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Existence of SRB-measures for some topologically hyperbolic diffeomorphisms

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Abstract. A diffeomorphism f of a compact manifold M is called *almost Axiom-A* if it is hyperbolic in a neighborhood of some compact f-invariant set Ω, except in some singular set of neutral points. We prove that if there exists some f-invariant set of hyperbolic points with positive *unstable*-Lebesgue measure and such that for every point in this set the stable and unstable leaves are 'long enough', then f admits either a probability Sinai–Ruelle–Bowen (SRB)-measure or a σ-finite SRB-measure.

1. *Introduction and statement of results*

Let us consider a smooth dynamical system (M, f), where M is a compact smooth Riemannian manifold (of dimension N) and f is a C^2 diffeomorphism acting on M. Let μ be an f-invariant ergodic probability measure on M. By the ergodic theorem, the set G_μ of generic points, i.e. the set of points x such that for every continuous function ϕ

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x) = \int \phi \, d\mu, \tag{1}$$

has full μ-measure. From the point of view of physics, this convergence can only actually be observed when this set G_μ has strictly positive measure for the measure of volume of the manifold, also called the Lebesgue measure on M and denoted by Leb_M.

Definition 1.1. We say that an f-invariant ergodic probability measure is a physical measure if and only if $\mathrm{Leb}_M(G_\mu) > 0$.

If f is Axiom-A, results about the existence of a physical measure are known: one can construct special Gibbs states which are the so-called Sinai–Ruelle–Bowen (SRB) measures, and which are special physical measures (see e.g. [**5**]). In the non-uniformly hyperbolic case, some results are known but there is no general theory, maybe because there are a lot of possibilities to make uniform hyperbolicity fail. However, several works

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give some conditions on a dynamical system (M, f) yielding the existence of a physical measure: in [**3**], it was proved that for the special case of Hénon's maps, many maps of this type admit a unique SRB measure. In [**7**], it was proved that every non-degenerate almost Anosov diffeomorphism on a surface admits a finite or σ-finite SRB-measure. In [**4**] or [**2**], it was proved that a partially hyperbolic diffeomorphism such that the central direction is mostly contracting or mostly expanding admits a SRB-measure.

In this paper we consider dynamical systems, in dimension ≥ 2, for which hyperbolicity comes from a point-wise behavior but fails on a globally f-invariant set S of indifferent points; this case is new in dimension higher than 3, and new problems arise. For instance, even the question of integrability of the hyperbolic splitting is not obvious, because the derivative has no uniform spectral gap near to the set of indifferent points; therefore, the classical method of the graph transform does not work. Hence, in this situation, the notion of SRB-measure is not obvious.

1.1. *Statement of results.* We first define the class of dynamical systems that we consider in the following.

Definition 1.2. Let f be in $\mathrm{Diff}^2(M)$. It is said to be almost Axiom-A if there exists an open set U which contains a non-empty f-invariant compact set $\Omega \subset U$ such that:

(i) for every $x \in U$, there is a df-invariant splitting (invariant where it makes sense) of the tangent space $T_xM = E^u(x) \oplus E^s(x)$ with $x \mapsto E^u(x)$ and $x \mapsto E^s(x)$ two Hölder continuous maps (with uniformly bounded Hölder constant);

(ii) there exist two continuous and non-negative functions $x \mapsto k^u(x)$ and $x \mapsto k^s(x)$ such that for all $x \in U$,

$$\forall v \in E^s(x) \quad \|df(x)v\|_{f(x)} \leq e^{-k^s(x)}\|v\|_x,$$
$$\forall v \in E^u(x) \quad \|df(x)v\|_{f(x)} \geq e^{k^u(x)}\|v\|_x,$$

where $\|\cdot\|_x$ denotes the Riemannian norm on T_xM;

(iii) the exceptional set, $S = \{x \in U,\ k^u(x) = k^s(x) = 0\}$, satisfies $f(S) = S$ and for every x in $U \setminus S$, $k^u(x)$ and $k^s(x)$ are positive real numbers.

Remark. In this definition, the set U can be considered as hyperbolic from a weak point of view. We probably do not really need hyperbolicity in U and we should obtain the same results without this assumption. However, we think it makes the proof easier and less technical than it would be without this assumption.

From now on, f will be almost Axiom-A; that is, f obeys the conditions of Definition 1.2. The sets S and Ω and the functions k^u and k^s are fixed as in the definition. The splitting $T_xM = E^u(x) \oplus E^s(x)$ is called the hyperbolic splitting.

Definition 1.3. Let $\lambda \in (0, +\infty)$. A point x in Ω is said to be λ-hyperbolic if:

(1) for all $v \in E^u(x) \setminus \{0\}$,

$$\limsup_{n \to +\infty} \frac{1}{n} \log \|df^{-n}(x)v\| < -\lambda;$$

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(2) for all $v \in E^s(x) \setminus \{0\}$,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|df^n(x)v\| < -\lambda.$$

An f-invariant set Λ_λ such that every point in Λ_λ is λ-hyperbolic is called a λ-hyperbolic set.

The notion of (non-uniform) hyperbolicity is not completely standard and several definitions exist in the literature. If hyperbolic currently means the existence of some splitting of the tangent space in one expanding direction and one contracting direction, the notion of 'expansion' is not clear. In our case, the definition of hyperbolicity is directly taken from the definition of Axiom-A, where expansion means contraction in negative time.

Definition 1.4. A point x in Ω is said to be a point of integration of the hyperbolic splitting if there exist $\varepsilon > 0$ and two C^1-disks $D^u_\varepsilon(x)$ and $D^s_\varepsilon(x)$ of size ε (and centered at x) such that for every y in $D^i_\varepsilon(x)$ $(i = u, s)$, $T_y D^i_\varepsilon(x) = E^i(y)$.

The set of points of integration is invariant by f. As usual, when we have two families of local unstable and stable manifolds we define $\mathcal{F}^u(x) = \bigcup_{n\geq 0} f^n D^u_{\varepsilon(-n)}(f^{-n}(x))$ and $\mathcal{F}^s(x) = \bigcup_{n\geq 0} f^{-n} D^s_{\varepsilon(n)}(f^n(x))$, where $\varepsilon(k)$ is the size of the disks associated with $f^k(x)$. They are the global unstable and stable manifolds.

As we said before, it is not obvious in our case and in dimension higher than 3 that there exist some points of integration of the hyperbolic splitting. Our first result is to prove their existence.

THEOREM A. *Let $\lambda > 0$. Any λ-hyperbolic point is a point of integration of the hyperbolic splitting.*

Remark. This result is well known when the hyperbolic splitting is dominated, because this condition yields the existence of a uniform spectral gap for the derivative df. A very close result is also well known in Pesin theory, but only on a set of full measure. Therefore, the precise topological characterization of the set of points of integration given by Pesin theory depends on the choice of the invariant measure. In our case the improvement consists of the fact that we prove integrability even in the presence of indifferent points for a set of points whose precise characterization does not depend on the ergodic properties of some invariant probability measure which would be given *a priori*.

Let x be some point of integration; the two manifolds $\mathcal{F}^u(x)$ and $\mathcal{F}^s(x)$ are also Riemannian manifolds. We denote by d^u and d^s their Riemannian metrics, and by Leb^u_x and Leb^s_x their Riemannian measures. If a measurable partition is subordinated to the unstable foliation $\mathcal{F}^u$ (see [**12**] and [**9**]), any f-invariant measure admits a unique system of conditional measures with respect to the given partition.

Definition 1.5. In this paper we refer to an invariant measure having absolutely continuous conditional measures on unstable leaves $\mathcal{F}^u(x)$ with respect to Leb^u_x as a SRB-measure.

Definition 1.6. Let two real numbers $\lambda > 0$ and $\varepsilon_0 > 0$ be fixed. A point x in Ω is said to be (ε_0, λ)-regular if the following conditions are satisfied:

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(i) x is λ-hyperbolic;
(ii) for $i = u, s$, $\mathcal{F}^i(x)$ contains a disk $D^i_{\varepsilon_0}(x)$ (centered at x) of size ε_0.
Let Λ be a f-invariant compact set in Ω. It is said to be (ε_0, λ)-regular if all points of Λ are (ε_0, λ)-regular points.

Markov partitions (see [**5**]) play an important role in the theory of uniformly hyperbolic diffeomorphisms. Unfortunately there is no general way to build some Markov partition in the non-uniformly hyperbolic case. In our case we are able to construct some Markov partition and, under an additional assumption, to produce SRB measures.

THEOREM B. *Let f be an almost Axiom-A diffeomorphism. Let Λ be a (ε_0, λ)-regular set. Then there exists a countable Markov partition of Λ. Moreover, if there exists some x_0 in Λ such that $\mathrm{Leb}^u_{x_0}(D^u_{\varepsilon_0}(x_0) \cap \Lambda) > 0$, then there exists a SRB measure for f which is finite or σ-finite.*

It is important to note that the hypothesis in Theorem B is very weak. If there exists some probability SRB-measure μ, then there exists some (ε_0, λ)-regular set Λ of full μ-measure such that for μ-almost every x in Λ, $\mathrm{Leb}^u_x(D^u_{\varepsilon_0}(x) \cap \Lambda) > 0$. However, a work due to Herman [**6**] proves that there exist some dynamical systems on the circle such that Lebesgue-almost-every point is 'hyperbolic' but there is no SRB-measure (even σ-finite).

The rest of this paper proceeds as follows: in §2 we prove Theorem A. The aim of §3 is to construct some special reduced dynamical system (R, g). This is a very technical part and certainly the key point in the proof of Theorem B. We first state and prove a shadowing lemma. Several shadowing lemmas already exist in the literature, but they introduce some invariant measure (which would be given *a priori*), or they hold for the so-called Pesin's set (the ergodic way). As far as we know, this is the first shadowing lemma for non-uniformly 'topologically' hyperbolic dynamical systems. Then, we use our shadowing lemma to construct several Markov covers of Λ. Finally, we produce the special reduced system. Section 4 is devoted to the rest of the proof of Theorem B: we prove the existence of some special measure for the reduced system and we extend it to the global system (M, f). This extended measure will be a SRB-measure.

The proofs are all based on the same key point: the estimates are all uniformly hyperbolic outside some fixed bad neighborhood $B(S, \varepsilon_1)$ of S. Moreover a λ-hyperbolic point cannot stay too long in the fixed neighborhood. The fact that f is almost Axiom-A implies that an incursion in $B(S, \varepsilon_1)$ cannot spoil the (uniformly) hyperbolic estimates of contractions or expansions too much. Obviously all the constants appearing are strongly correlated, and special care is taken in §3 in choosing them in the right order.

2. *Proof of Theorem A*

Let $\lambda > 0$ be fixed. For $\varepsilon_1 > 0$ we set $B(S, \varepsilon_1) \stackrel{\text{def}}{=} \{y \in M, d(S, y) < \varepsilon_1\}$. We pick some small $\varepsilon_1 > 0$: precise conditions are stated along the way. We first assume that ε_1 is small enough such that $B(S, \varepsilon_1) \subset U$. By continuity, we can choose ε_1 small enough such that, for every $x \in B(S, \varepsilon_1)$, $1 \leq \|df(x)_{|E^u(x)}\| < e^{\lambda/100}$ and $1 \leq \|df^{-1}(x)_{|E^s(x)}\| < e^{\lambda/100}$.

There exist some $\lambda^u > 0$ (namely $\min\{k^u(x) : x \in \Omega \setminus B(S, \varepsilon_1)\}$) and $\lambda^s > 0$ (namely $\min\{k^s(x) : x \in \Omega \setminus B(S, \varepsilon_1)\}$) such that for every $x \in \Omega \setminus B(S, \varepsilon_1)$:

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- for all $v \in E^u(x)$, $v \neq 0$,

$$\|df(x)v\|_{f(x)} > e^{\lambda^u}\|v\|_x,$$
$$\|df^{-1}(x)v\|_{f^{-1}(x)} < e^{-\lambda^u}\|v\|_x;$$

- for all $v \in E^s(x)$, $v \neq 0$,

$$\|df^{-1}(x)v\|_{f^{-1}(x)} > e^{\lambda^s}\|v\|_x,$$
$$\|df(x)v\|_{f(x)} < e^{-\lambda^s}\|v\|_x.$$

We denote the Euclidean norm on $\mathbb{R}^N$ by $|\cdot|$. By continuity we can always assume that the map $x \mapsto \dim E^i(x)$ are constant ($i = u, s$). From now on, we denote the space $\mathbb{R}^{\dim E^u} \times \{0\}^{\dim E^s}$ by $\mathbb{R}^u$. In the same way, $B^u(0,\rho)$, $B^s(0,\rho)$ and $\mathbb{R}^s$ denote the spaces $B^{\dim E^u}(0,\rho) \times \{0\}^{\dim E^s}$, $\{0\}^{\dim E^u} \times B^{\dim E^s}(0,\rho)$ and $\{0\}^{\dim E^u} \times \mathbb{R}^{\dim E^s}$.

