J. Plasma Physics (2001), *vol.* 65, *part* 3, *pp.* 181–195. © 2001 Cambridge University Press
DOI: 10.1017/S002237780101106 Printed in the United Kingdom

Stationary structures in multi-ion plasmas with differential streaming

J. F. McKENZIE

Max-Planck Institut für Aeronomie, Katlenburg–Lindau, Germany, and School of Pure and Applied Physics, University of Natal, Durban, South Africa

(Received 1 August 2000 and in revised form 23 October 2000)

Abstract. The nature of stationary-wave structures in multi-ion plasmas with differential streaming between the ions is elucidated by making use of the concept of a generalized sonic point for such plasmas and the Bernoulli equation for each flowing ion species. It is shown that the Bohm criterion for the existence of solitons and double layers is an expression of either the supersonic nature of the collective ion flow or the subsonic nature of individual ion flows. From the perspective of the dispersion equation for stationary waves, both of these requirements, and hence the Bohm criterion, merely ensure that the wave is evanescent, rather than sinusoidal, which, in turn, ensures that the system can evolve nonlinearly. The structure of the ion flows in these transitions is determined by the Bernoulli equation, a fundamental property of which is that the ion plasma energy exhibits a minimum where the ion flow speed equals the local ion sound speed. This implies that for positive flowing ions, a positive change in electric potential induces a deceleration (pile-up) if the flow is supersonic or an acceleration (spread-out) if the flow is subsonic. This feature limits the maximum strength of the potential in a soliton, and sheds light on the underlying reason why double layers can only exist in very special circumstances without the introduction of additional trapped or reflected particle populations.

1. Introduction

The multicomponent nature of many space and astrophysical plasmas (for example the solar-wind, which contains a non-neglible component of alpha particles as well as protons and other plasmas associated with comets and planetary magnetospheres where there can be a significant charged dust content) leads to interesting properties that do not arise in single-ion plasmas. These are manifested in the appearance of new, collective modes of the compound system by the coupling of the various ion species through the Lorentz force and the Poisson equation.

In this paper, we consider the special case of stationary-wave structures in a multi-ion plasma flowing with differential speeds along an ambient magnetic field or in an unmagnetized plasma. Although there is a considerable body of literature on the topic of solitons and double layers (see e.g. Deutsch and Räuchle 1992; Verheest and Hellberg 1997; Baboolal et al. 1991; Verheest 1993; Mace and Hellberg 1993; and references therein) the physical properties and structure of these stationary transitions has tended to be obscured by, for example, mathematical developments involving the expansion of the so-called full Sagdeev potential and by a lack of

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

appreciation of the concept of a collective sonic point, without which the Bohm criterion is bereft of its proper meaning.

In the first place, it is emphasized that the structure equation is an expression of total momentum (particles plus field) conservation in which the Sagdeev potential is the difference between the incoming particle momentum and its local value in the transition and that this difference is balanced by the momentum associated with the electric field tension and pressure. It is important to realize that the Bohm criterion for the existence of such a transition is an expression of the supersonic condition for the collective ion flow and also a subsonic condition for individual ion flows – both of which conditions follow from the requirement that the stationary wave of the system be of the evanescent type. Without this evanescent feature, the structure could not evolve nonlinearly. Furthermore, the Bernoulli equation for each ion flow reveals the nature of the flow in the transition through the property that the particle energy (enthalpy plus kinetic) exhibits a minimum where the local ion speed equals the ion sound speed.

These various interlinking themes are discussed in the paper, which is laid out as follows.

In Sec. 1, we introduce the idea of a sonic point in a multi-ion plasma. The concept of a sonic point of flow is generalized to a plasma system consisting of different ion species flowing with differential speeds u_i. This is done by using the linear dispersion equation to find the condition that waves, as viewed in a laboratory frame S relative to which the various ion species stream, appear stationary. For example, in a bi-ion plasma, this condition translates into two loci in the plane (u_1, u_2) of the ion speeds on which compressional waves are stationary in S. One locus is a closed curve encircling the origin, and corresponds to stationary waves associated with the slower mode, while the other locus is an open rectangular hyperbolic-like curve on which the faster mode waves appear stationary. Similarly in a three-ion plasma, stationary waves exist on certain surfaces in (u_1, u_2, u_3) space defined by the dispersion relation, and so on for an N-ion plasma.

In Sec. 3, we investigate finite-amplitude transitions and the criterion for their existence. This is done by casting the conservation law for the total momentum of the multi-ion beam system in the form of a classical energy integral of a particle moving in an effective potential well $V(\phi)$, the full Sagdeev potential, where ϕ is the electric potential. Three conditions for the existence of either a monotonic electrical potential transition or for a solitary pulse in ϕ must be met. The first two conditions merely express respectively total momentum balance and charge neutrality on either side of the transition/pulse, and therefore in no way impose any new restrictions. The third condition, relating to the curvature $V_{\phi\phi}$ of the potential well, yields the generalized Bohm criterion for a multi-ion flow. It is shown that this is equivalent to an expression of the 'supersonic' nature of the collective ion-beam system for which the loci, or surfaces, in ion-speed space on which waves in S are stationary provide the appropriate generalization of the idea of a sonic point to a multi-ion system. However, it also follows that individual subsonic ion flows satisfy the Bohm criterion, which in fact can be more properly interpreted as a requirement that the stationary wave be of the 'evanescent' type so as to allow for its nonlinear evolution.

