7

Discrete p-Laplacians

This chapter provides an introduction to graph p-Laplacians and their analogues on hypergraphs. It describes them in variational terms and derives many spectral properties of these nonlinear operators.

As a discrete version of the p-Laplacian treated in Section 3.3, the graph p-Laplacian has been successfully used in various applications, including data and image processing problems and spectral clustering. Many recent works indicate that the graph p-Laplacians, in particular the case $p = 1$, may enhance the performance of classical algorithms that are based on the standard graph Laplacian [39, 130]. This has contributed to progress on both the theoretical and the numerical aspects of p-Laplacians on graphs and networks. A remarkable development is that the second eigenvalue has a mountain-pass characterization, and thus it is a min–max eigenvalue, and more importantly, the second eigenvalue satisfies a Cheeger inequality that turns into an equality for $p = 1$.

This chapter develops the spectral properties and estimates for the spectra of various discrete versions of the p-Laplacian, which not only extends the discussions in Section 3.3 to discrete settings but also provides some important applications of the Lovász extension in Chapter 6. In addition, the contents in this chapter establish the foundations for Cheeger-type inequalities in Chapter 8 and also nodal domain theory in Chapter 9. Some general introduction has been given in Section 2.6.2, but we shall repeat the material necessary for the developments in the present chapter.

7.1 Graph p-Laplacians ($p > 1$)

For simplicity and because this fits best into our general framework (recall, for instance, Chapter 4), we mainly discuss the normalized p-Laplacian introduced

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by Amghibech [5]. Let $C(V)$ denote the collection of real functions on the vertex set $V = \{v_1, \dots, v_N\}$ of a simple graph. Clearly, we can identify $C(V)$ with $\mathbb{R}^N$ via a standard linear isomorphism. For $1 < p < \infty$, the *normalized p-Laplacian* $\Delta_p : C(V) \to C(V)$ is defined by

$$\Delta_p f(v) = \frac{1}{\deg v} \sum_{v' \in V \,:\, v' \sim v} |f(v) - f(v')|^{p-2}(f(v) - f(v')),$$

for $v \in V$ and $f \in C(V)$. The eigenvalue problem for Δ_p consists of finding an eigenfunction $f \neq 0$ and its corresponding eigenvalue $\lambda \in \mathbb{R}$ such that

$$\Delta_p f(v) = \lambda |f(v)|^{p-2} f(v), \quad \forall v \in V. \tag{7.1.1}$$

Just like the continuous p-Laplacian $-\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ in (3.3.2), we can express the graph p-Laplacian in terms of the incidence matrix $\mathcal{I}$ and the degree matrix D as

$$\Delta_p f = D^{-1}\mathcal{I}(|\mathcal{I}^\top f|^{p-2}\mathcal{I}^\top f). \tag{7.1.2}$$

Moreover, the eigenproblem for Δ_p can be rewritten as $\Delta_p f = \lambda\, |f|^{p-2} f$.

Remark 7.1.1. The operator Δ_p is *not* the subgradient of the functional

$$f \mapsto \sum_{\{v,v'\} \in E} |f(v) - f(v')|^p.$$

The latter is the numerator of the relevant nonlinear Rayleigh quotient (7.1.3). That would rather be the operator

$$\widehat{\Delta}_p f(v) = \deg v\, \Delta_p f(v),$$

for $v \in V$ and $f \in C(V)$. With this operator, the eigenvalue problem (7.1.1) becomes

$$\widehat{\Delta}_p f(v) = \lambda \deg v\, |f(v)|^{p-2} f(v),$$

and this then leads to a variational problem for the nonlinear Rayleigh quotient below. However, in order to be consistent with the normalization that we have used for $p = 2$ in this book, we always use Δ_p and its eigenproblem (7.1.1).

Having clarified that issue, we can now see how the eigenvalues and eigenfunctions of Δ_p correspond to the critical values and critical points of the Rayleigh quotient R_p defined by

$$R_p(f) := \frac{\sum_{\{v,v'\} \in E} |f(v) - f(v')|^p}{\sum_{v \in V} \deg v \cdot |f(v)|^p}. \tag{7.1.3}$$

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We shall use the general convention

$$\phi_p(t) \coloneqq |t|^{p-2}t, \tag{7.1.4}$$

so that (7.1.1) becomes

$$\Delta_p f(v) = \lambda\, \phi_p(f(v)),\ \forall v \in V \tag{7.1.5}$$

which can be reformulated as

$$\sum_{v' \in V\,:\, v' \sim v} \phi_p(f(v) - f(v')) = \lambda \deg v\, \phi_p(f(v)),\ \forall v \in V. \tag{7.1.6}$$

Since the eigenvalue equation for $p \neq 2$ is no longer linear, sums of eigenfunctions corresponding to the same eigenvalue no longer need to be eigenfunctions, and thus, the eigenspaces in general are not linear.