PROPOSITION 2.1. *Let ε be small compared to λ, λ^u or λ^s. There exist constants $\rho_1 > 0$, $0 < K_1 < K_2$, a positive function $\bar\rho$ and a family of embeddings $\phi_x : B_x(0,\rho_1) \subset \mathbb{R}^N \to M$, such that we have the following.*

(1) *$\phi_x(0) = x$ and $d\phi_x(0)$ map, respectively, $\mathbb{R}^u$ and $\mathbb{R}^s$ onto $E^u(x)$ and $E^s(x)$.*

(2) *Set $\widehat{f}_x = \phi^{-1}_{f(x)} \circ f \circ \phi_x$ and $\widehat{f}_x^{-1} = \phi^{-1}_{f^{-1}(x)} \circ f^{-1} \circ \phi_x$; then:*

(a) *if x is in $\Omega \setminus B(S,\varepsilon_1)$, then:*

- *for all $v \in \mathbb{R}^u$, $v \neq 0$,*

$$|d\widehat{f}_x(0)v| > e^{\lambda^u}|v|,$$
$$|d\widehat{f}_x^{-1}(0)v| < e^{-\lambda^u}|v|,$$

- *for all $v \in \mathbb{R}^s$, $v \neq 0$,*

$$|d\widehat{f}_x^{-1}(0)v| > e^{\lambda^s}|v|,$$
$$|d\widehat{f}_x(0)v| < e^{-\lambda^s}|v|;$$

(b) *if x is in $B(S,\varepsilon_1)$, then:*

- *for all $v \in \mathbb{R}^u$, $v \neq 0$,*

$$|v| \le |d\widehat{f}_x(0)v| < e^{\lambda/100}|v|,$$
$$|v| \ge |d\widehat{f}_x^{-1}(0)v| > e^{-\lambda/100}|v|,$$

- *for all $v \in \mathbb{R}^s$, $v \neq 0$,*

$$|v| \le |d\widehat{f}_x^{-1}(0)v| < e^{\lambda/100}|v|,$$
$$|v| \ge |d\widehat{f}_x(0)v| > e^{-\lambda/100}|v|.$$

(3) *For every x in $B(S,\varepsilon_1)$, $0 < \bar\rho(x) \le \rho_1$, and for every x in $\Omega \setminus B(S,\varepsilon_1)$, $\bar\rho(x) = \rho_1$.*

(4) *On the ball $B_x(0,\bar\rho(x))$ we have $\mathrm{Lip}(\widehat{f}_x - d\widehat{f}_x(0)) < \varepsilon$ and $\mathrm{Lip}(\widehat{f}_x^{-1} - d\widehat{f}_x^{-1}(0)) < \varepsilon$.*

(5) *For every x and for every $z, z' \in B_x(0,\rho_1)$,*

$$K_1|z - z'| \le d(\phi_x(z), \phi_x(z')) \le K_2|z - z'|.$$

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This is a simple consequence of the fibered map exp. However, it is important for the rest of the paper to understand that the two constants K_1 and K_2 do not depend on ε_1. They result from the distortion due to the angle between the two sub-spaces, E^u and E^s, plus the injectivity radius. These quantities are uniformly bounded.

Moreover, Ω is a compact set in U and, thus, we can choose $\rho_1 > 0$ such that $B(\Omega, \rho_1) \subset U$. As the maps $x \mapsto E^u(x)$ and $x \mapsto E^s(x)$ are continuous, we can also assume that ρ_1 is small enough such that for every x in Ω and y in $\phi_x(B(0, \rho_1))$, the slope of $d\phi_x^{-1}(y)(E^u(y))$ in $\mathbb{R}^N = \mathbb{R}^u \oplus \mathbb{R}^s$ is smaller than $\frac{1}{2}$.

For the rest of the paper, we set $\Omega_0 \stackrel{\text{def}}{=} \Omega \setminus B(S, \varepsilon_1)$, $\Omega_1 = \Omega_0 \cap f(\Omega_0) \cap f^{-1}(\Omega_0)$ and $\Omega_2 = \Omega_0 \setminus \Omega_1$.

2.1. *First step: the graph transform theory.* The idea to prove Theorem A is to use the graph transform theory (see [**11**]). We first give some general results.

PROPOSITION 2.2. *Let E be some Banach space and $T : E \to E$ be some linear map such that there exists a T-invariant splitting $E = E_1 \oplus E_2$. We set $T_i \stackrel{\text{def}}{=} T_{|E_i}$ and we assume that the norm on E is adapted to the splitting, i.e. $\|\cdot\|_E = \max(\|\cdot\|_{E_1}, \|\cdot\|_{E_2})$. We also assume that there exist two numbers $\lambda_2 < 0 < \lambda_1$ such that:*

- *for every v in E_1, $\|T_1^{-1}v\|_{E_1} \leq e^{-\lambda_1}\|v\|_{E_1}$;*
- *for every v in E_2, $\|T_2 v\|_{E_2} \leq e^{\lambda_2}\|v\|_{E_2}$.*

Let ρ be a positive number, let ε be a positive number very small compared to λ_1 and $-\lambda_2$ and let F be some C^1-map from E to E such that:

(i) $F(0) = 0$;

(ii) *on the ball $B(0, \rho)$, $\mathrm{Lip}(F - T) < \varepsilon$ and $\mathrm{Lip}(F^{-1} - T^{-1}) < \varepsilon$.*

Then we have the following.

(1) *The image by F of the graph of any map $g : B_1(0, \rho) \to B_2(0, \rho)$ satisfying $g(0) = 0$ is a graph of some map $\Gamma(g) : B_1(0, \rho e^{\lambda_1 - 2\varepsilon}) \to B_2(0, \rho e^{-\lambda_2 + 2\varepsilon})$.*

(2) *This induces some operator Γ on the set Lip_1 of 1-Lipschitz-continuous maps $g : B_1(0, \rho) \to B_2(0, \rho)$ satisfying $g(0) = 0$.*

(3) *On Lip_1 (with the standard norm on the set of Lipschitz-continuous maps) Γ is a contraction. Thus it admits some unique fixed point.*

For the proof of Theorem A, it is important to keep in mind the two key points of the proof of Proposition 2.2. On the one hand, the fact that Γ is a contraction on Lip_1 is essentially due to the spectral gap of dF. This is obtained by the properties of the T_i and the Lipschitz proximity of F and T. On the other hand, the fact that the image by F of any graph (from $B_1(0, \rho)$ to $B_2(0, \rho)$) extends beyond the boundary of the ball $B(0, \rho)$ is essentially due to expansion on E_1.

2.2. *Second step: estimations in our case.* We want to use Proposition 2.2 in the fibered case of $\widehat{f_x}$. If x is in $\Omega_0 \cap f^{-1}(\Omega_0)$, the spectral gap of $d\widehat{f_x}(0)$ is uniformly bounded from below in $B_x(0, \rho_1)$, and so we can apply the proposition with $d\widehat{f_x}(0)$, $\widehat{f_x}$ and $\rho = \rho_1$. In $B(S, \varepsilon_1)$, the value of $\bar{\rho}(x)$ has to decrease to zero when x tends to S: the spectral gap of df tends to zero as x tends to S because $k^u(x) + k^s(x)$ tends to zero. The idea is to

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apply Proposition 2.2 for $d\widehat{f}_x^n(0)$ and $\widehat{f}_x^n$ for some good n. First, we have to check that the hypothesis of the proposition hold.

PROPOSITION 2.3. *Let $x \in \Omega_0$ and $n \geq 2$ such that:*
(1) $f^n(x) \in \Omega_0$;
(2) *for all $0 < k < n$, $f^k(x) \in B(S, \varepsilon_1)$.*
Then, there exists some constant $C > 0$ such that for every $0 < r \leq 1$, Proposition 2.2 holds for $F = \widehat{f}_x^n$, $T = dF(0)$ and $\rho = Cre^{-9n\lambda/200}$.

Proof. Let us fix some r. We consider that the norm on $\mathbb{R}^N = \mathbb{R}^u \oplus \mathbb{R}^s$ is the adapted norm (with respect to the splitting); this norm is equivalent to the Euclidean norm.

We first control what happens in the bad area $B(S, \varepsilon_1)$ (for $f^k(x)$, with $1 \leq k \leq n-1$). For y in $B_{f(x)}(0, \rho_1)$ we have

$$|(d\widehat{f}^k_{f(x)}(0) - d\widehat{f}^k_{f(x)}(y))v| \leq \|d^2\widehat{f}\|\,|\widehat{f}^{k-1}_{f(x)}(y)|\,|d\widehat{f}^k_{f(x)}(y)v| + \|d\widehat{f}_{f(x)}(0)\|\,|(d\widehat{f}^{k-1}_{f(x)}(0) - d\widehat{f}^{k-1}_{f(x)}(y))v| \quad (2)$$

as soon as it makes sense to define $\widehat{f}^k_{f(x)}(y)$.

Let us assume that for every y in $B_{f(x)}(0, l)$ (with $l \leq \rho_1$) and for every $0 \leq k \leq p < n-1$ we have:
(i) $|\widehat{f}^k_{f(x)}(y)| \leq le^{(3/2)k(\lambda/100)} \leq \rho_1$;
(ii) $\|d\widehat{f}^k_{f(x)}(y)\| \leq e^{(3/2)k(\lambda/100)}$;
and let us pick some y in $B_{f(x)}(0, l)$. Then we can define $\widehat{f}^{p+1}_{f(x)}(y)$ and (2) gives

$$\|d\widehat{f}^{p+1}_{f(x)}(0) - d\widehat{f}^{p+1}_{f(x)}(y)\| \leq \|d^2\widehat{f}\|e^{3p(\lambda/100)}l + e^{(\lambda/100)}\|d\widehat{f}^p_{f(x)}(0) - d\widehat{f}^p_{f(x)}(y)\|. \quad (3)$$

As this formula also holds for every $k \leq p$, we obtain by induction

$$\|d\widehat{f}^{p+1}_{f(x)}(0) - d\widehat{f}^{p+1}_{f(x)}(y)\| \leq \|d^2\widehat{f}\|l\frac{e^{3p(\lambda/100)}}{1 - e^{-2(\lambda/100)}}.$$

We now have to compute how l must be, to ensure that conditions (i) and (ii) above still hold for $p+1$.

We first see that

$$\|d\widehat{f}^{p+1}_{f(x)}(y)\| \leq \|d\widehat{f}^{p+1}_{f(x)}(y) - d\widehat{f}^{p+1}_{f(x)}(0)\| + \|d\widehat{f}^{p+1}_{f(x)}(0)\|,$$

and so, in view of satisfying (ii), it is sufficient to have

$$\|d^2\widehat{f}\|l\frac{e^{3p(\lambda/100)}}{1 - e^{-2(\lambda/100)}} + e^{(p+1)(\lambda/100)} \leq e^{(3/2)(p+1)(\lambda/100)},$$

which is true for

$$l \leq (e^{\lambda/100} - 1)e^{\lambda/100}\frac{1 - e^{-2(\lambda/100)}}{\|d^2\widehat{f}\|}e^{-2p(\lambda/100)}.$$

Let us set

$$C_1 \stackrel{\text{def}}{=} (e^{\lambda/100} - 1)e^{\lambda/100}\frac{1 - e^{-2(\lambda/100)}}{\max(\|d^2\widehat{f}\|, \|d^2\widehat{f}^{-1}\|)},$$

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and pick some $l = \rho_1 l' C_1 e^{-2p(\lambda/100)}$, with $l' \leq 1$. Then (ii) holds for every point in the convex ball $B_{f(x)}(0, l)$ with $k = p + 1$. Hence, we have for every y in this ball

$$|\widehat{f}^k_{f(x)}(y)| = \left| \int_0^1 d\widehat{f}^k_{f(x)}(ty)y\,dt \right| \leq e^{(3/2)p(\lambda/100)} l$$

and (i) holds for $k = p + 1$.