Finally, in Sec. 4, we examine the structure of the ion flow in solitons and double layers. This is done by making use of a fundamental property of the Bernoulli equation, namely that the ion energy (enthalpy plus kinetic) exhibits a minimum

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

where the ion speed equals the ion sound speed. It follows that a positive change in potential induces an acceleration (depletion) in a positive ion if the flow is subsonic, whereas it causes a deceleration (enhancement) if the flow is supersonic. This property limits the maximum (positive) potential attainable in a soliton. It also sheds light on the reasons why a double layer can only exist in very special circumstances in the absence of trapped or reflected particle populations.

The basic results are summarized in Sec. 5.

2. The governing equations for a multicomponent unmagnetized plasma

2.1. The general equations

The macroscopic conservation laws of continuity, momentum and energy for each ion species i, with density n_i, velocity $\mathbf{u}_i$ and pressure p_i acted upon by an electric field $\mathbf{E}$, may be written

$$\frac{D_i n_i}{Dt} + n_i \boldsymbol{\nabla} \cdot \mathbf{u}_i = 0, \tag{1}$$

$$m_i n_i \frac{D_i \mathbf{u}_i}{Dt} = -\boldsymbol{\nabla} p_i + Z_i n_i \mathbf{E} + \mathbf{M}_i, \tag{2}$$

$$\frac{D_i p_i}{Dt} - c_i^2 m_i \frac{D n_i}{Dt} = (\gamma_i - 1) Q_i, \tag{3}$$

in which $i = 1, 2, \ldots, N$ and

$$\frac{D_i}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_i \cdot \nabla, \tag{4}$$

$$m_i c_i^2 = \gamma_i k T_i \equiv \gamma_i \frac{p_i}{n_i}.$$

Here $\mathbf{M}_i$, stands for momentum exchange (through, for example, Coulomb friction with other species), Q_i represents heating or dissipative processes, and γ_i is the 'appropriate' adiabatic index.

We shall assume that the electrons are massless (but this restriction can easily be lifted) and therefore obey the force-balance condition

$$\boldsymbol{\nabla} p_e = -e n_e \mathbf{E}. \tag{5}$$

An electron energy equation may be written in the same form as (3), namely

$$\frac{D_e p_e}{Dt} - \gamma_e k T_e \frac{D_e n_e}{Dt} = (\gamma_e - 1) Q_e. \tag{6}$$

Finally, the Poisson equation

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{e}{\varepsilon_0} \left(\sum_i Z_i n_i - n_e \right) \tag{7}$$

provides the coupling between the various ion species and the electrons through the electric field.

2.2. The dispersion equation for a homogenous plasma with differential ion streaming

We shall derive the dispersion equation in a laboratory frame S relative to which the various ions stream with constant velocities $\mathbf{u}_{i0}$, and assume that $\mathbf{M}_i$ is negligible

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

and also that $Q_i = Q_e = 0$, the latter being necessary to ensure that the background quantities n_{i0}, u_{i0} and p_{i0} are uniform and constant. The linearized equations for the perturbed quantities n_i, $\mathbf{u}_i$, p_i and $\mathbf{E}$ may be written

$$\boldsymbol{\nabla} \cdot \mathbf{u}_i = -\frac{1}{n_{i0}} \frac{D_i n_i}{Dt}, \tag{8}$$

$$m_i n_{i0} \frac{D_i \mathbf{u}_i}{Dt} = -\boldsymbol{\nabla} p_i + e Z_i n_{i0} \mathbf{E}, \tag{9}$$

$$\frac{D_i p_i}{Dt} = c_{i0}^2 m_i \frac{D n_i}{Dt}. \tag{10}$$

Taking the divergence of (9), using (8) to eliminate $\mathbf{u}_i$, and then taking D_i/Dt of the resultant equation and using (10) to eliminate p_i, gives

$$\frac{D_i}{D_t} \left[\left(\frac{D_i^2}{Dt^2} - c_{i0}^2 \nabla^2 \right) \frac{e Z_i n_i}{\varepsilon_0} - \omega_{pi}^2 \nabla^2 \phi \right] = 0, \tag{11}$$

in which we have introduced the electric potential ϕ, where

$$\mathbf{E} = -\boldsymbol{\nabla} \phi, \tag{12}$$

and the ion plasma frequency ω_{pi} is given by

$$\omega_{pi}^2 = \frac{(e Z_i)^2 n_{i0}}{\varepsilon_0 m_i}. \tag{13}$$

Compressional waves are described by the zero of the square brackets in (11). (The operator D_i/Dt, which may be factored out, merely corresponds to the convection of the perturbed entropy of each ion fluid).

With $Q_e = 0$, the perturbed electron equations (5) and (6) yield

$$\phi = \frac{\gamma_e T_{e0}}{e n_{e0}} n_e. \tag{14}$$

Finally, the linearized Poisson equation takes the form

$$\left(\frac{1}{\lambda_d^2} - \nabla^2 \right) \phi = \frac{e}{\varepsilon_0} \sum_i Z_i n_i, \tag{15}$$

in which we have used (14) and introduced the Debye length λ_d, where

$$\frac{1}{\lambda_d^2} = \frac{e^2 n_{e0}}{\varepsilon_0 \gamma_e k T_{e0}}. \tag{16}$$

