Probability in the Engineering and Informational Sciences, **18**, 2004, 185–190. Printed in the U.S.A.

CONVERGENCE RATES IN APPROXIMATING A COMPOUND DISTRIBUTION

JOHN E. ANGUS
Department of Mathematics
School of Mathematical Sciences
Claremont Graduate University
Claremont, California, 91711-3988
E-mail: john.angus@cgu.edu

In connection with the classical insurance risk problem, Ross [2] develops a recursive formula for approximating a tail probability in a geometrically compounded distribution. Here, we consider rates of convergence for this approximation.

1. BACKGROUND AND MOTIVATION

Ross [2] considered the following classical insurance risk problem. Claims are made against an insurance company according to a homogeneous Poisson process with rate $\lambda > 0$. The successive claim amounts are independent and identically distributed (i.i.d.) nonnegative random variables $Y_1, Y_2, \ldots$ which are, in turn, independent of the claim process. Claim amounts have the common cumulative distribution function F and finite mean μ, $0 < \mu < \infty$. The insurance company has initial capital x and receives premiums and other income at a constant rate $c > 0$ per unit time. The ruin probability is given by

$$p(x) = P\left\{\sum_{i=1}^{v(t)} Y_i > x + ct \text{ for some } t \geq 0\right\}, \tag{1}$$

where $v(t)$ is the number of claims made up to time t. We assume $\rho = \lambda\mu/c < 1$ to avoid the trivial case in which $p \equiv 1$.

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https://doi.org/10.1017/S026996480418203X Published online by Cambridge University Press

The ruin probability has been characterized (see [3] and [4]) by

$$p(x) = P\left\{\sum_{i=1}^{T} X_i > x\right\}, \tag{2}$$

where T has geometric distribution

$$P(T = n) = \rho^n(1-\rho), \qquad n \geq 0, \tag{3}$$

and $T, X_1, X_2, \ldots$ are mutually independent, with each X_i having the equilibrium distribution associated with the renewal process generated by $Y_1, Y_2, \ldots$; that is,

$$F_e'(x) = \frac{d}{dx} P\{X_i \leq x\} = \frac{1}{\mu}(1 - F(x)), \qquad x \geq 0. \tag{4}$$

Note that from Eq. (2) and (3), it follows immediately that $p(0) = \rho$ (any sum involving $\sum_{i=1}^{0}$ will be defined to be 0).

Let $\{N(t), t \geq 0\}$ be the renewal process generated by $X_1, X_2, \ldots$, and $Z_1, Z_2, \ldots, Z_r$ be mutually independent exponential random variables each having mean x/r and independent of $\{N(t), t \geq 0\}$. Ross [2] exploited the relation

$$p(x) = E(\rho^{N(x)+1}) \tag{5}$$

and the fact that $S_r = Z_1 + Z_2 + \cdots + Z_r$ converges in probability to x to approximate Eq. (5) by $\rho m(r)$ for large r, where

$$m(k) = E(\rho^{N(Z_1+Z_2+\cdots+Z_k)}), \qquad k = 1,2,\ldots,r, \tag{6}$$

and developed a recursive relation for computing $m(r)$ given by

$$m(k) = \frac{1 - a_0 + \sum_{j=1}^{k-1} (\rho m(k-j) - 1)a_j}{1 - \rho a_0}, \qquad k = 1,2,\ldots,r, \tag{7}$$

with

$$a_j = \int_0^\infty e^{-sr/x} \frac{(sr/x)^j}{j!} F_e'(s)\, ds \tag{8}$$

for $j = 0,1,\ldots,r$. A natural question is: What is the rate at which $|p(x) - \rho m(r)| \to 0$ as $r \to \infty$?

Before addressing this question, consider the general problem of approximating $p(x)$ in Eq. (2) when $X_i \geq 0$ has an arbitrary continuous cumulative distribution function $G(x) = P\{X_1 \leq x\}$ and ρ in Eq. (3) is an arbitrary fixed number in (0,1). This is the general problem of approximating a tail probability for a geometrically compounded continuous life distribution. The above method of Ross [2] for approximating $p(x)$ by $\rho m(r)$ still has merit, with the coefficients in the recursions (7) now given by

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$$a_j = \int_{[0,\infty)} e^{-sr/x} \frac{(sr/x)^j}{j!}\, dG(s) \tag{9}$$

for $j = 0,1,\ldots,r$. In the next section, the convergence to zero for $|p(x) - \rho m(r)|$ is first established in this setting for general continuous life distribution G, and then some convergence rates are given under various additional restrictions on G. These are then related back to the original insurance risk problem.

2. CONVERGENCE

First, consider the general problem with G continuous. We have $p(0) = \rho$, so we need only to consider $x > 0$. Fix an $x > 0$. Conditioning on S_r, we have $\rho m(r) = Ep(S_r)$, and, consequently, in view of the relation

$$|\rho m(r) - p(x)| \le E|p(S_r) - p(x)|,$$

it is sufficient, by the bounded convergence theorem and the fact that S_r converges in probability to x as $r \to \infty$, to show that p is continuous at x. Denoting by $G^{(n)}$ the n-fold convolution of G with itself, as $h \to 0$,

$$\begin{aligned} |p(x+h) - p(x)| &\le \sum_{n=1}^{\infty} \int |G(x+h-u) - G(x-u)|\, dG^{(n-1)}(u)\rho^n(1-\rho) \\ &\to 0 \end{aligned}$$

by the bounded convergence theorem, establishing the continuity of p at x. This establishes $\rho m(r) \to p(x)$ as $r \to \infty$ and confirms the utility of the Ross [2] method for this problem.

