$

(Here, we use

$$h_\mu(T, Q) + \int_X \phi(x)\, d\mu(x) + \int_X \log \frac{h}{hT}(x)\, d\mu(x) = 0$$

which follows from Lemma 4.1.)

(b) In the case when $m \in \mathcal{M}_T(X)$, $\mathcal{L}_m^n 1(x) = \mathcal{L}_\phi{}^n 1(x) = 1$ (μ-almost every $x \in X$) so that

$$\mathcal{F}(\phi, \mu) = \hat{h}_\mu(T, Q) + \liminf_{n\to\infty} \frac{\int_X \sum_{k=0}^{n-1} \phi T^k(x)}{n}\, d\mu(x) - e_T(\mu) - \limsup_{n\to\infty} \frac{1}{n} \min\{A_n, 0\}.$$

Therefore,

(c) if both μ and m are T-invariant, then $\mathcal{F}(\phi, \mu) = \int_X \log((hT)/h)\, d\mu$.

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Definition. For given $\phi : X \to \mathbb{R}$, we define

$$\mathcal{M}_T(X, \phi) := \left\{ \mu \in \mathcal{M}_T(X) \;\middle|\; I_\mu(Q \mid T^{-1}\epsilon) + \phi \in L^1(\mu), \text{ either } h_\mu(T) < \infty \right.$$
$$\left. \text{or } \int_X \phi \, d\mu > -\infty \text{ with } h_\mu(T) = \int_X I_\mu(Q \mid T^{-1}\epsilon) \, d\mu \text{ is satisfied} \right\}.$$

The *pressure of* ϕ is defined by

$$\mathcal{P}_T(\phi) := \sup \left\{ h_\mu(T) + \int_X \phi \, d\mu \;\middle|\; \mu \in \mathcal{M}_T(X, \phi) \right\}.$$

If $\mu \in \mathcal{M}_T(X, \phi)$ satisfies $h_\mu(T) + \int_X \phi \, d\mu = \mathcal{P}_T(\phi)$, then μ is called an *equilibrium state for* ϕ. Here $\mathcal{E}_T(\phi)$ denotes the set of all equilibrium states for ϕ.

The next result gives an answer to Question (A) in §1.

LEMMA 4.3. *If* $\mathcal{P}_T(\phi)$ *coincides with*

$$\hat{\mathcal{P}}_T(\phi) := \sup \left\{ h_\mu(T) + \int_X \phi \, d\mu \;\middle|\; \mu = hm \in \mathcal{M}_T(X, \phi), \right.$$
$$\left. \int_X \log \frac{hT}{h} \, d\mu = 0 \text{ and } \limsup_{n\to\infty} \int_X \frac{hT^n}{n} \, dm = 0 \right\},$$

then $\mathcal{P}_T(\phi) \le \mathcal{GP}_T(\phi)$.

We have the following answer to Question (B) in §1.

THEOREM 4.4. *Let* (T, X, Q) *be a piecewise invertible sofic system and let* ϕ *be a potential of WBV with* $\sup_{x\in X} \mathcal{L}_\phi 1(x) < \infty$. *Assume that there exists* $m \in \mathcal{N}_T(X)$ *with* $m(\bigcup_{U\in\mathcal{U}} U) = 1$ *satisfying* $\exp[\phi] = dm/d(m \circ T)$ *and possesses an indifferent periodic point with period* q *in a Bernoulli cylinder of rank* q. *If there exists an absolutely continuous weak Gibbs measure* $\mu \in M_T(X, \phi)$ *for* ϕ *(with* 0*) with respect to* m *and satisfies* $\lim_{n\to\infty}(1/n) \int_X \mathcal{L}_m^n 1 \, d\mu = 0$, *then we have:*

(i) $\mathcal{F}(\phi, \mu) = 0 = h_\mu(T) + \int_X \phi \, d\mu$ *and* $\mu \in \mathcal{WE}_T(\phi)$;

(ii) $\mu \in \mathcal{E}_T(\phi)$ *if and only if* $\mathcal{P}_T(\phi) = \hat{\mathcal{P}}_T(\phi)$ *and in this case* $P_T(\phi) = \mathcal{GP}_T(\phi)$;

(iii) *if* $\sup_{x\in X} \mathcal{L}_\phi^n 1(x) = o(n)$, *then* $\mathcal{E}_T(\phi) \subset \mathcal{WE}_T(\phi)$.

We will see in §5 that all examples admitting indifferent periodic points x_0 satisfy that μ, $(1/q)\sum_{k=0}^{n-1} \delta_{T^k x_0} \in \mathcal{E}_T(\phi)$ and $(1/q)\sum_{k=0}^{n-1} \delta_{T^k x_0}, m \in \mathcal{WE}_T(\phi)$. In order to clarify whether $\mu \in \mathcal{WE}_T(\phi)$, we need to verify finiteness of the second moment of the stopping time over hyperbolic regions (see Proposition 5.2).

COROLLARY 4.2. (The uniqueness of weak equilibrium states) *If the weak Gibbs measure* $\mu = hm$ *for* ϕ *satisfies* $\mu \sim m$ *and*

$$\|\mathcal{L}_m^n 1 - h\|_{L^\infty(m)} \longrightarrow 0 \quad (n \to \infty),$$

then

$$\sharp\mathcal{E}_T(\phi) = \sharp\mathcal{WE}_T(\phi) = 1.$$

Indeed, the assumption in Corollary 4.2 implies that μ and m are exact and hence ergodic [**Aa**]. Therefore, the condition $h/(hT) \equiv 1$ forces h to be m-almost everywhere constant.

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Remark (C). If $m \in \mathcal{N}_T(X)$ admits an indifferent periodic point x_0, then the uniform convergence of $\mathcal{L}_m^n 1$ typically fails. Even if the $L^1(m)$ convergence is valid, the invariant density h may be unbounded.