For plane waves varying as $\exp[i(\omega t - \mathbf{k} \cdot \mathbf{r})]$ the zero of the square brackets in (11) yields the following relation n_i and ϕ:

$$\frac{e Z_i n_i}{\varepsilon_0} = \frac{\omega_{pi}^2 k^2}{\omega_i'^2 - c_{i0}^2 k^2} \phi, \tag{17}$$

in which ω_i' is the Doppler-shifted wave frequency for each streaming ion species,

$$\omega_i' = \omega - \mathbf{u}_{i0} \cdot \mathbf{k}. \tag{18}$$

Substituting (17) into the Poisson equation (15) yields the following dispersion relation for the system:

$$\frac{1}{\lambda_d^2} + k^2 = \sum_i \frac{\omega_{pi}^2 k^2}{\omega_i'^2 - c_{io}^2 k^2}, \tag{19}$$

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

or

$$1 = \sum_i \frac{V_{ei}^2}{(V_{\rm ph} - u_{io}\cos\theta_i)^2 - c_{io}^2}, \tag{20}$$

in which

$$V_{\rm ph} \equiv \frac{\omega}{k} \text{ is the phase velocity,} \tag{21a}$$

$$V_{ei}^2 \equiv \frac{\lambda_d^2 \omega_{pi}^2}{1 + k^2\lambda_d^2}, \tag{21b}$$

$$\cos\theta_i \equiv \frac{\mathbf{k}\cdot\mathbf{u}_{i0}}{k_{i0}u_{i0}}. \tag{22}$$

The quantity $\sqrt{c_{i0}^2 + V_{ei}^2}$ is the 'ion acoustic' speed for each ion species. In our subsequent discussion of the relevance of the dispersion equation (20) to the Bohm criterion for double layers and solitons, we shall see how both the long- and short-wavelength versions make their appearance.

In a bi-ion plasma with no differential streaming, (20) shows that there are two compressional modes: one a slow mode with a phase speed $V_{\rm ph} < \sqrt{c_{20}^2 + V_{e2}^2}$ and the other a fast mode with phase speed $V_{\rm ph} > \sqrt{c_{10}^2 + V_{e1}^2}$ (where for the sake of definiteness we have taken $c_{20} < c_{10}$).

2.3. The 'sonic point' in a multicomponent plasma

In a single ion plasma streaming with positive speed u_{i0} the sonic point occurs where the flow speed matches the phase speed of a backward-propagating compressional wave, i.e. where

$$u_{i0} = \sqrt{V_{e1}^2 + c_1^2}, \tag{23}$$

since at that point the speed $V_{\rm ph}$ of waves in S is reduced to zero and therefore the wave appears stationary in S. In a multicomponent plasma, it follows from (20) that waves appear stationary in S (i.e. $V_{\rm ph} = 0$) if we have

$$1 = \sum_i \frac{V_{ei}^2}{(u_{i0}\cos\theta)^2 - c_{i0}^2}. \tag{24}$$

This provides the generalization of the idea of a sonic point to a multicomponent plasma. To elucidate its meaning, consider a bi-ion plasma in which the waves propagate parallel to u_{i0}, so that, with $\theta_i = 0$, (24) takes the form

$$1 = \frac{V_{e1}^2}{u_1^2 - c_1^2} + \frac{V_{e2}^2}{u_2^2 - c_2^2}, \tag{25a}$$

or

$$[u_1^2 - (c_1^2 + V_{e1}^2)][u_2^2 - (c_2^2 + V_{e2}^2)] = V_{e1}^2 V_{e2}^2, \tag{25b}$$

or

$$\begin{aligned} k^2\lambda_\alpha^2 &= \frac{a_1^2}{u_1^2 - c_1^2} + \frac{a_2^2}{u_2^2 - c_2^2} - 1 \\ &= \frac{a_1^2 - [u_1^2 - (c_1^2 + a_1^2)][u_2^2 - (c_2^2 + a_2^2)]}{(u_1^2 - c_1^2 - c_2^2)}, \end{aligned} \tag{25c}$$

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

in which

$$a_i^2 = \lambda_\alpha^2 \omega_{pi}^2 = \frac{kT_e}{m_i}\frac{n_i}{n_e}. \tag{25d}$$

Equations (25a–d) represent two loci in the (u_1, u_2) plane, on which the waves are stationary as viewed in the laboratory frame (see Fig. 1). The closed locus encircling the origin is associated with the slower of the two modes of the dispersion equation (20), while the other, a open rectangular-hyperbolic-like curve, corresponds to stationary waves of the faster mode. In this context, region 1 in Fig. 1 may be labelled 'subslow', region (2) 'superslow' but 'subfast', and region 3 'superfast'. This labelling is of significance in considerations of the evolutionarity of double layers or shock-like transitions, since in the different regions of (u_1, u_2) space, the number of waves that communicate upstream and downstream of the transition depend on the regions of the parameter space involved.

Similarly, in a three-ion plasma, stationary waves in S occur on the surfaces in (u_1, u_2, u_3) space defined by

$$1 = \frac{V_{e1}^2}{u_1^2 - c_1^2} + \frac{V_{e2}^2}{u_1^2 - c_2^2} + \frac{V_{e3}^2}{u_3^2 - c_3^2}, \tag{26}$$

and so on for an N-ion system.