To establish rates of convergence, additional assumptions are needed on G. Suppose first that G has a density g. Then, p is differentiable at $x > 0$. To see this, note that for $x > 0$, $p(0) - p(x)$ is given by

$$\begin{aligned} &\sum_{n=1}^{\infty} \int G(x-u)\, dG^{(n-1)}(u)\rho^n(1-\rho) \\ &\quad = \sum_{n=1}^{\infty} \iint_{-\infty}^{x} g(t-u)\, dt G^{(n-1)}(u)\rho^n(1-\rho) \\ &\quad = \int_{-\infty}^{x} \left(\sum_{n=1}^{\infty} \int g(t-u)\, dG^{(n-1)}(u)\rho^n(1-\rho) \right) dt \end{aligned}$$

by Fubini's theorem, since the integrand is nonnegative. It follows that the derivative of p is given by

$$p'(x) = -\sum_{n=1}^{\infty} \int g(x-u)\, dG^{(n-1)}(u)\rho^n(1-\rho). \tag{10}$$

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Now, if $g(x) \le M < \infty$ for $x \ge 0$, it follows easily that $|p'(x)| \le M\rho$ for $x > 0$. From the first-order Taylor expansion (with θ a number between 0 and 1),

$$p(S_r) = p(x) + p'(x + \theta(S_r - x))(S_r - x),$$

it follows then that

$$|p(x) - \rho m(r)| = |p(x) - Ep(S_r)| \le M\rho E|(S_r - x)|$$
$$= M\rho \frac{2x}{r} \frac{r^r e^{-r}}{(r-1)!},$$

the last equality following from the fact that S_r has the same distribution as

$$\left(\frac{x}{r}\right) \sum_{i=1}^{r} \xi_i,$$

where $\xi_1, \xi_2, \ldots$ are i.i.d. exponential random variables with mean 1, and the calculations

$$\int_0^\infty |x - r| \frac{x^{r-1} e^{-x}}{(r-1)!}\, dx = 2 \int_0^r (rx^{r-1} - x^r) \frac{e^{-x}}{(r-1)!}\, dx = \frac{2r^r e^{-r}}{(r-1)!}.$$

An application of Stirling's approximation then easily gives the rate

$$|p(x) - \rho m(r)| = O\left(M\rho x \sqrt{\frac{2}{\pi r}}\right)$$

as $r \to \infty$. Note that this case applies to the original insurance risk problem, since there $g(x) = (1/\mu)(1 - F(x))$ for $x \ge 0$, and so $g(x) \le (1/\mu)$ for all $x \ge 0$.

Finally, if G has a density g which itself has a bounded derivative g' for $x > 0$, then $p(x)$ will have a bounded second derivative for $x > 0$, given by

$$p''(x) = -\sum_{n=1}^{\infty} \int g'(x - u)\, dG^{(n-1)}(u) \rho^n (1 - \rho). \tag{11}$$

Assuming that this is true for the moment, taking $\sup_{x>0} |g'(x)| = 2Q < \infty$ and then taking expectations in a second-order Taylor expansion gives

$$\rho m(r) = Ep(S_r) = p(x) + p'(x)E(S_r - x) + R_n$$

or

$$|p(x) - \rho m(r)| \le |R_n|,$$

where

$$|R_n| \le \rho Q \operatorname{Var}(S_r) = \rho Q \frac{x^2}{r}$$

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so that

$$|p(x) - \rho m(r)| = O\left(\rho Q \frac{x^2}{r}\right)$$

as $r \to \infty$. Finally, to see that p will have a bounded second derivative for $x > 0$ if g' is bounded for $x > 0$, note that since G has a density, Eq. (10) gives

$$\frac{p'(x+h) - p'(x)}{h} = \sum_{n=1}^{\infty} \int \left(\frac{g(x-u) - g(x+h-u)}{h} \right) dG^{(n-1)}(u) \rho^n (1-\rho).$$

Now, since g' is bounded, then by the mean value theorem, where $x - u > 0$ and $x + h - u > 0$,

$$\left| \frac{g(x-u) - g(x+h-u)}{h} \right|$$

is bounded by a constant, so by the bounded convergence theorem, Eq. (11) follows, which is bounded for all $x > 0$.

For the original insurance risk problem, if F has a bounded derivative, then $g(x) = (1/\mu)(1 - F(x))$ will have a bounded derivative and we get a $O(1/r)$ convergence rate for $\rho m(r)$ to $p(x)$.

3. DISCUSSION

We may summarize the foregoing results in a theorem:

THEOREM 1: *Consider the approximation $\rho m(r)$ given by Eqs. (7) and (9) to $p(x)$ defined by Eqs. (2) and (3) with $X_1, X_2, \ldots$ i.i.d. positive random variables with continuous G and ρ a fixed number in (0,1). Then the following hold:*

1. *For any $x > 0$, $\rho m(r) \to p(x)$ as $r \to \infty$.*
2. *If G has a bounded density for $x > 0$, then for all $x > 0$, $|p(x) - \rho m(r)| = O(r^{-1/2})$.*
3. *If G has a density with a bounded derivative for $x > 0$,* then $|p(x) - \rho m(r)| = O(r^{-1})$.

Thus, for the general insurance risk problem posed at the beginning, the approximation $\rho m(r)$ always converges to $p(x)$ at least as $O(r^{-1/2})$. In addition, if the claim distribution F also has a bounded derivative, then $\rho m(r)$ will converge to $p(x)$ at least as $O(r^{-1})$.

It is hoped that this discussion will stimulate interest in applying and studying convergence rates for Ross' method, which has also been found useful in a number of other problems in renewal theory (see [1]). In particular, conditions for convergence and rates of convergence under fewer restrictions on G need further study.

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