5. *Applications to intermittent systems*

In this section, we apply our results in previous sections to intermittent phenomena caused by indifferent periodic points. Let (T, X, Q) be a piecewise invertible sofic system and let ϕ be a potential of WBV with $\sup_{x\in X} \mathcal{L}_\phi 1(x) < \infty$. Assume that there exists $m \in \mathcal{N}_T(X)$ with $m(\bigcup_{U\in\mathcal{U}} U) = 1$ satisfying $\exp[\phi] = dm/d(m \circ T)$ and possesses an indifferent periodic point with period q in a Bernoulli cylinder of rank q. Here $\mathcal{A}$ denotes the set of all admissible cylinders $\bigcup_{n=1}^\infty \{X_{i_1\dots i_n} \mid (i_1 \dots i_n) \in \mathcal{A}_n\}$. For $\mathcal{R} \subset \mathcal{A}$, we define the stopping time over $\mathcal{R}$, $R : X \to \mathbb{N} \cup \{\infty\}$ by $R(x) = \inf\{n \in \mathbb{N} \mid X_{i_1\dots i_n}(x) \in \mathcal{R}\}$. Then, for every $n \geq 1$, we define $D_n = \{x \in X \mid R(x) > n\}$ and $B_n = \{x \in X \mid R(x) = n\}$. Put $D_0 = X$. Define $I^* = \bigcup_{n=1}^\infty \{(i_1 \dots i_n) \mid X_{i_1\dots i_n} \subset B_n\}$. Then $Q^* = \{X_\alpha\}_{\alpha\in I^*}$ is a countable partition of $\bigcup_{i=1}^\infty B_i$. Now we define Schweiger's jump transformation $T^* : \bigcup_{i=1}^\infty B_i \to X$ by $T^*x = T^{R(x)}x$. Put $X^* = X \setminus (\bigcup_{m=0}^\infty T^{*-m}(\bigcap_{n\geq 0} D_n))$. Then (T^*, X^*, Q^*) is a piecewise invertible sofic system. We impose the following two conditions on $\mathcal{R}$:

(1) $X_{i_1\dots i_n j_1\dots j_m} \in \mathcal{R}$ whenever $X_{j_1\dots j_m} \in \mathcal{R}$ (*the strong playback property*);

(2) there exists $0 < \Gamma < \infty$, $\gamma^* > 1$ such that for every $(\alpha_1 \dots \alpha_n) \in I^{*n}$ with $X_{\alpha_1\dots\alpha_n} \in \mathcal{A}$ and for all $n \geq 1$,

$$d(v_{\alpha_1\dots\alpha_n}x,\, v_{\alpha_1\dots\alpha_n}y) \leq \Gamma\gamma^{*-n}\, d(x,\, y) \quad (\forall x,\, y \in T^{*n}X_{\alpha_1\dots\alpha_n}).$$

If the induced potential $\phi^* := \sum_{k=0}^{R(\cdot)-1} \phi T^k$ satisfies equi-Hölder continuity of $\{\phi^* v_\alpha\}_{\alpha\in I^*}$, then the measure theoretical bounded distortion is valid for $\exp\phi^*$ $(= dm/d(mT^*))$. Then we have a T^*-invariant measure μ^* of which density is bounded and piecewise Hölder continuous with respect to a finite partition generated by $\mathcal{U}$. Furthermore, if each $U \in \mathcal{U}$ contains a full cylinder in $\mathcal{R}$, then the density is bounded away from zero and μ^* is a Gibbs measure for ϕ^* which is exact and exponential mixing. Moreover, if $\lim_{n\to\infty} m(D_n) = 0$, then $m(X) = m(X^*)$ and the following formula gives a T-invariant σ-finite conservative measure $\mu \sim m$, which is exact [**Y1**] (cf. [**Sch**]):

$$\mu(E) = \sum_{n=0}^\infty \mu^*(D_n \cap T^{-n}E) \quad (\forall E \in \mathcal{F}).$$

Such μ is unique up to constant. In particular, if $\sum_{n=0}^\infty m(D_n) < \infty$, then μ is finite. We should recall that the distortion of $dm/d(mT^n)$ over cylinders of rank n touching indifferent periodic points diverges as $n \to \infty$ (cf. [**Y5**, Lemma 6.1]). Therefore, all indifferent periodic points with respect to m are contained in $\bigcap_{n\geq 0} D_n$. In particular, $\bigcap_{n\geq 0} D_n$ consists of only indifferent periodic points in our examples below. We have the following results.

PROPOSITION 5.1. [**Y4**, Theorem 3.2] *Suppose that:*

(i) *there exists* $0 < r_1 < r_2 < \infty$ *and there exists* $\alpha > 1$ *such that*

$$r_1 n^{-\alpha} \leq m(D_n) \leq r_2 n^{-\alpha};$$

(ii) *there exists* $0 < G_1 < \infty$ *such that for all* $X_{d_1\dots d_n} \in \mathcal{D}_n$, $m(D_n) \leq G_1 m(X_{d_1\dots d_n})$.

Then μ *is a weak Gibbs measure for* ϕ *with* 0.

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Definition. A cylinder $X_{i_1\dots i_n}$ is called a *Markov cylinder* if for every $U \in \mathcal{U}$ with $U \cap \operatorname{int} X_{i_1\dots i_n} \neq \emptyset$ it holds that $\operatorname{int} X_{i_1\dots i_n} \subset U$.

If (T, X, Q) is a piecewise invertible sofic system, then it follows from [**Y2**, Theorem 3.1] that there exists a Markov cylinder. We have the following result.

PROPOSITION 5.2. *If B_1 consists of Markov cylinders and $\int_X R^2\, dm < \infty$, then μ satisfies*

$$\lim_{n\to\infty} \frac{1}{n} \int_X \mathcal{L}_m^n 1 \, d\mu = 0.$$

Now we can apply all our results in §§3 and 4 and Propositions 5.1 and 5.2 to the following intermittent systems preserving absolutely continuous probability measures.

Example 1. (A one-parameter family of maps on the interval [0, 1]) Let $X = [0, 1]$ and let m be the normalized Lebesgue measure of [0, 1]. For $\beta > 0$ define

$$T_\beta(x) = \begin{cases} \dfrac{x}{(1 - x^\beta)^{1/\beta}} & \text{on } X_0 = [0, (1/2)^{1/\beta}), \\ \dfrac{x}{(1/2)^{1/\beta} - 1} + \dfrac{1}{1 - (1/2)^{1/\beta}} & \text{on } X_1 = [(1/2)^{1/\beta}, 1]. \end{cases}$$

The map T_β has an indifferent fixed point 0. It is similar in its properties to the more familiar Manneville–Pomeau maps: $x \to x + x^{1+\beta} \pmod 1$, in that it also has intermittent behavior. Here $(T, X, Q = \{X_0, X_1\})$ is a piecewise invertible Bernoulli system and $\phi = \log |T'|$ satisfies the WBV property. Therefore, m is a weak Gibbs measure for ϕ with 0. If $\beta < 1$, then T_β admits an invariant weak Gibbs equilibrium measure $\mu \sim m$. We see that $e_T(m) = e_T(\mu) = 0$ and $\mu, \delta_0 \in \mathcal{E}_T(\phi)$. We also know that $m, \delta_0 \in \mathcal{WE}_T(\phi)$. In particular, if $\beta < 1/2$, then $\mu \in \mathcal{WE}_T(\phi)$ too.