3. Finite-amplitude transitions in a multi-ion plasma and the Bohm sheath criterion

We consider stationary structures in a laboratory frame S relative to which each ion species streams steadily with speed u_i along the x axis. The relevant equations for the ions follow from (1)–(5), in which $\partial/\partial t = 0$, with the result that we have the equations

$$n_i u_i = \text{const} = j_i, \quad \text{say} \tag{27}$$

$$m_i n_i u_i \frac{du_i}{dx} = -\frac{dp_i}{dx} + eZ_i n_i E + M_i, \tag{28}$$

$$\frac{dp_i}{dx} = c_i^2 m_i \frac{dn_i}{dx} + (\gamma_i - 1)\frac{Q_i n_i}{j_i}, \tag{29}$$

$$0 = -\frac{dp_e}{dx} - en_e E, \tag{30}$$

coupled through Poisson's equation

$$\frac{dE}{dx} = \frac{e}{\varepsilon_0}\left(\sum_i Z_i n_i - n_e\right). \tag{31}$$

Summing (28) over all ions and adding (30) gives, after using the Poisson equation, the total momentum balance equation in the form

$$\frac{d}{dx}\left[\sum_i (m_i n_i u_1^2 + p_i) + p_e - \tfrac{1}{2}\varepsilon_0 E^2\right] = 0. \tag{32}$$

This form has already been used by McKenzie et al. (1977) and Zank and McKenzie (1988), in their discussions of the properties of transitions and waves in a plasma. It is worth noting that the appearance of $-\frac{1}{2}\varepsilon_0 E^2$ may be interpreted in terms of

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

the electric Maxwell stress tensor, which assigns an isotropic pressure $\frac{1}{2}\varepsilon_0 E^2$ and a tension $-\frac{1}{2}\varepsilon_0 E^2$ along the electric field direction. Alternatively (32) may be regarded as the first integral of the Poisson equation (Sagdeev 1966), and its interpretation in terms of momentum balance has been noted by several authors (see e.g. Andrews and Allan 1971; Torvén 1980; Raadu 1989):

$$\frac{\varepsilon_0}{2}\left(\frac{d\phi}{dx}\right)^2 = -V(\phi), \tag{33a}$$

where

$$V(\phi) = M_0 - \left[\sum(p + mnu^2) + p_e\right], \tag{33b}$$

in which M_0 represents the upstream momentum of the system. Thus (33a) is analogous to an energy integral of a particle moving in a potential well $V(\phi)$. The type of electric potential transition implied by (33a, b) depends on the shape of the equivalent potential well $V(\phi)$.

Let us seek a solution of (33a, b) in which $\phi(x)$ is a monotonic function of x going from $\phi = 0$ at one end ($x = -\infty$) and approaching asymptotically a constant value ϕ_0 at the other end ($x = +\infty$). Such a double layer or potential jump requires $V(\phi)$ to meet the following conditions at both ends:

$$V(0) = V(\phi_0) = 0, \tag{34a}$$

$$\left(\frac{dV}{d\phi}\right)_0 = \left(\frac{dV}{d\phi}\right)_{\phi 0} = 0, \tag{34b}$$

$$\frac{d^2V}{d\phi^2} < 0 \quad \text{at } \phi = 0 \quad \text{and} \quad \phi_0. \tag{34c}$$

Condition (34a) is equivalent to the total momentum jump relation, and therefore imposes no restrictions whatsoever. Condition (34b) ensures charge neutrality, $\sum_i Z_i n_i = n_e$, on each side. This can be proved by noting that (with $M_i = Q_i = 0$) (28)–(30) imply the following differential relations:

$$m_i(u_i^2 - c_i^2)\frac{1}{u_i}\frac{du_i}{d\phi} = -eZ_i \tag{35a}$$

$$\frac{dp_i}{d\phi} = -\gamma_i p_i \frac{1}{u_i}\frac{du_i}{d\phi}, \tag{35b}$$

$$dp_e = +en_e\, d\phi = \gamma_e kT_e\, dn_e. \tag{35c}$$

$dV/d\phi$ becomes

$$\frac{dV}{d\phi} = -\frac{dp_e}{d\phi} - \sum_i \left(m_i n_i \frac{du_i}{d\phi} + \frac{dp_i}{d\phi}\right)$$

$$= -en_e + \sum_i eZ_i n_i, \tag{36}$$

on using the relations (35a–c). Hence condition (34b) implies charge neutrality at each end. Similarly, condition (34c) follows from differentiating (36) with respect

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

to ϕ to obtain

$$\frac{d^2V}{d\phi^2} = -e\frac{dn_e}{d\phi} - \sum_i Z_i \frac{dn_i}{d\phi}$$

$$= -e\left(\frac{en_e}{\gamma_e kT_e} - \sum_i \frac{Z_i n_i e Z_i}{m_i n_i^2 - \gamma_i kT_i}\right)$$

$$= -\varepsilon_0\left(\frac{1}{\lambda_d^2} - \sum_i \frac{\omega_{pi}^2}{u_i^2 - c_i^2}\right). \tag{37}$$

Thus condition (34c), $d^2V/d\phi^2 < 0$ on each side, yields

$$\frac{1}{\lambda_d^2} > \sum_i \frac{\omega_{pi}^2}{u_i^2 - c_i^2}, \tag{38a}$$

or

$$1 > \sum_i \frac{V_{ei}^2}{u_i^2 - c_i^2}. \tag{38b}$$

This is the generalized Bohm criterion for the existence of a double layer (Block 1978) or solitary pulse. From the dispersion relations (20) and (24), the inequality (38a) or (38b) is evidently an expression of the required 'supersonic' nature of the collective multi-ion flow, in which the sonic condition corresponds to an equality in (38a, b). For the case of wall sheaths, Allen (1976) pointed out that the Bohm criterion is equivalent to the condition that an ion acoustic wave cannot propagate back into the ambient plasma. However, it is clear that (38a, b) are also satisfied if each ion flow is individually subsonic relative to its own sound speed, i.e. $u_i < c_i$. Figure 1 depicts the regions in the (u_2, u_1) plane of a bi-ion flow in which the Bohm criterion is satisfied. At first glance, it may seem somewhat surprising that the Bohm criterion also allows for individual subsonic flows as well as collective supersonic flows. The reason for this can be gleaned from the dispersion equation (20), which, for a stationary structure with spatial variation $\exp(\kappa x)$, may be written, in the simplest one-ion case, in the form

$$\lambda_d^2\kappa^2 = 1 - \frac{\lambda_d^2\omega_{p1}^2}{u_1^2 - c_1^2}. \tag{39}$$