In order to develop some intuition, we compute the eigenvalues of the graph p-Laplacian for general p for some examples. As we shall see, this turns out to be more difficult than in the case $p = 2$, even for simple graphs like a path or a star graph.

For the path graph P_3 with vertex set $\{v_1, v_2, v_3\}$ and edge set $\{\{v_1, v_2\}, \{v_2, v_3\}\}$, the p-Laplacian eigenvalue problem requires to solve

$$\begin{cases} \phi_p(f(v_1) - f(v_2)) = \lambda \phi_p(f(v_1)), \\ \phi_p(f(v_2) - f(v_1)) + \phi_p(f(v_2) - f(v_3)) = 2\lambda \phi_p(f(v_2)), \\ \phi_p(f(v_3) - f(v_2)) = \lambda \phi_p(f(v_3)). \end{cases}$$

By the preceding equations, we obtain

$$\begin{cases} f(v_1) - f(v_2) = \lambda^{\frac{1}{p-1}} f(v_1), \\ f(v_3) - f(v_2) = \lambda^{\frac{1}{p-1}} f(v_3), \\ \lambda(\phi_p(f(v_1)) + 2\phi_p(f(v_2)) + \phi_p(f(v_3))) = 0. \end{cases}$$

If $\lambda = 0$, then $f(v_1) = f(v_2) = f(v_3)$. For $\lambda > 0$, we obtain

$$\phi_p(f(v_1)) + 2\phi_p(f(v_2)) + \phi_p(f(v_3)) = 0.$$

If $\lambda = 1$, then $f(v_2) = 0$ and $f(v_1) = -f(v_3)$. Finally, if $\lambda > 0$ and $\lambda \neq 1$, then

$$f(v_1) = f(v_3) = (1 - \lambda^{\frac{1}{p-1}})^{-1} f(v_2) \neq 0$$

and thus

$$0 = (\phi_p(f(v_1)) + 2\phi_p(f(v_2)) + \phi_p(f(v_3)))\phi_p^{-1}(f(v_2)) = 2(1 - \lambda^{\frac{1}{p-1}})^{1-p} + 2,$$

which implies $\lambda = 2^{p-1}$. Therefore, the whole spectrum is $\{0, 2^{p-1}, 1\}$.

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We now look at the path graph P_4 with vertex set $\{v_1, v_2, v_3, v_4\}$ and edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}$. In this case, the p-Laplacian eigenvalue problem requires to solve the equations

$$\begin{cases} \phi_p(f(v_1) - f(v_2)) = \lambda \phi_p(f(v_1)), \\ \phi_p(f(v_2) - f(v_1)) + \phi_p(f(v_2) - f(v_3)) = 2\lambda \phi_p(f(v_2)), \\ \phi_p(f(v_3) - f(v_2)) + \phi_p(f(v_3) - f(v_4)) = 2\lambda \phi_p(f(v_3)), \\ \phi_p(f(v_4) - f(v_3)) = \lambda \phi_p(f(v_4)). \end{cases}$$

Since $\lambda = 0$ has, as always, a constant eigenfunction, we now look at the case $\lambda > 0$. Similarly to the case of P_3, we derive that

$$\begin{cases} f(v_1) - f(v_2) = \lambda^{\frac{1}{p-1}} f(v_1), \\ f(v_4) - f(v_3) = \lambda^{\frac{1}{p-1}} f(v_4), \\ \phi_p(f(v_1)) + 2\phi_p(f(v_2)) + 2\phi_p(f(v_3)) + \phi_p(f(v_4)) = 0. \end{cases}$$

Moreover, since it is easy to check that 1 is not an eigenvalue, we obtain

$$\begin{cases} f(v_1) = (1 - \lambda^{\frac{1}{p-1}})^{-1} f(v_2), \\ f(v_4) = (1 - \lambda^{\frac{1}{p-1}})^{-1} f(v_3), \\ \left(2 + \phi_p((1 - \lambda^{\frac{1}{p-1}})^{-1})\right)(\phi_p(f(v_2)) + \phi_p(f(v_3))) = 0. \end{cases}$$

One can then verify that

$$\lambda = (1 + 2^{\frac{1}{1-p}})^{p-1} \text{ and } f = (-2^{\frac{1}{p-1}}, 1, 1, -2^{\frac{1}{p-1}})$$

form an eigenpair. Now, if $\lambda \neq (1 + 2^{\frac{1}{1-p}})^{p-1}$, then $f(v_2) = -f(v_3)$ and

$$\begin{aligned} 2^{p-1}\phi_p(f(v_2)) &= \phi_p(f(v_2) - f(v_3)) \\ &= 2\lambda \phi_p(f(v_2)) + \lambda \phi_p(f(v_1)) \\ &= \lambda\left(2 + \phi_p((1 - \lambda^{\frac{1}{p-1}})^{-1})\right)\phi_p(f(v_2)), \end{aligned}$$

which implies

$$\lambda\left(2 + \phi_p((1 - \lambda^{\frac{1}{p-1}})^{-1})\right) = 2^{p-1}.$$