Example 2. (Inhomogeneous diophantine approximations [**Y2, Y4, Y5, Y6, Y7, Y8**]) We define $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1, -y \le x < -y + 1\}$ and $T : X \to X$ by

$$T(x, y) = \left(\frac{1}{x} - \left[\frac{1 - y}{x} \right] + \left[-\frac{y}{x} \right], -\left[-\frac{y}{x} \right] - \frac{y}{x} \right),$$

where $[x] = \max\{n \in \mathbb{Z} \mid n \le x\} (x \in \mathbb{N})$ and $[x] = \max\{n \in \mathbb{Z} \mid n < x\} (x \in \mathbb{Z} \setminus \mathbb{N})$. This map admits indifferent periodic points $(1, 0)$ and $(-1, 1)$ with period 2, i.e. $|\det DT^2(1, 0)| = |\det DT^2(-1, 1)| = 1$. Let $a(x, y) = [(1 - y)/x] - [-y/x]$ and $b(x, y) = -[-y/x]$. We can introduce an index set

$$I := \{(a, b) \in \mathbb{Z}^2 \mid a > b > 0, \text{ or } a < b < 0\}$$

and a partition $Q := \{X_{(a,b)\in I}\}$, where $X_{(a,b)} = \{(x, y) \in X \mid a(x, y) = a, b(x, y) = b\}$. Let m be the normalized Lebesgue measure of X. Here (T, X, Q) is a piecewise invertible Bernoulli system and $\phi = \log(dm/d(mT))$ satisfies the WBV property. Therefore, m is a weak Gibbs measure for ϕ with 0. There exists a T-invariant weak Gibbs equilibrium measure $\mu \sim m$. We see that $e_T(m) = e_T(\mu) = 0$ and $\mu, 2^{-1}(\delta_{(1,0)} + \delta_{(-1,1)}) \in \mathcal{E}_T(\phi)$. We also know that $m, 2^{-1}(\delta_{(1,0)} + \delta_{(-1,1)}) \in \mathcal{WE}_T(\phi)$.

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Example 3. (A complex continued fraction [**Y6, Y7**]) We can define a complex continued fraction transformation $T : X \to X$ on the diamond-shaped region $X = \{z = x_1\alpha + x_2\overline{\alpha} \mid -1/2 \le x_1, x_2 \le 1/2\}$, where $\alpha = 1 + i$, by $T(z) = 1/z - [1/z]_1$. Here $[z]_1$ denotes $[x_1 + 1/2]\alpha + [x_2 + 1/2]\overline{\alpha}$, where z is written in the form $z = x_1\alpha + x_2\overline{\alpha}$, $[x] = \max\{n \in \mathbb{Z} \mid n \le x\} (x \in \mathbb{N})$ and $[x] = \max\{n \in \mathbb{Z} \mid n < x\}$ $(x \in \mathbb{Z} - \mathbb{N})$. This transformation has an indifferent periodic orbit $\{1, -1\}$ of period 2 and two indifferent fixed points at i and $-i$. For each $n\alpha + m\overline{\alpha} \in I := \{m\alpha + n\overline{\alpha} \mid (m, n) \in \mathbb{Z}^2 - (0, 0)\}$, we define $X_{n\alpha+m\overline{\alpha}} := \{z \in X \mid [1/z]_1 = n\alpha + m\overline{\alpha}\}$. Then we have a countable Markov partition $Q = \{X_a\}_{a\in I}$ of X and (T, X, Q) is a transitive sofic system. For $\phi(z) = -\log|T'(z)| (= -2\log|z|)$, we can verify the WBV property of ϕ so that m is a weak Gibbs measure for 2ϕ with 0. There exists a T-invariant ergodic probability measure $\mu \sim m$ for 2ϕ, and we see that $e_T(m) = e_T(\mu) = 0$. Moreover, $\mu, \delta_i, \delta_{-i}, 2^{-1}(\delta_1 + \delta_{(-1)}) \in \mathcal{E}_T(\phi)$ and $m, \delta_i, \delta_{-i}, 2^{-1}(\delta_1 + \delta_{(-1)}) \in \mathcal{WE}_T(\phi)$.

In the case when $\sum_{n=0}^{\infty} m(D_n) = \infty$, Theorem 4.4 does not apply so that we need further observations. Let $\mathcal{M}_T^\infty(X)$ be the set of all σ-finite invariant measures on $(X, \mathcal{F})$ and define $\mathcal{E}_T^\infty(X) := \{\nu \in \mathcal{M}_T^\infty(X) \mid \nu$ is conservative and ergodic$\}$. For every $\nu \in \mathcal{E}_T^\infty(X)$, we can define the induced transformation T_A over $A \in \mathcal{F}$ with $0 < \nu(A) < \infty$ by $T_A(x) = T^{R_A(x)}(x)$, where $R_A(x) := \inf\{n \in \mathbb{N} \mid T^n(x) \in A\}$. Then $\nu_A := \nu|_A/\nu(A)$ is T_A-invariant and ergodic. Here ν can be represented by ν_A via the following Kac formula:

$$\int_X 1_E(x)\, d(x) = \int_A \sum_{k=0}^{R_A(x)-1} 1_E(T^k x)\, d\nu_A(x) \quad (\forall E \in \mathcal{F}).$$

Moreover, if $A_i \in \mathcal{F}$ satisfy $0 < \nu(A_i) < \infty$ $(i = 1, 2)$, then $T_{A_1} = (T_{A_1\cup A_2})_{A_1}$ so that

$$\frac{h_{\nu_1\cup\nu_2}(T_{A_1\cup A_2})}{\nu_{A_1\cup A_2}(A_1)} = h_{(\nu_1\cup\nu_2)_{A_1}}((T_{A_1\cup A_2})_{A_1}).$$

Hence, we have

$$\nu(A_1)h_{\nu_1}(T_{A_1}) = \nu(A_1 \cup A_2)h_{\nu_1\cup\nu_2}(T_{A_1\cup A_2}) = \nu(A_2)h_{\nu_2}(T_{A_2}).$$

This number is used as the entropy $h_\nu(T)$ of T with respect to ν (cf. [**T**]). For a function $f : X \to \mathbb{R}$, define $f_A(x) := \sum_{k=0}^{R_A(x)-1} fT^k(x)$. If $A_i \in \mathcal{F}$ with $0 < \nu(A_i) < \infty$ $(i = 1, 2)$, then the above Kac formula gives

$$\int_{A_1} f_{A_1}(x)\, d\nu_{A_1}(x) = \int_{A_1\cup A_2} f_{A_1\cup A_2}(x)\, d\nu_{A_1\cup A_2}(x) = \int_{A_2} f_{A_2}(x)\, d\nu_{A_2}(x)$$

whenever the integrals are well defined. Let $A \in Q$ be a Markov cylinder and let Q_A be a countable disjoint partition of $\{x \in A \mid R_A(x) < \infty\}$ consisting of all cylinders $X_{i_1\dots i_n} \subset \{x \in A \mid R_A(x) = n - 1\}$ with $n \ge 2$. We define

$$\begin{aligned}\mathcal{E}_T^\infty(X, \phi, A) := \Bigg\{ \nu \in \mathcal{E}_T^\infty(X) \;\Bigg|\; & 0 < \nu(A) < \infty,\ I_{\nu_A}(Q_A \mid T_A^{-1}\epsilon) + \phi_A \in L^1(\nu_A), \\ & \text{either } h_{\nu_A}(T_A) < \infty \text{ or } \int_A \phi_A\, d\nu_A > -\infty \text{ with } h_{\nu_A}(T_A) \\ & = \int_X I_{\nu_A}(Q_A \mid T_A^{-1}\epsilon)\, d\nu_A \Bigg\}\end{aligned}$$