This shows that exponentially growing (or decaying) structures are only possible if $\kappa^2 > 0$, requiring that the flow speed satisfies

$$u_1 > \sqrt{c_1^2 + V_{ei}^2}, \qquad V_{ei}^2 \equiv \lambda_d^2\omega_{pi}^2 \qquad \text{(supersonic)}, \tag{40a}$$

or

$$u_1 < c_1 \qquad \text{(subsonic)}. \tag{40b}$$

The generalization to a bi-ion or multi-ion plasma is obvious. Thus the Bohm criterion viewed from this perspective is merely an expression of the fact that near the endpoints of stationary structures, the spatial variations must be of the exponential type (i.e. 'evanescent') rather than sinusoidal, otherwise they could not possibly evolve. Alternatively, the Bohm criterion follows from the evanescent requirement ($k^2 < 0$), which from the dispersion equation translates into $V_{\rm ph} > \sqrt{u_1^2 + V_{ei}^2}$

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

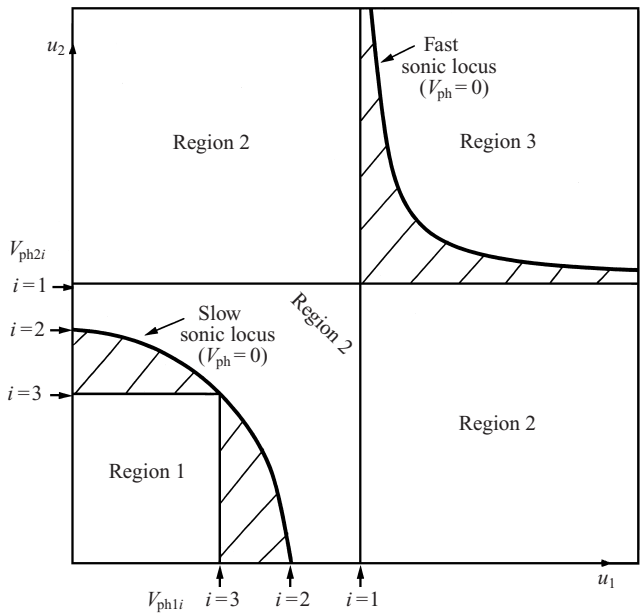


Figure 1. The generalized Bohm criterion for a bi-ion plasma in (u_2, u_1) space. There are two sonic loci (for $k = 0$); the one encircling the origin is associated with the slow mode, and the hyperbolic-like one corresponds to the fast mode. In region 3 the collective flow is 'superfast', while in region 2, the flow is 'superslow' but 'subfast'. The rectangular region 1 is not only 'subslow' but also subsonic for each species. The various speeds V_{ph1i} and V_{ph2i} are defined as (see (25c)) $V_{ph11} = \sqrt{c_1^2 + a_1^2}$, $V_{ph12} = \sqrt{c_1^2 + c_1^2/(1 + a_2^2/c_2^2)}$ and $V_{ph13} = c_1$ (and similarly for V_{ph21}). Thus $u_1 = V_{ph11}$ and $u_2 = V_{ph21}$ are the pair of asymptotes for the fast sonic locus for long wavelengths ($k = 0$), and $u_2 = V_{ph12}$ and $u_2 = V_{ph32}$ are the endpoints of the slow-mode locus (for $k = 0$). On the lines $u_1 = V_{ph13} = c_1$ and $u_2 = V_{ph23} = c_2$, $k = \infty$. In the hatched regions, the Bohm criterion is not fulfilled.

for 'long wavelengths' or $V_{ph} < u_1$ for 'short wavelengths'. These conclusions were arrived at by Bertrand and Feix (1968) for water-bag distribution and Raadu (1989) for general distribution functions.

4. The structure of finite-amplitude transitions

4.1. The Bernoulli equation

The structure of the flow in either a soliton or a double layer (monotonic transition in ϕ) is readily deduced from the Bernoulli-type energy equation for each species. For adiabatic transitions ($p_u \propto n_n^{\gamma_i} \propto u_i^{-\gamma_i}$), this follows as the integral of (35a), i.e.

$$\frac{\gamma_i}{\gamma_i - 1}\frac{p_i}{n_i} + \frac{1}{2}m_i u_i^2 + eZ_i\phi = \text{const} = E_i, \quad \text{say}, \tag{41}$$

which states that the sum of the enthalpy (the first term on the left-hand side), the kinetic energy (second term) per ion and the electrostatic potential is conserved. (In the isothermal limit $\gamma_i \to 1$, the enthalpy becomes $k_i \log n_i$). A fundamental property of one-dimensional flow is that the plasma energy E_{pi} associated with

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

each ion species, i.e.

