8

Cheeger Inequalities

This chapter systematically treats Cheeger type inequalities. Our theory combines higher-order Cheeger inequalities, Cheeger inequalities for signed graphs, and Cheeger inequalities for the 1-Laplacian. We introduce a new definition of a Cheeger constant for simplicial complexes. This leads us to a solution of the higher-order Cheeger problem on simplicial complexes.

Cheeger inequalities on discrete structures play an important role in expander theory, spectral graph theory, and spectral clustering methods. In the first two sections of this chapter, we present (multiway) Cheeger inequalities for simple graphs and signed graphs. In the third section, we then derive generalized inequalities for simplicial complexes, following mainly [160, 161]. Finally, in Section 8.4, we generalize the results in [214], and we prove new Cheeger-type inequalities for p-Laplacians on hypergraphs, which appear here for the first time in this general form.

Some Cheeger inequalities already occurred in Chapter 4. We first need to recall some of these results, in order to put them into the systematic perspective developed in the present chapter and also to build upon them.

8.1 Cheeger Inequalities on Graphs

We start by recalling the Cheeger constant (4.3.13) of a simple graph $\Gamma = (V, E)$,

$$h := \min_{\emptyset \neq S \subsetneq V} \frac{|E(S, S^c)|}{\min\{\mathrm{vol}\,(S), \mathrm{vol}\,(S^c)\}},$$

as well as the Cheeger inequalities from Theorem 4.3.1:

Theorem 8.1.1 (Cheeger inequalities on a graph)**.** *Let $\Gamma = (V, E)$ be a simple graph. Let h be its Cheeger constant, and let $\lambda_2(\Delta_2)$ be the second eigenvalue of the normalized Laplacian. Then,*

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

$$\frac{h^2}{2} \leq \lambda_2(\Delta_2) \leq 2h.$$

In fact, this is also a special case of Theorem 6.4.2, which shows Cheeger inequalities for p-Laplacians on chemical hypergraphs.

We point out that if (A, A^c) is a Cheeger cut of a connected graph Γ, then either A or A^c induces a connected subgraph. Moreover, there exists a Cheeger cut of Γ such that both A and A^c are connected. For the latter statement, a version for compact Riemannian manifolds was first proved by Yau [274], and it is collected in Buser's book [43, Lemma 8.3.6]. The idea of the original proof in the geometric setting is to use mathematical induction on the number of connected components of the boundary of A, but this idea does not work in the case of graphs. A combinatorial proof for the graph Cheeger cut was given by Mohar [211]. We refer to Theorem 7.2.3 for our new proof, which works in the graph case.

Miclo [206] introduced the higher-order Cheeger constant (7.2.7) and asked whether Cheeger-type inequalities can be proved for this constant. Lee, Oveis Gharan, and Trevisan [180] answered Miclo's question as follows.

Theorem 8.1.2. *There exists a universal constant C such that, for every graph, and for every natural number k,*

$$\frac{h_k^2}{Ck^4} \leq \lambda_k(\Delta_2) \leq 2h_k.$$

8.2 Cheeger Inequalities on Signed Graphs

Let now (Γ, s) be a signed graph, with $\Gamma = (V, E)$. Given two disjoint sets $V_1, V_2 \subset V$, let

$$E^+(V_1, V_2) \coloneqq \{\{u, v\} \in E \colon u \in V_1, v \in V_2, s(uv) = 1\},$$

and let

$$E^-(V_1) \coloneqq \{\{u, v\} \in E \colon u, v \in V_1, s(uv) = -1\}.$$

The *signed bipartiteness ratio* is defined as

$$\beta^s(V_1, V_2) \coloneqq \frac{2\Big(|E^-(V_1)| + |E^-(V_2)| + |E^+(V_1, V_2)|\Big) + |\partial(V_1 \sqcup V_2)|}{\operatorname{vol}(V_1 \sqcup V_2)}.$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

We recall that the *signed Cheeger constant* of (Γ, s) (see also (4.6.27)) is

$$h^s := \min_{(V_1,V_2)\neq(\emptyset,\emptyset)} \beta^s(V_1,V_2),$$

where the minimum is taken over all possible sub-bipartitions of V. Moreover, by Theorem 4.6.2,

Theorem 8.2.1. *Given a signed graph (Γ, s), we have*

$$\frac{\lambda_1(\Delta^s)}{2} \leq h^s \leq \sqrt{2\lambda_1(\Delta^s)}.$$

More generally, the *k-way signed Cheeger constant* has been introduced in [11] as

$$h_k^s := \min_{\{(V_{2i-1},V_{2i})\}_{i=1}^k} \max_{1\leq i\leq k} \beta^s(V_{2i-1},V_{2i}),$$

where the minimum is taken over the set of all possible k pairs of disjoint sub-bipartitions (V_1,V_2), (V_3,V_4), …, (V_{2k-1},V_{2k}). In particular, we note that h_k^s is switching invariant.

With these concepts, following [11], we can generalize and put into perspective the multiway Cheeger inequalities for simple graphs by Lee, Oveis Gharan, and Trevisan [180]. Before stating the main result, we need the following preliminary lemmata.

Lemma 8.2.1. *Let (Γ, s) be a signed graph. Let $k \in \{1,\dots,N\}$ and let $f_1,\dots,f_k$ be an orthonormal system of eigenfunctions corresponding to $\lambda_1(\Delta^s),\dots,\lambda_k(\Delta^s)$. Given a vertex v, let*

$$F(v) := (f_1(v),\dots,f_k(v)) \in \mathbb{R}^k,$$

and let $V_F := \{v \in V : F(v) \neq (0,\dots,0)\}$. Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^k$, and consider the pseudometric

$$d_F(v,v') := \min\left\{\left\|\frac{F(v)}{\|F(v)\|} - \frac{F(v')}{\|F(v')\|}\right\|, \left\|\frac{F(v)}{\|F(v)\|} + \frac{F(v')}{\|F(v')\|}\right\|\right\}.$$

Then, there exist k nonempty, mutually disjoint subsets $S_1,\dots,S_k \subset V_F$, and a universal constant $C > 0$ (which is the same for all signed graphs), such that $d_F(S_i,S_j) \geq k^{-\frac{5}{2}}/C$, for all $i \neq j$, and

$$\sum_{v\in S_i} \deg v \cdot \|F(v)\|^2 \geq \frac{1}{2k} \sum_{v\in V} \deg v \cdot \|F(v)\|^2,$$

for all $i \in \{1,\dots,k\}$.

For the *proof*, which is based on random partition theory, we refer to [11, 187].

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

Lemma 8.2.2. *Consider the setting of Lemma 8.2.1. For any subset $S \subset V$, and for any $\varepsilon > 0$, we define the cut-off function $\eta_S : V \to \mathbb{R}$ by*

$$\eta_S(v) := \begin{cases} 0, & \text{if } F(v) = 0, \\ \max\left\{0, 1 - \frac{1}{\varepsilon} d_F(v, S)\right\}, & \text{otherwise.} \end{cases}$$

Then, for any $0 < \varepsilon < 2$, and for any $\{v, v'\} \in E$, we have that

$$\|\eta_S(v)F(v) - s(vv')\eta_S(v')F(v')\| \leq \left(1 + \frac{2}{\varepsilon}\right) \|F(v) - s(vv')F(v')\|.$$

For the *proof* we refer to [11].

Theorem 8.2.2. *There exists a universal constant C such that, for any signed graph (Γ, s) and for any $k \in \{1, \ldots, N\}$,*

$$\frac{\lambda_k(\Delta^s)}{2} \leq h_k^s \leq Ck^3 \sqrt{\lambda_k(\Delta^s)}.$$

Proof We first prove the lower bound. Let $\{(V_{2i-1}, V_{2i})\}_{i=1}^k$ be a k-sub-bipartition of V that realizes h_k^s, that is,

$$\max_{1 \leq i \leq k} \beta^s(V_{2i-1}, V_{2i}) = h_k^s,$$

and let $\mathbf{1}_{V_i}$ be the indicator function of V_i. For any

$$g \in \operatorname{span}(\mathbf{1}_{V_1} - \mathbf{1}_{V_2}, \ldots, \mathbf{1}_{V_{2k-1}} - \mathbf{1}_{V_{2k}}),$$

we can write

$$g(v) = \sum_{i=1}^{k} t_i (\mathbf{1}_{V_{2i-1}}(v) - \mathbf{1}_{V_{2i}}(v)),$$

for some $t_1, \ldots, t_k \in \mathbb{R}$. We then have

$$\begin{aligned} &(g(v) - s(vv')g(v'))^2 \\ &\leq 2\left|g(v)^2 \operatorname{sign}(g(v)) - s(vv')g(v')^2 \operatorname{sign}(g(v'))\right| \\ &= 2\left|\sum_{i=1}^{m} t_i^2 \operatorname{sign}(t_i)\, (\mathbf{1}_{V_{2i-1}}(v) - \mathbf{1}_{V_{2i}}(v) - s(vv')(\mathbf{1}_{V_{2i-1}}(v') - \mathbf{1}_{V_{2i}}(v')))\right| \\ &\leq 2\sum_{i=1}^{m} t_i^2 \left|\mathbf{1}_{V_{2i-1}}(v) - \mathbf{1}_{V_{2i}}(v) - s(vv')(\mathbf{1}_{V_{2i-1}}(v') - \mathbf{1}_{V_{2i}}(v'))\right|, \end{aligned}$$

where the first inequality follows from the fact that

$$(a \pm b)^2 \leq 2|a^2 \operatorname{sign}(a) \pm b^2 \operatorname{sign}(b)|.$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

Therefore,

$$\begin{aligned} R(g) &\le 2 \cdot \frac{\sum_{i=1}^m t_i^2 \sum_{\{v,v'\}\in E} \left|\mathbf{1}_{V_{2i-1}}(v) - \mathbf{1}_{V_{2i}}(v) - s(vv')(\mathbf{1}_{V_{2i-1}}(v') - \mathbf{1}_{V_{2i}}(v'))\right|}{\sum_{i=1}^m t_i^2 \sum_{v\in V} |\mathbf{1}_{V_{2i-1}}(v) - \mathbf{1}_{V_{2i}}(v)|} \\ &\le 2 \max_{i=1,\dots,k} \frac{\sum_{\{v,v'\}\in E} \left|\mathbf{1}_{V_{2i-1}}(v) - \mathbf{1}_{V_{2i}}(v) - s(vv')(\mathbf{1}_{V_{2i-1}}(v') - \mathbf{1}_{V_{2i}}(v'))\right|}{\sum_{v\in V} |\mathbf{1}_{V_{2i-1}}(v) - \mathbf{1}_{V_{2i}}(v)|} \\ &= 2 \max_{1\le i\le k} \beta^s(V_{2i-1}, V_{2i}) \\ &= 2h_k^s. \end{aligned}$$

By the min–max principle, we obtain that $\lambda_k^s \le 2h_k^s$. This proves the lower bound,

$$\frac{\lambda_k(\Delta^s)}{2} \le h_k^s.$$

To prove the upper bound, consider the setting of Lemma 8.2.1. We have that

$$\begin{aligned} \lambda_k(\Delta^s) &= \max_{i=1,\dots,k} R(f_i) \\ &\ge \frac{\sum_{i=1}^k \sum_{v\sim v'} (f_i(v) - s(vv')f_i(v'))^2}{\sum_{i=1}^k \sum_{v\in V} \deg(v) f_i(v)^2} \\ &= \frac{\sum_{v\sim v'} \|F(v) - s(vv')F(v')\|^2}{\sum_{v\in V} \deg(v)\|F(v)\|^2} \\ &:= R(F). \end{aligned}$$

Let now $\varepsilon := k^{-\frac{5}{2}}/C$. Then, by Lemma 8.2.2,

$$\begin{aligned} &\sum_{\{v,v'\}\in E} \|\eta_{S_i}(v)F(v) - s(vv')\eta_{S_i}(v')F(v')\|^2 \\ &\quad\le \left(1 + \frac{2}{\varepsilon}\right)^2 \sum_{\{v,v'\}\in E} \|F(v) - s(vv')F(v')\|^2. \end{aligned}$$

Moreover, by Lemma 8.2.1,

$$\sum_{v\in V} \deg v \cdot \|\eta_{S_i}(v)F(v)\|^2 \ge \sum_{v\in S_i} \deg v \cdot \|F(v)\|^2 \ge \frac{1}{2k} \sum_{v\in V} \deg v \cdot \|F(v)\|^2.$$

Therefore, for $i = 1, \dots, k$,

$$R(\eta_{S_i}F) \le 2k(1 + Ck^{\frac{5}{2}})^2 R(F) \le C'k^6 R(F) \le C'k^6 \lambda_k(\Delta^s).$$

We now write

$$\eta_{S_i}F = (g_i^1, \dots, g_i^k) \colon V \to \mathbb{R}^k,$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

and we observe that we can always find $j_i \in \{1,\dots,k\}$ such that $R(g_i^{j_i}) \leq R(\eta_{S_i}F)$, for all $i = 1,\dots,k$. Now, similarly to (6.4.10) in the proof of Theorem 6.4.1, one can show that there exist disjoint subsets $\Omega_i^+, \Omega_i^- \subset \operatorname{supp}(g_i^{j_i})$ such that

$$\beta^s(\Omega_i^+, \Omega_i^-) \leq \sqrt{2R(g_i^{j_i})}.$$

As a consequence,

$$\begin{aligned} h_k^s &\leq \max_{i=1,\dots,k} \beta^s(\Omega_i^+, \Omega_i^-) \\ &\leq \max_{i=1,\dots,k} \sqrt{2R(g_i^{j_i})} \\ &\leq \max_{i=1,\dots,k} \sqrt{2R(\eta_{S_i}F)} \\ &\leq Ck^3 \sqrt{\lambda_k(\Delta^s)}, \end{aligned}$$

for some constant $C > 0$. □

8.3 Cheeger Inequalities on Simplicial Complexes

This section addresses the fundamental problem of generalizing the classical Cheeger inequalities that are known for graphs to simplicial complexes. Estimating the first nontrivial eigenvalue of the Hodge Laplacian on a simplicial complex is a major long-standing open problem that has been formulated in the field of expander graph theory. Also, the analogue in Riemannian geometry, which is to find Cheeger inequalities for differential k-forms, is still far from being understood and solved.