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and

$$P^{\infty}(T, \phi, A) := \sup_{\nu \in \mathcal{E}_T^{\infty}(X,\phi,A)} \left\{ \nu(A) \left(h_{\nu_A}(T_A) + \int_A \phi_A \, d\nu_A \right) \right\}$$
$$\left(= \sup_{\nu \in \mathcal{E}_T^{\infty}(X,\phi,A)} \left\{ h_{\nu}(T) + \nu(A) \int_A \phi_A \, d\nu_A \right\} \right).$$

By using a similar argument as in [**Y1**, Lemma 7.1] (cf. [**T**]), we can prove the following result.

THEOREM 5.1. *Let B_1 be a Markov cylinder of rank one. If $\phi_{B_1} \in L^1(\mu_{B_1})$, then $P^{\infty}(T, \phi, B_1) = h_{\mu}(T) + \int_X \phi \, d\mu = 0$.*

The following example gives a countable non-Markovian sofic system preserving an absolutely continuous invariant measures to which Theorem 5.1 can apply.

Example 4. (A two-dimensional map related to a negative continued fraction) Let $X = \{(x, y) \mid 0 < x \leq 1, 0 \leq y \leq 1\}$ and T is defined by

$$T(x, y) = \left(-\frac{1}{x} - \left[-\frac{1}{x} \right], \frac{y}{x} - \left[\frac{y}{x} \right] \right).$$

The first component of T is known as a map related to a negative continued fraction which preserves a σ-finite infinite ergodic invariant measure equivalent to the normalized Lebesgue measure of [0, 1]. We define

$$I = \{(a, b) \in \mathbb{N} \times (\mathbb{N} \cup \{0\}) \mid a \geq 2, a > b\}$$

and a countable disjoint partition $Q = \{X_{(a,b)}\}_{(a,b)\in I}$, where $X_{(a,b)}$ is defined by

$$(x, y) \in X_{(a,b)} \quad \text{if and only if } -[-1/x] = a \quad \text{and} \quad [y/x] = b.$$

Then (T, X, Q) is a piecewise invertible non-Markovian sofic system. Indeed, we see that $\mathcal{U}$ consists of $U_0 = X$ and $U_1 = \{(x, y) \in X \mid x + y \leq 1\}$. Let m be the normalized Lebesgue measure of X. We can verify that $\phi = \log(dm/d(mT))$ is a potential of WBV and m is a weak Gibbs measure for ϕ with 0. Let D_n be the union of cylinders of rank n touching $\{1\} \times [0, 1]$ which consists of indifferent fixed points with respect to m. Then we see that $\sum_{n\geq 0} m(D_n) = \infty$ and T preserves a σ-finite infinite conservative ergodic measure $\mu \sim m$. The invariant density is given by

$$\frac{d\mu}{dm}(x, y) = \begin{cases} \dfrac{2 - x}{2(1 - x)^2} & \text{if } x + y < 1, \\ \dfrac{1}{2(1 - x)} & \text{if } x + y > 1, \end{cases}$$

and there is no finite absolutely continuous invariant measure with respect to m (see [**Y1, Y2**]).

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6. *Proofs*

Proof of Lemma 3.1. First we show that $\bigcup_{U\in\mathcal{U}^{(n)}} U \subseteq \bigcup_{U\in\mathcal{U}^{(n-1)}} U$. Indeed,

$$\begin{aligned}\bigcup_{U\in\mathcal{U}^{(n)}} U &= T^{n-1}\Bigg(\bigcup_{(i_1\dots i_n)\in\mathcal{A}_n}(T\ \text{int}\, X_{i_1}\cap \text{int}\, X_{i_2\dots i_n})\Bigg)\\ &= T^{n-1}\Bigg(\bigcup_{(i_2\dots i_n)\in\mathcal{A}_{n-1}}\Bigg(\bigcup_{i_1\in I;(i_1i_2\dots i_n)\in\mathcal{A}_n}(T\ \text{int}\, X_{i_1}\cap \text{int}\, X_{i_2\dots i_n})\Bigg)\Bigg)\\ &\subseteq T^{n-1}\Bigg(\bigcup_{(i_2\dots i_n)\in\mathcal{A}_{n-1}}\text{int}\, X_{i_2\dots i_n}\cap\bigcup_{U\in\mathcal{U}^{(1)}}U\Bigg)\subseteq\bigcup_{U\in\mathcal{U}^{(n-1)}}U.\end{aligned}$$

Next we assume $x\in\bigcup_{U\in\mathcal{U}^{(n-1)}}U$. Then there exists $(i_1\dots i_{n-1})\in\mathcal{A}_{n-1}$ such that $x\in T^{n-1}(\text{int}\, X_{i_1\dots i_{n-1}})$. Therefore, by (C1) we have $v_{i_1\dots i_{n-1}}(x)\in \text{int}\, X_{i_1\dots i_{n-1}}\subset\bigcup_{U'\in\mathcal{U}^{1)}}U'$. This implies that there exists $X_j\in\mathcal{Q}$ with $\text{int}X_j\neq\emptyset$ satisfying

$$v_{i_1\dots i_{n-1}}(x)\in \text{int}\, X_{i_1\dots i_{n-1}}\cap T(\text{int}\, X_j)=T(\text{int}\, X_{ji_1\dots i_{n-1}}).$$

Since $x\in T^n(\text{int}\, X_{ji_1\dots i_{n-1}})\in\mathcal{U}^{(n)}$, we complete the proof. □

Proof of Proposition 3.1. First we note that the weak Gibbs property of m gives

$$-\frac{1}{n}\log K_n\le\frac{1}{n}\int_X\Bigg(\log m(X_{i_1\dots i_n}(x))-\sum_{k=0}^{n-1}\phi\circ T^k(x)\Bigg)\,dm(x)\le\frac{1}{n}\log K_n.$$

Therefore, if m is T-invariant, then for every $n\ge 1$

$$-\frac{1}{n}\log K_n\le\frac{1}{n}\sum_{(i_1\dots i_n)\in\mathcal{A}_n}m(X_{i_1\dots i_n})\log m(X_{i_1\dots i_n})-\int_X\phi(x)\,dm(x)\le\frac{1}{n}\log K_n.$$

