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# Bohr–Sommerfeld orbits and quantizable symplectic manifolds

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1. *Introduction.* The phase space of a finite dimensional classical Hamiltonian system is a $C^\infty$ differentiable manifold $M$ which carries a $C^\infty$ differential 2-form $\omega$ which is closed, $d\omega = 0$, and non-singular in the sense that there is a bijective map $\alpha \to X_\alpha$ from covariant vector fields to contravariant vector fields satisfying the identity

$$\omega(X, X_\alpha) = \alpha(X).$$

Such a pair $(M, \omega)$ is called a *symplectic manifold.*

The 2-form $\omega$ defines a real cohomology class $[\omega] \in H^2(M, R)$. The symplectic manifold $(M, \omega)$ is said to be *quantizable* if $[\omega]$ lies in the image of the natural map $H^2(M, Z) \to H^2(M, \mathbb{R})$. For compact $M$ this is equivalent to the statement that the periods of $\omega$ (the integrals of $\omega$ over integral 2-cycles) are all integers. This quantization condition plays a fundamental role in Kostant's theory (1) of unitary representations of Lie groups, and in Souriau's theory (6) of geometric quantization.

In this note we show that the quantization condition, when applied to the 4-dimensional compact manifold of Kepler orbits of fixed negative energy $E$, yields energy levels which are proportional to those of Bohr–Sommerfeld–Wilson.

For the calculation it will be convenient to use the notion of the *Poisson bracket* of two $C^\infty$ functions $\phi$ and $\psi$ on a symplectic manifold $(M, \omega)$. This is defined as the function

$$\{\phi, \psi\} = X_\phi(\psi) = \omega(X_\phi, X_\psi),$$

where $X_\phi$ is the unique contravariant field on $M$ which satisfies the identity

$$\omega(X, X_\phi) = X\phi.$$

2. *The Kepler manifold for negative energy E.* The phase space corresponding to a free particle in $R^3 - \{0\}$ is

$$M = \{(\mathbf{p}, \mathbf{q}) | \mathbf{p}, \mathbf{q} \in R^3, \mathbf{q} \neq 0\} \quad \text{with} \quad \omega = \sum_{i=1}^{3} dp_i \wedge dq_i.$$

The Poisson bracket on $M$ is then given by

$$\{\phi, \psi\} = \sum_{i=1}^{3} \left( \frac{\partial \phi}{\partial p_i} \frac{\partial \psi}{\partial q_i} - \frac{\partial \phi}{\partial q_i} \frac{\partial \psi}{\partial p_i} \right).$$

The seven functions

$$h = \frac{1}{2m} p^2 - \frac{K}{q}, \quad \mathbf{l} = \mathbf{q} \times \mathbf{p}, \quad a = \mathbf{l} \times \mathbf{p} + mK \frac{\mathbf{q}}{q},$$



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where $m$ and $K$ are constants, satisfy $\mathbf{a}.\mathbf{l} = 0$ and $a^3 - 2mhl^2 = m^2K^2$. They also satisfy Poisson bracket relations of the type:

$$\{h, l_i\} = \{h, a_i\} = \{l_i, a_i\} = 0, \quad \{l_1, l_2\} = -l_3, \quad \{l_1, a_2\} = -a_3, \quad \{a_1, a_2\} = 2mhl_3.$$

In particular the functions $h$ (energy), $\mathbf{l}$ (angular momentum) and $\mathbf{a}$ (Runge–Lenz vector) are constant on the trajectories of the vector field

$$X_h = \sum_{i=1}^{3} \left( \frac{\partial h}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial h}{\partial q_i} \frac{\partial}{\partial p_i} \right).$$

The trajectories of $X_h$ are solutions of

$$\frac{dq_i}{dt} = \frac{\partial h}{\partial p_i} = \frac{1}{m} p_i \quad \text{and} \quad \frac{dp_i}{dt} = \frac{-\partial h}{\partial q_i} = \frac{-Kq_i}{q^3}$$

and correspond to solutions of $m(d^2\mathbf{q}/dt^2) = -K\mathbf{q}/q^3$ in $R^3 - \{0\}$, which is the motion of a particle of mass $m$ in the inverse square law field $-K\mathbf{q}/q^3$. We therefore call the trajectories of $X_h$ the *Kepler orbits*.