This can be reformulated as

$$\begin{cases} \lambda(2 + (1 - \lambda^{\frac{1}{p-1}})^{1-p}) = 2^{p-1} & \text{, for } \lambda \in (0, 1), \\ \lambda(2 - (\lambda^{\frac{1}{p-1}} - 1)^{1-p}) = 2^{p-1} & \text{, for } \lambda \in [1, +\infty). \end{cases} \tag{7.1.7}$$

One can then check that this has exactly two roots in $(0, +\infty)$: one is $\lambda = 2^{p-1}$ and the other one lies in the interval $(0, 1)$. In summary, the whole spectrum is

$$\{0, \lambda_2, (1 + 2^{\frac{1}{1-p}})^{p-1}, 2^{p-1}\},$$

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where λ_2 is the unique root of (7.1.7) in the interval $(0,1)$.

The next example is the star graph S_3 with vertex set $\{v_1,v_2,v_3,v_4\}$ and edge set $\{\{v_1,v_2\},\{v_1,v_3\},\{v_1,v_4\}\}$. Its p-Laplacian eigenvalue problem requires to solve the equations

$$\begin{cases} \phi_p(f(v_2)-f(v_1)) = \lambda\phi_p(f(v_2)), \\ \phi_p(f(v_3)-f(v_1)) = \lambda\phi_p(f(v_3)), \\ \phi_p(f(v_4)-f(v_1)) = \lambda\phi_p(f(v_4)), \\ \phi_p(f(v_1)-f(v_2)) + \phi_p(f(v_1)-f(v_3)) + \phi_p(f(v_1)-f(v_4)) = 3\lambda\phi_p(f(v_1)). \end{cases}$$

Again, we only need to consider the case $\lambda > 0$; therefore, the above eigenequation can be simplified as

$$\begin{cases} f(v_2)-f(v_1) = \lambda^{\frac{1}{p-1}} f(v_2), \\ f(v_3)-f(v_1) = \lambda^{\frac{1}{p-1}} f(v_3), \\ f(v_4)-f(v_1) = \lambda^{\frac{1}{p-1}} f(v_4), \\ 3\phi_p(f(v_1)) + \phi_p(f(v_2)) + \phi_p(f(v_3)) + \phi_p(f(v_4)) = 0. \end{cases}$$

For $\lambda = 1$, the corresponding eigenfunctions are the nonzero functions satisfying $f(v_1) = 0$ and

$$\phi_p(f(v_2)) + \phi_p(f(v_3)) + \phi_p(f(v_4)) = 0.$$

Thus, the multiplicity of $\lambda = 1$ is 2. If $\lambda \neq 1$ and $\lambda > 0$, then

$$f(v_2) = f(v_3) = f(v_4) = (1-\lambda^{\frac{1}{p-1}})^{-1} f(v_1),$$

therefore

$$(3 + 3\phi_p((1-\lambda^{\frac{1}{p-1}})^{-1}))\phi_p(f(v_1)) = 0,$$

which implies $(1-\lambda^{\frac{1}{p-1}})^{-1} = -1$ and thus $\lambda = 2^{p-1}$. In summary, we conclude that the whole spectrum of the star graph S_3 is $\{0,1,2^{p-1}\}$.

From the preceding examples, we can already see that the spectrum of the graph p-Laplacian, for general $p \neq 2$, is quite difficult to determine.

We shall now present some first properties of the spectrum of graph p-Laplacians. In contrast to the difficulties with the examples, the proofs are very similar to the case $p = 2$. Before that, we need to clarify the concept of the multiplicity of an eigenvalue of the p-Laplacian. For an eigenvalue λ of Δ_p, its multiplicity is defined to be the genus of the set of eigenfunctions corresponding to λ, which is formally written as

$$\operatorname{genus}\{f\colon (\lambda,f) \text{ is an eigenpair of } \Delta_p,\ \|f\|_2 = 1\}.$$

In the linear case $p = 2$, this agrees with the usual algebraic multiplicity of λ.

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Proposition 7.1.1. *The spectrum of Δ_p satisfies the following properties:*

- *The eigenvalues lie in the interval $[0, 2^{p-1}]$.*
- *The multiplicity of the eigenvalue 0 equals the number of connected components of the graph.*
- *The largest eigenvalue is 2^{p-1} if and only if the graph has a bipartite connected component. The multiplicity of 2^{p-1} equals the number of bipartite connected components.*

The proof of the above proposition is easy, and thus, we omit it.

Proposition 7.1.2. *Let (λ, f) be an eigenpair of Δ_p. Then, for any $A \subset V$,*

$$\sum_{v\in A,\, v'\notin A,\, v'\sim v} \phi_p(f(v) - f(v')) = \lambda \sum_{v\in A} \deg v \cdot \phi_p(f(v)).$$

In particular, if f is an eigenfunction corresponding to a positive eigenvalue, then

$$\sum_{v\in V} \deg v \cdot \phi_p(f(v)) = 0.$$

Moreover, if $A = \{v \in V : f(v) > 0\} \neq \emptyset$, then

$$\lambda \geq \frac{\sum_{v\in A} \deg_{V\setminus A}(v) \cdot (f(v))^{p-1}}{\sum_{v\in A} \deg v \cdot (f(v))^{p-1}},$$

where $\deg_{V\setminus A}(v)$ denotes the number of edges connecting v with the vertices in $V \setminus A$.