Therefore, if $C \stackrel{\text{def}}{=} \rho_1 C_2 C_1 e^{-\lambda^u - 2\varepsilon}$ (with $C_2 \leq 1$), for every y in $B_x(0, Cl''re^{-3n(\lambda/100)})$ (with $l'' \leq 1$), we can define $\widehat{f}^n_x(y)$, and (3) gives

$$\|d\widehat{f}^n_x(0) - d\widehat{f}^n_x(y)\| \leq C_2 l'' r \rho_1.$$

This formula yields

$$\mathrm{Lip}(\widehat{f}^n_x - d\widehat{f}^n_x(0)) \leq C_2 l'' r \rho_1. \tag{4}$$

Moreover, the previous discussion proves that $\widehat{f}^n_x(y)$ belongs to $B_{f^n(x)}(0, C_2 l'' r \rho_1 e^{-(3/2)n(\lambda/100)})$. On the other hand, we have $|d\widehat{f}^n_x(0)v| \geq e^{\lambda^u}|v|$ for every v in $\mathbb{R}^u$ and $|d\widehat{f}^n_x(0)v| \leq e^{-\lambda^s}|v|$ for every v in $\mathbb{R}^s$. For the same reasons we have $|d\widehat{f}^{-n}_{f^n(x)}(0)v| \geq e^{\lambda^s}|v|$ for every v in $\mathbb{R}^s$ and $|d\widehat{f}^{-n}_{f^n(x)}(0)v| \leq e^{-\lambda^u}|v|$ for every v in $\mathbb{R}^u$. Hence, the previous discussion gives some formula which is equivalent to (4), but for $\widehat{f}^{-n}_{f^n(x)}$ as soon as $l'' \leq e^{-(3/2)n(\lambda/100)}$:

$$\mathrm{Lip}(\widehat{f}^{-n}_{f^n(x)} - d\widehat{f}^{-n}_{f^n(x)}(0)) \leq C_2 r \rho_1. \tag{5}$$

Therefore, (4) and (5) prove that for $C_2 \leq \varepsilon$ we can apply Proposition 2.2 with $F = \widehat{f}^n_x$, $T = d\widehat{f}^n_x(0)$ and $\rho = Cre^{-9n(\lambda/200)}$, with $C = C_3 C_2 C_1$, where C_3 is a constant that is introduced because the norm on $\mathbb{R}^N$ is not the adapted norm. □

We remark that Proposition 2.2 also holds with $F = \widehat{f}^{-n}_{f^n(x)}$, $T = d\widehat{f}^{-n}_{f^n(x)}(0)$ and $\rho = Cre^{-(9/2)n(\lambda/100)}$.

Definition 2.4. Let $x \in \Omega_0$.

- If $f(x)$ is in $B(S, \varepsilon_1)$ we call the forward length of stay of x the integer (in $\overline{\mathbb{N}}$)

$$n^+(x) \stackrel{\text{def}}{=} \sup\{n\colon \forall 0 < k < n,\ f^k(x) \in B(S, \varepsilon_1)\}.$$

- If $f^{-1}(x)$ is in $B(S, \varepsilon_1)$, we call the backward length of stay of x the value

$$n^-(x) \stackrel{\text{def}}{=} \sup\{n\colon \forall 0 < k < n,\ f^{-k}(x) \in B(S, \varepsilon_1)\}.$$

Let $x \in \Omega_0$.

- If $f(x)$ is in $B(S, \varepsilon_1)$ and $n \stackrel{\text{def}}{=} n^+(x)(< +\infty)$, then we set $l^{\mathrm{f}}(x) \stackrel{\text{def}}{=} Ce^{-9n\lambda/200}$,
- If $f^{-1}(x)$ is in $B(S, \varepsilon_1)$ and $n \stackrel{\text{def}}{=} n^-(x)(< +\infty)$, then we set $l^{\mathrm{b}}(x) \stackrel{\text{def}}{=} Ce^{-9n\lambda/200}$.

For the rest of the proof, if x is in $\Omega_0 \cap f^{-1}(\Omega_0)$, we call the *one-step graph transform* the graph transform due to Proposition 2.2 with $F = \widehat{f}_x$, $T = d\widehat{f}_x(0)$ and $\rho = \rho_1$. If x belongs to $\Omega_0 \cap f^{-1}(B(S, \varepsilon_1))$, the graph transform that results from Proposition 2.3 will be referred to as the $n^+(x)$*-steps graph transform*. Both are denoted by Γ_x.

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2.3. *Third step: end of the proof.* Let x in Ω_0 be some λ-hyperbolic point. We are going to construct a piece of unstable leaf as some set $\phi_x(\mathrm{graph}(g_x))$, where g_x would be some special map from $B_x^u(0, l(x))$ to $B_x^s(0, l(x))$ satisfying $g_x(0) = 0$ (for some $l(x) \leq \rho_1$). For that purpose we use the graph transform along the backward orbit of x.

For any ξ in Ω_0, we set $\widehat{0}_\xi$ the null-map from $B_\xi^u(0, \rho_1)$ to $B_\xi^s(0, \rho_1)$. We also set $\xi_n \stackrel{\text{def}}{=} f^{-n}(\xi)$.

First we check that every λ-hyperbolic point must return infinitely many times in Ω_0 (in the future and in the past).

LEMMA 2.5. *Let ξ in Ω_0 be some λ-hyperbolic point such that $f(\xi) \notin \Omega_0$ (respectively $f^{-1}(\xi) \notin \Omega_0$). Then $n^+(\xi) < +\infty$ (respectively $n^-(\xi) < +\infty$).*

Proof. Let us assume that $f(\xi) \notin \Omega_0$ and $n^+(\xi) = +\infty$, i.e. for all $n \geq 1$, $f^n(\xi) \in B(S, \varepsilon_1)$. This yields

$$\|df(f^n(\xi))v\| \geq e^{-n\lambda/100}\|v\| \quad \forall n \geq 1,\ \forall v \in E^s(f^n(\xi)). \tag{6}$$

Hence, we get

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|df^n_{|E^s}(\xi)\| \geq -\frac{\lambda}{100},$$

which contradicts the fact that ξ is λ-hyperbolic.

The proof is of the same kind for the other case. □

Therefore, there exist two sequences $0 \leq q_0 < p_0 \leq q_1 < p_1 \leq \cdots$ of integers such that:

- for $0 \leq k \leq q_0$, $x_k \in \Omega_0$;
- for every $i \geq 0$, for $q_i < k < p_i$, $x_k \in B(S, \varepsilon_1)$;
- for every $i > 1$, for every $p_{i-1} \leq k \leq q_i$, $x_k \in \Omega_0$.

We set $y_i \stackrel{\text{def}}{=} x_{q_i} = f^{-q_i}(x)$, $z_i \stackrel{\text{def}}{=} x_{p_i} = f^{-p_i}(x)$ and $m_i \stackrel{\text{def}}{=} p_i - q_i$.

Let n be any integer such that x_n belongs to Ω_0. Then, we define Γ_x^n as the composition of the graph transforms along the piece of orbit $x_n, x_{n-1}, \ldots, x_1$, where we take the one-step graph transform Γ_{x_k} if x_k and x_{k-1} are in Ω_0 and the m_i-steps graph transform if x_k is one of the z_i.

The goal is to prove that the sequence of maps $\Gamma_x^n(\widehat{0}_{x_n})$ converges to some map. This is well-known for uniformly hyperbolic dynamical systems, but here the critical set S influences the graph transform: by construction, Γ_x^n is a contraction (of ratio smaller than $e^{-(n-\sum_{i,p_i\leq n}(m_i-1))(\lambda^u+\lambda^s-4\varepsilon)}$), but it is not clear that the length of the graph associated to $\Gamma_x^n(\widehat{0}_{x_n})$ is uniformly (in n) bounded away from zero. Now we explain how S influences the graph transform and prove that the lengths of the graphs associated to the $\Gamma_x^n(\widehat{0}_{x_n})$ are uniformly bounded away from zero.

Let n be any integer such that x_n belongs to Ω_0. There exists some integer k such that $p_k \leq n \leq q_{k+1}$.

For $p_k + 1 \leq i \leq n$ we just apply the one-step graph transform and so we get some graph in $B_{z_k}(0, \rho_1)$. At this moment we apply the m_k-steps graph transform to the graph restricted to the ball $B_{z_k}(0, l^f(z_k))$, and we get some graph in $B_{y_k}(0, l^p(y_k))$. We call this

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phenomenon a 'truncation'. For $p_{k-1} \leq i \leq q_k$, we can again apply the one-step graph transform, but to the small piece of graph (with length $2l^p(y_k)$).

However, the length increases along this piece of orbit (as long as it is smaller than $2\rho_1$) because, at each (one-)step, we can take the whole of the image-graph. For $i = p_{k-1}$ three cases may occur.

(i) The length of the graph is $2\rho_1$ and so there is a new truncation. Thus there is a 'past-stabilization' (a notion due to S. Newhouse).

(ii) The length of the graph is strictly smaller than $2\rho_1$, but is bigger than $2l^f(z_{k-1})$. Again, there is a new truncation and again there is a 'past-stabilization'.

(iii) The length of the graph, $2l$, is strictly smaller than $2l^f(z_{k-1})$. We apply the m_{k-1}-steps graph transform with $\rho = l$, and we obtain some graph in $B_{y_{k-1}}(0, l)$. From x, we can see the truncation due to Γ_{z_k}.

And so on, along the piece of orbit $y_{k-1}, \ldots, x_{-1}$.

Hence, the length of the graph $\Gamma_x^n(\widehat{0}_{x_n})$ is (at least) equal to $2l^f(z_j)$, where j is such that p_j is the smallest integer where a truncation occurs. We set $i(n) \stackrel{\text{def}}{=} j$ and we want now to give a more precise estimate for this length.

LEMMA 2.6. *For every λ-hyperbolic point ξ and for every v in $E^u(\xi)$,*

$$\lim_{j \to +\infty} e^{9m_j\lambda/200} \|df^{-q_j}(\xi)v\| = 0.$$

Proof. There exists $n = n(\varepsilon_1, \xi)$ such that for all $p \geq n$, $\|df^{-p}_{|E^u}(\xi)\| < e^{-p(99\lambda/100)}$.

Pick j such that $q_j \geq n$. Then,

$$\|df^{-(q_j+m_j)}_{|E^u}(\xi)\| < e^{-(q_j+m_j)(99\lambda/100)}. \tag{7}$$

Moreover, by definition of ε_1 we have

$$\|df^{m_j}(f^{-q_j-m_j}(\xi))v\| \leq e^{m_j\lambda/100}\|v\| \quad \forall v \in E^u(f^{-(m_j+q_j)}(\xi)). \tag{8}$$

Hence, inequalities (7) and (8) give for every v in $E^u(\xi)$

$$\|df^{-q_j}(\xi)v\| \leq e^{-m_j(98\lambda/100)}e^{-q_j(99\lambda/100)}\|v\|. \qquad \square$$

With the previous notation, the length l_n of the graph associated to $\Gamma_x^n(\widehat{0}_{x_n})$ is smaller than $2\rho_1$ but at least

$$2l^{\mathrm{f}}(z_{i(n)})\frac{1}{\|df^{-q_{i(n)}}_{|E^u}(x)\|}$$

(if this quantity is smaller than $2\rho_1$). Lemma 2.6 proves that there exists some integer p such that if $i(n) \geq p$, then

$$l^{\mathrm{f}}(z_{i(n)})\frac{1}{\|df^{-q_{i(n)}}_{|E^u}(x)\|} > \rho_1.$$

Therefore, x only sees the truncations due to p_j with $j \leq p$, which proves that the sequences of lengths (l_n) are bounded away from zero. The family of maps $\Gamma_x^n(\widehat{0}_{x_n})$

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converges to some map $g_x : B^u_x(0, l(x)) \to B^s_x(0, l(x))$, with $g_x(0) = 0$, and where $l(x)$ is such that for every n, $0 < l(x) \leq l_n$.

For the rest of the paper we set $\mathcal{F}^u_{\text{loc}}(x) \stackrel{\text{def}}{=} \phi_x(\text{graph}(g_x))$.

Lemma 2.5 also proves that the backward orbit of x returns infinitely often in Ω_0. Therefore, if $f^{-k}(x)$ belongs to $B(S, \varepsilon_1)$ we set

$$\mathcal{F}^u_{\text{loc}}(f^{-k}(x)) \stackrel{\text{def}}{=} f^n(\mathcal{F}^u_{\text{loc}}(f^{-(n+k)}(x))),$$

where n is the smallest positive integer such that $f^{-(n+k)}(x)$ belongs to Ω_0. Then, we set

$$\mathcal{F}^u(x) = \bigcup_{n,\ f^{-n}(x)\in\Omega_0} f^n(\mathcal{F}^u_{\text{loc}}(f^{-n}(x))).$$

The uniqueness of the map g_x and its construction prove that $\mathcal{F}^u(x)$ is an immersed manifold.