The first issue in any attempt of extending Cheeger-type inequalities to higher-order structures, like simplicial complexes or differential k-forms, is to identify the right quantities to be estimated. Here, we shall focus on this problem and build upon Sections 4.8 and 8.2. The key is to convert a Cheeger problem for higher-dimensional simplices into one for signed graphs. For this purpose, we shall describe a relation between simplicial complexes and signed graphs that will allow us to interpret the Laplace operator Δ_d^{up} as the Laplacian of a signed graph.

We first recall some constructions from Section 2.5.1. We consider a simplicial complex Σ, and we denote the collection of its d-dimensional simplices by Σ_d. When σ and σ' are two orientations on the same underlying d-simplex, we write $\operatorname{sgn}(\sigma, \sigma') = \pm 1$ when those orientations agree/differ. And $\partial\sigma$ is the

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

cellular boundary of a simplex; that is, the collection of its facets. When the simplices ρ carry weights $w(\rho)$, the *degree* of a d-simplex σ of Σ is

$$\deg \sigma := \sum_{\rho \in \Sigma_{d+1}(\Sigma) \colon \sigma \in \partial \rho} w(\rho). \tag{8.3.1}$$

Moreover, since we want to work with the normalized up-Laplacian, we assume that the weight function satisfies

$$w(\sigma) = \deg \sigma, \tag{8.3.2}$$

for every $\sigma \in \Sigma_d$, and that the weights of the facets of Σ are equal to 1. Under these conditions, the normalized combinatorial up-Laplacian, according to (2.5.21), is given by

$$\begin{aligned}(\Delta_{up}^d f)([\sigma]) &= f([\sigma]) \\ &+ \frac{1}{\deg \sigma} \sum_{\sigma' \in \Sigma_d \colon \sigma' \neq \sigma, \sigma, \sigma' \in \partial \rho} w(\rho) \operatorname{sgn}([\sigma], \partial[\rho]) \operatorname{sgn}([\sigma'], \partial[\rho]) f([\sigma']).\end{aligned} \tag{8.3.3}$$

Recalling the constructions of Section 4.8, we can express Δ_d^{up} in terms of the Laplacian of an associated signed graph, rewriting it as

$$(\Delta_d^{up} f)([\sigma]) = f([\sigma]) - \frac{1}{\deg \sigma} \sum_{\substack{\sigma' \in \Sigma_d \colon \sigma \neq \sigma', \\ \sigma, \sigma' \in \partial \rho}} w(\rho) s([\sigma], [\sigma']) f([\sigma']), \tag{8.3.4}$$

with

$$s([\sigma], [\sigma']) := -\operatorname{sgn}([\sigma], \partial[\rho]) \operatorname{sgn}([\sigma'], \partial[\rho]). \tag{8.3.5}$$

This naturally suggests a signed graph (Γ_d, s). Its vertex set consists of the d-simplices of our simplicial complex, and there is an edge $\sigma \sim \sigma'$ between vertices $\sigma \neq \sigma'$ if there exists a $(d+1)$-simplex ρ in Σ with $\sigma, \sigma' \in \partial \rho$. By Theorem 4.8.2 and by evaluating the weights $w(\rho)$ in (8.3.4), we have that the Laplacian Δ_d^{up} in (8.3.4) is related to the Laplacian $\Delta_{(\Gamma_d, s)}$ of the signed graph (Γ_d, s) via

$$\Delta_d^{up} = (d+1)\Delta_{(\Gamma_d, s)} - d \operatorname{Id}. \tag{8.3.6}$$

Remark 8.3.1. A similar construction is used in [98] to define the signed adjacency matrix of a triangulation on a surface.

By (8.3.6), in particular, the eigenvalues μ_j of Δ_d^{up} and the eigenvalues λ_j of $\Delta_{(\Gamma_d, s)}$ are related by

$$\mu_j = (d+1)\lambda_j - d. \tag{8.3.7}$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

Hence, since the eigenvalues of $\Delta_{(\Gamma_d,s)}$ lie in the interval $[0,2]$, those of Δ_d^{up} lie in the interval $[0,d+2]$. In fact, since $\mu_j \geq 0$ in (8.3.7), the eigenvalues of $\Delta_{(\Gamma_d,s)}$ are $\geq d/(d+1)$. Equality holds if and only if there is some nontrivial f with $\delta^d f = 0$. More precisely, the multiplicity of the eigenvalue $d/(d+1)$ of $\Delta_{(\Gamma_d,s)}$ equals the dimension of the kernel of the coboundary operator δ^d. In particular, for $d > 0$, the graph (Γ_d, s) is never balanced.

Now, given disjoint $A, A' \subset \Sigma_d$, let

$$|E^+(A,A')| := \#\{\{\tau,\tau'\}\colon \tau \in A, \tau' \in A', \operatorname{sgn}([\tau],[\tau']) = 1\},$$

and let

$$|E^-(A)| := \#\{\{\tau,\tau'\}\colon \tau,\tau' \in A, \operatorname{sgn}([\tau],[\tau']) = -1\}.$$

Let also

$$\beta(A,A') := \frac{2\left(|E^-(A)| + |E^-(A')| + |E^+(A,A')|\right) + |\partial(A \sqcup A')|}{\operatorname{vol}(A \sqcup A')},$$

where $|\partial A|$ is the number of the edges of (Γ_d, s) that cross A and $\Sigma_d \setminus A$,

$$\operatorname{vol}(A) := \sum_{\tau \in A} \deg \tau,$$

and $\deg \tau := \#\{\sigma \in \Sigma_{d+1}\colon \tau \subset \sigma\}$. We define the *kth Cheeger constant* of Σ_d as

$$h_k(\Sigma_d) := \min_{\substack{A_1,\ldots,A_{2k} \\ \text{disjoint, in } \Sigma_d}} \max_{1 \leq i \leq k} \beta(A_{2i-1}, A_{2i}).$$

In order to have $h_k(\Sigma_d) = 0$, we need to find $2k$ disjoint sets $A_1,\ldots,A_{2k}$ such that $|E^-(A_i)| = 0$ and $|E^+(A_{2i-1},A_{2i})| = 0$ for all of them. Thus, all edges inside any A_i have to be positive, whereas all edges between A_{2i-1} and A_{2i} have to be negative, and also $A_{2i-1} \cup A_{2i}$ has to be disconnected from the rest of the graph so that there are no boundary terms. By Lemma 4.6.3, this means that the graph $G_{up}^-(\Sigma_d) := (\Gamma_d, -s)$ with the opposite signs has at least k balanced connected components.

Remark 8.3.2. For $d = 0$, the constant $h_k(\Sigma_0)$ reduces to the k-way Cheeger constant (7.2.7) of a graph [180].

Theorem 8.3.1. *For every simplicial complex Σ and for every $d \geq 0$,*

$$\frac{h_1(\Sigma_d)^2}{2(d+1)} \leq d + 2 - \lambda_n(\Delta_d^{up}) \leq 2h_1(\Sigma_d), \tag{8.3.8}$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

where $n = \#\Sigma_d$. Moreover, there exists a universal constant C such that, for any simplicial complex Σ and for any $k \geq 1$,

$$\frac{h_k(\Sigma_d)^2}{Ck^6(d+1)} \leq d + 2 - \lambda_{n+1-k}(\Delta_d^{up}) \leq 2h_k(\Sigma_d). \tag{8.3.9}$$

Proof We first observe that

$$\begin{aligned} d + 2 - & \frac{\sum_{\sigma \in \Sigma_{d+1}} \left(\sum_{\tau \in \Sigma_d, \tau \subset \sigma} \operatorname{sgn}([\tau], \partial[\sigma]) f(\tau) \right)^2}{\sum_{\tau \in \Sigma_d} \deg_\tau f(\tau)^2} \\ &= (d+1) \frac{\sum_{[\tau] \sim [\tau']} \left(f(\tau) - \operatorname{sgn}(\tau, \tau') f(\tau') \right)^2}{\sum_{\tau \in \Sigma_d} \widetilde{\deg}_\tau f(\tau)^2}, \end{aligned}$$

where $[\tau] \sim [\tau']$ represents an edge in $G_{up}^-(\Sigma_d)$ and $\widetilde{\deg}_\tau := (d+1)\deg_\tau$ is the degree of τ in $G_{up}^-(\Sigma_d)$. It follows that

$$d + 2 - \lambda_{n-i+1}(\Delta_d^{up}) = (d+1)\lambda_i(\Delta(G_{up}^-(\Sigma_d))),$$

for $i = 1, \ldots, n$. Moreover, since $h_k(\Sigma_d)/(d+1)$ is the kth Cheeger constant of the signed graph $G_{up}^-(\Sigma_d)$, by Theorem 8.2.1, we have

$$\frac{\lambda_1(\Delta(G_{up}^-(\Sigma_d)))}{2} \leq \frac{h_1(\Sigma_d)}{d+1} \leq \sqrt{2\lambda_1(\Delta(G_{up}^-(\Sigma_d)))}.$$

And by Theorem 8.2.2, there exists a universal constant C such that, for any signed graph and for any $k \geq 1$,

$$\frac{\lambda_k(\Delta(G_{up}^-(\Sigma_d)))}{2} \leq \frac{h_k(\Sigma_d)}{d+1} \leq Ck^3 \sqrt{\lambda_k(\Delta(G_{up}^-(\Sigma_d)))}.$$

Altogether, we obtain

$$\frac{d + 2 - \lambda_n(\Delta_d^{up})}{2} \leq h_1(\Sigma_d) \leq \sqrt{2(d+1)(d+2-\lambda_n(\Delta_d^{up}))},$$

and

$$\frac{d + 2 - \lambda_{n+1-k}(\Delta_d^{up})}{2} \leq h_k(\Sigma_d) \leq Ck^3 \sqrt{(d+1)(d+2-\lambda_{n+1-k}(\Delta_d^{up}))}.$$

This completes the proof of (8.3.8) and (8.3.9). □

As a consequence of Theorem 8.3.1, we have that $\lambda_n(\Delta_d^{up}) = d + 2$ if and only if $h_1(\Sigma_d) = 0$ and if and only if $G_{up}^-(\Sigma_d)$ has a balanced component.

Moreover, Theorem 8.3.1 provides the estimates for the spectral gap from $d + 2$. The more important estimate, however, is the one for the spectral gap from 0, namely, the Cheeger-type estimates for the first nontrivial eigenvalue

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

of the Hodge Laplacian. This was proposed as an open problem in the original paper by Cheeger [56], in the setting of differential k-forms on a closed Riemannian manifold. Its discrete version, a Cheeger-type inequality on simplicial complexes, has also been a long-standing open problem in the area of high-dimensional expanders [84, 121, 194, 225, 254].

While some partial answers have been proposed and developed in the literature, it seems that none of them gives a complete solution to the problem. In fact, there are many different definitions of Cheeger constants on simplicial complexes. For example, an upper Cheeger-like bound for the Cheeger constant has been suggested by Parzanchevski, Rosenthal, and Tessler [121, 225]. In particular, in the field of higher-dimensional expanders, people use the so-called $\mathbb{Z}_2$-*expander* to indicate the Cheeger constants on a simplicial complex.

Steenbergen, Klivans, and Mukherjee [254] introduced the Cheeger constant

$$h^d(\Sigma) \coloneqq \min_{\phi \in C^d(\Sigma,\mathbb{Z}_2)\setminus \operatorname{im}\delta} \frac{\|\delta\phi\|}{\min_{\psi\in \operatorname{im}\delta} \|\phi+\psi\|},$$

for all $d \geq 0$. Here, $\|\cdot\|$ is the Hamming norm on $C^d(\Sigma,\mathbb{Z}_2)$, that is, the l^1-norm on $\mathbb{Z}_2^n$, where $n = \#\Sigma_d$. This constant satisfies

$$h^d(\Sigma) = 0 \iff \tilde{H}^d(\Sigma,\mathbb{Z}_2) \neq 0.$$

According to the examples and theorems in [84, 121, 122, 184, 205, 225, 254], all Cheeger constants defined using cohomology (or homology) with $\mathbb{Z}_2$-coefficients cannot satisfy both Cheeger inequalities as in the graph setting. This follows from the fact that, on the one hand, we have the well-known relation

$$\lambda(\Delta_d^{up}) = 0 \iff \lambda(L_d^{up}) = 0 \iff \tilde{H}^d(\Sigma,\mathbb{R}) \neq 0,$$

for $d \geq 0$, while on the other hand, for $d \geq 1$, $\tilde{H}^d(\Sigma,\mathbb{R}) \neq 0$ is not equivalent to $\tilde{H}^d(\Sigma,\mathbb{Z}_2) \neq 0$.

We shall now present a Cheeger constant that appears in [160, 161] and solves the problem formulated above of generalizing both Cheeger inequalities. In particular, we shall propose four equivalent definitions, denoted by (D1)–(D4). In Theorem 8.3.2, we shall then prove that they are all equivalent.

(D1) A *(generalized) multiset* is a pair $\mu = (S,m)$, where S is a set (given by distinct elements), and $m\colon S \to \mathbb{Z}$ is an integer-valued function, which gives the *multiplicity* of the elements in the multiset. For simplicity, in this section, we shall simply call μ a *multiset*. We also let

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

$$|\mu| := \sum_{s \in S} |m(s)|$$

denote the *size* of μ.

In our context, we consider multisets on $S = \Sigma_d$, and we use the notation $\mu \subset_M \Sigma_d$ to indicate that $\mu = (\Sigma_d, m)$ and the multiplicities are in $\{-M, \dots, 0, \dots, M\}$. We also define the *coboundary*[1] $\partial^*_{d+1}\mu$ of μ as the multiset of all $(d+1)$-simplices that have a member of μ in its boundary, together with the corresponding multiplicities. Thus, an element $\sigma \in \Sigma_{d+1}$ has multiplicity

$$\sum_{\tau \in \Sigma_d} m(\tau) \operatorname{sgn}([\tau], \partial[\sigma]),$$

where $m(\tau)$ is the multiplicity of τ in μ. And the *support* of $\partial^*_{d+1}\mu$ then consists of all such simplices with nonzero multiplicity. We also define the *volume* of μ as

$$\operatorname{vol}(\mu) := \sum_{\tau \in \Sigma_d} \deg \tau |m(\tau)|. \tag{8.3.10}$$

Moreover, for $d \geq 0$, we let

$$h(\Sigma_d) := \min_{\substack{\mu \subset_M \Sigma_d : \\ \mu \neq \partial^*_d(\nu), \\ \forall \nu \subset_M \Sigma_{d-1}}} \frac{|\partial^*_{d+1}\mu|}{\min\limits_{\substack{\mu' \neq 0 : \\ \partial^*_{d+1}\mu' = \partial^*_{d+1}\mu}} \operatorname{vol}(\mu')}. \tag{8.3.11}$$

We have that $h(\Sigma_d)$ is constant when M is sufficiently large. For such sufficiently large M, we call $h(\Sigma_d)$ the *Cheeger constant* of Σ_d.