From the above inequalities, we conclude $h_m(T,\,Q)+\int_X\phi\,dm=0$. □

Proof of Theorem 3.1. First we note that $\sigma(T^{-n}\epsilon)=T^{-n}\mathcal{F}$. Therefore, for all $A\in\sigma(T^{-n}\epsilon)$, there exists $E\in\mathcal{F}$ such that $A=T^{-n}E$. It follows from the property of $\mathcal{L}_\phi$ that

$$\begin{aligned}\int_A\Bigg(\frac{\mathcal{L}_\phi{}^n 1_{X_{i_1\dots i_n}}}{\mathcal{L}_\phi{}^n 1}\Bigg)\circ T^n(x)\,dm(x)&=\int_X\mathcal{L}_\phi{}^n 1(x)\Bigg(\frac{\mathcal{L}_\phi{}^n 1_{X_{i_1\dots i_n}}(x)}{\mathcal{L}_\phi{}^n 1(x)}\Bigg)1_E(x)\,dm(x)\\ &=m(A\cap X_{i_1\dots i_n}).\end{aligned}$$

Since $((\mathcal{L}_\phi{}^n 1_{X_{i_1\dots i_n}})/(\mathcal{L}_\phi{}^n 1))\circ T^n$ gives a $\sigma(T^{-n}\epsilon)$-measurable function, we complete the proof of (i). Next we note that for all $X_{i_1\dots i_n}$ with $(i_1\dots i_n)\in\mathcal{A}_n$

$$m(X_{i_1\dots i_n})=\int_{T^nX_{i_1\dots i_n}}\frac{dm}{d(mT^n)}(v_{i_1\dots i_n}y)\,dm(y)=\int_{T^nX_{i_1\dots i_n}}\exp\Bigg[\sum_{k=0}^{n-1}\phi\circ T^k(y)\Bigg]dm(y).$$

Then (ii) follows from the WBV property of ϕ. □

Proof of Lemma 4.1. It follows from Theorem 3.1(ii) that m satisfies the (WG-1) property for every $\phi:X\to\mathbb{R}$ satisfying $\exp\phi=dm/d(mT)$. Therefore, we obtain for the unique cylinder $X_{i_1\dots i_n}(x)$ of rank n containing x

$$m(X_{i_1\dots i_n}(x)\mid T^{-n}\epsilon)(x)=\frac{(dm/d(mT^n))(x)}{\mathcal{L}_m^n 1\circ T^n(x)}$$

and this equality gives the desired result. □

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Proof of Lemma 4.2. Let x_0 be a fixed point with $x_0 \in X_{i_0}$. First we note that $i_0(n) := \underbrace{i_0 \dots i_0}_{n} \in \mathcal{A}_n$ (for all $n \geq 1$) and $T^n \operatorname{int} X_{i_0(n)} = \bigcup_{U \in \mathcal{U}} U$. Since $m(\bigcup_{U \in \mathcal{U}} U) = 1$, we have for m-almost every $x \in X$

$$\begin{aligned}\mathcal{L}_m^n 1(x) &= \sum_{(i_1 \dots i_n) \in \mathcal{A}_n} \frac{dm}{d(m \circ T^n)}(v_{i_1 \dots i_n}(x)) 1_{T^n X_{i_1 \dots i_n}}(x) \\ &\geq \frac{dm}{d(m \circ T^n)}(v_{i_0 \dots i_0}(x)) = \exp\left[\sum_{k=0}^{n-1} \phi \circ T^k(v_{i_0 \dots i_0}(x))\right] \\ &= \frac{\exp[\sum_{k=0}^{n-1} \phi \circ T^k(v_{i_0 \dots i_0}(x))]}{\exp[\sum_{k=0}^{n-1} \phi \circ T^k(v_{i_0 \dots i_0}(x_0))]} \geq C_n^{-1}.\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}(*) \quad \mathcal{L}_m^n 1(x) &\leq C_n \sum_{(i_1 \dots i_n) \in \mathcal{A}_n} \inf_{y \in T^n X_{i_1 \dots i_n}} \frac{dm}{d(m \circ T^n)}(v_{i_1 \dots i_n}(y)) \\ &\leq C_n \sum_{(i_1 \dots i_n) \in \mathcal{A}_n} m(X_{i_1 \dots i_n}) \leq C_n.\end{aligned}$$

If all cylinders are Bernoulli cylinders, then we have that

$$\mathcal{L}_m^n 1(x) \geq C_n^{-1} \sum_{(i_1 \dots i_n) \in \mathcal{A}_n} m(X_{i_1 \dots i_n}) \geq C_n^{-1}.$$

These observations give us $\|(1/n) \log \mathcal{L}_m^n 1\|_{L^\infty(m)} \leq (1/n) \log C_n$ when either (i) or (ii) with $q = 1$ is satisfied. Moreover, the decay rate $(1/n) \log C_n$ is subexponential. Let x_0 be a periodic point with period q in a Bernoulli cylinder of rank q. Then, as we have observed in the above, for every $n = sq + r (0 \leq r \leq q - 1)$ we have $C_{sq}^{-1} \leq \mathcal{L}_\phi^{sq} 1(x) \leq C_{sq}$ which allows one to see that

$$\mathcal{L}_\phi^n 1(x) \leq C_{sq} \mathcal{L}_\phi^r 1(x) \leq C_{sq} C_r$$

and

$$\mathcal{L}_\phi^n 1(x) \geq C_{sq}^{-1} \mathcal{L}_\phi^r 1(x) \geq C_{sq}^{-1} C_r^{-1} \sum_{(i_1 \dots i_r), x \in T^r X_{i_1 \dots i_r}} m(X_{i_1 \dots i_r}).$$

We complete the proof. □

Proof of Theorem 4.1. The first assertion follows from the WBV property for ϕ directly. Indeed we can take the WBV sequence $\{C_n\}_{n \geq 1}$ for ϕ as the weak Gibbs sequence $\{K_n\}_{n \geq 1}$ for m. As we have observed the inequalities $(*)$ in the proof of Lemma 4.2,

$$\frac{1}{n} \int_X \mathcal{L}_m^n 1(x) \log \mathcal{L}_m^n 1(x) \, dm(x) \leq \frac{1}{n} \log C_n$$

holds for all $n \geq 1$. Therefore,

$$\limsup_{n \to \infty} \frac{1}{n} \int_X \mathcal{L}_m^n 1(x) \log \mathcal{L}_m^n 1(x) \, dm(x) \leq 0.$$