Proof Since f and λ satisfy the eigenequation (7.1.6), we have

$$\begin{aligned}\lambda \sum_{v\in A} \deg v \cdot \phi_p(f(v)) &= \sum_{v\in A,\, v'\in V,\, v'\sim v} \phi_p(f(v) - f(v')) \\ &= \sum_{v\in A,\, v'\notin A,\, v'\sim v} \phi_p(f(v) - f(v'))\end{aligned}$$

where we used the fact that

$$\phi_p(f(v) - f(v') + \phi_p(f(v') - f(v)) = 0$$

when $v \in A$, $v' \in A$, and $v' \sim v$. If $\lambda > 0$ and $A = V$, we simply obtain

$$\sum_{v\in A} \deg v \cdot \phi_p(f(v)) = \sum_{v\in V} \deg v \cdot \phi_p(f(v)) = 0.$$

If

$$A = \{v \in V : f(v) > 0\} \neq \emptyset,$$

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then $f(v) - f(v') \geq f(v) > 0$ for any $v \in A$ and $v' \notin A$ such that $v' \sim v$. Therefore,

$$\sum_{v\in A,\, v'\notin A,\, v'\sim v} \phi_p(f(v) - f(v')) \geq \sum_{v\in A,\, v'\notin A,\, v'\sim v} \phi_p(f(v)) = \sum_{v\in A} \deg_{V\setminus A}(v) \cdot (f(v))^{p-1}.$$

□

Proposition 7.1.3. *For a connected graph, the second eigenvalue λ_2 of Δ_p is positive, and it satisfies*

$$\lambda_2 = \min_{f \text{ nonconstant}} \frac{\sum_{\{v,v'\}\in E} |f(v) - f(v')|^p}{\min_{t\in\mathbb{R}} \sum_{v\in V} \deg v \cdot |f(v) - t|^p} \tag{7.1.8}$$

$$= \inf_{\substack{\text{curve } \gamma\colon [-1,1]\to\mathbb{R}^N\setminus\{0\},\\ \gamma(\pm 1)=\pm\mathbf{1}}} \sup_{f\in\gamma([-1,1])} R_p(f). \tag{7.1.9}$$

The *proof* is the same as for the case $p = 2$, which was treated in Section 4.1.

For more elaborated results, we now recall the variational characterization of the eigenvalues of p-Laplacians in terms of the Krasnoselskii $\mathbb{Z}_2$ genus, which we introduced in Definition 2.6.2. The Krasnoselskii genus of a centrally symmetric set S in $\mathbb{R}^N\setminus\{0\}$ was defined as the smallest k for which there exists an odd continuous map from S into the sphere $\mathbb{S}^{k-1}$. To start with a trivial example, the genus of a sphere $\mathbb{S}^{k-1}$ is k. If we consider the centrally symmetric subset

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon |x_1|^3 + |x_2|^3 + |x_3|^3 = 1,\ x_1 + x_2 + x_3 = 0\},$$

then, S is homeomorphic to $\mathbb{S}^1$ and $\text{genus}(S) = \text{genus}(\mathbb{S}^1) = 2$. Another interesting centrally symmetric set is

$$S(p) := \{\vec{x} \in \mathbb{R}^4 \colon \|\vec{x}\|_2 = 1,\ x_1 = 0,\ \phi_p(x_2) + \phi_p(x_3) + \phi_p(x_4) = 0\},$$

for $p > 1$. The set $S(p)$ is also homeomorphic to $\mathbb{S}^1$ and $\text{genus}(S(p)) = 2$. If we consider again the above example where we have computed the p-Laplacian spectrum of the star graph S_3, we conclude that the genus of the eigenspace corresponding to the eigenvalue 1 of the p-Laplacian on S_3 is 2.

For $k \geq 1$, let now

$$\text{Gen}_k := \{S \subset \mathbb{R}^n \colon S \text{ centrally symmetric with } \text{genus}(S) \geq k\}.$$

As mentioned in Section 2.6.2, for $k = 1, \ldots, N$, the constants

$$\lambda_k(\Delta_p) := \inf_{S\in\text{Gen}_k} \sup_{f\in S\setminus\{\mathbf{0}\}} R_p(f) \tag{7.1.10}$$

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are eigenvalues of Δ_p. They satisfy

$$\lambda_1 \leq \ldots \leq \lambda_N$$

and, if $\lambda = \lambda_{k+1} = \ldots = \lambda_{k+l}$ for $0 \leq k < k+l \leq N$, then

$$\text{genus}(\{\text{eigenfunctions corresponding to } \lambda\}) \geq l.$$

In addition, $\{\lambda_k(\Delta_p)\}_{k=1}^N$ defines a sequence of N critical values of the Rayleigh quotient R_p.