Remark 8.3.3. In this definition, we need to look at quotients of the form $|\partial^*_{d+1}\mu|/\operatorname{vol}(\mu')$, where μ has to be homologically nontrivial, that is, it cannot be of the form $\partial^*_d(\nu)$ and where $\partial^*_{d+1}\mu' = \partial^*_{d+1}\mu$. The new aspect here, compared to the graph case, is that we need to consider multisets of simplices. In fact, when we were looking at graphs, all edges had weight or degree 1, but now $(d+1)$-dimensional simplices may have degrees $\neq 1$.

(D2) We now describe the Cheeger constant in terms of the $\mathbb{Z}$-expander. In particular, we let

$$h(\Sigma_d) := \min_{\phi \in C^d(K, \mathbb{Z}) \setminus \operatorname{im} \delta} \frac{\|\delta\phi\|_1}{\min\limits_{\psi \in \operatorname{im} \delta} \|\phi + \psi\|_{1,\deg}},$$

[1] This is different from the notion of coboundary used above when we defined the coboundary as the dual of the boundary operator in homology theory.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

where

$$\|\phi\|_{1,\deg} \coloneqq \sum_{\tau\in\Sigma_d} \deg\tau|\phi(\tau)|.$$

Remark 8.3.4. Compared to the definition of $h^d(\Sigma)$, here we use $\mathbb{Z}$- instead of $\mathbb{Z}_2$-coefficients and we use the (weighted) l^1-norm instead of the Hamming norm.

(D3) We define the *up* 1*-Laplacian eigenproblem* on Σ_d as the eigenvalue problem of the function pair $(\|B_{d+1}^\top\cdot\|_1, \|\cdot\|_{1,\deg})$, which consists in finding λ and $\mathbf{x}$ such that

$$\mathbf{0}\in\partial\|B_{d+1}^\top\mathbf{x}\|_1 - \lambda\partial\|\mathbf{x}\|_{1,\deg}. \tag{8.3.12}$$

Moreover, let

$$I_d \coloneqq \dim\operatorname{im}(B_d^\top) + 1 = \operatorname{rank}(B_d) + 1,$$

and let $\lambda_{I_d}(\Delta_{d,1}^{up})$ be the smallest nontrivial eigenvalue of the up 1-Laplacian. Given a norm $\|\cdot\|$ on a real linear space with an inner product $\langle\cdot,\cdot\rangle$, we say that $\mathbf{x}$ is $\|\cdot\|$-*orthogonal* to $\mathbf{y}$ if there exists $\mathbf{u}\in\nabla\|\mathbf{x}\|$ satisfying $\langle\mathbf{u},\mathbf{y}\rangle = 0$. We say $\mathbf{x}$ is $\|\cdot\|$-*orthogonal* to a nonempty set Y if $\mathbf{x}$ is $\|\cdot\|$-orthogonal to all $\mathbf{y}\in Y$.

We define

$$h(\Sigma_d) \coloneqq \lambda_{I_d}(\Delta_{d,1}^{up}) = \min_{\mathbf{x}\perp^1\operatorname{im}(B_d^\top)} \frac{\|B_{d+1}^\top\mathbf{x}\|_1}{\|\mathbf{x}\|_{1,\deg}},$$

where $\mathbf{x}\perp^1\operatorname{im}(B_d^\top)$ indicates that $\mathbf{x}$ is $\|\cdot\|_{1,\deg}$-orthogonal to $\operatorname{im}(B_d^\top)$, that is, $\mathbf{u}\in\operatorname{im}(B_d^\top)^\perp$, for some $\mathbf{u}\in\nabla\|\mathbf{x}\|_{1,\deg}$.

Remark 8.3.5. In this definition, we draw upon the insights developed in Sections 6.4 and 7.2, which suggest to consider L^1- instead of L^2-quantities in order to get equalities instead of inequalities when relating Cheeger-type constants and eigenvalues.

Remark 8.3.6. If $\|\cdot\| = \|\cdot\|_2$ is the standard l^2-norm, then the $\|\cdot\|_2$-orthogonality coincides with the classical orthogonality with respect to the standard inner product.

(D4) The norm $\|\cdot\|_{1,\deg}$ on $C^d(\Sigma)$ induces a quotient norm on $C^d(\Sigma)/\operatorname{im}(\delta_{d-1})$, which we denote by $\|\cdot\|$, for simplicity. More precisely, given an equivalence class $[\mathbf{x}]\in C^d(\Sigma)/\operatorname{im}(\delta_{d-1})$, we let

$$\|[\mathbf{x}]\| \coloneqq \inf_{\mathbf{x}'\in[\mathbf{x}]}\|\mathbf{x}'\|_{1,\deg}.$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

We use it to define

$$h(\Sigma_d) \coloneqq \min_{0\neq[\mathbf{x}]\in C^d(\Sigma)/\operatorname{im}(\delta_{d-1})} \frac{\|\delta_d \mathbf{x}\|_1}{\|[\mathbf{x}]\|} = \min_{0\neq[\mathbf{x}]\in C^d(\Sigma,\mathbb{Z})/\operatorname{im}(\delta_{d-1})} \frac{\|\delta_d \mathbf{x}\|_1}{\|[\mathbf{x}]\|}.$$

In the case when $\tilde{H}^d(\Sigma,\mathbb{R}) = 0$, we then have

$$\begin{aligned} h(\Sigma_d) &= \min_{\mathbf{y}\in\operatorname{im}(\delta_d)} \frac{\|\mathbf{y}\|_1}{\|\mathbf{y}\|_{\text{fil}}} \\ &= \frac{1}{\max\limits_{\mathbf{y}\in\operatorname{im}(\delta_d)} \|\mathbf{y}\|_{\text{fil}}/\|\mathbf{y}\|_1} \\ &= \frac{1}{\|\delta_d^{-1}\|_{\text{fil}}}, \end{aligned}$$

where

$$\|\mathbf{y}\|_{\text{fil}} \coloneqq \inf_{x\in\delta_d^{-1}(\mathbf{y})} \|\mathbf{x}\|_{1,\deg}$$

is the filling norm of $\mathbf{y}$, and

$$\|\delta_d^{-1}\|_{\text{fil}} \coloneqq \max_{\mathbf{y}\in\operatorname{im}(\delta_d)} \frac{\|\mathbf{y}\|_{\text{fil}}}{\|\mathbf{y}\|_1}$$

is called the *filling profile* by Gromov (see Section 2.3 in [117]).

Remark 8.3.7. This definition has a perspective that is in some sense opposite to (D1). Here, instead of first considering a set S and then looking at its boundary or coboundary, we first consider a boundary Y, and we then try to find the smallest set for which Y is a boundary.

Theorem 8.3.2. *The four definitions in (D1)–(D4) are all equivalent.*

Proof The idea of the proof is to show that (D1), (D2), and (D4) are all equivalent to (D3). In order to do so, we need to apply some technical results from [160].

Since $\operatorname{im}(B_d^\top) \subset \ker(B_{d+1}^\top)$, by Theorem 2.1 in [160], we have that

$$\lambda_{I_d}(\Delta_{d,1}^{up}) = \inf_{\mathbf{x}\in\mathbb{R}^n\setminus\operatorname{im}(B_d^\top)} \frac{\|B_{d+1}^\top \mathbf{x}\|_1}{\inf\limits_{\mathbf{z}\in\operatorname{im}(B_d^\top)} \|\mathbf{x}+\mathbf{z}\|_{1,\deg}} \tag{8.3.13}$$

$$= \inf_{[\mathbf{x}]\in\mathbb{R}^n/\operatorname{im}(B_d^\top)} \frac{\|B_{d+1}^\top \mathbf{x}\|_1}{\|[\mathbf{x}]\|} \tag{8.3.14}$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

$$= \inf_{\substack{\mathbf{x}\in\mathbb{R}^n:\\ \nabla\|\mathbf{x}\|_{1,\deg}\bigcap \operatorname{im}(B_d^\top)^\perp\neq\emptyset}} \frac{\|B_{d+1}^\top \mathbf{x}\|_1}{\|\mathbf{x}\|_{1,\deg}}, \tag{8.3.15}$$

where $n := \#\Sigma_d$,

$$[\mathbf{x}] := \left\{\mathbf{y}\in\mathbb{R}^n : \mathbf{y}-\mathbf{x}\in\operatorname{im}(B_d^\top)\right\},$$

and

$$\|[\mathbf{x}]\| := \inf_{\mathbf{x}'\in[\mathbf{x}]} \|\mathbf{x}+\mathbf{z}\|_{1,\deg}. \tag{8.3.16}$$

In fact, the definition of the norm $\|\cdot\|$ on the quotient space $\mathbb{R}^n/\operatorname{im}(B_d^\top)$ implies that

$$\|[\mathbf{x}]\| = \inf_{\mathbf{z}\in\operatorname{im}(B_d^\top)} \|\mathbf{x}+\mathbf{z}\|_{1,\deg}.$$

Moreover, Proposition 2.3 in [160] yields

$$\|[\mathbf{x}]\| = \|\mathbf{x}\|_{1,\deg} \iff \nabla\|\mathbf{x}\|_{1,\deg}\bigcap \operatorname{im}(B_d^\top)^\perp \neq \emptyset.$$

Hence, the minimization problem

$$\inf_{\mathbf{x}'\in\mathbb{R}^n\,:\,\mathbf{x}'-\mathbf{x}\in\operatorname{im}(B_d^\top)} \|\mathbf{x}'\|_{1,\deg}$$

reaches its minimum for some points in the set

$$\left\{\mathbf{x}\in\mathbb{R}^n : \nabla\|\mathbf{x}\|_{1,\deg}\bigcap \operatorname{im}(B_d^\top)^\perp \neq \emptyset\right\}.$$

This shows that (8.3.13), (8.3.14), and (8.3.15) coincide.

Now, let $\mathbf{x}\perp^1 \operatorname{im}(B_d^\top)$ denote that $\mathbf{u}\perp \operatorname{im}(B_d^\top)$ for some $\mathbf{u}\in\nabla\|\mathbf{x}\|_{1,\deg}$. Then,

$$\left\{\mathbf{x}\in\mathbb{R}^n : \nabla\|\mathbf{x}\|_{1,\deg}\bigcap \operatorname{im}(B_d^\top)^\perp \neq \emptyset\right\}$$

in (8.3.15) can be reduced to

$$\left\{\mathbf{x}\in\mathbb{R}^n : \mathbf{x}\perp^1 \operatorname{im}(B_d^\top)\right\},$$

as shown in (D3).

Similar to the proof of Proposition 6.3.1, we can apply Lemma 6.1.1 to derive that every eigenvalue of the function pair $(\|B_{d+1}^\top\cdot\|_1, \|\cdot\|_{1,\deg})$ has an eigenvector in the set of the extreme points associated with $(\|B_{d+1}^\top\cdot\|_1, \|\cdot\|_{1,\deg})$, since both $\|B_{d+1}^\top\cdot\|_1$ and $\|\cdot\|_{1,\deg}$ are piecewise linear.

Now, let k be the smallest integer such that the unit l^1-sphere,

$$\left\{\mathbf{x}\in\mathbb{R}^n : \|\mathbf{x}\|_{1,\deg} = 1\right\},$$

can be represented as the union $P_1\cup\ldots\cup P_k$ of k convex polytopes of dimension $(n-1)$, on which both $\|B_{d+1}^\top\cdot\|_1$ and $\|\cdot\|_{1,\deg}$ are linear. Moreover,

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

for $i = 1, \ldots, k$, let $\mathrm{Ext}(\|B_{d+1}^\top \cdot\|_1, \|\cdot\|_{1,\deg})$ be the vertex set of P_i. Then, $\mathrm{Ext}(\|B_{d+1}^\top \cdot\|_1, \|\cdot\|_{1,\deg})$ is a finite set, and its elements are called the *extreme points* determined by the function pair $(\|B_{d+1}^\top \cdot\|_1, \|\cdot\|_{1,\deg})$.

Since all entries of the matrix $B_{d+1}^\top$ and the degrees are rational numbers, by the theory of systems of linear equations, we have that

$$\mathrm{Ext}(\|B_{d+1}^\top \cdot\|_1, \|\cdot\|_{1,\deg}) \subset \mathbb{Q}^n.$$

Now, let M be a natural number that is greater than the least common multiple of all the denominators of the components of all points in $\mathrm{Ext}(\|B_{d+1}^\top \cdot\|_1, \|\cdot\|_{1,\deg})$. Then,

$$\mathrm{Ext}(\|B_{d+1}^\top \cdot\|_1, \|\cdot\|_{1,\deg}) \subset \{t\mathbf{x} \colon t \geq 0 \text{ and } \mathbf{x} \in \{-M, \ldots, -1, 0, 1, \ldots, M\}^n\},$$

and thus every eigenvalue has an eigenvector in

$$\{t\mathbf{x} \colon t \geq 0 \text{ and } \mathbf{x} \in \{-M, \ldots, -1, 0, 1, \ldots, M\}^n\}.$$

Since both $\|B_{d+1}^\top \cdot\|_1$ and $\|\cdot\|_{1,\deg}$ are positively one-homogeneous, we further derive that every eigenvalue has an eigenvector in the set

$$\{-M, \ldots, -1, 0, 1, \ldots, M\}^n.$$

Moreover, the minimizations in (8.3.13) and (8.3.15) can reach their minima at some points in $\{-M, \ldots, -1, 0, 1, \ldots, M\}^n$, while the minimization problem in (8.3.14) achieves its minima at some equivalence class $[\mathbf{x}]$, for some

$$\mathbf{x} \in \{-M, \ldots, -1, 0, 1, \ldots, M\}^n.$$

This implies that we can use $\{-M, \ldots, -1, 0, 1, \ldots, M\}^n$ instead of $\mathbb{R}^n$ in the constraints of the minimization problems in (8.3.13), (8.3.14), and (8.3.15). Since, furthermore,

$$\{-M, \ldots, -1, 0, 1, \ldots, M\}^n \subset \mathbb{Z}^n \subset \mathbb{R}^n,$$

it follows that we can also replace $\mathbb{R}^n$ by $\mathbb{Z}^n$, in the constraints of these three minimization problems.