To complete the proof of Theorem A, we must check that the $\mathcal{C}^1$-disks that have been constructed are tangent to the correct spaces. Let y be in $\mathcal{F}^u_{\text{loc}}(x)$. By the construction of $\mathcal{F}^u_{\text{loc}}(x)$ we have for every n such that $f^{-n}(x) \in \Omega_0$, $f^{-n}(y) \in \mathcal{F}^u_{\text{loc}}(f^{-n}(x))$. For such an integer n, we pick some map $g_{n,y} : B^u(0, \rho_1) \to B^s(0, \rho_1)$ such that $f^{-n}(y) \in \phi_{f^{-n}(x)}(\text{graph}(g_{n,y}))$ and $T_{f^{-n}(y)}\phi_{f^{-n}(x)}(\text{graph}(g_{n,y})) = E^u(f^{-n}(y))$. As the map g_x is obtained as some unique fixed-point for the graph transform, the sequence $(\Gamma^n_x(g_{n,y}))$ converges to g_x. By the df-invariance of E^u (until the orbit of y leaves U) we must have $T_y\mathcal{F}^u(x) = E^u(y)$.

We can also do the same construction with f^{-1} to obtain some immersed manifold $\mathcal{F}^s(x)$. Then, $x \in \mathcal{F}^u(x) \cap \mathcal{F}^s(x)$.

However, it can be important to have an estimate for the length of $\mathcal{F}^u_{\text{loc}}(x)$ or $\mathcal{F}^u(x)$. Actually, we are not able to give a lower bound for such estimates.

3. *Shadowing lemma and Markov rectangles*

Let Λ be some fixed (ε_0, λ)-regular set satisfying the hypothesis of Theorem B. The goal of this section is to construct some Markov rectangle to use Young's method (see, for instance, [**8**]). The principal steps of this section are the following. In §3.1 we define the pseudo-orbits and we state (and prove) a shadowing lemma. In §3.2 we prove that the dynamical system (Λ, f) is conjugate to some symbolic dynamical system (Σ_0, σ). In §3.3 we define two generations of rectangles. In §3.4 we fix the constants and construct a third generation of rectangles. In §3.5 we define the special reduced system (R, g), where R is some rectangle satisfying the Markov property and g the return map in R.

We first assume that ε_1 has been chosen so small that for every x in Ω_2, $1 \leq \|df(x)_{|E^u(x)}\| < e^{2(\lambda/3)}$ and $1 \leq \|df^{-1}(x)_{|E^s(x)}\| < e^{2(\lambda/3)}$.

3.1. *Shadowing lemma.*

For convenience, we briefly recall how Bowen constructed a Markov partition in [**5**] (for Axiom-A).

A pseudo-orbit is a sequence of points (x_n) such that for every n, $f(x_n)$ is 'close enough' to x_{n+1}. Therefore, every sequence $(y_{n+1,k})_k$ of points on the local piece of unstable manifold $W^u_\varepsilon(x_{n+1})$ gives a new sequence $(z_{n+1,k})_k$ on the local piece of unstable

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manifold $W^u_\varepsilon(f(x_n))$ by sliding along the stable leaves. By contraction, this gives a sequence $(y_{n,k})_k$ of points on the local piece of unstable manifold $W^u_\varepsilon(x_n)$.

One of the key points in this proof is to control the distortion due to this stable holonomy: with the notations from above, it is untrue that the 'unstable distance' along the unstable leaves, $d^u(f(x_n), z_{n+1,k})$ is exactly $d^u(f(x_n), x_{n+1}) + d^u(x_{n+1}, y_{n+1,k})$. However, we have

$$d^u(f(x_n), z_{n+1,k}) \leq \kappa\, d(x_n, x_{n+1}) + R\, d^u(x_{n+1}, y_{n+1,k}), \tag{9}$$

where κ and R are some distortion constants. Therefore, to be able to use the contraction (in negative time) along the unstable leaves, Bowen considers pseudo-orbits for some f^k, where k is such that $Re^{-k\lambda^u} < 1$ (see [**5**, pp. 74–75]).

3.1.1. *Control of the distortion in our case.* Here we cannot control the distortion of the system of local coordinates by the dynamic because we do not have any control on how long a point stays in Ω_0. Hence we must compute this distortion and control it at each step.

Moreover we want to use the family of charts $(\phi_x, B_x(0, \rho_1))$, which introduce another distortion (that appears in κ and R in (9)). This distortion is not really correlated to the dynamics, firstly because it occurs uniformly and secondly because its effects do not add up along the orbits. Hence, for convenience, *we will assume that these charts introduce no distortion*, namely $K_1 = K_2 = 1$.

For x in Λ we set $D^i_\rho(x)$ $(i = u, s)$ the disk in $\mathcal{F}^i(x)$ of radius ρ (with $\rho \leq \varepsilon_0$). We also set $D^i_\rho(y)$ for every y in $\mathcal{F}^i(x)$ when it makes sense. Let $0 < \alpha < \frac{1}{8}$. For x in Ω_0 we define the unstable and stable cones of angle α as

$$C^u_\alpha(x) = \left\{ v = v^u + v^s \in \mathbb{R}^u \oplus \mathbb{R}^s \colon \frac{|v^u|}{|v^s|} \geq \frac{1}{\alpha} \right\},$$
$$C^s_\alpha(x) = \left\{ v = v^u + v^s \in \mathbb{R}^u \oplus \mathbb{R}^s \colon \frac{|v^u|}{|v^s|} \leq \alpha \right\}.$$

By the continuity of E^u and E^s, if ρ is small enough, we have for every $x \in \Omega_0 \cap \Lambda$ and for every $y \in B(x, 10\rho)$

$$d\phi_x^{-1}(y)E^s(y) \subset C^s_\alpha(x),$$
$$d\phi_x^{-1}(y)E^u(y) \subset C^u_\alpha(x).$$

Therefore, for sufficiently small ρ (namely $\rho < \varepsilon_0/100$ and $\rho < \rho_1/100$), for every x in Ω_0 and for every y and z in $B(x, \rho)$ such that $D^u_{2\rho}(y)$ and $D^s_{2\rho}(z)$ exist, we can define

$$[z, y] \stackrel{\text{def}}{=} D^s_{2\rho}(z) \cap D^u_{2\rho}(y),$$

and we get a system of local coordinates. However, this system introduces some distortion, because the holonomies (stable and unstable) are not isometries. Thus we have formulae like (9): for x in Λ, the following hold:

$$d^u(x, [z, x]) \leq r_3\, d(x, y) + r_1\, d^u(y, [z, y]), \tag{10}$$
$$d^s(x, [x, y]) \leq r_4\, d(x, z) + r_2\, d^s(z, [z, y]), \tag{11}$$
$$d^u(x, [z, x]) \leq r_3\, d(x, z), \tag{12}$$
$$d^s(x, [x, y]) \leq r_4\, d(x, y). \tag{13}$$

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By the definition of ρ the disks $D^u_{2\rho}(y)$ and $D^s_{2\rho}(z)$ must stay in some cones, and a standard computation in $\mathbb{R}^N$ proves that the distortion constants r_i $(i = 1, 2, 3, 4)$ depend continuously on $d(x, y)$ and $d(x, z)$. Obviously we have $r_1 = 1$ if $x = y$ and $r_2 = 1$ if $x = z$. Moreover, the continuity of E^u and E^s proves that r_3 and r_4 tend to 1 when y and z go to x. On the other hand, if ρ is small enough (namely $\rho < \varepsilon_1$), there exist constants $\gamma^u > 0$ and $\gamma^s > 0$ (which depend on ρ) such that for every $x \in \Omega_0 \cap \Lambda$ and for every $y \in B(x, \rho)$:

- for all $v \in E^s(y)$,
$$|d\widehat{f}_x(y)v| \le e^{-\gamma^s}|v|,$$
$$|d\widehat{f}_x^{-1}(y)v| \ge e^{\gamma^s}|v|;$$
- for all $v \in E^u(y)$,
$$|d\widehat{f}_x(y)v| \ge e^{\gamma^u}|v|,$$
$$|d\widehat{f}_x^{-1}(y)v| \le e^{-\gamma^u}|v|.$$

We also have $\gamma^i \to \lambda^i$ when ρ goes to zero. Hence, we assume that ρ is sufficiently small such that for every $x \in \Omega_0 \cap \Lambda$ and for every y and z in $B(x, \rho)$ such that $D^u_{2\rho}(y)$ and $D^s_{2\rho}(z)$ exist, the distortion constants in (10), (11), (12) and (13) satisfy, for every $i = 1, 2, 3, 4$,
$$r_i e^{-(1/2)\gamma^u} < 1 \quad \text{and} \quad r_i e^{-(1/2)\gamma^s} < 1. \tag{14}$$

3.1.2. *Definition of a pseudo-orbit.* We can now define more precisely what a pseudo-orbit is. Let us pick some $\rho_2 \le \rho$ sufficiently small. We set
$$\Lambda_2^+(n) = \{x \in \Omega_2 \cap \Lambda,\ f(x) \notin \Omega_0 \text{ and } n^+(x) = n\},$$
$$\Lambda_2^-(n) = \{x \in \Omega_2 \cap \Lambda,\ f^{-1}(x) \notin \Omega_0 \text{ and } n^-(x) = n\}.$$
By the definitions of $n^+(x)$ and $n^-(x)$ we have $\Lambda_2^i(m) \cap \Lambda_2^i(n) = \emptyset$ if $n \ne m$ with $i = +, -$.

Definition 3.1. Let $\delta > 0$ be such that:

(1) $\delta < \dfrac{\rho_2}{16}(1 - e^{-(1/2)\gamma^u})$;

(2) $\delta < \dfrac{\rho_2}{16}(1 - e^{-(1/2)\gamma^s})$;

(3) $\delta < \dfrac{\varepsilon_0}{4}(1 - e^{-\gamma^u/2})e^{-(1/2)\gamma^u}$;

(4) $\delta < \dfrac{\varepsilon_0}{4}(1 - e^{-\gamma^s/2})e^{-(1/2)\gamma^s}$;

(5) $\delta < \dfrac{\varepsilon_0}{4}\dfrac{1}{e^{(1/2)\gamma^s} + \dfrac{1}{1 - e^{-\gamma^u/2}}}$;

(6) $\delta < \dfrac{\varepsilon_0}{4}\dfrac{1}{e^{(1/2)\gamma^u} + \dfrac{1}{1 - e^{-\gamma^s/2}}}$.

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A sequence $(x_n)_{n\in\mathbb{Z}}$ of points in Λ is called a δ-pseudo-orbit if:
(1) for all $n \in \mathbb{Z}$, $d(f(x_n), x_{n+1}) < \delta$ and $d(x_n, f^{-1}(x_{n+1})) < \delta$;
(2) for all $n \in \mathbb{Z}$, if $x_n \in \Lambda_2^+(m)$ then $x_{n+m} \in \Lambda_2^-(m)$ and either
- for all $0 < k < m$, $x_{n+k} = f^k(x_n)$; or
- for all $0 < k < m$, $x_{n+k} = f^{-m+k}(x_{n+m})$.

In both cases we have $d(x_n, f^{-k}(x_{n+k})) < \delta$ and $d(x_{n+k}, f^k(x_n)) < \delta$.

We say that the point x β-tracks the δ-pseudo-orbit (x_n) if and only if for all $n \in \mathbb{Z}$, $d(f^n(x), x_n) < \beta$.

Definition 3.2. A point x in Ω is said to be weakly hyperbolic if:
(i) it is a point of integration of the hyperbolic splitting;
(ii)
$$\limsup_{n\to+\infty} \frac{1}{n} \log \|df^n_{|E^s}(x)\| \le 0 \quad \text{and} \quad \limsup_{n\to+\infty} \frac{1}{n} \log \|df^{-n}_{|E^u}(x)\| \le 0.$$

PROPOSITION 3.3. (Shadowing lemma) *Let $\delta > 0$ be as in Definition 3.1 and let (x_n) be some δ-pseudo-orbit. Let*
$$\beta(\delta) = \max\left(\frac{2\delta}{(1-e^{-(1/2)\gamma^u})}, \frac{2\delta}{(1-e^{-(1/2)\gamma^s})}\right).$$
Then, there exists a unique point $x \in U$ such that x $\beta(\delta)$-tracks the δ-pseudo-orbit (x_n). Moreover, x is weakly hyperbolic.

Proof. The positive part $(x_n)_{n\ge0}$ and the negative part $(x_n)_{n\le0}$ play symmetric roles. Therefore, we are just going to study the positive part. The goal is to 'repatriate' by f^{-n}, for each n, a piece of $D^s_{\varepsilon_0}(x_n)$ close to x_0. We have to control that such a piece of stable leaf is long enough to be sure that it intersects $D^u_{\varepsilon_0}(x_0)$ in one point. This will allow us to define a family of pieces of stable leaves close to x_0 which will accumulate themselves on some limit piece of stable leaf. Doing the same in negative time, we will also obtain a piece of unstable leaf close to x_0. The two pieces of leaves must intersect themselves in exactly one point, which must track the pseudo-orbit. To be sure that this point tracks (x_n) the repatriation has to be done by induction.