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The second assertion is proved. It follows from Lemma 4.2 that

$$\begin{aligned} 0 = e_T(m) &= \lim_{n\to\infty} \frac{1}{n} \int_X \log \mathcal{L}_m^n 1 \circ T^n(x)\, dm(x) \\ &= \limsup_{n\to\infty} \frac{1}{n} \int_X I_m\Big(\bigvee_{k-0}^{n-1} T^{-k} Q \,\Big|\, T^{-n}\epsilon \Big)(x)\, dm(x) \\ &\quad + \liminf_{n\to\infty} \frac{1}{n} \int_X \log \frac{dm}{d(mT^n)}(x)\, dm(x). \end{aligned}$$

Hence, we have

$$\begin{aligned} \hat{h}_m(T, Q) &= -\liminf_{n\to\infty} \frac{1}{n} \int_X \log \frac{dm}{d(mT^n)}(x)\, dm(x) \\ &= \limsup_{n\to\infty} \frac{1}{n} \int_X \left(-\log \frac{dm}{d(mT^n)}(x) \right) dm(x). \end{aligned}$$

On the other hand, the weak Gibbs property of m for ϕ allows us to see that

$$\limsup_{n\to\infty} \frac{1}{n} \int_X (-\log m(X_{i_1\dots i_n}(x))\, dm(x) = \limsup_{n\to\infty} \frac{1}{n} \int_X \left(-\log \frac{dm}{d(mT^n)}(x) \right) dm(x).$$

We complete the proof. □

Proof of Theorem 4.2. As we have already observed in the proof of Theorem 4.1, the weak Gibbs property allows us to have

$$\begin{aligned} \hat{h}_m(T, Q) &= \limsup_{n\to\infty} \int_X \frac{1}{n} \sum_{k=0}^{n-1} (-\phi \circ T^k(x))\, dm(x) \\ &= -\liminf_{n\to\infty} \int_X \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ T^k(x)\, dm(x). \end{aligned}$$

Thus (i) is proved. For the proof of (ii), first we note the next equality.

$$\begin{aligned} &\int_X \left(I_\mu\Big(\bigvee_{k=0}^{n-1} T^{-k} Q \,\Big|\, T^{-n}\epsilon \Big)(x) + \sum_{k=0}^{n-1} \phi \circ T^k(x) \right) d\mu(x) \\ &= \int_X \left(I_\mu\Big(\bigvee_{k=0}^{n-1} T^{-k} Q \,\Big|\, T^{-n}\epsilon \Big)(x) + \log \frac{d\mu}{d(\mu T^n)}(x) \right. \\ &\quad \left. + \sum_{k=0}^{n-1} \log \frac{\exp[\phi]}{d\mu/d(\mu T)} \circ T^k(x) \right) d\mu(x). \end{aligned}$$

Then by using the property that $\log x \le x - 1$ (for all $x > 0$) we have

$$\begin{aligned} &\int_X \left(I_\mu\Big(\bigvee_{k=0}^{n-1} T^{-k} Q \,\Big|\, T^{-n}\epsilon \Big)(x) + \sum_{k=0}^{n-1} \phi \circ T^k(x) \right) d\mu(x) \\ &\le \int_X \left(I_\mu\Big(\bigvee_{k=0}^{n-1} T^{-k} Q \,\Big|\, T^{-n}\epsilon \Big)(x) + \log \frac{d\mu}{d(\mu T^n)}(x) \right) d\mu(x) + \min\{A_n, B_n\}, \end{aligned}$$

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where

$$A_n := \int_X \sum_{k=0}^{n-1} \left(\frac{\exp[\phi]}{d\mu/d(\mu T)} \circ T^k - 1 \right) d\mu,$$

and

$$B_n := \int_X \left(\frac{\exp[\sum_{k=0}^{n-1} \phi \circ T^k]}{d\mu/d(\mu T^n)} - 1 \right) d\mu(x).$$

Moreover, we can write

$$\begin{aligned} B_n &= \int_X \mathcal{L}_\mu^n \left(\frac{\exp[\sum_{k=0}^{n-1} \phi \circ T^k(x)]}{d\mu/d(\mu T^n)(x)} - 1 \right) d\mu(x) \\ &= \int_X \sum_{y \in T^{-n}x} \frac{d\mu}{d(\mu T^n)}(y) \left(\frac{\exp[\sum_{k=0}^{n-1} \phi \circ T^k(y)]}{d\mu/d(\mu T^n)(y)} - 1 \right) d\mu(x) \\ &= \int_X \sum_{y \in T^{-n}x} \left(\exp\left[\sum_{k=0}^{n-1} \phi \circ T^k(y) \right] - \frac{d\mu}{d(\mu T^n)}(y) \right) d\mu(x) \\ &= \int_X (\mathcal{L}_\phi{}^n 1(x) - 1)\, d\mu(x). \end{aligned}$$

Then we obtain the desired inequality immediately. □

Proof of Theorem 4.3. The result $\mathcal{GP}_T(\phi) = 0$ follows from Theorem 4.2 directly. By Theorem 3.1(i), we see that $\mu = hm$ with $h/(hT) \equiv 1$ is a (WG-1) measure for the common potential ϕ. Therefore, for all $n \geq 1$ and μ-almost every $x \in X$

$$(**) \quad I_\mu \left(\bigvee_{k=0}^{n-1} T^{-k} Q \,\middle|\, T^{-n}\epsilon \right)(x) + \sum_{k=0}^{n-1} \phi \circ T^k(x) = \log \mathcal{L}_\mu^n 1 \circ T^n(x)$$

because of Lemma 4.1, and for $\mu \in \mathcal{N}_T(X, \phi)$ we have

$$\begin{aligned} &\frac{1}{n} \int_X I_\mu \left(\bigvee_{k=0}^{n-1} T^{-k} Q \,\middle|\, T^{-n}\epsilon \right)(x)\, d\mu(x) + \frac{1}{n} \int_X \sum_{k=0}^{n-1} \phi \circ T^k(x)\, d\mu(x) \\ &\quad = \frac{1}{n} \int_X \log \mathcal{L}_\mu^n 1 \circ T^n(x)\, d\mu(x). \end{aligned}$$

Since $\mu \leq m$, $\mu(\bigcup_{U \in \mathcal{U}} U) = 1$ holds so that $\mathcal{F}(\phi, \mu) = 0$ follows from Theorem 4.1. Thus part (i) is proved. To proving part (ii), without loss of generality we assume that x_0 is a fixed point. Then $\mu = \delta_{x_0} \in M_T(X)$ satisfies $d\mu/d\mu T = 1$ and $\mathcal{L}_\mu f(x) = f(x_0)$ so that $A_n = 0$. Also we have $B_n \geq 0$ because x_0 is an indifferent fixed point for m. We complete the proof. □