By using $\mathbb{Z}^n$ instead of $\mathbb{R}^n$ in (8.3.13) and by equivalently replacing $\mathbb{Z}^n$ by $C^d(\Sigma, \mathbb{Z})$ and $\mathrm{im}(B_d^\top)$ by $\mathrm{im}\,\delta$, we obtain that (D2) is a reformulation of (8.3.13). Similarly, (D4) is a reformulation of (8.3.14). Moreover, if $\tilde{H}^d(\Sigma, \mathbb{R}) = 0$, then $\mathrm{im}(\delta_{d-1}) = \ker(\delta_d)$, which implies that

$$C^d(\Sigma)/\mathrm{im}(\delta_{d-1}) = C^d(\Sigma)/\ker(\delta_d) \cong \mathrm{im}(\delta_d).$$

Therefore, by considering the equivalence classes, we have that $\delta_d^{-1}(\mathbf{y}) = [\mathbf{x}]$ when $\mathbf{y} = \delta_d \mathbf{x}$. Note that, by (8.3.16), the filling norm

$$\|\mathbf{y}\|_{\mathrm{fil}} \coloneqq \inf_{x \in \delta_d^{-1}(\mathbf{y})} \|\mathbf{x}\|_{1,\deg}$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

coincides with $\|[\mathbf{x}]\|$, and

$$\|\mathbf{y}\|_1 = \|\delta_d \mathbf{x}\|_1 = \|B_{d+1}^\top \mathbf{x}\|_1.$$

Therefore, (D4) and (8.3.14) coincide.

By using $\{-M,\dots,-1,0,1,\dots,M\}^n$ instead of $\mathbb{R}^n$ in (8.3.13), we can similarly identify every multiset $\mu \subset_M \Sigma_d$ with a unique

$$\mathbf{x} \in \{-M,\dots,-1,0,1,\dots,M\}^n,$$

by identifying x_τ with $m(\tau)$ for any $\tau \in \Sigma_d$, where $m(\tau)$ is the multiplicity of τ in μ. For such μ and $\mathbf{x}$, we then have $\mathrm{vol}\,(\mu) = \|\mathbf{x}\|_{1,\deg}$, and

$$|\partial_{d+1}^* \mu| = \|B_{d+1}^\top \mathbf{x}\|_1.$$

If $\tilde{H}^d(\Sigma,\mathbb{R}) \neq 0$, then $\mathrm{im}(B_d^\top)$ is a proper subset of $\ker(B_{d+1}^\top)$, and thus (8.3.13) is zero. In this case, there exists $\mu' \neq \emptyset$ such that

$$\partial_{d+1}^* \mu' = \partial_{d+1}^* \mu = \emptyset,$$

which means that (8.3.11) equals zero. If $\tilde{H}^d(\Sigma,\mathbb{R}) = 0$, then $\mathrm{im}(B_d^\top) = \ker(B_{d+1}^\top)$, and thus for such μ and $\mathbf{x}$ with $\mathbf{x} \notin \ker(B_{d+1}^\top)$, we have

$$\begin{aligned}\inf_{\mathbf{z}\in\mathrm{im}(B_d^\top)} \|\mathbf{x}+\mathbf{z}\|_{1,\deg} &= \inf_{\substack{\mathbf{x}'\in\mathbb{R}^n:\\ \mathbf{x}'-\mathbf{x}\in\ker(B_{d+1}^\top)}} \|\mathbf{x}'\|_{1,\deg} \\ &= \min_{\substack{S'\neq\emptyset:\\ \partial_{d+1}^* S' = \partial_{d+1}^* S}} \mathrm{vol}\,(S').\end{aligned}$$

Therefore, (8.3.11) and (8.3.13) represent the same quantity, which has been denoted by $h(\Sigma_d)$. This proves the equivalence between (D1), (D2), (D3), and (D4). □

Hence, the four equivalent definitions in (D1)–(D4) represent the same Cheeger constant $h(\Sigma_d)$ from different viewpoints. In particular,

1. (D1) uses the language of multisets and provides a combinatorial formulation of $h(\Sigma_d)$.
2. (D2) presents $h(\Sigma_d)$ as a $\mathbb{Z}$-expander and makes it clear that

$$h(\Sigma_d) = 0 \iff \tilde{H}^d(\Sigma,\mathbb{R}) \neq 0,$$

for all $d \geq 0$. In particular, as we have discussed, when the Cheeger constant is defined as a $\mathbb{Z}_2$-expander, then the Cheeger inequalities cannot hold on general simplicial complexes. However, as a $\mathbb{Z}$-expander, it is possible to get Cheeger inequalities.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

3. (D3) shows that $h(\Sigma_d)$ coincides with the smallest nontrivial 1-Laplacian eigenvalue. This was already known in both the graph and the domain settings, which are treated in Sections 7.2 and 3.3, respectively.
4. Finally, (D4) reveals the nonobvious fact that $h(\Sigma_d)$ has a deep relation with Gromov's filling profile.

   In addition, for sufficiently large numbers $M \in \mathbb{Z}_+$, if $\tilde{H}^d(\Sigma,\mathbb{R}) = 0$, then

$$h(\Sigma_d) = \min_{\substack{S \subset_M \Sigma_d \\ \partial^*_{d+1} S \neq \emptyset}} \frac{|\partial^*_{d+1} S|}{\min\limits_{S' \colon \partial^*_{d+1} S' = \partial^*_{d+1} S} \operatorname{vol}(S')} > 0.$$

   For the case of $d = 0$, we can take $M = 1$, and then $h(\Sigma_0)$ reduces to the usual Cheeger constant on graphs.

For example, consider the usual tetrahedron Σ, which is viewed as a three-dimensional simplicial complex on $V = \{1,2,3,4\}$. It is not difficult to check $h(\Sigma_0) = \frac{2}{3}$, $h(\Sigma_1) = 1$, and $h(\Sigma_2) = 1$.

The following result shows that the constant $h(\Sigma_d)$ satisfies Cheeger-type inequalities and therefore provides a solution to the problem that was formulated at the beginning of this section.

Theorem 8.3.3. *If* $\deg \tau > 0$ *for all* $\tau \in \Sigma_d$*, then*

$$\frac{h^2(\Sigma_d)}{\#\Sigma_{d+1}} \leq \lambda_{I_d}(\Delta^{up}_d) \leq \operatorname{vol}(\Sigma_d) h(\Sigma_d),$$

and, for all $p \geq 1$*,*

$$\frac{h^p(\Sigma_d)}{|\#\Sigma_{d+1}|^{p-1}} \leq \lambda_{I_d}(\Delta^{up}_{d,p}) \leq \operatorname{vol}(\Sigma_d)^{p-1} h(\Sigma_d).$$

Proof For simplicity, we let $h = h(\Sigma_d)$ and $\lambda = \lambda_{I_d}(\Delta^{up}_d)$. To prove the first statement, it suffices to prove that

$$\frac{\min\limits_{\tau \in \Sigma_d} \deg \tau}{\#\Sigma_{d+1}} h^2 \leq \lambda \leq \operatorname{vol}(\Sigma_d) h^2, \tag{8.3.17}$$

by using the fact that, for all $\tau \in \Sigma_d$, we have $h \leq 1 \leq \deg \tau$.

Note that λ and h are the I_dth min–max eigenvalues of the dth normalized up Laplacian and the dth up 1-Laplacian, respectively. One can show that

$$\min_\tau \deg \tau \leq \frac{\|\mathbf{x}\|^2_{1,\deg}}{\|\mathbf{x}\|^2_{2,\deg}} \leq \sum_{\tau \in \Sigma_d} \deg \tau \quad \text{and} \quad 1 \leq \frac{\|B^\top_{d+1}\mathbf{x}\|_1^2}{\|B^\top_{d+1}\mathbf{x}\|_2^2} \leq \#\Sigma_{d+1} \tag{8.3.18}$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

by applying Cauchy's inequality

$$\|\mathbf{x}\|_{2,\deg}^2 = \sum_{\tau\in\Sigma_d} \deg\tau \cdot x_\tau^2 \geq \frac{(\sum_{\tau\in\Sigma_d} \deg\tau\,|x_\tau|)^2}{\sum_{\tau\in\Sigma_d}\deg\tau} = \frac{\|\mathbf{x}\|_{1,\deg}^2}{\sum_{\tau\in\Sigma_d}\deg\tau},$$

as well as

$$\#\Sigma_{d+1}\,\|B_{d+1}^\top\mathbf{x}\|_2^2 \geq \|B_{d+1}^\top\mathbf{x}\|_1^2 \geq \|B_{d+1}^\top\mathbf{x}\|_2^2,$$

and the inequality

$$\left(\sum_{\tau\in\Sigma_d}\deg\tau\,|x_\tau|\right)^2 \geq \sum_{\tau\in\Sigma_d}(\deg\tau)^2\,x_\tau^2 \geq \min_\tau \deg\tau \sum_{\tau\in\Sigma_d}\deg\tau\cdot x_\tau^2.$$

By (8.3.18), we then obtain

$$\frac{1}{\sum_{\tau\in\Sigma_d}\deg\tau}\frac{\|B_{d+1}^\top\mathbf{x}\|_2^2}{\|\mathbf{x}\|_{2,\deg}^2} \leq \frac{\|B_{d+1}^\top\mathbf{x}\|_1^2}{\|\mathbf{x}\|_{1,\deg}^2} \leq \frac{\#\Sigma_{d+1}}{\min_\tau\deg\tau}\frac{\|B_{d+1}^\top\mathbf{x}\|_2^2}{\|\mathbf{x}\|_{2,\deg}^2}. \tag{8.3.19}$$

Since the kth min–max eigenvalue of the dth up 1-Laplacian is

$$\lambda_k(\Delta_{d,1}^{up}) = \inf_{S\in\mathrm{Gen}_k}\sup_{\mathbf{x}\in S\setminus\{\mathbf{0}\}}\frac{\|B_{d+1}^\top\mathbf{x}\|_1}{\|\mathbf{x}\|_{1,\deg}},$$

while the kth smallest eigenvalue of dth normalized up Laplacian is

$$\lambda_k(\Delta_d^{up}) = \inf_{S\in\mathrm{Gen}_k}\sup_{\mathbf{x}\in S\setminus\{\mathbf{0}\}}\frac{\|B_{d+1}^\top\mathbf{x}\|_2^2}{\|\mathbf{x}\|_{2,\deg}^2},$$

we can apply (8.3.19) to derive

$$\frac{1}{\sum_{\tau\in\Sigma_d}\deg\tau}\lambda_k(\Delta_d^{up}) \leq \left(\lambda_k(\Delta_{d,1}^{up})\right)^2 \leq \frac{\#\Sigma_{d+1}}{\min_\tau\deg\tau}\lambda_k(\Delta_d^{up}). \tag{8.3.20}$$

This is a general inequality relating $\lambda_k(\Delta_{d,1}^{up})$ and $\lambda_k(\Delta_d^{up})$. By taking $k = I_d$ in (8.3.20), and by observing that $h = h(\Sigma_d) = \lambda_{I_d}(\Delta_{d,1}^{up})$, we finally obtain

$$\begin{aligned}\frac{1}{\mathrm{vol}\,(\Sigma_d)}\lambda &= \frac{1}{\sum_{\tau\in\Sigma_d}\deg\tau}\lambda \\ &\leq h^2 \\ &\leq \frac{\#\Sigma_{d+1}}{\min_\tau\deg\tau}\lambda,\end{aligned}$$

which completes the proof of (8.3.17). The case of $\Delta_{d,p}^{up}$ is similar, and thus we omit the proof of the second statement. □

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

For $d \geq 1$, we also define the *down Cheeger constant* as

$$h_{down}(\Sigma_d) \coloneqq \min_{\mathbf{x} \perp^1 \operatorname{im}(B_{d+1})} \frac{\|B_d \mathbf{x}\|_1}{\|\mathbf{x}\|_{1,\deg}} = \lambda_{I_{d+1}}(\Delta_{d,1}^{down}),$$

where

$$I_{d+1} \coloneqq \dim \operatorname{im}(B_{d+1}) + 1 = \operatorname{rank}(B_{d+1}) + 1.$$

Remark 8.3.8. Let Σ be a d-dimensional *combinatorial manifold*, that is, a topological manifold that has a triangulation as a simplicial complex. As a manifold, we assume that Σ is connected and has no boundary. We then have that B_{d+1} is a $|\Sigma_d| \times 1$ matrix of rank 1, therefore

$$I_{d+1} = \operatorname{rank}(B_{d+1}) + 1 = 1 + 1 = 2.$$

Therefore, in particular, $\lambda_{I_{d+1}} = \lambda_2$. Let now $\sigma \overset{\text{down}}{\sim} \sigma'$ indicate that σ and σ' are *down adjacent*, that is, they share a common facet. Then, the down adjacency relation induces a graph on Σ_d, and we can infer the following Cheeger inequality:

$$\frac{h_{down}^2(\Sigma_d)}{2} \leq \lambda_2(\Delta_d^{down}) \leq 2h_{down}(\Sigma_d).$$

Definition 8.3.1. Let M be a d-dimensional, orientable, compact, and closed Riemannian manifold, and let $c > 1$. A triangulation T of M is *c-uniform* if, for any two d-simplexes $\triangle$ and $\triangle'$ in the triangulation T,

$$\frac{1}{c} \leq \frac{\operatorname{diam}(\triangle)}{\operatorname{diam}(\triangle')} \leq c \tag{8.3.21}$$

and

$$\frac{1}{c} \leq \frac{\operatorname{diam}(\triangle)}{\operatorname{vol}(\triangle)^{\frac{1}{d}}} \leq c. \tag{8.3.22}$$

A set $\mathcal{T}$ of triangulations of M is *uniform* if there exist $N > 1$ and $c > 1$ such that, for each triangulation in $\mathcal{T}$, either its number of vertices is smaller than N, or it is c-uniform. In this case, these triangulations are simply said to be *uniform*, and the constants N and c are called the *uniform parameters* of these triangulations.