We now define the independence number of a graph as follows.

Definition 7.1.1. A subset $U \subseteq V$ of the vertex set of a graph $\Gamma = (V, E)$ is *independent* if $\#(U \cap e) \leq 1$ for all $e \in E$. The *independence number* of Γ is

$$\alpha := \max\{\#U \colon U \subseteq V \text{ is independent}\}.$$

Thus, a subset of the vertex set is independent if it contains at most one endpoint for each edge. Among all connected graphs with N vertices, the star graph $S_{N-1} = K_{1,N-1}$ has the largest independence number, $\alpha = N-1$. More generally, a complete bipartite graph $K_{m,n}$ has independence number $\alpha = \max(m,n)$. The complete graph K_N has the smallest independence number, $\alpha = 1$.

We now prove two results on the independence number of a graph, in relation with the eigenvalues of its p-Laplacian.

Proposition 7.1.4. *Let α be the independence number of a graph. Then,*

$$\alpha \leq \min\{\#\{i \colon \lambda_i(\Delta_p) \leq 1\}, \#\{i \colon \lambda_i(\Delta_p) \geq 1\}\}, \tag{7.1.11}$$

and the inequality is sharp.

The *proof* of this nonlinear inertia bound is a special case of the proof of Theorem 7.4.1; hence, we do not present it here.

Proposition 7.1.5. *Let Γ be a graph and let α be its independence number. If Γ has $n \geq 2$ duplicate vertices, then for any $p > 1$, 1 is an eigenvalue of Δ_p with multiplicity at least*

$$\max\{n-1, 2\alpha - N\}.$$

Proof If $2\alpha - N > 0$, then $\alpha \geq N - \alpha + 1$ and therefore

$$\lambda_{N-\alpha+1} \leq \ldots \leq \lambda_\alpha.$$

Combining this with the inequalities $1 \leq \lambda_{N-\alpha+1}$ and $\lambda_\alpha \leq 1$, which are shown in (7.1.11), we infer that

$$\lambda_{N-\alpha+1} = \ldots = \lambda_\alpha = 1,$$

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and this implies that the multiplicity of the eigenvalue 1 is at least $\alpha - (N - \alpha + 1) + 1 = 2\alpha - N$. Now, let $v_1, \ldots, v_n$ be duplicate vertices and let

$$X = \Big\{ f \in C(V) \colon |f(v_1)|^{p-2} f(v_1) + \ldots + |f(v_n)|^{p-2} f(v_n) = 0 \\ \operatorname{supp}(f) \subset \{v_1, \ldots, v_n\} \Big\}.$$

Then, it is not difficult to check that f is an eigenfunction corresponding to the eigenvalue 1 and that $\operatorname{genus}(X) = n - 1$. This implies that the multiplicity of the eigenvalue 1 is at least $n - 1$. □

If Γ has no duplicate vertices and its independence number is smaller than or equal to $N/2$, then Γ may have a large spectral gap around 1. For instance, as we have seen in Theorem 4.1.2 for $p = 2$, petal graphs (see Figure 4.6) and book graphs (see Figure 4.7) do not have duplicate vertices, they have independence number $\alpha = [N/2]$, and they have a large spectral gap around 1.

Remark 7.1.2. For $p \neq 2$ and for some graphs, there may exist eigenvalues of the p-Laplacian that are not min–max (variational) eigenvalues. That is, the variational scheme does not necessarily produce all the eigenvalues. We refer to [278] for a refined analysis of such nonvariational eigenvalues of graph p-Laplacians. In this reference it is also shown that, for $k = 1, \ldots, N$,

- The function
$$p \mapsto p(2\lambda_k(\Delta_p))^{\frac{1}{p}}$$
is increasing on $[1, +\infty)$;
- The function
$$p \mapsto 2^{-p} \lambda_k(\Delta_p)$$
is decreasing on $[1, +\infty)$.

7.2 The Graph 1-Laplacian

The 1-Laplacian on graphs is an analogue of the 1-Laplacian in partial differential equations discussed in Section 3.3. The latter has been successfully applied for modelling the interface of two different media and the segmentation in image processing. The spectral theory for the graph 1-Laplacian was proposed by Hein and Bühler [130] for 1-spectral clustering and was later studied by Chang [52] from a variational point of view. In some sense, the graph 1-Laplacian is much closer to the essence of a graph than the linear 2-Laplacian. For example, the Cheeger constant, which has only some upper

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and lower bounds in linear spectral theory, equals the second eigenvalue of the graph 1-Laplacian [52, 130].