Let us assume that x_0 is in Ω_0 and pick $n > 0$ such that $x_n \in \Omega_0$.

Step 1. We first study how we can repatriate one piece of stable leaf close to the 'previous' point.

Case one: $x_{n-1} \in \Omega_0$. Because δ is sufficiently small (conditions with ε_0) and the x_i are (ε_0, λ)-regular, we can define $[x_n, f(x_{n-1})]$ and (12) gives
$$d^u(f(x_{n-1}), [x_n, f(x_{n-1})]) \le r_3\delta.$$
Moreover, (13) and (14) give $d^s(x_n, [x_n, f(x_{n-1})]) \le r_4\delta \le e^{\frac{1}{2}\gamma^s}\delta$.

Thus we can define $D^s_{2\beta(\delta)}([x_n, f(x_{n-1})])$, because $[x_n, f(x_{n-1})]$ is far away from the boundary of $\mathcal{F}^s(x_n)$. Then, by expansion along the stable leaves in negative times, we can define $D^s_{2\beta(\delta)}(f^{-1}([x_n, f(x_{n-1})]))$ and by contraction along the unstable leaves in negative times (14) yields
$$d(x_{n-1}, f^{-1}([x_n, f(x_{n-1})])) \le e^{-(1/2)\gamma^u}\delta.$$

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Case two: $x_{n-1} \notin \Omega_0$. Let m be such that $x_m \in \Omega_0$ and for all k, $0 < k < n-m$, $x_{m+k} \notin \Omega_0$. By definition of a δ-pseudo-orbit there are two sub-cases; either:

- Case (2.1): for all $0 < k < n-m$, $x_{m+k} = f^k(x_m)$; or
- Case (2.2): for all $0 < k < n-m$, $x_{n+k} = f^{-n+k}(x_n)$.

In both cases we can define $[x_n, f^{n-m}(x_m)]$ and as in the first case we can define $D^s_{2\beta(\delta)}([x_n, f^{n-m}(x_m)])$. We set $\xi_{m,n} \stackrel{\text{def}}{=} f^{-(n-m)}([x_n, f^{n-m}(x_m)])$. The unstable foliation is contracted and the stable foliation is expanded in negative time and both points x_m and x_n are in Ω_0 (where there is strong contractions). Thus:

(i) $d(x_m, \xi_{m,n}) \le r_3 e^{-2\gamma^u}\delta \le e^{-(3/2)\gamma^u}\delta$;

(ii) $f^k(\xi_{m,n}) \in B(x_{m+k}, \frac{1}{2}\beta(\delta))$ for every $0 < k < n-m$;

(iii) we can define $D^s_{2\beta(\delta)}(\xi_{m,n})$.

Step 2. We now study how we can induce the construction.

On $D^u_{\varepsilon_0}(x_n) \cap B(x_n, \frac{1}{2}\beta(\delta))$ we have a family of points $(\xi_{n,j})_j$. For each j we can define $D^s_{2\beta(\delta)}(\xi_{n,j})$.

In case one above, we set $\xi_{n-1,j} \stackrel{\text{def}}{=} f^{-1}([\xi_{n,j} f(x_{n-1})])$. Then (10) and (14) give

$$d^u(x_{n-1}, \xi_{n-1,j}) \le e^{-\gamma^u}(r_3\delta + r_1\tfrac{1}{2}\beta(\delta)) \le \delta + e^{-\frac{1}{2}\gamma^u}\tfrac{1}{2}\beta(\delta) \le \tfrac{1}{2}\beta(\delta).$$

Moreover $d^s(\xi_{n,j}, [f(\xi_{n-1,j})]) \le e^{(1/2)\gamma^s}\delta$. Therefore, we can define $D^s_{2\beta(\delta)-e^{(1/2)\gamma^s}\delta}(f(\xi_{n-1,j}))$ and by expansion along the stable leaves in the negative times we can define $D^s_{2\beta(\delta)}(\xi_{n-1,j})$ (there is 'strong' expansion close to Ω_0).

In cases (2.1) and (2.2), we set $\xi_{m,j} \stackrel{\text{def}}{=} f^{-(n-m)}([\xi_{n,j}, f^{n-m}(x_m)])$ and for $0 < k < n-m$, $\xi_{m+k,j} = f^k(\xi_{m,j})$.

In the both cases, we obtain

$$d^u(x_m, \xi_{m,j}) \le e^{-2\gamma^u}(r_3\delta + r_1\tfrac{1}{2}\beta(\delta)).$$

Thus we get

$$d^u(x_m, \xi_{m,j}) \le \tfrac{1}{2}\beta(\delta).$$

As in the first case, we can define $D^s_{2\beta(\delta)}(\xi_{n-1,j})$ because of the expansion in the stable leaves in negative times (which occurs at each step and at least twice 'strongly' because both points x_n and x_m are in Ω_0). We also set $\xi_{m,j} = \xi_{m,n}$ for $m+1 < j < n$.

Step 3. We can iterate this construction. By induction this gives a sequence of points $(\xi_{0,j})_{j\ge 1}$ in $D^u_{\varepsilon_0}(x_0) \cap B(x_0, \frac{1}{2}\beta(\delta))$, such that for each j we can define $D^s_{2\beta(\delta)}(\xi_{0,j})$. Such an s-disk can be viewed in $B_{x_0(0,\rho_2)}$ as the graph of some map $g_{0,j} : B^s(0, \frac{1}{2}\beta(\delta)) \to B^u(0, \beta(\delta))$. The family of maps $(g_{0,j})_j$ is equicontinuous in the $\mathcal{C}^1$-topology because E^s is Hölder continuous. Therefore, by Ascoli's theorem, it converges (up to some sub-sequence) to some map $g^s : B^s(0, \frac{1}{2}\beta(\delta)) \to B^u(0, \beta(\delta))$. This proves that $\phi_{x_0}(\text{graph}(g^s))$ is a $\mathcal{C}^1$-disk of size $\frac{1}{2}\beta(\delta)$. This disk must contain a point in $B(x_0, \frac{1}{2}\beta(\delta))$ (obtained as an accumulation point of the sequence $(\xi_{0,j})$). Moreover for every point y in $\phi_{x_0}(\text{graph}(g^s))$ we have

$$T_y\phi_{x_0}(\text{graph}(g^s)) = E^s(y).$$

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Doing the same with the negative part of the pseudo-orbit, we get some $\mathcal{C}^1$-u-disk of size $\frac{1}{2}\beta(\delta)$, which contains a point in $B(x_0, \frac{1}{2}\beta(\delta))$. The two disks must intersect each other in exactly one point x. By the construction of x we have for every integer n,

$$f^n(x) \in B(x_n, \beta(\delta)).$$

Moreover x is a point of integration of the hyperbolic splitting and at least weakly hyperbolic because it returns infinitely many times (in the past and in the future) in $B(\Omega_0, \rho_2)$ (where there are uniform expansions and contractions). □

3.2. *Symbolic dynamics.* We now have to prove the converse of Proposition 3.3: let δ be fixed as above. We want some countable set, $\Gamma \subset \Lambda$, such that every point in Λ is $\beta(\delta)$-tracked by some sequence (that should be some δ-pseudo-orbit) in $\Gamma^{\mathbb{Z}}$.

Let κ be

$$\max\left(\sup_{x\in M}\|df(x)\|, \sup_{x\in M}\|df^{-1}(x)\|\right).$$

We build a cover of Λ in the following way.

(1) We take a finite cover of $\Lambda \cap \Omega_0$ by balls $B(p_i, \delta/3\kappa)$, where p_i is in $\Lambda \cap \Omega_0$ and i describes $\{0, \dots, P_0\} \subset \mathbb{N}$. This can be done because $\Lambda \cap \Omega_1$ is totally bounded.

(2) For each $n > 0$, we take a finite cover of $\Lambda_2^+(n)$ by balls $B_n(q_{i,n}, \delta/3\kappa)$, where $q_{i,n}$ is in $\Lambda_2^+(n)$, $B_n(y, \epsilon)$ denotes $\bigcap_{0\le j<n} f^{-j}(B(f^j(y), \epsilon))$ and i describes $\{0, \dots, P_n\} \subset \mathbb{N}$. We can find such a cover because each dynamical ball $B_n(y, \delta/3\kappa)$ contains the ball $B(y, \delta/3\kappa^{n+1})$.

(3) For each $n > 0$, we take a finite cover of $\Lambda_2^-(n)$ by balls $B_{-n}(r_{i,n}, \delta/3\kappa)$, where $r_{i,n}$ is in $\Lambda_2^+(n)$, $B_{-n}(y, \epsilon)$ denotes $\bigcap_{0\le j<n} f^{j}(B(f^{-j}(y), \epsilon))$ and i describes $\{0, \dots, P_{-n}\} \subset \mathbb{N}$.

For $m \le 0$, let us set $\mathbb{N}_m \overset{\text{def}}{=} \{0, \dots, m\}$; then we define

$$\Gamma \overset{\text{def}}{=} \{p_i : i \in \mathbb{N}_{P_0}\} \bigcup_{n>0, i\in\mathbb{N}_{P_n}} \{q_{i,n}, f(q_{i,n}), \dots, f^{n-1}(q_{i,n})\} \bigcup_{n>0, i\in\mathbb{N}_{P_{-n}}} \{r_{i,n}, f^{-1}(r_{i,n}), \dots, f^{-n+1}(r_{i,n})\}.$$

The set Γ is a countable set. For convenience we set $\Gamma = \{\xi_k, k \in \mathbb{N}\}$.

PROPOSITION 3.4. *For every x in Λ, there exists some sequence $(x_n)_{n\in\mathbb{Z}}$ in $\Gamma^{\mathbb{Z}}$ such that:*

(i) *$(x_n)_{n\in\mathbb{Z}}$ is a δ-pseudo-orbit;*

(ii) *x $\beta(\delta)$-tracks $(x_n)_{n\in\mathbb{Z}}$.*

Proof. Pick x in Λ. We may always assume that x is in Ω_0 (every point in Λ comes in Ω_0 along its orbit). The main idea of the proof is to construct the sequence in a recursive way. It will be constructed by 'blocks'.

Step 1. We first define the family of times, $(\tau_n)_{n\in\mathbb{Z}}$, such that the τ_nth-iterate of x are in Ω_0: $\tau_0 = 0$ and if we have built $\tau_{-n}, \dots, \tau_0, \dots, \tau_n$, then we define $\tau_{n+1} = \tau_n + \min\{k : 0 < k, f^{\tau_n+k}(x) \in \Omega_0\}$ and $\tau_{-(n+1)} = \tau_{-n} - \min\{k : 0 < k, f^{\tau_{-n}-k}(x) \in \Omega_0\}$. The τ_n are called the stopping times.

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Step 2. We construct the first block of point(s) x_k. Each block will have a length $\tau_k - \tau_{k-1}$. There are several cases.

(C1) If x is in Ω_1 and there is some n_0 such that $x \in B(\xi_{n_0}, \delta/3\kappa)$ (ξ_{n_0} is one of the p_i), then we set $x_0 = \xi_{n_0}$. The first block is $\{x_0\}$.

(C2) If x is in $\Lambda_2^+(m)$, then $\tau_1 = m$. Hence, there exists some n_0 such that $x \in B_m(\xi_{n_0}, \delta/3\kappa)$ (ξ_{n_0} is one of the $q_{i,m}$). We set $x_k = f^k(\xi_{n_0})$, for every $0 \le k < m$. The first block is $\{x_0, \dots, x_{m-1}\}$.

(C3) If x is in Ω_2 but does not belong to any $\Lambda_2^+(j)$, then $\tau_{-1} = -m$. Hence, there exists some n_0 such that $x \in B_{-m}(\xi_{n_0}, \delta/3\kappa)$ (ξ_{n_0} is one of the $r_{i,m}$). We set $x_k = f^k(\xi_{n_0})$, for every $-m < k \le 0$. The first block is $\{x_{-m+1}, \dots, x_0\}$.

Step 3. We construct new blocks in a recursive way. Let us assume that we have constructed $(x_i, \dots, x_j)$ with $i \le j$. We study how we can construct a block $(x_{j+1}, \dots, ?)$ (called an upper block) and a block $(?, \dots, x_{i-1})$ (called a lower block).

Upper block.