Thus, (8.3.21) requires that all the diameters of the simplices of the triangulation are compatible with each other, while (8.3.22) asks that they do not become too thin. To illustrate the concept of uniform triangulations, we consider the following example. Let $S \subset \mathbb{R}^3$ be the unit sphere with center at $\mathbf{0}$, and let T_1 be a regular tetrahedron inscribed in the sphere. Using central projection from $\mathbf{0}$ onto S, the one-dimensional skeleton of T_1 can be projected onto the sphere to obtain a geodesic triangulation of the sphere, which we denote by S_1.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

By induction, having constructed T_n, we obtain T_{n+1} by connecting the midpoints of the edges of each triangle, and by centrally projecting the resulting triangulation onto the sphere, we obtain a triangulation S_{n+1}. The geometric simplicial complexes $\{T_n\}_{n=1}^{\infty}$ are then $2/\sqrt[4]{3}$-uniform, and the resulting triangulations $\{S_n\}_{n=1}^{\infty}$ are c-uniform, for some $c \leq 4$. Of course, this is a family of particularly nice triangulations, as uniform as they could possibly be. Other families of triangulations that one may encounter may not be so nice, but the concept of uniformity excludes degenerations within families that would make our estimates invalid.

Theorem 8.3.4. *Let M be an orientable, compact, and closed Riemannian manifold of dimension $(d+1)$. There exists a constant C such that, for all simplicial complexes Σ that are combinatorially equivalent to some given uniform triangulations of M,*

$$\frac{h^2(\Sigma_d)}{C} \leq \lambda_{I_d}(\Delta_d^{up}) \leq C \cdot h(\Sigma_d). \tag{8.3.23}$$

Moreover, $h(\Sigma_d) > 0$ if and only if $H_1(\Sigma) = 0$ (or equivalently, $H_1(M) = 0$).

Proof By Theorem 8.3.3, we have that

$$\lambda_{I_d}(\Delta_d^{up}) = 0 \iff h(\Sigma_d) = 0.$$

Therefore, it suffices to assume that $h(\Sigma_d) > 0$, that is,

$$\tilde{H}^d(M) = \tilde{H}^d(\Sigma) = 0.$$

Since M and Σ are of dimension $(d+1)$, Poincaré duality implies that

$$\tilde{H}_1(M) = \tilde{H}^d(M) = 0.$$

Moreover, since there are only finitely many simplicial complexes with less than a given number of vertices, the existence of the constant $C > 0$ in (8.3.23) for these simplicial complexes follows immediately from Theorem 8.3.3.

We may assume, without loss of generality, that M is simply connected. Since M is compact, the triangulation is c-uniform for some $c > 1$, and Σ_d has n elements for some positive integer n. Also. for any $\varepsilon > 0$, there exists $N > 0$ such that any c-uniform triangulation with at least N facets satisfies

$$\frac{1}{2c}\varepsilon' < \operatorname{diam}(\triangle) < \varepsilon', \tag{8.3.24}$$

for all $\triangle$, and for some $\varepsilon' \in (0, \varepsilon)$. This follows since for sufficiently large N, there must be one simplex of small diameter, and then by (8.3.21), all diameters must be compatible with that one. We shall also consider the uniform triangulation as a uniform ε-net.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

Now, we let $\varepsilon := \max_{\triangle \in T} \operatorname{diam}(\triangle)$ denote the maximum diameter of the facets in the triangulation T. In other words, we fix a small $\varepsilon > 0$ and a sufficiently large $N > 0$ in (8.3.24); we consider any simplicial complex Σ, which is combinatorially equivalent to a c-uniform triangulation with at least N faces, and we let $\varepsilon := \max_{\sigma \in \Sigma_{d+1}} \operatorname{diam}(|\sigma|)$.

We split the rest of the proof into multiple steps.

1. The down Cheeger constant satisfies

$$\frac{d+2}{4} h^2_{down}(\Sigma_{d+1}) \leq \lambda_{I_d}(\Delta^{up}_d) \leq (d+2) h_{down}(\Sigma_{d+1}).$$

This is derived by the Cheeger inequalities from Remark 8.3.8,

$$\frac{h^2_{down}(\Sigma_{d+1})}{2} \leq \lambda_2(\Delta^{down}_{d+1}) \leq 2 h_{down}(\Sigma_{d+1}),$$

and by the equality

$$\lambda_{I_d}(\Delta^{up}_d) = \frac{1}{2}\lambda_{I_d}(L^{up}_d) = \frac{1}{2}\lambda_2(L^{down}_{d+1}) = \frac{d+2}{2}\lambda_2(\Delta^{down}_{d+1}). \quad (8.3.25)$$

Here, $\lambda_{I_d}(L^{up}_d)$ and $\lambda_2(L^{down}_{d+1})$ denote the smallest nonzero eigenvalues of the unnormalized up-Laplacian L^{up}_d and the unnormalized down-Laplacian L^{down}_{d+1}, respectively. By (4.8.6), we infer the second equality in (8.3.25). Moreover, the first and the last equality in (8.3.25) are based on the fact that the degrees appearing in the expression of the dth normalized up-Laplacian Δ^{up}_d are all equal to 2, while the degrees in the expression of the $(d+1)$-th normalized down-Laplacian Δ^{down}_{d+1} are all equal to $d+2$. In fact, since M is a compact manifold without boundary, each d-face is contained in exactly two $(d+1)$-faces, and every $(d+1)$-face contains exactly $(d+2)$ different d-faces.

2. The Cheeger constant $h(\Sigma_d)$ and the down Cheeger constant $h_{down}(\Sigma_{d+1})$ satisfy $h(\Sigma_d) \sim h_{down}(\Sigma_{d+1})$, that is, there exists a uniform constant $C > 1$ such that

$$\frac{1}{C} h_{down}(\Sigma_{d+1}) \leq h(\Sigma_d) \leq C h_{down}(\Sigma_{d+1}).$$

The proof of this step is further divided into the following two claims:

(a)

$$\frac{1}{\varepsilon} h_{down}(\Sigma_{d+1}) \sim h(M).$$

Proof Let G be the graph on $n := \#\Sigma_{d+1}$ vertices, which is located in the barycenters of all $(d+1)$-simplexes and is such that two vertices form an edge if and only if the corresponding two $(d+1)$-simplexes are

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

down adjacent. We call G the *underlying graph of the triangulation*, and we observe that $h_{down}(\Sigma_{d+1})$ coincides with the Cheeger constant of G. Now, an approximation approach developed in [264, 265] implies that the Cheeger constant of a uniform triangulation approximates the Cheeger constant of the manifold when we equip the edges of the underlying graph of the triangulation with appropriate weights (related to ε). In fact, since G is the underlying graph of the uniform triangulation, we may assume that G is embedded in the manifold M, and that the distribution of the vertices of G is uniform. Then, according to such approximation results, by putting appropriate weights[2] on the edges of G, the Cheeger constant of G together with those edge weights approximates $h(M)$. (More precisely, the difference between $h(M)$ and the Cheeger constant of the weighted graph G is bounded by $h(M)/2$, whenever ε is sufficiently small.) We can then adopt such an approximation approach to derive that

$$\frac{1}{\varepsilon} h_{down}(\Sigma_{d+1}) \sim h(M).$$

(b) If $H_1(M) = 0$, then

$$\frac{1}{\varepsilon} h(\Sigma_d) \sim h(M).$$

Proof By Poincaré duality,

$$H_1(M) = 0 \iff H^d(M) = 0 \iff \ker(\delta_d) = \mathrm{im}(\delta_{d-1}),$$

since M is a compact closed manifold of dimension $(d+1)$. Thus, by $\mathrm{im}(B_d^\top) = \ker(B_{d+1}^\top)$, $\delta_d := B_{d+1}^\top$, and by applying (8.3.13), we have that

$$h(\Sigma_d) = \min_{x \notin \ker(\delta_d)} \frac{\sum_{\sigma \in \Sigma_{d+1}} \left| \sum_{\tau \in \Sigma_d} \mathrm{sgn}(\tau, \partial\sigma) x_\tau \right|}{\min_{z \in \ker(\delta_d)} \sum_{\tau \in \Sigma_d} 2|x_\tau + z_\tau|} > 0.$$

Now, note that $h(\Sigma_d)$ equals the smallest nonzero eigenvalue of $\Delta_{d,1}^{up}$; therefore, the smallest nonzero eigenvalue of the nonlinear eigenproblem in (8.3.12). By the Duality Theorem on nonlinear eigenvalue problems (Lemma 2.5 in [160], Theorem 1 in [267]), the nonzero spectrum of the eigenproblem (8.3.12) coincides with the nonzero spectrum

[2] The weight of an edge $\{u, v\}$ is determined by the distance of u and v in M, which is about $O(\varepsilon)$.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

of its *dual* eigenproblem:

$$\mathbf{0} \in \partial \|B_{d+1}\mathbf{y}\|_{1,\deg}^* - \lambda \partial \|\mathbf{y}\|_1^*, \tag{8.3.26}$$

where $\|\cdot\|_{1,\deg}^*$ indicates the dual norm of $\|\cdot\|_{1,\deg}$. Since every d-face is a facet of exactly two $(d+1)$-faces, we have that $\deg \equiv 2$, and (8.3.26) is equivalent to

$$\mathbf{0} \in \partial \frac{1}{2}\|B_{d+1}\mathbf{y}\|_\infty - \lambda \partial \|\mathbf{y}\|_\infty, \tag{8.3.27}$$

which can also be written as

$$\mathbf{0} \in \partial \frac{1}{2} \max_{\sigma \overset{\text{down}}{\sim} \sigma'} |y_\sigma - y_{\sigma'}| - \lambda \partial \max_{\sigma \in \Sigma_{d+1}} |y_\sigma|. \tag{8.3.28}$$

where we recall that $\sigma \overset{\text{down}}{\sim} \sigma'$ indicates that σ and σ' are down adjacent, that is, they share a common facet. By Theorem 2.1 in [160], the smallest nonzero eigenvalue of (8.3.28) can then be formulated as

$$\min_{\mathbf{y} \text{ nonconstant}} \frac{\max_{\sigma \overset{\text{down}}{\sim} \sigma'} \frac{1}{2}|y_\sigma - y_{\sigma'}|}{\min_{t \in \mathbb{R}} \max_{\sigma \in \Sigma_{d+1}} |y_\sigma + t|}.$$

As a consequence, $h(\Sigma_d)$ is the smallest nonzero eigenvalue of both the 1-Laplacian eigenvalue problem (8.3.12) and its dual eigenproblem (8.3.28). Therefore,

$$h(\Sigma_d) = \min_{\mathbf{y} \text{ nonconstant}} \frac{\max_{\sigma \overset{\text{down}}{\sim} \sigma'} \frac{1}{2}|y_\sigma - y_{\sigma'}|}{\min_{t \in \mathbb{R}} \max_{\sigma \in \Sigma_{d+1}} |y_\sigma + t|}.$$

Now, we have that

$$\min_{t \in \mathbb{R}} \max_{\sigma \in \Sigma_{d+1}} |y_\sigma + t| = \max_{\sigma \in \Sigma_{d+1}} |y_\sigma + \widetilde{t}|,$$

where

$$\widetilde{t} := -\frac{\max_{\sigma \in \Sigma_{d+1}} y_\sigma + \min_{\sigma \in \Sigma_{d+1}} y_\sigma}{2}.$$

By taking $\widetilde{y_\sigma} = y_\sigma + \widetilde{t}$, one can check that $\min_\sigma \widetilde{y_\sigma} + \max_\sigma \widetilde{y_\sigma} = 0$. Hence,

$$h(\Sigma_d) = \min_{\min_\sigma \widetilde{y_\sigma} + \max_\sigma \widetilde{y_\sigma} = 0} \frac{\max_{\sigma \overset{\text{down}}{\sim} \sigma'} |\widetilde{y_\sigma} - \widetilde{y_{\sigma'}}|}{2 \max_\sigma |\widetilde{y_\sigma}|} = \frac{1}{\operatorname{diam}(G)},$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

where $\operatorname{diam}(G)$ indicates the diameter of G. Hence, we can rewrite $h(\Sigma_d)$ as $1/2$ times the smallest nontrivial eigenvalue of the ∞-Laplacian, which is equal to $1/\operatorname{diam}(G)$. This argument is similar to Lemma 1.5 in [163].

Finally, since the triangulation is C-uniform, we have that

$$\frac{1}{\varepsilon} h(\Sigma_d) = \frac{1}{\varepsilon \cdot \operatorname{diam}(G)} \sim \frac{1}{\operatorname{diam}(M)}.$$

Hence,

$$\frac{1}{\varepsilon} h(\Sigma_d) \sim h(M).$$

The proof then follows by combining all the steps above. □

Remark 8.3.9. We conclude this section with some observations regarding the last theorem.

- The constant C in Theorem 8.3.4 depends on the parameters of the given triangulations of the ambient manifold. In future work, it would be good to find a new constant that only depends on the dimension d.
- Under the same condition as in Theorem 8.3.4, with the same proof, one can show that
$$\frac{\lambda_{k_d}(\Delta^{up}_{d,1})^2}{C} \leq \lambda_{k_d}(\Delta^{up}_d) \leq C \lambda_{k_d}(\Delta^{up}_{d,1}),$$
where $k_d := \dim \ker(\delta_d) + 1$. This inequality coincides with the Cheeger inequality in Theorem 8.3.4 if and only if $H_1(M) = 0$.
- With a modification of the proof of Theorem 8.3.4, we can show that $1/\operatorname{diam}(G) \sim \lambda_2(G)$ whenever G can be uniformly embedded into such a manifold, where $\lambda_2(G)$ is the second smallest eigenvalue of the normalized Laplacian of G.
- Inspired by the approximation theory for Laplacians on triangulations of manifolds that has been proposed by Dodziuk in [81] and Dodziuk and Patodi in [83], we hope that it is possible to develop an approximation theory for Cheeger constants on triangulations of manifolds.