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Proof of Lemma 4.3. First we note the following inequalities:

$$\begin{aligned}
\mathcal{GP}_T(\phi) &\geq \sup\{\mathcal{F}(\phi,\mu) \mid \mu \in \mathcal{M}_T(X,\phi)\} \\
&= \sup\left\{ h_\mu(T) + \int_X \phi\, d\mu - \limsup \frac{1}{n} \min\{A_n, B_n\} \,\middle|\, \mu \in \mathcal{M}_T(X,\phi) \right\} \\
&\geq \sup\left\{ h_\mu(T) + \int_X \phi\, d\mu - \limsup \frac{1}{n} \min\{A_n, B_n\} \,\middle|\, \mu \in \mathcal{M}_T(X,\phi),\ \mu \leq m \right\} \\
&= \sup\left\{ h_\mu(T) + \int_X \phi\, d\mu - \min\left\{ \int_X hT\, dm - 1, \limsup_{n\to\infty} \int_X \frac{hT^n}{n}\, dm \right\} \,\middle|\, \mu \right. \\
&\qquad \left. = hm \in \mathcal{M}_T(X,\phi) \right\} \\
&\geq \sup\left\{ h_\mu(T) + \int_X \phi\, d\mu - \min\left\{ \int_X hT\, dm - 1, \limsup_{n\to\infty} \int_X \frac{hT^n}{n}\, dm \right\} \,\middle|\, \mu \right. \\
&\qquad \left. = hm \in \mathcal{M}_T(X,\phi), \text{ and } \int_X \log\frac{hT}{h}\, d\mu = 0 \right\}.
\end{aligned}$$

As $\int_X \log((hT)/h)\, d\mu \leq \int_X (((hT)/h) - 1)\, d\mu$ holds, we have

$$\begin{aligned}
\mathcal{GP}_T(\phi) \geq \sup\Bigg\{ h_\mu(T) + \int_X \phi\, d\mu \,\Bigg|\, \mu = hm \in \mathcal{M}_T(X,\phi),\ \int_X \log\frac{hT}{h}\, d\mu = 0, \\
\text{and } \limsup_{n\to\infty} \int_X \frac{hT^n}{n}\, dm = 0 \Bigg\} = \hat{\mathcal{P}}_T(\phi).
\end{aligned}$$

□

Proof of Theorem 4.4. We first show that $\int_X \log(h/(hT))\, d\mu = 0$. By Theorem 3.1(i), μ is a (WG-1) measure for $\hat{\phi} : X \to \mathbb{R}$ with

$$\hat{\phi} := \log\frac{d\mu}{d(\mu T)} = \log\frac{dm}{d(mT)} + \log\frac{h}{hT}.$$

Therefore, the equality (∗∗) in the proof of Theorem 4.3 holds for $\hat{\phi}$ so that

$$\int_X (I_\mu(Q \mid T^{-1}\epsilon)(x)\, d\mu(x) + \hat{\phi}(x))\, d\mu(x) = 0.$$

On the other hand, since the weak Gibbs property of μ for ϕ gives $H_\mu(Q) < \infty$, from Proposition 3.1 we see that

$$\begin{aligned}
&\int_X (I_\mu(Q \mid T^{-1}\epsilon)(x)\, d\mu(x) + \phi(x))\, d\mu(x) \\
&\quad = H_\mu(Q \mid T^{-1}\epsilon) + \int_X \phi(x)\, d\mu(x) = h_\mu(T) + \int_X \phi(x)\, d\mu(x) = 0.
\end{aligned}$$

This implies $\int_X \log(h/(hT))\, d\mu = 0$. The observation (a) and the inequality

$$\int_X \log\frac{hT}{h}\, d\mu \leq \int_X \left(\frac{hT}{h} - 1\right) d\mu$$

allow us to have

$$\limsup_{n\to\infty} \frac{1}{n} \min\{A_n, B_n\} = \min\left\{ \int_X hT\, dm - 1, 0 \right\} = 0.$$

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Hence, we have $\mathcal{F}(\phi, \mu) = 0 = h_\mu(T) + \int_X \phi(x)\, d\mu(x)$. On the other hand, from Theorem 4.3 we know $\mathcal{GP}_T(\phi) = 0$. We have thus proved (i). If $\hat{\mathcal{P}}_T(\phi) = \mathcal{P}_T(\phi)$, then $\mathcal{P}_T(\phi) \le 0$ which implies $\mu \in \mathcal{E}_T(\phi)$. Conversely, if $\mu \in \mathcal{E}_T(\phi)$, then $\mathcal{P}_T(\phi) = 0 \ge \mathcal{GP}_T(\phi)$. Since μ attains $\hat{\mathcal{P}}_T(\phi)$, we have $\hat{\mathcal{P}}_T(\phi) = \mathcal{P}_T(\phi) = \mathcal{GP}_T(\phi)$. We have thus proved (ii). Let $\nu \in \mathcal{E}_T(\phi)$. Then we see that

$$\frac{A_n}{n} = \int_X \left(\frac{\exp \phi}{d\nu/d(\nu T)} - 1 \right) d\nu \ge \int_X \log \frac{\exp \phi}{d\nu/d(\nu T)}\, d\nu = h_\nu(T) + \int_X \phi\, d\nu = 0.$$

Hence, we have $\mathcal{F}(\phi, \nu) = 0$. We complete the proof. □

Proof of Proposition 5.2. By [**Y4**, Theorem 3.1], there exists $0 < H < \infty$ so that

$$\begin{aligned}\int_X \mathcal{L}_\phi^n 1(x)\, d\mu(x) &= \sum_{k=1}^\infty \int_{B_k} \mathcal{L}_\phi^n 1(x) h(x)\, dm(x) \le H \sum_{k=1}^\infty k \int_{B_k} \mathcal{L}_\phi^n 1(x)\, dm(x) \\ &= H \sum_{k=1}^\infty k \int_X 1_{B_k} \circ T^n(x)\, dm(x) = H \sum_{k=1}^\infty k \times m(T^{-n} B_k).\end{aligned}$$

Let $\mathcal{V}$ be a finite disjoint partition of X generated by $\mathcal{U}$. Then we can write

$$B_k = \bigcup_{V \in \mathcal{V}} B_k^V \quad \text{where } B_k^V := \bigcup_{X_{b_1 \dots b_k} \subset B_k \cap V} X_{b_1 \dots b_k}.$$

Define $\mathcal{A}_n^V := \{(i_1 \dots i_n) \in \mathcal{A}_n \mid T^n X_{i_1 \dots i_n} \supset V\}$. For every $X_{b_1 \dots b_k} \subset B_k \cap V$, we can define a finite disjoint partition of $\mathcal{A}_n^V$ by $\mathcal{A}_n^V = \bigcup_{l=0}^n \mathcal{A}_n^{V,l}(b_1 \dots b_k)$, where

$$\mathcal{A}_n^{V,l}(b_1 \dots b_k) := \{(i_1 \dots i_n) \in \mathcal{A}_n^V \mid X_{i_1 \dots i_{n-l}} \in \mathcal{R},\ X_{i_{n-l+1} \dots i_n b_1 \dots b_k} \subset B_{k+l}\}.$$