Let us thus present the definition of the graph 1-Laplacian that we shall utilize. This definition is in line with the more abstract setting developed in Section 2.6.2, but here we can identify the relevant set as the interval $[-1,1]$. The reason is essentially that $\frac{f}{|f|}$ becomes undetermined for $f = 0$, and we therefore admit the entire range of values from that interval. This issue did not arise for the p-Laplacian for $p > 1$, because in that case, the quantity $|f|^{p-2}f$ can be safely set to 0 when $f = 0$.

The 1-Laplacian Δ_1 is the set-valued map defined by

$$\Delta_1 f(v) = \left\{ \frac{1}{\deg v} \sum_{v' \in V\,:\, v' \sim v} z_{vv'} \,\middle|\, z_{vv'} \in \operatorname{Sgn}(f(v) - f(v')),\ z_{vv'} = -z_{v'v} \right\},$$

in which

$$\operatorname{Sgn}(t) := \begin{cases} \{1\} & \text{if } t > 0, \\ [-1,1] & \text{if } t = 0, \\ \{-1\} & \text{if } t < 0. \end{cases}$$

The eigenvalue problem for Δ_1 is to find $\lambda \in \mathbb{R}$ and $f \neq 0$ such that

$$\Delta_1 f(v) \bigcap \lambda \operatorname{Sgn}(f(v)) \neq \emptyset, \quad \forall v \in V,$$

which can be interpreted as the differential inclusion $0 \in \Delta_1 f - \lambda \operatorname{Sgn}(f)$, where '$-$' is understood in the sense of Minkowski addition. As in (7.1.3), we have the Rayleigh quotient

$$R_1(f) := \frac{\sum_{\{v,v'\} \in E} |f(v) - f(v')|}{\sum_{v \in V} \deg v \cdot |f(v)|}. \tag{7.2.1}$$

The Lusternik–Schnirelman theory (see for instance [152, 277]) leads to the sequence of min–max eigenvalues of Δ_1:

$$\lambda_k(\Delta_1) := \inf_{\operatorname{genus}(S) \geq k} \sup_{f \in S} \frac{\sum_{\{v,v'\} \in E} |f(v) - f(v')|}{\sum_{v \in V} \deg v |f(v)|}, \quad k = 1, 2, \ldots \tag{7.2.2}$$

where $\operatorname{genus}(S)$ is the Krasnoselskii genus of a centrally symmetric compact subset S in $C(V) \cong \mathbb{R}^V$ (see Definition 2.6.2).[1] But we now have an additional powerful tool at our disposition, the Lovász extension. In fact, the graph 1-Laplacian is a typical example of a Lovász extension. Recalling the discussion in Chapter 6, the denominator of the Rayleigh quotient $R_1(\cdot)$ is the disjoint-pair

[1] We should point out, however, that in addition to the variational eigenvalues λ_k as defined here, there may exist other eigenvalues, sitting between those λ_k, that do not arise by the scheme in (7.2.2). An example will be given below.

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Lovász extension of the volume function $(A, B) \mapsto \mathrm{vol}\,(A) + \mathrm{vol}\,(B)$, while the numerator of the Rayleigh quotient is the disjoint-pair Lovász extension of the boundary measure function $(A, B) \mapsto |\partial A| + |\partial B|$. Theorems 6.3.1, 6.3.3, and 6.3.4 then cover many important properties of Δ_1. For example,

$$\lambda_k(\Delta_1) = \inf_{S \subset 1^\perp, \mathrm{genus}(S) \geq k-1} \sup_{f \in S} \frac{\sum_{\{v,v'\} \in E} |f(v) - f(v')|}{\min_{t \in \mathbb{R}} \sum_{v \in V} \deg v |f(v) - t|}, \quad k = 2, 3, \ldots$$

Just like the continuous 1-Laplacian $-\mathrm{div}\,(\nabla u/|\nabla u|)$ in (3.3.7), we can express the graph 1-Laplacian Δ_1 as a normalized gradient-like operator, that is, a nonlinear operator involving the incidence matrix:

$$\Delta_1 f = D^{-1} \mathcal{I} \cdot \frac{\mathcal{I}^\top f}{|\mathcal{I}^\top f|}$$

is analogous to (7.1.2). To clarify the meaning of $\frac{\mathcal{I}^\top f}{|\mathcal{I}^\top f|}$ when $\mathcal{I}^\top f = 0$, we use the set-valued function $t \mapsto \mathrm{Sgn}(t)$ to replace the function $t \mapsto \frac{t}{|t|}$. Finally, we define

$$\Delta_1 f = D^{-1} \mathcal{I} \cdot \mathrm{Sgn}(\mathcal{I}^\top f),$$

which coincides with the definition in coordinate form above. In fact, the graph 1-Laplacian agrees with the subgradient of the functional

$$f \mapsto \sum_{\{v,v'\} \in E} |f(v) - f(v')|,$$

up to the inverse of the degree matrix, D^{-1}. Another remarkable relation between the cases of $p > 1$ and $p = 1$ is that $\Delta_1 f$ is actually the collection of all the limit points of $\Delta_p \widehat{f}$ as p tends to 1^+ and $\widehat{f}$ converges to f. The situation for the eigenvalues is more subtle, however.