8.4 Cheeger Inequalities on Hypergraphs

In this section, we shall derive Cheeger inequalities for hypergraph p-Laplacians, with a theory that is very different from that for simplicial complexes that we developed in the previous section.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

Fix a hypergraph $\Gamma = (V, E)$ on N vertices. Let $\Theta := \max_{e \in E} |e|$ and, given $v \in V$, let $d_v := \sum_{e \ni v} |e|/\Theta$. The Laplacian $\tilde{\Delta}_2$ that we consider in this section is the operator $\tilde{\Delta}_2 \colon C(V) \to C(V)$ such that, for given $f \colon V \to \mathbb{R}$ and $v \in V$,

$$\tilde{\Delta}_2 f(v) := \frac{1}{d_v} \sum_{e \ni v} \sum_{v' \in e} f(v'). \tag{8.4.1}$$

In the case of uniform hypergraphs, $\tilde{\Delta}_2$ coincides with the Laplacian for uniform oriented hypergraphs with only inputs from Chapters 2 and 4. In the general case, the two operators might be different. We let $\lambda_{N-1}(\tilde{\Delta}_2)$ denote the second largest eigenvalue of $\tilde{\Delta}_2$. Given a vertex measure $\mu_v > 0$ for each $v \in V$, we let $\mu(S) := \sum_{v \in S} \mu_v$ for each $S \subseteq V$, and we let

$$h := \min_{S \neq \emptyset, V} \frac{\sum_{e \in E} |e \cap S| \cdot |e \setminus S|}{\min\{\mu(S), \mu(V \setminus S)\}}.$$

Note that the quantity $\sum_{e \in E} |e \cap S| \cdot |e \setminus S|$ was introduced in [279] as a *hypergraph cut measure*, and it is also called the *clique expansion* [133].

In particular, if the hypergraph is Θ-uniform, for some integer $\Theta \geq 2$, and μ_v is the usual vertex degree, then h coincides with the Cheeger constant for uniform hypergraphs that were introduced in [214]. The main result in [214] states that, in this case,

$$\frac{h^2}{2(\Theta - 1)} \leq \Theta - \lambda_{N-1}(\tilde{\Delta}_2) \leq 2h. \tag{8.4.2}$$

And even though this might seem counterintuitive at first, the above inequalities generalize the classical Cheeger inequalities for simple graphs. In fact, the classical Cheeger inequalities involve the second eigenvalue $\lambda_2(\Delta)$ of the graph Laplacian Δ. But, by Definition 4.9.5 and Remark 4.9.2, we know that $\lambda_2(\Delta) = 2 - \lambda_{N-1}(\Delta^+)$, where Δ^+ is the signless Laplacian of the graph Γ, or, equivalently, the Laplacian of the oriented graph Γ^+, which is obtained from Γ by letting each vertex be an input for all edges in which it is contained. Hence, in particular, the classical Cheeger inequalities can be equivalently rewritten in terms of $\lambda_{N-1}(\Delta^+)$, which coincides with our $\lambda_{N-1}(\tilde{\Delta}_2)$.

We now generalize the inequalities in (8.4.2) as follows.

Theorem 8.4.1. *Let $\Gamma = (V, E)$ be a hypergraph. Then,*

$$ch^2 \leq \Theta - \lambda_{N-1}(\tilde{\Delta}_2) \leq Ch, \tag{8.4.3}$$

where

$$c := \min_{v \in V} \frac{\sum_{e \ni v} (|e| - 1)}{2 \sum_{e \ni v} |e|/\Theta} \cdot \min_{v \in V} \frac{\mu_v^2}{\left(\sum_{e \ni v} (|e| - 1) \right)^2},$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

and

$$C := \max_{v\in V} \frac{\sum\limits_{e\ni v}(|e|-1)}{\sum\limits_{e\ni v}|e|/\Theta} \cdot \max_{v\in V} \frac{2\mu_v}{\sum\limits_{e\ni v}(|e|-1)}.$$

Proof Let G be the weighted graph that has vertex set V, edge weight

$$w_{vv'} := \#\{e \in E\colon e \supset \{v, v'\}\},$$

and vertex weight

$$w_v = \sum_{v'\in V} w_{vv'} = \sum_{e\ni v}(|e|-1).$$

Let $\hat{\lambda}_2$ denote the second eigenvalue of the normalized Laplacian on G and let $\hat{h}$ denote the standard Cheeger constant on G. Then,

$$\begin{aligned}
\Theta - \lambda_{N-1}(\tilde{\Delta}_2) &= \Theta - \max_{\dim(X)\geq 2} \min_{f\in X} \frac{\sum_{e\in E}(\sum_{v\in e} f(v))^2}{\sum_{v\in V} d_v f(v)^2} \\
&= \min_{\dim(X)\geq 2} \max_{f\in X} \frac{\Theta \sum_{v\in V} d_v f(v)^2 - \sum_{e\in E}(\sum_{v\in e} f(v))^2}{\sum_{v\in V} d_v f(v)^2} \\
&= \min_{\dim(X)\geq 2} \max_{f\in X} \frac{\sum_{v\in V}\sum_{e\ni v} |e| f(v)^2 - \sum_{e\in E}(\sum_{v\in e} f(v))^2}{\sum_{v\in V} d_v f(v)^2} \\
&= \min_{\dim(X)\geq 2} \max_{f\in X} \frac{\sum_{e\in E}\sum_{v\in e} |e| f(v)^2 - \sum_{e\in E}(\sum_{v\in e} f(v))^2}{\sum_{v\in V} d_v f(v)^2} \\
&= \min_{\dim(X)\geq 2} \max_{f\in X} \frac{\sum_{e\in E}\left(|e|\sum_{v\in e} f(v)^2 - (\sum_{v\in e} f(v))^2\right)}{\sum_{v\in V} d_v f(v)^2} \\
&= \min_{\dim(X)\geq 2} \max_{f\in X} \frac{\sum_{e\in E}\sum_{\{v,v'\}\subset e}(f(v)-f(v'))^2}{\sum_{v\in V} d_v f(v)^2} \\
&= \min_{\dim(X)\geq 2} \max_{f\in X} \frac{\sum_{\{v,v'\}\subset V} w_{vv'}(f(v)-f(v'))^2}{\sum_{v\in V} d_v f(v)^2} \\
&= \min_{\dim(X)\geq 2} \max_{f\in X} \frac{\sum_{\{v,v'\}\subset V} w_{vv'}(f(v)-f(v'))^2}{\sum_{v\in V} w_v f(v)^2} \cdot \frac{\sum_{v\in V} w_v f(v)^2}{\sum_{v\in V} d_v f(v)^2} \\
&\in [m\hat{\lambda}_2, M\hat{\lambda}_2] \subset [m\hat{h}^2/2, M2\hat{h}] \subset [m\tilde{c}^2 h^2/2, 2M\tilde{C}h],
\end{aligned}$$

where

$$m := \min_v \frac{w_v}{d_v}, \quad M := \max_v \frac{w_v}{d_v}, \quad \tilde{c} := \min_v \frac{\mu_v}{w_v}, \quad \text{and} \quad \tilde{C} := \max_v \frac{\mu_v}{w_v}.$$

In the preceding formulae, we use the inequality

$$m \leq \frac{\sum_{v\in V} w_v x_v^2}{\sum_{v\in V} d_v x_v^2} \leq M,$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

the Cheeger inequality $\hat{h}^2/2 \le \hat{\lambda}_2 \le 2\hat{h}$ for the graph G, and the relation $\tilde{c}h \le \hat{h} \le \tilde{C}h$, which follows by

$$\begin{aligned}\hat{h} &= \min_{S\neq\emptyset,V} \frac{\sum\limits_{v\in S, v'\notin S} w_{vv'}}{\min\left\{\sum\limits_{v\in S} w_v, \sum\limits_{v\in V\setminus S} w_v\right\}} \\ &= \min_{S\neq\emptyset,V} \frac{\sum\limits_{e\in E} |e\cap S|\cdot|e\setminus S|}{\min\left\{\sum\limits_{v\in S}\mu_v, \sum\limits_{v\in V\setminus S}\mu_v\right\}} \cdot \frac{\min\left\{\sum\limits_{v\in S}\mu_v, \sum\limits_{v\in V\setminus S}\mu_v\right\}}{\min\left\{\sum\limits_{v\in S} w_v, \sum\limits_{v\in V\setminus S} w_v\right\}}.\end{aligned}$$

The claim follows by taking $c := m\tilde{c}^2/2$ and $C := 2M\tilde{C}$. □

Now, we also define a multiway version of the hypergraph Cheeger constant, as follows. Given $S \subseteq V$, let

$$\psi(S) := \frac{\sum_{e\in E} |e\cap S|\cdot|e\setminus S|}{\mu(S)}.$$

We define the *lth Cheeger constant* as

$$h_l := \min_{\substack{S_1,\dots,S_l \\ \text{pairwise disjoint}}} \max_{i=1,\dots,l} \psi(S_i).$$

By the higher-order Cheeger inequalities for graphs from Lee–Oveis Gharan–Trevisan [180], and by using a similar technique as in the preceding proof, we obtain:

Theorem 8.4.2. *There exists a universal constant C such that, for any $l = 1,\dots,N$,*

$$c_l h_l^2 \le \Theta - \lambda_{N-l+1} \le C_l h_l,$$

where

$$\begin{aligned}C_l &:= \max_v \frac{\sum\limits_{e\ni v}(|e|-1)}{\sum\limits_{e\ni v}|e|/\Theta} \cdot \max_v \frac{\mu_v}{2\sum\limits_{e\ni v}(|e|-1)}, \quad \textit{and} \\ c_l &:= \min_v \frac{\sum\limits_{e\ni v}(|e|-1)}{C^2 l^4 \sum\limits_{e\ni v}|e|/\Theta} \cdot \min_v \frac{\mu_v^2}{\left(\sum\limits_{e\ni v}(|e|-1)\right)^2}.\end{aligned}$$

In order to prove Theorem 8.4.3, we need the following preliminary lemma.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

Lemma 8.4.1. *Let $\Gamma = (V, E)$ be a Θ-uniform hypergraph. Fix $p \in (1, +\infty)$ and fix $e \in E$. Then, there exist $m_{p,\Theta} > 0$ (if $p \geq 2$) and $M_{p,\Theta} > 0$ (if $1 < p \leq 2$) such that, for any $\mathbf{x} \in \mathbb{R}^n$,*

$$m_{p,\Theta} \cdot \sum_{\{v,v'\} \subset e} |x_v - x_{v'}|^p \leq |e|^{p-1} \sum_{v \in e} |x_v|^p - \left| \sum_{v \in e} x_v \right|^p,$$

$$M_{p,\Theta} \cdot \sum_{\{v,v'\} \subset e} |x_v - x_{v'}|^p \geq |e|^{p-1} \sum_{v \in e} |x_v|^p - \left| \sum_{v \in e} x_v \right|^p.$$

Proof Since Γ is Θ-uniform, we have $|e| = \Theta$ for any $e \in E$. And for a fixed $e \in E$, we can assume without loss of generality that $e = \{1, \cdots, \Theta\}$. Then

$$\sum_{\{v,v'\} \subset e} |x_v - x_{v'}|^p = \frac{1}{2} \sum_{i,j=1}^{\Theta} |x_i - x_j|^p.$$

It suffices to prove that

$$m'_{p,\Theta} := \inf_{\mathbf{x} \text{ nonconstant}} \frac{\Theta^{p-1} \sum_{i=1}^{\Theta} |x_i|^p - \left| \sum_{i=1}^{\Theta} x_i \right|^p}{\sum_{i,j=1}^{\Theta} |x_i - x_j|^p} > 0,$$

for $p \geq 2$, and

$$M'_{p,\Theta} := \sup_{\mathbf{x} \text{ nonconstant}} \frac{\Theta^{p-1} \sum_{i=1}^{\Theta} |x_i|^p - \left| \sum_{i=1}^{\Theta} x_i \right|^p}{\sum_{i,j=1}^{\Theta} |x_i - x_j|^p} < +\infty,$$

for $p \in (1, 2]$. In fact, we can then simply take $m_{p,\Theta} := 2m'_{p,\Theta}$ and $M_{p,\Theta} := 2M'_{p,\Theta}$ to prove Lemma 8.4.1.

Clearly,

$$\sum_{i,j=1}^{\Theta} |x_i - x_j|^p > 0 \iff \mathbf{x} \text{ is nonconstant.}$$

By Hölder's inequality,

$$\Theta^{p-1} \sum_{i=1}^{\Theta} |x_i|^p \geq \left| \sum_{i=1}^{\Theta} x_i \right|^p,$$

and equality holds if and only if $\mathbf{x}$ is constant. Therefore,

$$\sum_{i,j=1}^{\Theta} |x_i - x_j|^p > 0 \iff \Theta^{p-1} \sum_{i=1}^{\Theta} |x_i|^p - \left| \sum_{i=1}^{\Theta} x_i \right|^p > 0.$$

Now, given $\delta > 0$, for any vector $\mathbf{x}$ satisfying

$$\max_i x_i - \min_i x_i = \delta > 0,$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

we have that

$$\delta^p \le \sum_{i,j=1}^{\Theta} |x_i - x_j|^p \le \Theta^2 \delta^p .$$

Given $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x} \perp \mathbf{1}$, $t \in \mathbb{R}$, and $p \ge 1$, let now

$$g(\mathbf{x},t,p) := \Theta^{p-1} \sum_{i=1}^{\Theta} |x_i + t|^p - \left| \sum_{i=1}^{\Theta} x_i + \Theta t \right|^p .$$

Since $\mathbf{x}$ is nonconstant and $p > 1$, by Hölder's inequality, we have that $g(\mathbf{x},t,p) > 0$. Moreover,

$$\partial_t g(\mathbf{x},t,p) = p\Theta^{p-1} \sum_{i=1}^{\Theta} |x_i + t|^{p-1} \operatorname{sign}(x_i + t) - p\Theta \, |\Theta t|^{p-1} \operatorname{sign}(\Theta t) ,$$

where we used the assumption $\mathbf{x} \perp \mathbf{1}$, that is, $\sum_{i=1}^{\Theta} x_i = 0$. Fix $p \ge 2$. If $t > \delta$, then $x_i + t > 0$ for any $i = 1, \dots, \Theta$, and by Hölder's inequality, we have that $\partial_t g(\mathbf{x},t,p) \ge 0$. Similarly, if $t < -\delta$, then $\partial_t g(\mathbf{x},t,p) \le 0$. Therefore, $\mathbb{R} \ni t \mapsto g(\mathbf{x},t,p)$ reaches its minimum at some point in $[-\delta,\delta]$, implying that

$$\min_{t \in \mathbb{R}} g(\mathbf{x},t,p) = \min_{-\delta \le t \le \delta} g(\mathbf{x},t,p)$$

is a continuous function on the compact set

$$\left\{ \mathbf{x} \in \mathbb{R}^{\Theta} \colon \sum_{i=1}^{\Theta} x_i = 0, \max_i x_i - \min_i x_i = \delta \right\} .$$