Now fix $E < 0$ and let $M_E$ denote the set of Kepler orbits on which $h$ has constant value $E$. Put $\rho = \sqrt{(-2mE)}$ and let $\mathbf{x}$ and $\mathbf{y}$ be the functions on $M_E$ induced by $\rho\mathbf{l} + \mathbf{a}$ and $\rho\mathbf{l} - \mathbf{a}$ respectively. We then have $x^2 = y^2 = m^2K^2$, and the pair $(\mathbf{x}, \mathbf{y})$ maps $M_E$ bijectively onto the product $S^2 \times S^2$ of two 2-spheres of radius $mK$. This gives $M_E$ the structure of a manifold diffeomorphic to $S^2 \times S^2$, called the *Kepler manifold* for energy $E$.

The symplectic form $\omega$ on $M$ restricts to a closed 2-form on the 5-dimensional submanifold given by $h = E$. The kernel of this 2-form at each point is 1-dimensional and tangential to the Kepler orbit. It therefore induces a closed non-singular 2-form $\omega_E$ on the 4-dimensional Kepler manifold $M_E$. Thus $(M_E, \omega_E)$ is a symplectic manifold diffeomorphic to $S^2 \times S^2$.

3. *Quantization of the Kepler manifold.* The Poisson brackets on $(M_E, \omega_E)$ of the functions $\mathbf{x}$ and $\mathbf{y}$ are the functions on $M_E$ induced by the Poisson bracket on $(M, \omega)$ of the functions $\rho\mathbf{l} + \mathbf{a}$ and $\rho\mathbf{l} - \mathbf{a}$. We therefore have relations of the form

$$\{x_1, x_2\} = -2\rho x_3, \quad \{x_i, y_j\} = 0, \quad \{y_1, y_2\} = -2\rho y_3.$$

Taking $x_1, x_2, y_1, y_2$ as coordinate functions in $M_E$ we have, in the domain of these coordinates,

$$X_\phi = \sum_{i=1}^{2} \left[ \{\phi, x_i\} \frac{\partial}{\partial x_i} + \{\phi, y_i\} \frac{\partial}{\partial y_i} \right].$$

Therefore

$$X_{x_1} = -2\rho x_3 \frac{\partial}{\partial x_2}, \quad X_{x_2} = 2\rho x_3 \frac{\partial}{\partial x_1},$$

which gives

$$\omega_E \left( 2\rho x_3 \frac{\partial}{\partial x_1}, 2\rho x_3 \frac{\partial}{\partial x_2} \right) = \omega_E(X_{x_2}, -X_{x_1}) = -\{x_2, x_1\} = 2\rho x_3.$$

Thus

$$\omega_E \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) = \frac{1}{2\rho x_3} \quad \text{and similarly} \quad \omega_E \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right) = 0 \quad \text{and} \quad \omega_E \left( \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \right) = \frac{1}{2\rho y_3}.$$

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On this coordinate patch we therefore have

$$\omega_E = \frac{dx_1 \wedge dx_2}{2\rho x_3} + \frac{dy_1 \wedge dy_2}{2\rho y_3}.$$

The 2-form $\omega_E$ has integral periods if and only if the integrals of $\omega_E$ over both factors $S^2$ in $M_E = S^2 \times S^2$ are integers. The restriction of $\omega_E$ to the first factor is $1/2\rho mK$ times the area element $mK(dx_1 \wedge dx_2)/x_3$ of the 2-sphere $S^2$ of radius $mK$. The integral of $\omega_E$ over each factor is therefore equal to $4\pi(mK)^2/2\rho mK = (2\pi mK)/\rho = N$, say.

The Kepler manifold is therefore quantizable if and only if

$$E = \frac{-\rho^2}{2m} = \frac{-1}{2m}\left(\frac{2\pi mK}{N}\right)^2 = \frac{-2\pi^2 mK^2}{N^2},$$

where $N$ is a non-zero integer.

4. *Literature.* The structure $S^2 \times S^2$ of the Kepler manifold $M_E$ has been pointed out by Souriau (6), page 157, and by Moser (2). Quantization of the Riemann sphere $S^2$ as a symplectic manifold is considered by Souriau (6), page 324, and also by Renouard (3) and Rawnsley (4) in connexion with quantization of the Poincaré and de Sitter groups.

I should like to thank the referee for his useful comments and in particular for pointing out that the relation between the integrality of $[\omega]$ and the Bohr–Sommerfeld condition can in a general situation be deduced from an exact sequence associated with a fibration by circles. He also remarks that the next step in the Kostant programme, using the complex polarization of $S^2 \times S^2$ yields the correct quantization of the hydrogen atom. These aspects are also discussed in (5).

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