In fact, while already the properties of the eigenvalues and eigenfunctions of Δ_p for $p > 1$ were different from those of Δ_2 when $p \neq 2$, when we turn to Δ_1, almost everything ceases to hold, as we shall see in this section and also in Section 9.2. The smallest eigenvalue is still 0, with a constant eigenfunction when Γ is connected. This will be established in Theorem 7.2.1, among with other basic properties, for instance that the largest eigenvalue is at most 1. However, the number of eigenvalues need no longer be equal to the number N of vertices, but can be substantially larger, and the sum of the eigenvalues will no longer be N. Eigenfunctions f no longer satisfy orthogonality relations, and in particular, we no longer have the relation $\sum_v \deg v f(v) = 0$ for nonconstant eigenfunctions. In fact, eigenfunctions need not even change sign. Importantly, and very usefully, when f is an eigenfunctions, so is $\frac{f}{|f|}$, as follows for instance from the characterization (7.2.2) by a Rayleigh quotient. That means that eigenfunctions

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can be characteristic functions of sets, and we can then in turn try to identify those sets whose characteristic functions are eigenfunctions. In Section 9.2, we shall consider these sets as nodal domains, realizing Cheeger cuts, that is, optimal decompositions of the underlying graph. Thus, not only will λ_2 of Δ_1 equal the Cheeger constant, see Theorem 7.2.2 below, but even better, the corresponding eigenfunction will yield an optimal decomposition. So, let us start with the observation that eigenfunctions can be assumed to be characteristic functions of vertex sets.

Proposition 7.2.1. *For any eigenfunction f corresponding to an eigenvalue λ of Δ_1,*

$$(\lambda, \mathbf{1}_{\{f>0\}}) \quad (\textit{respectively, } (\lambda, \mathbf{1}_{\{f<0\}}))$$

is also an eigenpair if $\{f > 0\} \neq \emptyset$ (respectively, $\{f < 0\} \neq \emptyset$). In particular, every eigenvalue of Δ_1 has an eigenfunction of the form $\mathbf{1}_A$ for some $A \subset V$.

Proof This is a consequence of the proof of Proposition 6.3.1 and Theorem 6.3.1. □

This result is of course different from what holds in the case $p > 1$, and it is an indication that the 1-Laplacian, in spite of the additional technical difficulties, possesses some very useful properties. In particular, since eigenfunctions can be chosen as indicator functions of vertex sets, we can then relate the properties of such sets to the eigenvalues.

Corollary 7.2.1. *The eigenvalue corresponding to the eigenfunction $\mathbf{1}_A$ of Proposition 7.2.1 is*

$$\lambda = \frac{|\partial A|}{\operatorname{vol} A}. \tag{7.2.3}$$

Proof This follows from the variational characterization (7.2.2) in terms of the Rayleigh quotient (7.2.1). □

Theorem 7.2.1. *The spectrum of Δ_1 satisfies the following properties.*

- *The eigenvalues lie in the interval* $[0, 1]$.
- *The multiplicity of the eigenvalue* 0 *equals the number of connected components of the graph.*
- *The largest eigenvalue is* 1*, and it is realized by eigenvectors of the form $\mathbf{1}_v$, where v is a vertex.*

The last property, in particular, will be very useful for evaluating the Cheeger constant, which is one of our main topics.

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Proof We split the proof into four parts.

- Observe that, for any $z_{vv'} \in \mathrm{Sgn}(f(v) - f(v'))$ with $z_{vv'} = -z_{v'v}$, we have

$$\begin{aligned}\sum_{v\in V} f(v) \sum_{v'\in V\,:\, v'\sim v} z_{vv'} &= \sum_{v\in V} \sum_{v'\in V\,:\, v'\sim v} z_{vv'} f(v) \\ &= \sum_{\{v,v'\}\in E} (z_{vv'} f(v) + z_{v'v} f(v')) \\ &= \sum_{\{v,v'\}\in E} z_{vv'}(f(v) - f(v')) \\ &= \sum_{\{v,v'\}\in E} |f(v) - f(v')|,\end{aligned}$$

  and

$$\sum_{v\in V} f(v) \deg v \,\mathrm{Sgn}(f(v)) = \sum_{v\in V} \deg v\, f(v)\mathrm{Sgn}(f(v)) = \sum_{v\in V} \deg v\, |f(v)|.$$