Therefore, we have

$$\begin{aligned} m(T^{-n} B_k) &\le \sum_{V \in \mathcal{V}} \sum_{X_{b_1 \dots b_k} \subset B_k \cap V} m\left(\bigcup_{(i_1 \dots i_n) \in \mathcal{A}_n^V} X_{i_1 \dots i_n b_1 \dots b_k} \right) \\ &= \sum_{V \in \mathcal{V}} \sum_{X_{b_1 \dots b_k} \subset B_k \cap V} \left(\sum_{l=0}^n \sum_{(i_1 \dots i_n) \in \mathcal{A}_n^{V,l}(b_1 \dots b_k)} m(X_{i_1 \dots i_{n-l}} \cap T^{-(n-l)} X_{i_{n-l+1} \dots i_n b_1 \dots b_k}) \right). \end{aligned}$$

Since $dm/d(mT^*)$ satisfies the uniformly bounded distortion property, there exists $1 \le C < \infty$ such that

$$m(X_{i_1 \dots i_n b_1 \dots b_k}) \le C m(X_{i_1 \dots i_{n-l}}) m(X_{i_{n-l+1} \dots i_n b_1 \dots b_k}).$$

Hence, we have

$$\begin{aligned} &m(T^{-n} B_k) \\ &\quad \le C \sum_{V \in \mathcal{V}} \sum_{X_{b_1 \dots b_k} \subset B_k \cap V} \left(\sum_{l=0}^n \sum_{(i_1 \dots i_n) \in \mathcal{A}_n^{V,l}(b_1 \dots b_k)} m(X_{i_1 \dots i_{n-l}}) m(X_{i_{n-l+1} \dots i_n b_1 \dots b_k}) \right). \end{aligned}$$

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The right-hand side is bounded from above by

$$C\sum_{V\in\mathcal{V}}\sum_{l=0}^{n}\sum_{\substack{X_{i_1\dots i_{n-l}}\\ \subset D_{n-l}^c}} m(X_{i_1\dots i_{n-l}})\Bigg(\sum_{\substack{X_{b_1\dots b_k}\\ \subset B_k\cap V}}\Bigg(\sum_{\substack{(i_{n-l+1}\dots i_n):\\ (i_1\dots i_n)\in\mathcal{A}_n,\\ X_{i_{n-l+1}\dots i_n b_1\dots b_k}\subset B_{k+l}}} m(X_{i_{n-l+1}\dots i_n b_1\dots b_k})\Bigg)\Bigg)$$

$$= C\sum_{V\in\mathcal{V}}\sum_{l=0}^{n}\Bigg(\sum_{X_{i_1\dots i_{n-l}}\subset D_{n-l}^c} m(X_{i_1\dots i_{n-l}})\Bigg)m(B_{k+l}).$$

Therefore, we have $m(T^{-n}B_k)\le C\sharp\mathcal{V}\sum_{l=0}^{n} m(B_{k+l})$. Since

$$\bigcup_{l=0}^{n} B_{k+l}\subset D_{k-1},\quad n^{-1}\int_X \mathcal{L}_\phi^n 1\,d\mu$$

is bounded from above by $HC\sharp\mathcal{V}\sum_{k=1}^{\infty} k(m(D_{k-1}))$. We remark that there exists $1\le r_1'<r_2'<\infty$ such that for all $n\ge 1$

$$r_1' n^{-(\alpha+1)}\le m(B_n)\le r_2' n^{-(\alpha+1)}.$$

Finally we see that $\int_X R^2\,dm=\sum_{k=1}^{\infty} k^2 m(B_k)<\infty$ implies $\sum_{k=1}^{\infty} km(D_k)<\infty$. We have thus completed the proof. □

References

[Aa] J. Aaronson. *An Introduction to Infinite Ergodic Theory*. American Mathematical Society, Providence, RI, 1997.

[GR] G. Gallavotti and D. Ruelle. SRB states and nonequilibrium statistical mechanics close to equilibrium. *Commun. Math. Phys.* **190** (1997), 279–285.

[MRTMV] C. Maes, F. Reidig, F. Takens, A. van Moffaert and E. Verbitski. Intermittency and weak Gibbs states. *Nonlinearity* **13** (2000), 1681–1689.

[PS1] Y. Pesin and S. Santi. Themodynamical formalism associated with inducing schemes for one-dimensional maps. *Mosc. Math. J.* **5**(3) (2005), 669–678.

[PS2] Y. Pesin and S. Santi. Equilibrium measures for maps with inducing schemes. *J. Modern. Dynam.* **2**(3) (2008), 1–31.

[PZ] Y. B. Pesin and Ke Zhang. Thermodynamics associated with inducing schemes and liftability of measures. *Proc. Fields Inst.* **51** (2007), 289–305.

[R1] D. Ruelle. Entropy production in nonequilibrium statistical mechanics. *Commun. Math. Phys.* **189** (1997), 365–371.

[R2] D. Ruelle. Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics. *J. Stat. Phys.* **95** (1999), 393–468.

[Sch] F. Schweiger. *Ergodic Theory and Fibred Systems and Metric Number Theory*. Oxford University Press, Oxford, 1995.

[T] M. Thaler. Transformations on [0, 1] with infinite invariant measures. *Israel J. Math.* **46** (1983), 67–97.

[Y1] M. Yuri. Multi-dimensional maps with infinite invariant measures and countable state sofic shifts. *Indag. Math. (N.S.)* **6** (1995), 355–383.

[Y2] M. Yuri. Zeta functions for certain nonhyperbolic systems and topological Markov approximations. *Ergod. Th. & Dynam. Sys.* **18** (1998), 1589–1612.

[Y3] M. Yuri. Thermodynamic formalism for certain nonhyperbolic maps. *Ergod. Th. & Dynam. Sys.* **19** (1999), 1365–1378.

https://doi.org/10.1017/S0143385708000667 Published online by Cambridge University Press

[Y4] M. Yuri. Weak Gibbs measures for certain nonhyperbolic systems. *Ergod. Th. & Dynam. Sys.* **20** (2000), 1495–1518.

[Y5] M. Yuri. On the speed of convergence to equilibrium states for multi-dimensional maps with indifferent periodic points. *Nonlinearity* **15** (2002), 429–445.

[Y6] M. Yuri. Multifractal analysis of weak Gibbs measures for intermittent systems. *Commun. Math. Phys.* **230** (2002), 365–388.

[Y7] M. Yuri. Thermodynamic Formalism for countable to one Markov systems. *Trans. Amer. Math.* **355** (2003), 2949–2971.

[Y8] M. Yuri. Weak Gibbs measures for Intermittent systems and weakly Gibbsian states in Statistical mechanics. *Commun. Math. Phys.* **241** (2003), 453–466.

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