Hence,

$$\min_{\substack{\mathbf{x} \perp \mathbf{1}, \\ \max_i x_i - \min_i x_i = \delta}} \min_{t \in \mathbb{R}} g(\mathbf{x},t,p) > 0.$$

Thus,

$$\begin{aligned} \inf_{\mathbf{x} \text{ nonconstant}} \frac{\Theta^{p-1} \sum_{i=1}^{\Theta} |x_i|^p - |\sum_{i=1}^{\Theta} x_i|^p}{\sum_{i,j=1}^{\Theta} |x_i - x_j|^p} &= \inf_{\substack{\mathbf{x} \perp \mathbf{1}, \\ \max_i x_i - \min_i x_i = \delta}} \min_{t \in \mathbb{R}} \frac{g(\mathbf{x},t,p)}{\sum_{i,j=1}^{\Theta} |x_i - x_j|^p} \\ &\ge \frac{1}{\Theta^2 \delta^p} \min_{\substack{\mathbf{x} \perp \mathbf{1}, \\ \max_i x_i - \min_i x_i = \delta}} \min_{t \in \mathbb{R}} g(\mathbf{x},t,p) \\ &> 0. \end{aligned}$$

Now,

$$g(\mathbf{x},t,2) = \sum_{\{i,j\} \subset \{1,\dots,\Theta\}} (x_i - x_j)^2 \quad \text{and} \quad g(\mathbf{x},t,1) \ge 0.$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

Furthermore,

$$\partial_p g(\mathbf{x},t,p) = \frac{1}{\Theta}\sum_{i=1}^{\Theta} |\Theta x_i + \Theta t|^p \ln|\Theta x_i + \Theta t| - \left|\sum_{i=1}^{\Theta} x_i + \Theta t\right|^p \ln\left|\sum_{i=1}^{\Theta} x_i + \Theta t\right|.$$

Since $s \mapsto s^p \ln s$ is a convex and increasing function on $s \in (1,+\infty)$, by Jensen's inequality for convex functions, we have that $\partial_p g(\mathbf{x},t,p) > 0$ whenever $|t| > \delta + 1/\Theta$ and $p > 1$. Therefore,

$$g(\mathbf{x},t,1) < g(\mathbf{x},t,p) \le \sum_{\{i,j\}\subset\{1,\ldots,\Theta\}} (x_i - x_j)^2,$$

whenever $|t| > \delta + 1/\Theta$ and $1 < p \le 2$. Consequently,

$$\begin{aligned}\sup_{\mathbf{x}\ \text{nonconstant}} \frac{\Theta^{p-1}\sum_{i=1}^{\Theta}|x_i|^p - |\sum_{i=1}^{\Theta} x_i|^p}{\sum_{i,j=1}^{\Theta}|x_i - x_j|^p} &= \sup_{\substack{\mathbf{x}\perp\mathbf{1},\\ \max_i x_i - \min_i x_i = \delta}} \max_{t\in\mathbb{R}} \frac{g(\mathbf{x},t,p)}{\sum_{i,j=1}^{\Theta}|x_i - x_j|^p} \\ &\le \frac{1}{\delta^p} \max_{\substack{\mathbf{x}\perp\mathbf{1},\\ \max_i x_i - \min_i x_i = \delta}} \max_{t\in\mathbb{R}} g(\mathbf{x},t,p) \\ &< +\infty.\end{aligned}$$

□

We need Lemma 8.4.1 to show an inequality involving the Cheeger constant for uniform hypergraphs that was introduced in [214] and that we presented in the beginning of this section, as well as the hypergraph p-Laplacian $\tilde{\Delta}_p$ that was introduced in [157] and discussed in Section 7.4. In particular, we consider

$$\widetilde{\lambda}_{N-1}(\tilde{\Delta}_p) = \sup_{\mathrm{gen}(A)\ge 2} \inf_{\mathbf{x}\in A} \frac{\sum_{e\in E} |\sum_{v\in e} x_v|^p}{\sum_{v\in V}\deg(v)|x_v|^p},$$

that is, the second largest max–min eigenvalue of $\tilde{\Delta}_p$.

Theorem 8.4.3. *Let $\Gamma = (V,E)$ be a Θ-uniform hypergraph. Given $p \in (1,2]$, we have*

$$\Theta^{p-1} - \widetilde{\lambda}_{N-1}(\tilde{\Delta}_p) \le C_{p,\Theta} h,$$

and given $p \in [2,+\infty)$, we have

$$c_{p,\Theta} h^p \le \Theta^{p-1} - \widetilde{\lambda}_{N-1}(\tilde{\Delta}_p),$$

where the universal positive constants $c_{p,\Theta}$ and $C_{p,\Theta}$ only depend on p and Θ. In particular, in the case of $p = 2$, one can let $C_{2,\Theta} = 2$ and $c_{2,\Theta} = 2/2(\Theta - 1)$.

Proof Let G be the weighted graph that has vertex set V, edge weight

$$w_{vv'} \coloneqq \#\{e \in E \colon e \supset \{v,v'\}\},$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

and vertex weight

$$w_v \coloneqq \sum_{v' \in V} w_{vv'} = \sum_{e \ni v} (|e| - 1) = (\Theta - 1)\deg(v).$$

Let $\hat{\lambda}_2(\Delta_p)$ denote the second eigenvalue of the p-Laplacian of G and let $\hat{h}$ denote the Cheeger constant of G. Since $|e| = \Theta$ for all $e \in E$, we have

$$\begin{aligned}
\Theta^{p-1} - \widetilde{\lambda}_{N-1}(\tilde{\Delta}_p) &= \Theta^{p-1} - \sup_{\mathrm{gen}(A)\geq 2} \inf_{\mathbf{x}\in A} \frac{\sum_{e\in E} |\sum_{i\in e} x_v|^p}{\sum_{v\in V} \deg(v)|x_v|^p} \\
&= \inf_{\mathrm{gen}(A)\geq 2} \sup_{\mathbf{x}\in A} \frac{\Theta^{p-1}\sum_{v\in V} \deg(v)|x_v|^p - \sum_{e\in E} |\sum_{v\in e} x_v|^p}{\sum_{v\in V} \deg(v)|x_v|^p} \\
&= \inf_{\mathrm{gen}(A)\geq 2} \sup_{\mathbf{x}\in A} \frac{\Theta^{p-1}\sum_{e\in E}\sum_{v\in e} |x_v|^p - \sum_{e\in E} |\sum_{v\in e} x_v|^p}{\sum_{v\in V} \deg(v)|x_v|^p} \\
&= \inf_{\mathrm{gen}(A)\geq 2} \sup_{\mathbf{x}\in A} \frac{\sum_{e\in E}\left(|e|^{p-1}\sum_{v\in e} |x_v|^p - |\sum_{v\in e} x_v|^p\right)}{\sum_{v\in V} \deg(v)|x_v|^p}.
\end{aligned}$$

By Lemma 8.4.1,

$$\begin{aligned}
\Theta^{p-1} - \widetilde{\lambda}_{N-1}(\tilde{\Delta}_p) &\leq \inf_{\mathrm{gen}(A)\geq 2} \sup_{x\in A} \frac{\sum_{e\in E} M_{p,\Theta} \sum_{\{v,v'\}\subset e} |x_v - x_{v'}|^p}{\sum_{v\in V} \deg(v)|x_v|^p} \\
&= M_{p,\Theta}(\Theta - 1) \inf_{\mathrm{gen}(A)\geq 2} \sup_{x\in A} \frac{\sum_{\{v,v'\}\subset V} w_{vv'}|x_v - x_{v'}|^p}{\sum_{v\in V} w_v|x_v|^p} \\
&= M_{p,\Theta}(\Theta - 1)\hat{\lambda}_2(\Delta_p) \\
&\leq M_{p,\Theta}(\Theta - 1)2^{p-1}\hat{h} \\
&= M_{p,\Theta}2^{p-1}h.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Theta^{p-1} - \widetilde{\lambda}_{N-1}(\tilde{\Delta}_p) &\geq m_{p,\Theta}(\Theta - 1)\hat{\lambda}_2(\Delta_p) \\
&\geq m_{p,\Theta}(\Theta - 1)2^{p-1}\frac{\hat{h}^p}{p^p} \\
&= m_{p,\Theta} \cdot \frac{h^p}{p^p}\left(\frac{2}{\Theta - 1}\right)^{p-1}.
\end{aligned}$$

Therefore, we can choose

$$c_{p,\Theta} = \frac{m_{p,\Theta}2^{p-1}}{p^p(\Theta - 1)^{p-1}} \quad \text{and} \quad C_{p,\Theta} = 2^{p-1}M_{p,\Theta}.$$

For the case of $p = 2$ (see also Theorem 8.4.1), since

$$|e|\sum_{v\in e} x_v^2 - \left(\sum_{v\in e} x_v\right)^2 = \sum_{\{v,v'\}\subset e} (x_v - x_{v'})^2,$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

it follows that we can choose $m_{2,\Theta} = M_{2,\Theta} = 1$, $C_{2,\Theta} = 2$, and $c_{2,\Theta} = 1/2(\Theta - 1)$. □

Remark 8.4.1. For $p = 2$, Theorem 8.4.3 enhances the main result in [214]. Moreover, for $p > 2$,

$$\sup_{x \text{ nonconstant}} \frac{\Theta^{p-1} \sum_{i=1}^{\Theta} |x_i|^p - |\sum_{i=1}^{\Theta} x_i|^p}{\sum_{i,j=1}^{\Theta} |x_i - x_j|^p} = +\infty,$$

and for $1 < p < 2$,

$$\inf_{x \text{ nonconstant}} \frac{\Theta^{p-1} \sum_{i=1}^{\Theta} |x_i|^p - |\sum_{i=1}^{\Theta} x_i|^p}{\sum_{i,j=1}^{\Theta} |x_i - x_j|^p} = 0.$$

Hence, in these cases, we only obtain one bound, namely

$$c_{p,\Theta} h^p \leq \Theta^{p-1} - \widetilde{\lambda}_{N-1}(\tilde{\Delta}_p) \text{ if } p > 2,$$

and

$$\Theta^{p-1} - \widetilde{\lambda}_{N-1}(\tilde{\Delta}_p) \leq C_{p,\Theta} h \text{ if } 1 \leq p < 2.$$

Similarly to Theorem 8.4.3, we can prove the following

Theorem 8.4.4. *Let Γ be a hypergraph such that $2 \leq |e| \leq \Theta$, for each edge e. Then, given $p \in (1, +\infty)$,*

$$c_{p,\Theta} h^p \leq \Theta^{p-1} - \widetilde{\lambda}_{N-1}(\tilde{\Delta}_p) \text{ when } p \geq 2,$$

$$\Theta^{p-1} - \widetilde{\lambda}_{N-1}(\tilde{\Delta}_p) \leq C_{p,\Theta} h \text{ when } 1 < p \leq 2.$$

where the universal positive constants $c_{p,\Theta}$ and $C_{p,\Theta}$ only depend on p and Θ. In the particular case of $p = 2$, the constants satisfy $C_{2,\Theta} \leq 2$ and $c_{2,\Theta} \geq 1/2(\Theta - 1)$.

We conclude this chapter by presenting Cheeger inequalities for an oriented hypergraph $\Gamma = (V, E, \varphi)$. Given two disjoint sets $A, B \subseteq V$, let

$$|E_+(A,B)| := \sum_{v \in A,\, v' \in B} \max\left\{ \sum_{e \in E} o_e(v,v'), 0 \right\},$$

$$|E_-(A)| := \sum_{\{v,v'\} \subset A} \max\left\{ -\sum_{e \in E} o_e(v,v'), 0 \right\},$$

$$|\partial(A \cup B)| := \sum_{v \in A \cup B,\, v' \notin A \cup B} \left| \sum_{e \in E} o_e(v,v') \right|,$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

and

$$\mathrm{vol}_w(A\cup B) \coloneqq \sum_{v\in A\cup B}\left(\sum_{e\ni v}(|e|-1) - \sum_{v'\in V}\left|\sum_{e\in E} o_e(v,v')\right|\right),$$

where

$$o_e(v,v') \coloneqq \begin{cases} 1, & \text{if } v,v'\in e, v \text{ and } v' \text{ are co-oriented in } e\\ -1, & \text{if } v,v'\in e, v \text{ and } v' \text{ are anti-oriented in } e\\ 0, & \text{otherwise.}\end{cases}$$

We define

$$h_B \coloneqq \min_{A\cap B=\emptyset\neq A\cup B} \frac{2(|E_+(A,B)| + |E_-(A)| + |E_-(B)|) + |\partial(A\cup B)| + \mathrm{vol}_w(A\cup B)}{\mathrm{vol}\,(A\cup B)}.$$

Before proving the next two theorems, we need the following preliminary results.