$$E_{pi} = \frac{\gamma_i}{\gamma_i - 1}\frac{p_i}{n_i} + \frac{1}{2}m_i u_i^2, \tag{42}$$

possesses a minimum when the flow speed u_i equals the sound speed $c_i\,(= \sqrt{\gamma_i kT_i/m_i})$ for the species. Therefore the question of whether a flow is accelerated or decelerated by a change in potential depends not only on the sign of the charge Z_i but also on whether the flow is subsonic or supersonic. If the flow is supersonic (i.e. the kinetic energy dominates) then a positive change in potential induces a deceleration (acceleration) if the ion is positively (negatively) charged. This is the familiar ('intuitive') result that follows from the ('cold' energy equation, and implies pile-up. On the other hand, if the flow is subsonic (i.e. the enthalpy dominates) then exactly the opposite result follows – namely that a positive change in the potential induces an acceleration (deceleration) if the ion is positively (negatively) charged and hence there is a spread-out. This arises because the ion pressure gradient must balance the electric field force, which is negative ($Z_i > 0$), and hence the flow goes from higher pressure (density) to lower pressure (density) and is thereby accelerated. Thus we must expect that the minimum value of E_{pi} will play a critical role in determining the maximum permissible strength of the potential in the structure.

This reasoning can be put in a more analytical way by writing the Bernoulli equation (41) in the normalized form

$$L_i \equiv \frac{1}{\gamma_i - 1}\left[\left(\frac{u_{0i}}{u_i}\right)^{\gamma_i - 1} - 1\right] + \frac{M_{0i}^2}{2}\left[\left(\frac{u_i}{u_{0i}}\right)^2 - 1\right] = \frac{-eZ_i\phi}{\gamma_i kT_{0i}}, \tag{43}$$

where L_i is the change in the normalized ion plasma energy accompanying the change in potential ϕ. Here u_{0i} is the entry speed and M_{0i} the entry Mach number,

$$M_{0i} = u_{0i}/c_{i0}.$$

L_i has a minimum (negative) value

$$L_{i\,\min} = \frac{1}{\gamma_i - 1}[(M_{0i}^2)^{(\gamma_i - 1)(\gamma_i + 1)} - 1] + \frac{M_{0i}^2}{2}\left[\left(\frac{1}{M_{0i}^2}\right)^{2/(\gamma + 1)} - 1\right], \tag{44a}$$

occurring at

$$\frac{u_i}{u_{i0}} = (M_{0i}^{-2})^{1/(\gamma_i + 1)}, \tag{44b}$$

which corresponds to the local flow speed u_i matching the local sound speed c_i.

Equation (43) lends itself to the simple graphical construction shown in Figs 2(a) and (b) for subsonic and supersonic entry flows. The interaction between the horizontal lines representing the right-hand side of (43) with the curve $L_i(u_i)$ provides the value of the flow speed u_i as a function of the potential ϕ. This construction clearly illustrates the fundamental property of such flows – namely that positive changes in ϕ induce accelerations in subsonic flow and decelerations in supersonic flows for $Z_i > 0$, and vice versa if $Z_i < 0$. We can use this feature to build up a picture of the flow structure in any given potential structure and ask if it is compatible with the charge-neutrality and generalized Bohm conditions. We also note from Fig. 2 the maximum value of the potential $\phi_{\max}$, say, permitted in the structure occurs where the horizontal lines touch L_i at its minimum, and is

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

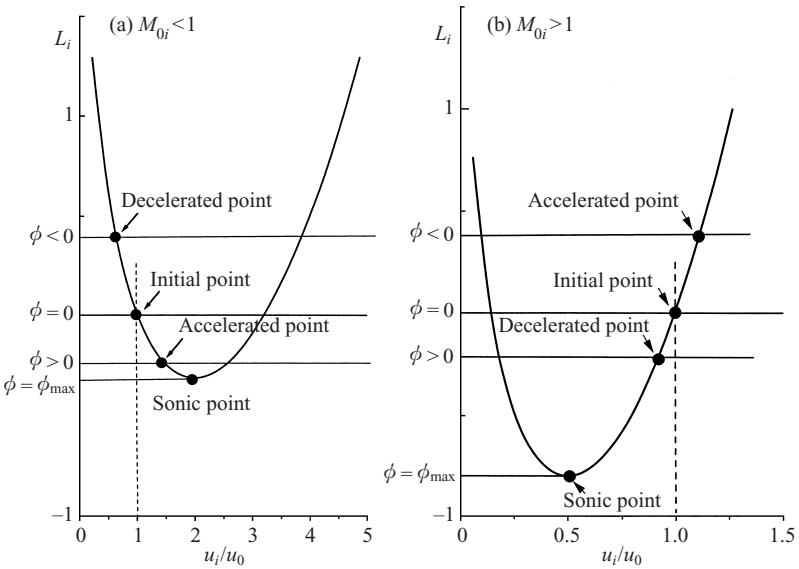


Figure 2. Graphical construction of the solution of the Bernoulli equation for u_i/u_0 as a function of ϕ for positive ($Z_i > 0$) ions. (a) In subsonic flows, positive changes in ϕ induce an acceleration (depletion), whereas negative changes in ϕ cause a deceleration (enhancement). (b) In supersonic flows, positive changes in ϕ lead to a deceleration (pile-up), whereas negative charges in ϕ induce an acceleration (spread-out).