- If j is not a stopping time, then $j+1$ must be some stopping time τ_k and $f^{j+1}(x) \in \Lambda_2^-(\tau_k - \tau_{k-1})$.
  - * If $\tau_{k+1} = \tau_k + 1 = j + 2$, then we just construct x_{j+1}; $f^{j+1}(x)$ belongs to Ω_0 and there exists some $p = p_{m(j+1)}$ such that $f^{j+1}(x) \in B(p, \delta/3\kappa)$. We set $x_{j+1} = p$.
  - * If $\tau_{k+1} > \tau_k + 1$, then $f^{j+1}(x) \in \Lambda_2^+(\tau_{k+1} - \tau_k)$, and there exists some $q = q_{m(j+1),\tau_{k+1}-\tau_k}$ such that $f^{j+1}(x) \in B_{\tau_{k+1}-\tau_k}(q, \delta/3\kappa)$. Hence, we set $x_{j+l} = f^l(q)$, for $1 \le l \le \tau_{k+1} - \tau_k$.
- If $j = \tau_k$, then necessarily $j + 1 = \tau_{k+1}$.
  - * If $f^{j+1}(x)$ belongs to Ω_1, then we construct x_{j+1} by choosing any $p = p_{m(j+1)}$ such that $f^{j+1}(x) \in B(p, \delta/3\kappa)$. We set $x_{j+1} = p$.
  - * If $f^{j+1}(x)$ does not belong to Ω_1, then $f^{j+1}(x) \in \Lambda_2^+(\tau_{k+2} - \tau_{k+1})$ and there exists some $q = q_{m(j+1),\tau_{k+2}-\tau_{k+1}}$ such that $f^{j+1}(x) \in B_{\tau_{k+2}-\tau_{k+1}}(q, \delta/3\kappa)$. Hence, we set $x_{j+l} = f^l(q)$, for $1 \le l \le \tau_{k+2} - \tau_{k+1}$.

Lower block.

- If i is not a stopping time, then $i - 1$ must be some stopping time τ_k.
  - * If $\tau_{k-1} = \tau_k - 1$, then we just construct x_{i-1}. There exists some $p = p_{m(i-1)}$ such that $f^{i-1}(x) \in B(p, \delta/3\kappa)$. We set $x_{i-1} = p$.
  - * If $\tau_{k-1} < \tau_k - 1$, then $f^{i-1}(x) \in \Lambda_2^-(|\tau_{k-1} - \tau_k|)$ and there exists some $r = r_{m(i-1),|\tau_{k-1}-\tau_k|}$ such that $f^{i-1}(x) \in B_{\tau_{k-1}-\tau_k}(r, \delta/3\kappa)$. Hence, we set $x_{i-l} = f^l(r)$, for $1 \le l \le |\tau_{k-1} - \tau_k|$.
- If $i = \tau_k$, we have the following.
  - * If $f^{i-1}(x) \notin \Omega_0$, then $\tau_{k-1} < \tau_k - 1$ and $f^{\tau_{k-1}}(x)$ belongs to $\Lambda_2^+(\tau_k - \tau_{k-1})$. Hence, there exists some $q = q_{m(i-1),\tau_k-\tau_{k-1}}$ such that $f^{\tau_{k-1}}(x) \in B_{\tau_k-\tau_{k-1}}(q, \delta/3\kappa)$. Hence, we set $x_{\tau_{k-1}+l} = f^l(q)$, for $0 \le l < \tau_k - \tau_{k-1}$.
  - * If $f^{i-1}(x) \in \Omega_1$, then we just construct x_{i-1} by choosing any $p = p_{m(i-1)}$ such that $f^{i-1}(x) \in B(p, \delta/3\kappa)$. We set $x_{i-1} = p$.

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* If $f^{i-1}(x) \in \Omega_2$, then $f^{i-1}(x) \in \Lambda_2^-(|\tau_{k-2}-\tau_{k-1}|)$ but does not belong to any $\Lambda_2^+(m)$. Then, there exists some $r = r_{m(i-1),|\tau_{k-2}-\tau_{k-1}|}$ such that $f^{i-1}(x) \in B_{\tau_{k-2}-\tau_{k-1}}(q, \delta/3\kappa)$. Hence, we set $x_{i-l} = f^l(q)$, for $1 \leq l \leq \tau_{k-1} - \tau_{k-2}$.

We have now constructed $x_{i'}, \ldots, x_i, \ldots, x_j, \ldots, x_{j'}$.

We let the reader check that if i' (respectively j') is not a stopping time, then $i' - 1$ (respectively $j' + 1$) must be a stopping time.

Step 4. Hence, by construction, we obtain for every n,

$$d(f^n(x), x_n) < \frac{\delta}{3\kappa} < \beta(\delta).$$

The sequence $(x_n)_{n\in\mathbb{Z}}$ is a δ-pseudo-orbit in $\Gamma^{\mathbb{Z}}$ and x $\beta(\delta)$-tracks $(x_n)_{n\in\mathbb{Z}}$. □

We denote the set of δ-pseudo-orbit in $\Gamma^{\mathbb{Z}}$ by Σ_0. Proposition 3.3 proves that there is a canonical map $\Theta : \Gamma^{\mathbb{Z}} \to M$ which maps every δ-pseudo-orbit $\underline{x} = (x_n)_{n\in\mathbb{Z}}$ to the unique x which $\beta(\delta)$-tracks $\underline{x}$. Conversely, Proposition 3.4 proves that Λ is contained in the image of Θ. Hence, there is a conjugacy of the dynamics on M with some symbolic dynamical system:

$$\begin{array}{ccc} \sigma : & \Sigma_0 & \longrightarrow \Sigma_0 \\ & \downarrow & \downarrow \Theta \\ f : & M & \longrightarrow M \end{array}$$

where σ is the shift on Σ_0.

3.3. *Rectangles.* From now on, δ is some fixed constant as above (relatively to the choice of ε_1). Moreover, its exact magnitude will also depend on several (but finitely many) conditions that will follow. In Σ_0, there is the canonical partition into 1-cylinders: for ξ_n in Γ we set

$$[\xi_n] \stackrel{\text{def}}{=} \{\underline{x} \in \Sigma_0,\ x_0 = \xi_n\}.$$

There is a canonical map, $[\cdot, \cdot]$, from $\bigcup_{n\in\mathbb{N}}[\xi_n]^2$ to Σ_0 defined in the following way.

If $\underline{y} = (y_n)_{n\in\mathbb{Z}}$ and $\underline{x} = (x_n)_{n\in\mathbb{Z}}$ are in $[\xi_p]$, then $[\underline{x}, \underline{y}]$ is the sequence $\underline{z} = (z_n)_{n\in\mathbb{Z}}$ defined by $z_n = x_n$ for every $n \geq 0$ and $z_n = y_n$ for every $n \leq 0$. The first goal of this section is to study the push forward of the 1-cylinders on M by Θ. In particular, we prove that the map $\Theta \circ [\cdot, \cdot]$ is the map $[\Theta(\cdot), \Theta(\cdot)]$ where the second $[\cdot, \cdot]$ is the system of local coordinates on the manifold. Then we produce several generations of rectangles.

LEMMA 3.5. *Let $\underline{y} = (y_n)_{n\in\mathbb{Z}}$ and $\underline{x} = (x_n)_{n\in\mathbb{Z}}$ be two sequences in Σ_0 such that for every $n \geq 0$, $x_n = y_n$. Then, if δ is small enough such that $\beta(\delta) < \rho_2/8\kappa$, we have $\Theta(\underline{x}) \in D^s_{2\beta(\delta)}(\Theta(\underline{y}))$.*

Proof. For convenience we set $x = \Theta(\underline{x})$ and $y = \Theta(\underline{y})$. By construction there exist two points z_1 and z_2 such that $\{z_1\} = D^u_{2\beta(\delta)}(x) \cap D^s_{2\beta(\delta)}(y)$ and $\{z_2\} = D^u_{2\beta(\delta)}(y) \cap D^s_{2\beta(\delta)}(x)$. Let us assume that $x \neq z_1$. We are going to find some contradiction.

Because $x \neq z_1$, we must have $y \neq z_2$. For $n \geq 0$, we have $d(f^n(x), x_n) \leq \beta(\delta)$, $d(f^n(y), x_n) \leq \beta(\delta)$ and $\lim_{n\to+\infty} d(f^n(x), f^n(z_2)) = 0$. Hence, we may assume that

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for every $n \geq 0$, $d(f^n(x), f^n(z_2)) \leq \beta(\delta)$, and so for all $n \geq 0$,

$$f^n(z_2) \in B(x_n, 2\beta(\delta)) \quad \text{and} \quad f^n(y) \in B(x_n, \beta(\delta)). \tag{15}$$

Moreover, there are infinitely many n_k in $\mathbb{N}$ such that $x_{n_k} \in \Omega_0$. Therefore, property (15) implies that, if n_k is such that for every $j \leq n_k$, $f^j(z_2)$ belongs to the connected component of $D^u_{2\beta(\delta)}(f^j(y)) \cap B(x_j, 2\beta(\delta))$ that contains $f^j(y)$, then we must have

$$d^u(f^{n_k}(y), f^{n_k}(z_2)) \geq e^{k\gamma^u} d^u(z_2, y). \tag{16}$$

Therefore, there exists some n such that for every $0 \leq k \leq n$, $f^k(z_2)$ belongs to the connected component of $\mathcal{F}^u(f^k(y)) \cap B(x_k, 2\beta(\delta))$ that contains $f^k(y)$ and $f^{n+1}(z_2)$ does not belong to the connected component of $\mathcal{F}^u(f^{n+1}(y)) \cap B(x_{n+1}, 2\beta(\delta))$ which contains $f^{n+1}(y)$.

Because δ has been chosen small enough, $f^{n+1}(z_2)$ belongs to the connected component of $\mathcal{F}^u(f^{n+1}(y)) \cap B(x_{n+1}, \rho_2/2)$ that contains $f^{n+1}(y)$ and so $d(f^{n+1}(z_2), x_{n+1})$ must be greater than $2\beta(\delta)$, which is absurd. Therefore, $x = z_1$ and $y = z_2$. □

3.3.1. *First generation of rectangles.* For n in $\mathbb{N}$, we set $T_n \overset{\text{def}}{=} \Theta([\xi_n])$. If x is in T_n, we set $W^u(x, T_n) \overset{\text{def}}{=} D^u_{2\beta(\delta)}(x) \cap T_n$ and $W^s(x, T_n) \overset{\text{def}}{=} D^s_{2\beta(\delta)}(x) \cap T_n$. As a direct consequence of Lemma 3.5 we obtain the following.

PROPOSITION 3.6. *Let n be an integer and x and y be in T_n. Let $\underline{x}$ and $\underline{y}$ be in $[\xi_n]$ such that $\Theta(\underline{x}) = x$ and $\Theta(\underline{y}) = y$. Then, $W^u(x, T_n)$ (respectively $W^s(x, T_n)$) is exactly the image by Θ of $\{[\underline{z}, \underline{x}],\ \underline{z} \in [\xi_n]\}$ (respectively $\{[\underline{x}, \underline{z}],\ \underline{z} \in [\xi_n]\}$). Moreover, $W^u(x, T_n) \cap W^s(y, T_n)$ exists and $[y, x] = \Theta([\underline{y}, \underline{x}]) = W^u(x, T_n) \cap W^s(y, T_n)$.*

This proposition allows us to define T_n as a rectangle in the sense of Bowen (see [**5**]). The T_n are called rectangles of first generation and their set is denoted by $\mathcal{T}$. They also satisfy some restricted Markov property as follows.

PROPOSITION 3.7. *Let T_i and T_j be in $\mathcal{T}$ such that some point x belongs to $T_j \cap f^{-1}(T_i)$. If there exists some $\underline{x}$ in Σ_0 such that:*

(i) $\Theta(\underline{x}) = x$;

(ii) $x_0 = \xi_j$;

(iii) $x_1 = \xi_i$;

then $W^u(f(x), T_i) \subset f(W^u(x, T_j))$ and $W^s(f(x), T_i) \supset f(W^s(x, T_j))$.

Proof. Let y be in $W^s(x, T_j)$. There exists some $\underline{y}$ in $\Theta^{-1}(y) \cap [\xi_j]$. Moreover, Proposition 3.6 proves that $y = [x, y]$, and so $y = \Theta([\underline{x}, \underline{y}])$. If $\underline{z}$ denotes $[\underline{x}, \underline{y}]$, then $z_0 = \xi_j$ and $z_1 = \xi_i$. Hence, $f(y) = \Theta \circ \sigma(\underline{z})$ belongs to T_i.