Proposition 8.4.1. *For any* $\mathbf{x}\neq\mathbf{0}$, *there exists* $s\geq 0$ *such that*

$$\frac{\sum_{vv'}\left(w^+_{vv'}|\hat{x}^s_v + \hat{x}^s_{v'}| + w^-_{vv'}|\hat{x}^s_v - \hat{x}^s_{v'}|\right)}{\sum_{v\in V}\deg(v)|\hat{x}^s_v|}$$
$$\leq \frac{p\sum_{vv'}\left(w^+_{vv'}|x_v + x_{v'}| + w^-_{vv'}|x_v - x_{v'}|\right)\left(|x_v|^p + |x_{v'}|^p\right)^{\frac{1}{p^*}}}{2^{1/p^*}\sum_v \deg(v)|x_v|^p},$$

where

$$\hat{x}^s \coloneqq \mathbf{1}_{\{v\in V:\, x_v > s^{1/p}\}} - \mathbf{1}_{\{v\in V:\, x_v < -s^{1/p}\}} \in \{-1,0,1\}^N.$$

Proof First, one can prove that

$$\int_0^\infty |\hat{x}^t_v|dt = |x_v|^p,$$

$$\int_0^\infty |\hat{x}^t_v + \hat{x}^t_{v'}|dt = \begin{cases} |x_v|^p + |x_{v'}|^p, & \text{if } x_v x_{v'}\geq 0,\\ ||x_v|^p - |x_{v'}|^p|, & \text{if } x_v x_{v'}\leq 0,\end{cases}$$

$$\int_0^\infty |\hat{x}^t_v - \hat{x}^t_{v'}|dt = \begin{cases} |x_v|^p + |x_{v'}|^p, & \text{if } x_v x_{v'}\leq 0,\\ ||x_v|^p - |x_{v'}|^p|, & \text{if } x_v x_{v'}\geq 0,\end{cases}$$

$$|a|^p + |b|^p \leq \begin{cases} |a+b|(|a|^p + |b|^p)^{1/p^*} & \text{if } ab\geq 0,\\ |a-b|(|a|^p + |b|^p)^{1/p^*} & \text{if } ab\leq 0,\end{cases}$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

and

$$||a|^p - |b|^p| \leq \begin{cases} \frac{p}{2^{1/p^*}}|a-b|(|a|^p + |b|^p)^{1/p^*} & \text{if } ab \geq 0, \\ \frac{p}{2^{1/p^*}}|a+b|(|a|^p + |b|^p)^{1/p^*} & \text{if } ab \leq 0, \end{cases}$$

where the last inequality has been used in [5]. Since $p/2^{1/p^*} \geq 1$ for $p \geq 1$, we have that

$$\int_0^\infty |\hat{x}_v^t \pm \hat{x}_{v'}^t| dt \leq \frac{p}{2^{1/p^*}}|x_v \pm x_{v'}|(|x_v|^p + |x_{v'}|^p)^{1/p^*}.$$

Also, one can show that there exists $s \geq 0$ such that

$$\begin{aligned} &\frac{\sum_{vv'}\left(w_{vv'}^+|\hat{x}_v^s + \hat{x}_{v'}^s| + w_{vv'}^-|\hat{x}_v^s - \hat{x}_{v'}^s|\right)}{\sum_{v\in V}\deg(v)|\hat{x}_v^s|} \\ \leq\ &\frac{\int_0^\infty \sum_{vv'}\left(w_{vv'}^+|\hat{x}_v^t + \hat{x}_{v'}^t| + w_{vv'}^-|\hat{x}_v^t - \hat{x}_{v'}^t|\right)dt}{\int_0^\infty \sum_{v\in V}\deg(v)|\hat{x}_v^t|dt} \\ \leq\ &\frac{p\sum_{vv'}\left(w_{vv'}^+|x_v + x_{v'}| + w_{vv'}^-|x_v - x_{v'}|\right)\left(|x_v|^p + |x_{v'}|^p\right)^{\frac{1}{p^*}}}{2^{1/p^*}\sum_{v\in V}\deg(v)|x_v|^p}. \end{aligned}$$

□

An immediate consequence is the following

Corollary 8.4.1. *For any* $\mathbf{x} \neq \mathbf{0}$*, there exists* $t \geq 0$ *such that*

$$\begin{aligned} &\frac{\sum_{vv'} w_{vv'}|\hat{x}_v^t - s_{vv'}\hat{x}_{v'}^t| + \sum_{v\in V} w_v|\hat{x}_v^t|}{\sum_{v\in V}\deg(v)|\hat{x}_v^t|} \\ &\quad\leq \frac{\sum_{vv'} w_{vv'}|x_v - s_{vv'}x_{v'}|(|x_v| + |x_{v'}|) + \sum_v w_v x_v^2}{\sum_v \deg(v)x_v^2}, \end{aligned}$$

where

$$\hat{x}^t := \mathbf{1}_{V_+^t(\mathbf{x})} - \mathbf{1}_{V_-^t(\mathbf{x})}$$

and $V_\pm^t(\mathbf{x}) := \{v \in V : \pm x_v > t\}$.

Theorem 8.4.5. *Let* $\Gamma = (V, E, \varphi)$ *be the connected,* Θ*-uniform oriented hypergraph. Then,*

$$\frac{h_B^2}{2(\Theta - 1)} \leq \Theta - \lambda_N(\tilde{\Delta}_2) \leq 2h_B\ .$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

Proof Let G be a signed graph that has vertex set V, edge signature

$$s_{vv'} \coloneqq \operatorname{sign}\left(\sum_{e\in E} \varphi(e,v)\varphi(e,v')\right),$$

and edge weight

$$\begin{aligned} w_{vv'} &\coloneqq \left|\sum_{e\in E} \varphi(e,v)\varphi(e,v')\right| \\ &= \left|\#\{e\colon \varphi(e,v)\varphi(e,v') = 1\} - \#\{e\colon \varphi(e,v)\varphi(e,v') = -1\}\right|. \end{aligned}$$

Then, it is easy to see that h_B also represents the Cheeger constant of G, and

$$\Theta \sum_{v\in V} \deg(v)x_v^2 - \sum_{e\in E}\left(\sum_{v\in V}\varphi(e,v)x_v\right)^2 = \sum_{vv'} w_{vv'}\,(x_v - s_{vv'}x_{v'})^2 + \sum_{v\in V} w_v x_v^2,$$

where

$$\deg(v) = \sum_{e\in E} \varphi(e,v)^2,$$

and

$$w_v \coloneqq (\Theta - 1)\deg(v) - \sum_{v'} w_{vv'}.$$

Therefore,

$$\begin{aligned} \Theta - \lambda_N &= \Theta - \max_{\mathbf{x}\neq 0} \frac{\sum_{e\in E}(\sum_{v\in V}\varphi(e,v)x_v)^2}{\sum_{v\in V}\deg(v)x_v^2} \\ &= \min_{\mathbf{x}\neq 0} \frac{\sum_{v\sim v'} w_{vv'}(x_v - s_{vv'}x_{v'})^2 + \sum_{v\in V} w_v x_v^2}{\sum_{v\in V}\deg(v)x_v^2} \\ &\le \min_{\mathbf{x}\in\{-1,0,1\}^N} \frac{\sum_{v\sim v'} w_{vv'}(x_v - s_{vv'}x_{v'})^2 + \sum_{v\in V} w_v x_v^2}{\sum_{v\in V}\deg(v)x_v^2} \\ &\le 2 \min_{\mathbf{x}\in\{-1,0,1\}^N} \frac{\sum_{v\sim v'} w_{vv'}|x_v - s_{vv'}x_{v'}| + \sum_{v\in V} w_v|x_v|}{\sum_{v\in V}\deg(v)|x_v|} \\ &= 2h_B. \end{aligned}$$

Moreover,

$$\begin{aligned} h_B &= \min_{\mathbf{x}\in\{-1,0,1\}^N} \frac{\sum_{v\sim v'} w_{vv'}|x_v - s_{vv'}x_{v'}| + \sum_{v\in V} w_v|x_v|}{\sum_{v\in V}\deg(v)|x_v|} \\ &\le \min_{\mathbf{x}\neq\mathbf{0}} \frac{\sum_{v\sim v'} w_{vv'}|x_v - s_{vv'}x_{v'}|(|x_v| + |x_{v'}|) + \sum_v w_v x_v^2}{\sum_v \deg(v)x_v^2} \qquad (8.4.4) \\ &\le \min_{\mathbf{x}\neq\mathbf{0}} \frac{\sqrt{\sum_{v\sim v'} w_{vv'}|x_v - s_{vv'}x_{v'}|^2 + \sum_v w_v x_v^2}\sqrt{\sum_{v\sim v'} w_{vv'}(|x_v| + |x_{v'}|)^2 + \sum_v w_v x_v^2}}{\sum_v \deg(v)x_v^2} \end{aligned}$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

$$\leq \min_{\mathbf{x}\neq\mathbf{0}} \sqrt{C\frac{\sum_{v\sim v'} w_{vv'}|x_v - s_{vv'}x_{v'}|^2 + \sum_v w_v x_v^2}{\sum_v \deg(v)x_v^2}}$$
$$= \sqrt{C(\Theta - \lambda_N)},$$

where

$$C \coloneqq \max_{v\in V} \frac{2\sum_{v'} w_{vv'} + w_v}{\deg(v)} \leq 2(\Theta - 1),$$

and the inequality (8.4.4) follows from Corollary 8.4.1. The claim follows.

Finally, we now consider the Cheeger constant

$$h \coloneqq \min_{A\cap B=\emptyset\neq A\cup B} \frac{2(|E_+(A,B)| + |E_-(A)| + |E_-(B)|) + |\partial(A\cup B)|}{\mathrm{vol}\,(A\cup B)}.$$

Theorem 8.4.6. *Let $\Gamma = (V,E,\varphi)$ be a connected, Θ-uniform oriented hypergraph. For any $p \in (1,+\infty)$, there exist universal constants $C_{p,\Theta} \geq c_{p,\Theta} > 0$ such that*

$$c_{p,\Theta}h^p \leq \Theta^{p-1} - \widetilde{\lambda}_{N-1}(\tilde{\Delta}_p) \text{ when } p \geq 2,$$

$$\Theta^{p-1} - \widetilde{\lambda}_{N-1}(\tilde{\Delta}_p) \leq C_{p,\Theta}h \text{ when } 1 < p \leq 2.$$

Proof The technique is a combination of the proofs of Theorems 8.4.3 and 8.4.5. We have that

$$\begin{aligned}
\Theta^{p-1} - \lambda_N(\Delta_p) &= \Theta^{p-1} - \max_{\mathbf{x}\neq 0} \frac{\sum_{e\in E} |\sum_{v\in V}\varphi(e,v)x_v|^p}{\sum_{v\in V}\deg(v)|x_v|^p} \\
&= \min_{\mathbf{x}\neq 0} \frac{\sum_{e\in E}(|e|^{p-1}\sum_{v\in e}|x_v|^p - |\sum_{v\in e}\varphi(e,v)x_v|^p)}{\sum_{v\in V}\deg(v)|x_v|^p} \\
&\leq M_{p,\Theta} \min_{\mathbf{x}\neq 0} \frac{\sum_{vv'}(w^+_{vv'}|x_v + x_{v'}|^p + w^-_{vv'}|x_v - x_{v'}|^p)}{\sum_{v\in V}\deg(v)|x_v|^p} \\
&\leq M_{p,\Theta} \min_{\mathbf{x}\in\{-1,0,1\}^N} \frac{\sum_{vv'}(w^+_{vv'}|x_v + x_{v'}|^p + w^-_{vv'}|x_v - x_{v'}|^p)}{\sum_{v\in V}\deg(v)|x_v|^p} \\
&\leq 2^{p-1}M_{p,\Theta} \min_{\mathbf{x}\in\{-1,0,1\}^N} \frac{\sum_{vv'}(w^+_{vv'}|x_v + x_{v'}| + w^-_{vv'}|x_v - x_{v'}|)}{\sum_{v\in V}\deg(v)|x_v|} \\
&= 2^{p-1}M_{p,\Theta}h.
\end{aligned}$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012

Moreover,

$$
\begin{aligned}
h &= \min_{\mathbf{x}\in\{-1,0,1\}^N} \frac{\sum_{vv'}(w^+_{vv'}|x_v + x_{v'}| + w^-_{vv'}|x_v - x_{v'}|)}{\sum_{v\in V}\deg(v)|x_v|} \\
&\le \frac{p}{2^{1/p^*}} \min_{\mathbf{x}\neq\mathbf{0}} \frac{\sum_{vv'}(w^+_{vv'}|x_v + x_{v'}| + w^-_{vv'}|x_v - x_{v'}|)(|x_v|^p + |x_{v'}|^p)^{1/p^*}}{\sum_v \deg(v)|x_v|^p} \\
&\le \frac{p}{2^{1/p^*}} \min_{\mathbf{x}\neq\mathbf{0}} \\
&\frac{\left(\sum_{vv'}(w^+_{vv'}|x_v + x_{v'}|^p + w^-_{vv'}|x_v - x_{v'}|^p)\right)^{\frac{1}{p}} \left(\sum_{vv'}(w^+_{vv'} + w^-_{vv'})(|x_v|^p + |x_{v'}|^p)\right)^{\frac{1}{p^*}}}{\sum_v \deg(v)|x_v|^p} \\
&\le \frac{Cp}{2^{1/p^*}} \min_{\mathbf{x}\neq\mathbf{0}} \frac{\left(\sum_{vv'}(w^+_{vv'}|x_v + x_{v'}|^p + w^-_{vv'}|x_v - x_{v'}|^p)\right)^{\frac{1}{p}} \left(\sum_v \deg(v)|x_v|^p\right)^{\frac{1}{p^*}}}{\sum_v \deg(v)|x_v|^p} \\
&\le C \min_{\mathbf{x}\neq\mathbf{0}} \left(\frac{\sum_{vv'}(w^+_{vv'}|x_v + x_{v'}|^p + w^-_{vv'}|x_v - x_{v'}|^p)}{\sum_v \deg(v)|x_v|^p}\right)^{\frac{1}{p}} \\
&\le C \min_{\mathbf{x}\neq\mathbf{0}} \left(\frac{1}{m_{p,\Theta}} \frac{\sum_{e\in E}(|e|^{p-1}\sum_{v\in e}|x_v|^p - |\sum_{v\in e}\varphi(e,v)x_v|^p)}{\sum_{v\in V}\deg(v)|x_v|^p}\right)^{\frac{1}{p}} \\
&= \frac{C}{m^{1/p}_{p,\Theta}} \left(\Theta^{p-1} - \max_{\mathbf{x}\neq 0} \frac{\sum_{e\in E}|\sum_{v\in V}\varphi(e,v)x_v|^p}{\sum_{v\in V}\deg(v)|x_v|^p}\right)^{\frac{1}{p}} \\
&= \tilde{C}\left(\Theta^{p-1} - \lambda_N(\Delta_p)\right)^{\frac{1}{p}},
\end{aligned}
$$

where the first inequality follows from Proposition 8.4.1. This proves the claim. □

We conclude by observing that, by Theorem 8.4.5 and by taking $p = 2$ in Theorem 8.4.6, we can also infer that

$$\frac{h^2}{2(\Theta - 1)} \le \frac{h_B^2}{2(\Theta - 1)} \le \Theta - \lambda_N(\Delta_2) \le 2h \le 2h_B.$$

This improves Theorem 1 in [214].

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 21:33:31, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009641821.012