given by

$$\frac{cZ_i}{\gamma_i k T_{0i}}\phi_{\max} = -L_{i\,\min}. \tag{45}$$

4.2. Soliton structures

A soliton pulse in ϕ of the classical type

$$\phi = \phi_0 \,\mathrm{sech}^2\,(kx) \tag{46a}$$

is a solution of the total momentum equation (33a) if the effective potential well has the form

$$-V(\phi) \sim k^2\phi^2(\phi_0 - \phi). \tag{46b}$$

In the following discussion it is not necessary to restrict the potential to be of the form (44a) so long as it is pulse-like in the sense that ϕ rises from zero (with zero derivative), reaches a maximum within the transition, and falls to zero at $x = \infty$. The corresponding flow trajectories for the ions in the transition follow directly from Fig. 2, and are shown in Fig. 3 for the cases of subsonic and supersonic entry speeds for $\phi < \phi_{\max}$ as defined by (45). If $\phi_0 = \phi_{\max}$, there is a critical solution in which the flow undergoes a smooth subsonic–supersonic (or vice versa) transition in which the exit speed u_∞ is given by the zero (other than $u_i/u_0 = 1$) of L_i, i.e. where

$$\frac{u_\infty}{u_0} \approx \begin{cases} \left[\dfrac{2}{(\gamma_i - 1)M_0^2}\right]^{1/2} & (M_{0i} \ll 1), \\ \left[\dfrac{2}{(\gamma_i - 1)M_{0i}^2}\right]^{1/(\gamma_i - 1)} & (M_{0i} \gg 1). \end{cases} \tag{47}$$

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

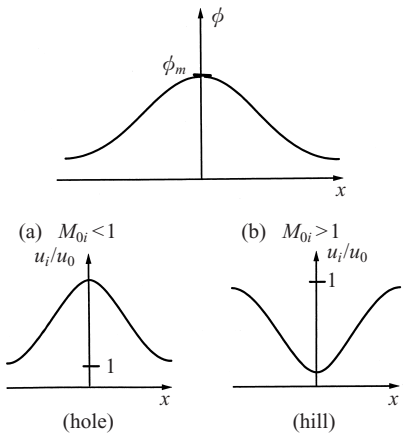


Figure 3. A solitary pulse (positive) in ϕ gives rise to either (a) an initial acceleration followed by a deceleration (a hole in n_i) if the flow is supersonic or (b) a deceleration–acceleration (a hill in n_i) if the flow is supersonic. The maximum strength of the pulse is limited by $L_{i\,\min}$, which reflects the minimum in the ion particle energy occurring where the local flow is sonic. There is no such limitation for a negative pulse in ϕ, which is accompanied by either a hill in n_i (if $M_{0i} < 1$) or a hole in n_i (if $M_{0i} > 1$).

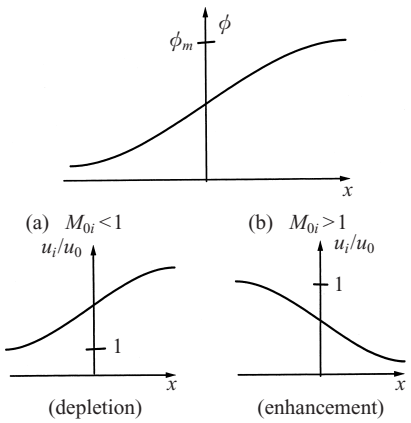


Figure 4. A monotonic transition (positive) in ϕ induces either (a) an acceleration (a depletion in u_2) if $M_{0i} < 1$ or (b) a deceleration (an enhancement in n_i) if $M_{0i} > 1$.

However, in this critical solution, the charge-neutrality requirement downstream cannot be met, since there is either a depletion or an enhancement of ions – unless, of course, a trapped ion species is added downstream to compensate for the depletion or enhancement. Finally, if $\phi_0 > \phi_{\max}$, the ion flow is choked and there are no smooth single-valued ion trajectories connecting upstream with downstream. In summary, any pulse-like potential in the transition is permitted, provided that $\phi_0 < \phi_{\max}$ and the corresponding ion trajectories can be of either the type shown in Fig. 3(a) if $M_0 < 1$ or the type shown in Fig. 3(b) if $M_{0i} > 1$. Also note that if ϕ is negative throughout the transition, arbitrary-amplitude pulses are possible in which subsonic ions are first decelerated and then accelerated back to their entry values, and vice versa for supersonic ions.

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

4.3. Monotonic structures (double layers)

Monotonic variations in ϕ of the type

$$\phi = \frac{\phi_0 \exp(kx)}{1 + \exp(kx)}, \tag{48a}$$

are solutions of the momentum equation (38a) if the potential well takes the form