The other part of the proof is similar. □

Remark. It is important to note that Proposition 3.7 does not prove that all the rectangles satisfy a complete Markov property. If x is in $T_j \cap f^{-1}(T_i)$ and if there is no $\underline{x}$ in $\Theta^{-1}(x) \cap [\xi_j] \cap \sigma^{-1}[\xi_i]$, the result of the proposition would probably be false. In view to obtain the global Markov property we have to cut these rectangles. Before we do this, we need to make them thinner.

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3.3.2. *Second generation of rectangles.* The main idea of this part is to restrict the f-invariant set $\bigcup_{i\in\mathbb{N}} T_i$ $(= \Theta(\Sigma_0))$ to some subset satisfying good properties.

PROPOSITION 3.8. *There exists some λ-hyperbolic set $\Delta \subset \Lambda$ such that:*
(1) $\mathrm{Leb}^u(\Delta) > 0$;
(2) *every point x in Δ is a density point of Δ for Leb^u_x;*
(3) *there exist some $\zeta > 0$ such that for every x in Δ,*

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|df^{-1}_{|E^u(f^j(x))}(f^j(x))\| < -\zeta. \tag{17}$$

Proof. We first note that, if (17) holds for some x in Λ (and for some fixed ζ in $\mathbb{R}^*_+$), then it also holds for every $f^n(x)$, when n describes $\mathbb{Z}$.

The first part of the proof is to show that for every x in Λ,

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|df^{-1}_{|E^u(f^j(x))}(f^j(x))\| < 0. \tag{18}$$

Let x be in Λ, ε_2 be some fixed real number in $(0, 1)$ and $\Omega_3 = \Omega_3(\varepsilon_2)$ be the set of point x in Ω such that

$$\min(\log \|df_{|E^u(x)}\|, -\log \|df^{-1}_{|E^u(x)}\|, -\log \|df_{E^s(x)}\|, \log \|df^{-1}_{|E^s(x)}\|) \geq \varepsilon_2\lambda.$$

We set $\delta_{\varepsilon_2} \overset{\text{def}}{=} \liminf(1/n)\#\{0 \leq k < n\colon f^k(x) \in \Omega_3\}$ (where $\#A$ denotes the cardinality of the finite set A). Because x is λ-hyperbolic, it must spend enough time in Ω_3, and we have

$$\delta_{\varepsilon_2} \geq \frac{1-\varepsilon_2}{\log\kappa/\lambda - \varepsilon_2} > 0. \tag{19}$$

Hence, inequality (19) yields

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|df^{-1}_{|E^u(f^j(x))}(f^j(x))\| < -\delta_{\varepsilon_2}\varepsilon_2\lambda,$$

which proves that (18) holds for every x in Λ.

Let us set

$$\Lambda_{|\zeta} \overset{\text{def}}{=} \left\{x \in \Lambda\colon \limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|df^{-1}_{|E^u(f^j(x))}(f^j(x))\| < -\zeta\right\}.$$

Then, Λ is the increasing union of all $\Lambda_{|\zeta}$ (when ζ decreases to zero) and so there exists some ζ such that $\mathrm{Leb}^u(\Lambda_{|\zeta}) > 0$. We pick such a ζ.

Let us set $\Delta_0 \overset{\text{def}}{=} \bigcup_{i\in\mathbb{N}}\{x \in T_i\colon \exists(y, z) \in \Lambda^2_{|\zeta}, x \in \mathcal{F}^u(y) \cap \mathcal{F}^s(z)\}$. Then, Δ_0 is an f-invariant set of λ-hyperbolic points in $\bigcup_{i\in\mathbb{N}} R_i$ such that (17) holds for every x in Δ_0. Moreover, $\mathrm{Leb}^u(\Delta_0) > 0$.

Now, if x is a density point of Δ_0 for Leb^u_x, we check that every y in $\mathcal{F}^s(x) \cap \Delta_0$ is also a density point of Δ_0 for Leb^u_y.

This is a well-known fact in Pesin theory that the stable holonomy is absolutely continuous with respect to Lebesgue measure on the unstable manifolds. This again holds

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in our case by the definition of $\mathcal{F}^s(x)$: there exists some n such that for every $k \geq n$, $f^k(y)$ belongs to $\mathcal{F}^s_{\text{loc}}(f^k(x))$. Then we can use Mañé's method (see [**10**]).

Therefore, we denote the set of density points in Δ_0 for Leb^u by Δ. This set is f-invariant and stable by intersections of stable and unstable leaves. Hence, every point in Δ is also a density point of Δ for Leb^u. □

For $i \in \mathbb{N}$, we denote by S_i the restriction of the rectangle T_i to Δ, i.e. $S_i = T_i \cap \Delta$. If x is in S_i we set $W^u(x, S_i) \stackrel{\text{def}}{=} W^u(x, T_i) \cap \Delta$ and $W^s(x, S_i) \stackrel{\text{def}}{=} W^s(x, T_i) \cap \Delta$. By the construction of Δ, if x and y are in S_i, then $[x, y]$ is also in S_i. As before we have

$$\{[x, y]\} = W^s(x, S_i) \cap W^u(y, S_i) = D^s_{2\beta(\delta)}(x) \cap D^u_{2\beta(\delta)}(y).$$

The sets S_i are called rectangles of second generation and their set is denoted by $\mathcal{S}$.

3.4. *Third generation of rectangles.*

3.4.1. *Constants.* Since the beginning we have introduced several constants. It is now time to fix some of them. For that purpose we first want to summarize how these constants depend on each other.

The constant κ is fixed by f. The two constants K_1 and K_2 are also fixed. We fix the set Λ and so the two constants ε_0 and λ are also fixed. Then we fix ζ as above (note that this is done before fixing ε_1). We can pick some ε, very small compared to λ and ζ (namely $\varepsilon < \zeta/10$).

Now, as soon as ε_1 is fixed (sufficiently small as asked in the proof of Theorem A but only in relation with f and Λ), the constant ρ_1 can be adjusted. Therefore, we can choose the constants ρ, ρ_2, γ^u and γ^s. After this we can choose δ such that δ and $\beta(\delta)$ are sufficiently small to satisfy the several asked conditions. Several new conditions will also be stated later.

3.4.2. *Choice of ε_i.* We first fix ε_2 sufficiently small such that every x in Δ such that $\log \|df^{-1}_{|E^u(x)}\| < -\zeta/3$ belongs to Ω_3 (this set has been defined in Proposition 3.8). This is possible because k^u and k^s vanish at the same time. Hence, we fix ε_1 sufficiently small such that Ω_3 is a closed subset of Ω which does not intersect $\overline{\Omega_2 \cup B(S, \varepsilon_1)}$. Therefore, there exist some $\rho_3 > 0$ such that $d(\Omega_3, \overline{\Omega_2 \cup B(S, \varepsilon_1)}) > \rho_3$, and we assume that δ satisfies

$$\beta(\delta) < \frac{\rho_3}{10}.$$

3.4.3. *Third generation of rectangles.* If S_i is in $\mathcal{S}$, we say that it has (or it is of) order 0 if ξ_i is in Ω_1. We say that it has order n if either $\xi_i = f^k(\xi_j)$ with ξ_j in $\Lambda^+_2(n)$ and $0 \leq k < n$, or $\xi_i = f^{-k}(\xi_j)$ with ξ_j in $\Lambda^-_2(n)$ and $0 \leq k < n$. Because $\beta(\delta)$ is small enough, none of the rectangles of second generation and of order 0 which intersects Ω_3 can intersect some rectangle of order $n > 0$. We say that a rectangle of order 0 which intersects Ω_3 is of order 00. Then, each rectangle of order 00 only intersects a finite number of other

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rectangles. Hence, we can cut them as in [**5**]: let S_i be a rectangle of order 00 and S_j be any other rectangle such that $S_i \cap S_j \neq \emptyset$. We set

$$S_{ij}^1 = \{x \in S_i : W^u(x, S_i) \cap S_j = \emptyset \text{ and } W^s(x, S_i) \cap S_j = \emptyset\},$$
$$S_{ij}^2 = \{x \in S_i : W^u(x, S_i) \cap S_j \neq \emptyset \text{ and } W^s(x, S_i) \cap S_j = \emptyset\},$$
$$S_{ij}^3 = \{x \in S_i : W^u(x, S_i) \cap S_j \neq \emptyset \text{ and } W^s(x, S_i) \cap S_j \neq \emptyset\},$$
$$S_{ij}^4 = \{x \in S_i : W^u(x, S_i) \cap S_j = \emptyset \text{ and } W^s(x, S_i) \cap S_j \neq \emptyset\}.$$

Let us set $\mathcal{S}_0 = \{x \in M : \exists S_i \ni x \text{ of order } 00\}$. Then, for x in $\mathcal{S}_0$ we set

$$\mathcal{R}(x) \stackrel{\text{def}}{=} \{y \in M : \forall i, j \in \mathbb{N}, x \in S_{ij}^k \Rightarrow y \in S_{ij}^k\}.$$

This defines a partition $\mathcal{R}$ of $\mathcal{S}_0$. By construction it is finite, and each of its elements, R_i, is stable by the map $[\cdot, \cdot]$. Hence, each R_i is a rectangle; the sets R_i are called rectangles of third generation. If x is in R_i, we set $W^u(x, R_i) \stackrel{\text{def}}{=} D^u_{2\beta(\delta)}(x) \cap R_i$ and $W^s(x, R_i) \stackrel{\text{def}}{=} D^s_{2\beta(\delta)}(x) \cap R_i$. It is easy to check that if x is in the rectangles of second generation $S_{i_1}, \ldots S_{i_p}$, then $W^{u,s}(x, R_i) = W^{u,s}(x, S_{i_k}) \cap R_i$ for $1 \leq k \leq p$. Moreover, we let the reader check that Proposition 3.7 yields the following result (as in [**5**]).

PROPOSITION 3.9. *Let R_i and R_j be two rectangles of third generation, let n be some integer in $\mathbb{N}$ and let x be in $R_i \cap f^{-n}(R_j)$. Then,*

$$f^n(W^u(x, R_i)) \supset W^u(f^n(x), R_j),$$

and

$$f^n(W^s(x, R_i)) \subset W^s(f^n(x), R_j).$$

This means that $\mathcal{R}$ is a Markov partition of $\mathcal{S}_0$.

Remark. This construction can also be done with the first generation of rectangles because every T_i which intersects Ω_3 intersects only a finite number of T_j.

3.5. *Reduced dynamical systems.*

3.5.1. *Hyperbolic times.* Let x be in $\mathcal{S}_0$. It is a λ-hyperbolic point and so it must return infinitely often in Ω_3. Hence, every point in $\mathcal{S}_0$ returns infinitely often in $\mathcal{S}_0$. Let x be in $\mathcal{S}_0$ and n be in $\mathbb{N}^*$ such that $f^n(x)$ is in $\mathcal{S}_0$. There exist two rectangles of third generation R_l and R_k such that x is in R_l and $f^n(x)$ is in R_k. Then, the Markov property (Proposition 3.9) implies that

$$f^{-n}(W^u(f^n(x), R_k)) \subset W^u(x, R_l).$$

We need some uniform distortion bound, i.e. some uniform control on

$$\prod_{k=0}^{n-1} \frac{J^u(f^k(x))}{J^u(f^k(y))},$$

where y belongs to $f^{-n}(W^u(f^n(x), R_k))$ and $J^u(z)$ is the unstable Jacobian $\det df_{|E^u(z)}(z)$.

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If the orbit of x comes into $B(S, \varepsilon_1)$ between x and $f^n(x)$, then we do not have good control of $\prod_{k=0}^{n-1} J^u(f(x))$ and, in particular, on the distortion ratio

$$\prod_{k=0}^{n-1} \frac{J^u(f^k(x))}{J^u(f^k(y))}.$$

To obtain good estimates we have to use the notion of hyperbolic times which was introduced in [**1**].

Definition 3.10. Given $0 < r < 1$, we say that n is a r-hyperbolic time for x if for every $1 \leq k \leq n$

$$\prod_{i=n-k+1}^{n} \|df^{-1}_{|E^u(f^i(x))}\| \leq r^k.$$

Moreover, it is known that, if

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|df^{-1}_{|E^u(f^j(x))}(f^j(x))\| < 2 \log r,$$

then there exist infinitely many r-hyperbolic times for x (see [**2**, §3, Corollary 3.2]). Therefore, by the construction of Δ, for $\rho_4 = e^{-(1/3)\zeta}$ and for every x in $\mathcal{S}_0$, there exist infinitely many ρ_4-hyperbolic times for x.