$$-V(\phi) = k^2\phi^2(\phi_0 - \phi)^2. \tag{48b}$$

Again this is merely illustrative, and it is permissible to choose any monotonic potential variation, provided that it is a solution of the equations. Once again, the associated ion flow trajectories in such a transition follow from Fig. 2, and are shown in Fig. 4. In the subsonic case and for $\phi_0 < \phi_{\max}$, ions are accelerated to a final speed $u_{if}(< c_i)$, so that such ions are depleted at the end of the transition, while in the supersonic case, ions are decelerated to a final speed $u_{if}(> c_i)$, with the consequence that such ions are enhanced. Since n_e increases throughout such a transition, charge neutrality downstream cannot be fulfilled in general. Thus such transitions cannot exit in general unless the incoming ion speeds are so finely tuned as to ensure that the sum of their exit depletions and/or enhancements satisfy the downstream charge-neutrality requirement. However, these restrictions can be relaxed somewhat by including the effects of trapped particles, which exert pressure and have a corresponding enthalpy but no net kinetic energy. For example, if there is a depletion of streaming ions downstream then charge neutrality there would require a population of (positive) trapped particles in that region. Clearly, there is also the possibility of including trapped-particle populations both upstream and downstream (see McKenzie et al. 1976), which allow for momentum deficits (surpluses) in total momentum and charge enhancements or depletions in order to maintain charge neutrality at either end of the transition. In this way, it is possible to escape the conclusion drawn by Verheest and Hellberg (1997) that arbitrary-amplitude double layers cannot exist in plasmas with one Boltzmann species (i.e. subsonic ions) and any number of cold (i.e. supersonic) species. The conclusion was based on expanding $V(\phi)$ in powers of ϕ, using a combination of Boltzmann ion species and cold ions, and examining the nature of the solutions of the structure equation (33). This procedure tends to obscure the underlying physical reason for the non-existence of double layers, in the absence of trapped-particle populations – namely that if the ion flow is subsonic, the flow is decelerated and the ion density is enhanced, with the result that, in general, charge neutrality at both ends cannot be maintained without introducing trapped or reflected particles.

5. Summary and conclusions

We have used the linear dispersion equation for coupled ion acoustic waves in a plasma consisting of different ions flowing with differential speeds to generalize the concept of the sonic point in a single-ion plasma to a multi-ion plasma. For example, in a bi-ion plasma in which the ion speeds are u_1 and u_2 relative to a laboratory frame S, the requirement that the waves appear stationary in S results in two loci in the (u_2, u_1) plane – one being associated with the slow mode of the compound system and the other belonging to the fast mode, on which the wave system appears stationary.

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

The conservation of the total momentum of a multi-ion system can be cast in the form of a classical energy integral of a particle moving in an effective Sagdev potential well $V(\phi)$, where ϕ is the electric potential and $V(\phi)$ reflects the difference between the entry momentum and the momentum at any point in the transition. For either a monotonic electric potential transition or a solitary pulse transition to exist, three conditions must be met. The first two express that total momentum and charge neutrality must be maintained, and therefore do not impose any new restrictions. The third condition, relating to the curvature $V_{\phi\phi}$ of the effective potential well, yields the generalized Bohm criterion for a multi-ion flow. In certain regions of flow space, for example (u_2, u_1) space in a bi-ion plasma, this condition is an expression of the required 'supersonic' nature of the collective ion-beam system for which the loci in ion-speed space on which waves in S are stationary provide the generalization of the idea of a sonic point to a multi-ion system. However, this condition is not exclusively a supersonic one, since it can also be met if the ion flows are individually subsonic. Figure 1 depicts the regions in a bi-ion flow that satisfy this generalized Bohm criterion, which is merely the evanescent requirement in the stationary-wave dispersion equation.

The structure of the ion flow within solitons or monotonic potential transitions can be analysed by using the Bernoulli equation for each species. A fundamental property of an ion flow is that the ion plasma energy (enthalpy plus kinetic) exhibits a minimum where the local ion speed matches the ion sound speed. The consequence of this basic feature is that a positive change in potential induces an acceleration in subsonic (positive) ions and a deceleration in supersonic (positive) ions, and vice versa for negative ions. In a solitary pulse in ϕ for which the charge-neutrality constraint at either end is automatically satisfied, its maximum strength is given by the difference between the incoming ion energy and its minimum value attained at the ion sound speed. However, for negative-potential pulses, there are no constraints on the strength of the pulse, and in this case a negative change in potential induces a deceleration in subsonic flow and an acceleration in supersonic flow. In a monotonic transition in ϕ (a double layer), the charge-neutrality constraint restricts their existence to very special cases without the introduction of trapped- or reflected-particle populations, which can provide both momentum defects and the necessary additional enhancements or depletion in charge to maintain charge neutrality at the ends of the transition.

References

Allen, J. E. 1976 *J. Phys. D: Appl. Phys.* **9**, 2331–2332.
Andrews, J. G. and Allen, J. E. 1971 *Proc. R. Soc. Lond.* **A320**, 124–140.
Baboolal, S., Bahruthram, R. and Hellberg, M. A. 1991 *J. Plasma. Phys.* **46**, 247–254.
Bertrand, P. and Feix, M. R. 1968 *Phys. Lett.* **28A**, 68–71.
Block, L. P. 1978 *Astrophys. Space Sci.* **55**, 59–72.
Deutsch, R. and Räuchle, E. 1992 *Phys. Rev.* **A46**, 3442–3453.
Mace, R. L. and Hellberg, M. A. 1993 *Planet. Space Sci.* **41**, 235–244.
McKenzie, J. F., Lee, H.-J. and Axford, W. I. 1977 *Space Sci.* **51**, 3–32.
Raadu, M. A. 1989 *Phys. Rep.* **178**, 25–97.

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press

Sagdeev, R. Z. 1966 In: *Reviews of Plasma Physics*, Vol. 4 (ed. M. A. Leontovich), pp. 23–91. Consultants Bureau, New York.

Torvén, S. 1980 In: *Proceedings of International Conference on Plasma Physics, Nagoya, Japan, 7–11 April, 1980*, Vol. II, pp. 89–94. Fusion Research Association of Japan.

Verheest, F. 1993 *Planet. Space Sci.* **42**, 395–406.

Verheest, F. and Hellberg, M. A. 1997 *J. Plasma Phys.* **57**, 465–477.

Zank, G. P. and McKenzie, J. F. 1987 *J. Plasma Phys.* **39**, 183–191.

https://doi.org/10.1017/S0022377801001106 Published online by Cambridge University Press