LEMMA 3.11. *There exists $\delta' > 0$ such that, if $\delta < \delta'$, then for every x in $\mathcal{S}_0$, for all ρ_4-hyperbolic times for x, n, and for every y in $B_{n+1}(x, 4\beta(\delta))$, the integer n is also a $\sqrt{\rho}_4$-hyperbolic time for y.*

Proof. By the continuity of df and E^u, there exists some δ'' such that for every x in U and for every y in $B(x, \delta'')$,

$$e^{-\varepsilon} < \frac{\|df^{-1}_{|E^u(x)}(x)\|}{\|df^{-1}_{|E^u(y)}(y)\|} < e^{\varepsilon}.$$

The definition of $\beta(\delta)$ proves that if δ is sufficiently small (namely $\delta < \delta'$ for some δ'), then $4\beta(\delta)$ is smaller than δ''. Let us assume that $\delta < \delta'$; let x be some point in $\mathcal{S}_0$ and n be some ρ_4-hyperbolic time for x. Let y be in $B_{n+1}(x, 4\beta(\delta))$. Then for every $0 \leq k \leq n$, y belongs to $B(x, 4\beta(\delta)) \subset B(x, \delta'')$, which means that for every $0 \leq k \leq n$

$$e^{-\varepsilon} < \frac{\|df^{-1}_{|E^u(f^k(x))}(f^k(x))\|}{\|df^{-1}_{|E^u(f^k(y))}(f^k(y))\|} < e^{\varepsilon}. \tag{20}$$

The real number ε is very small compared to ζ, thus (20) proves that for every $0 \leq k \leq n$,

$$\|df^{k-n}_{E^u(f^k(y))}\| < (\sqrt{\rho_4})^{n-k}.$$

This proves that n is a $\sqrt{\rho}_4$-hyperbolic time for every y. □

For the rest of the proof, we assume that δ is smaller than δ'.

LEMMA 3.12. *Let x be in $\mathcal{S}_0$ and n be some ρ_4-hyperbolic time for x. Then, $f^n(x)$ is in $\mathcal{S}_0$.*

Proof. By the definition of the ρ_4-hyperbolic times, $\|df^{-1}_{|E^u(f^n(x))}\| < \rho_4 = e^{-(1/3)\zeta}$, which implies that $f^n(x)$ is in Ω_3 (by the definition of ε_2). Hence, $f^n(x)$ is in $\mathcal{S}_0$. □

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Conversely, if x and $f(x)$ belong to $\mathcal{S}_0$, then 1 is a ρ_4-hyperbolic time. Therefore, we say that a ρ_4-hyperbolic time for x is a *hyperbolic return time in* $\mathcal{S}_0$. Moreover, the Markov property of $\mathcal{R}$ proves that, if n is a ρ_4-hyperbolic time for x, then there exists $R_k \in \mathcal{R}$ such that $f^n(x)$ is in R_k and we have

$$f^{-n}(W^u(f^n(x), R_k)) \subset \mathcal{S}_0.$$

3.5.2. *First return time map and special rectangle R.* For x in $\mathcal{S}_0$, we define some return time $\tau(x)$ by

$$\tau(x) = \inf\{n \in \mathbb{N}^* : \exists y \in f^{-n}(W^u(f^n(x), \mathcal{R}(f^n(x)))), n \text{ is a hyperbolic return time for } y\},$$

where $\mathcal{R}(f^n(x))$ denotes the rectangle of third generation which contains $f^n(x)$. This defines a map g_0 from $\mathcal{S}_0$ to itself by setting $g_0(x) \stackrel{\text{def}}{=} f^{\tau(x)}(x)$.

Remark. Because of the Markov property, the map g_0 is well defined. It is an injection but not necessary a bijection.

PROPOSITION 3.13. *There exists at least one rectangle of third generation R_k in $\mathcal{R}$ such that:*

(1) $\text{Leb}^u(R_k) > 0$;

(2) *for each x in R_k, the set of points y in $W^u(x, R_k)$ which return infinitely often in R_k by the iteration of g_0 has positive Leb^u_x-measure.*

Proof. We first note that Δ has positive Leb^u-measure, thus $\text{Leb}^u(\mathcal{S}_0) > 0$. Moreover, $\mathcal{S}_0$ is covered by a finite number of rectangles of third generation, every point in $\mathcal{S}_0$ is a density point for Δ and every point in $\mathcal{S}_0$ returns infinitely many times in $\mathcal{S}_0$.

Therefore, there exists one rectangle R_k with positive Leb^u-measure and such that for one x in R_k, the set of points y in $W^u(x, R_k)$ which return infinitely often to R_k by the iteration of g_0 has positive Leb^u_x-measure. Now, the Markov property and the absolutely continuous property of the foliations yield (2) for every z in R_k. □

Let R_k be one rectangle of third generation as in Proposition 3.13. We denote by R the set of points y in R_k which return infinitely often to R_k (by the iteration of g_0). We also define $W^u(y, R) \stackrel{\text{def}}{=} W^u(y, R_k) \cap R$ and $W^s(y, R) \stackrel{\text{def}}{=} W^s(y, R_k) \cap R = W^s(y, R_k)$. Because of Proposition 3.9, R is a rectangle, i.e. for all $x, y \in R$,

$$\{[x, y]\} = W^s(x, R) \cap W^u(y, R). \tag{21}$$

Moreover, for all $x, y \in R$,

$$\text{Leb}^u_x(R) > 0, \tag{22}$$

and for all $x \in R$, for all $n \in \mathbb{N}$ such that $f^n(x) \in R$,

$$W^u(f^n(x), R) \subset f^n(W^u(x, R)) \quad \text{and} \quad W^s(f^n(x), R) \supset f^n(W^s(x, R)). \tag{23}$$

We denote the first return map in R (by the iteration of g_0) by g. By construction, there exists some first return time map r such that for every x in R, $g(x) = f^{r(x)}(x)$.

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As usually we denote by $r^n(x)$ the nth-return time in R by the iteration of g, namely $g^n(x) = f^{r^n(x)}(x)$ (this time can be different from the nth return time of x in R by the iteration of f).

4. *Proof of Theorem B*

4.1. *SRB-measure for* (R, g). We copy the method of [**8**]. Let x be some point in R. We denote by μ_n the measure

$$\mu_n \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=0}^{n-1} g_*^i \operatorname{Leb}_x^u,$$

and pick μ some accumulation point of the family (μ_n). To prove that μ is a SRB-measure, it is sufficient to prove that there exists some constant χ, such that for every integer n, for every y in $g^n(W^u(x, R))$ and for every z in $W^u(y, R)$,

$$e^{-\chi} \leq \frac{\prod_{i=0}^{r^n-1} J^u(f^{-i}(y))}{\prod_{i=0}^{r^n-1} J^u(f^{-i}(z))} \leq e^{\chi}. \tag{24}$$

We have chosen the map g in relation to hyperbolic times. Hence, we have some distortion bounds.

LEMMA 4.1. *There exists* $0 < \omega_0 < 1$ *such that for every* z *in* $\mathcal{S}_0$ *and* y *in* $g_0^{-1}(W^u(g_0(z), \mathcal{R}(g_0(z))))$, *and for every* $0 \leq k \leq \tau(z)$,

$$d^u(f^k(z), f^k(y)) \leq (\omega_0)^{\tau(z)-k} d^u(g_0(z), g_0(y)).$$

Proof. If z is in $\mathcal{S}_0 \cap f^{-1}(\mathcal{S}_0)$, then $g_0(z) = f(z)$ and z is in Ω_3 (far away from S).

If $g_0(z) = f^{\tau(z)}(z)$ with $\tau(z) > 1$, then by the definition of g_0, there exists z' such that:

(i) $\tau(z)$ is a ρ_4-hyperbolic time for z';

(ii) z' is in $\mathcal{C}(z) \stackrel{\text{def}}{=} f^{-\tau(z)}(W^u(f^{\tau(z)}(z), \mathcal{R}(f^{\tau(z)}(z))))$.

The diameter of $\mathcal{R}(f^{\tau(z)}(z))$ is smaller than $2\beta(\delta)$; hence, by construction of the third generation of rectangles, $\mathcal{C}(z)$ is included into $B_{\tau(z)+1}(z, 4\beta(\delta))$, which yields the fact that $\tau(z)$ is a $\sqrt{\rho_4}$-hyperbolic time for every point in $\mathcal{C}(z)$.

Therefore, the map $f^{k-\tau(z)} : W^u(f^{\tau(z)}(z), \mathcal{R}(f^{\tau(z)}(z))) \to f^k(\mathcal{C}(z))$ is a contraction and satisfies $\|df^{k-\tau(z)}_{|E^u}\| \leq (\sqrt{\rho_4})^{\tau(z)-k}$. Thus,

$$d^u(f^k(z), f^k(y)) \leq (\rho_4)^{(\tau(z)-k)/2} d^u(g_0(z), g_0(y)). \qquad \square$$

LEMMA 4.2. *There exist some constants* $\chi_1 > 0$ *and* $0 < \omega < 1$ *such that for every* $n \geq 1$, *for every* y *in* $W^u(x, R)$, *for every* z *in* $g^{-n}(W^u(g^n(y), R))$ *and for every* $m \leq n$, *we obtain*

$$\sum_{j=0}^{r^m(z)-1} |\log(J^u(f^j(z))) - \log(J^u(f^j(y)))| \leq \chi_1 \omega^{n-m}.$$

Proof. The map $x \mapsto J^u(x)$ is Hölder-continuous because the map E^u is Hölder-continuous; moreover, it has its values in $[1, +\infty)$. The map $t \mapsto \log(t)$ is Lipschitz-continuous on $(1, +\infty)$. Thus, there exist some constants χ_2 and α, such that

$$\sum_{j=0}^{r^m(z)-1} |\log(J^u(f^j(z))) - \log(J^u(f^j(y)))| \leq \chi_2 \sum_{j=0}^{r^m(z)-1} (d^u(f^j(z), f^j(y)))^{\alpha}. \tag{25}$$

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Hence, Lemma 4.1 and (25) give

$$\sum_{j=0}^{r^m(z)-1} |\log(J^u(f^j(z))) - \log(J^u(f^j(y)))| \leq \chi_2 \left[\sum_{j=0}^{+\infty} \rho_4^{j\alpha/2} \right] (d^u(g^m(z), g^m(y)))^\alpha.$$

Lemma 4.1 also gives $d^u(g^m(z), g^m(y)) \leq \rho_4^{(n-m)/2} \operatorname{diam}(R)$. □

Lemma 4.2 proves that (24) holds for every n, for every y in $g^n(W^u(x, R))$ and for every z in $W^u(y, R)$, with $\chi := \chi_1 + 2\log\kappa$. Then the measure μ is a SRB-probability measure for (R, g).

4.2. *SRB-measure for* (M, f). The map g satisfies $g(z) = f^{r(z)}(z)$ for every z in R. Let $R(i) = \{z \in R : r(z) = i\}$ be the set of points in R such that the first return time equals i. We set

$$m = \sum_{i=1}^{+\infty} \sum_{j=0}^{i-1} f_*^j(\mu | R(i)).$$

Then m is at least a σ-finite SRB-measure. It is finite if and only if $\int_F r(x) d\operatorname{Leb}_F < \infty$. In this case, we may normalize the measure to obtain some probability measure.

4.3. *Markov partitions.* There are several possibilities to construct a Markov partition from our cover. The simplest is to use the iterates of R: if x is in $R(n)$, then for every $0 < k < n$ we set

$$C(f^k(x)) = f^k([f^{-n}(W^u(f^n(x)), R), W^s(x, R)]).$$

This defines a countable family of rectangles which satisfy the Markov property. Each rectangle has a diameter smaller than $2\beta(\delta)$. This is a Markov partition of Δ but Δ is strictly included in Λ.

However, we could do the same with the partition $\mathcal{R}$ (instead of one of its elements) and the map g_0. This also defines a partition of Δ.

In fact, we can do it with our first generation of rectangles: we have made the rectangles of the first generation thinner because we needed some control on the distortion of the unstable-Jacobian, but the key point is that, rectangles which intersect Ω_3 must intersect only a finite number of other rectangles. This holds for the rectangles of the first generation. Therefore we can cut all the rectangles which intersect Ω_3 like Bowen did. This gives a Markov partition of $\Omega_3 \cap \Lambda$ in rectangles (T_i'). For x in an element of this partition (T_j'), there exists an integer $n > 0$ such that $f^n(x)$ is again in such an element (T_i'). If n is the smallest positive integer which satisfies this condition, then we set for $0 < k < n$,

$$C(f^k(x)) = f^k([f^{-n}(W^u(f^n)(x), T_i'), W^s(x, T_j')]).$$

This defines a Markov partition of Λ with diameter smaller than the diameter of $\mathcal{T}$.

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