  Thus,

$$\langle \Delta_1 f, Df\rangle = \langle D\Delta_1 f, f\rangle = \sum_{\{v,v'\}\in E} |f(v) - f(v')|$$

  and

$$\langle \mathrm{Sgn}(f), Df\rangle = \langle D\mathrm{Sgn}(f), f\rangle = \sum_{v\in V} \deg v\, |f(v)|,$$

  where $\langle\cdot,\cdot\rangle$ denotes the inner product. Given any eigenpair (λ, f) of Δ_1, by taking the inner product of both sides of the eigenequation $0 \in \Delta_1 f - \lambda D\mathrm{Sgn}(f)$ with Df, we then obtain

$$\sum_{\{v,v'\}\in E} |f(v) - f(v')| = \lambda \sum_{v\in V} \deg v\, |f(v)|,$$

  which implies $\lambda \geq 0$. Also, since

$$\sum_{\{v,v'\}\in E} |f(v) - f(v')| \leq \sum_{\{v,v'\}\in E} (|f(v)| + |f(v')|) = \sum_{v\in V} \deg v\, |f(v)|, \tag{7.2.4}$$

  we must have $\lambda \leq 1$. Hence, the spectrum of Δ_1 is contained in the interval $[0, 1]$.
- The eigenfunctions corresponding to the eigenvalue 0 of Δ_1 are determined by the equation

$$\sum_{\{v,v'\}\in E} |f(v) - f(v')| = 0.$$

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Its solutions form the linear subspace $\operatorname{span}(\mathbf{1}_{V_1}, \ldots, \mathbf{1}_{V_k})$, where $V_1, \ldots, V_k$ are the connected components of the vertex set. Consequently, the multiplicity of the eigenvalue 0 equals k, the number of connected components of the graph.

- The vector $\mathbf{1}_v$ is an eigenvector corresponding to the eigenvalue 1 of Δ_1 because it realizes the supremum in (7.2.2). Since, by the first statement, no eigenvalues are larger than 1, we conclude that the maximum eigenvalue is 1.

□

Now, given a graph $\Gamma = (V, E)$, a nonempty proper subset $A \subset V$ is called a *Cheeger set* of Γ if $\operatorname{vol} A \leq \operatorname{vol} A^c$ and

$$h = \frac{|\partial A|}{\min\{\operatorname{vol} A, \operatorname{vol} A^c\}}.$$

In this case, (A, A^c) is called a *Cheeger cut* of Γ.

Theorem 7.2.2. *For any graph, $h = \lambda_2(\Delta_1)$. Moreover,*

- *If A is a Cheeger set, then, $\mathbf{1}_A$ is an eigenfunction corresponding to $\lambda_2(\Delta_1)$.*
- *If f is an eigenfunction corresponding to $\lambda_2(\Delta_1)$, then either $\{f > 0\}$ or $\{f < 0\}$ is a Cheeger set.*

Proof The fact that the second eigenvalue coincides with the Cheeger constant is a direct consequence of Theorem 6.3.1.

For the results on eigenfunctions, by Proposition 7.2.1, we only need to prove that A is a Cheeger set if and only if $\mathbf{1}_A$ is an eigenfunction corresponding to $\lambda_2(\Delta_1)$. In fact, by the third statement of Theorem 6.3.1 (or Theorems 6.3.3 and 6.3.4),

$$\begin{aligned}\lambda_2(\Delta_1) &= \min_{f \text{ nonconstant}} \frac{\sum_{\{v,v'\}\in E} |f(v) - f(v')|}{\min_{t\in\mathbb{R}} \sum_{v\in V} |f(v) - t|} = \min_{f \text{ nonconstant}} \max_{t\in\mathbb{R}} R_1(f - t) \\ &= \min_{\emptyset \neq A \subseteq V} \frac{|\partial A|}{\min\{\operatorname{vol} A, \operatorname{vol} A^c\}} = h \end{aligned} \tag{7.2.5}$$

and thus, any nonconstant function f satisfying $R_1(f) = \max_{t\in\mathbb{R}} R_1(f - t) = \lambda_2(\Delta_1)$ is an eigenfunction corresponding to $\lambda_2(\Delta_1)$. Moreover, note that for any nonempty proper subset $A \subset V$,

$$\max_{t\in\mathbb{R}} R_1(\mathbf{1}_A - t) = \frac{|\partial A|}{\min\{\operatorname{vol} A, \operatorname{vol} A^c\}}. \tag{7.2.6}$$

The equalities (7.2.5) and (7.2.6) imply that A is a Cheeger set if and only if $\mathbf{1}_A$ is an eigenfunction corresponding to $\lambda_2(\Delta_1)$. □

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The above theorem is closely related to some more general results in Chapter 9, for instance, Theorem 9.2.1 and its consequences.

Using the above results, one can compute the entire spectrum of the 1-Laplacian for graphs, although the process can be very complicated. Interestingly, the variational eigenvalues of the 1-Laplacian have deep relations with the multiway Cheeger constants,

$$h_k(\Gamma) \coloneqq \min_{\text{disjoint } S_1,\dots,S_k} \max_{1\le i\le k} \frac{|\partial S_i|}{\operatorname{vol}(S_i)}, \tag{7.2.7}$$

for $k = 1,\dots,N$. By Theorem 2 in [54], the 1-Laplacian spectrum of the path graph P_N is

$$\left\{0, 1, \frac{1}{3}, \dots, \frac{1}{2[\frac{N}{2}]-1}, \frac{1}{2}\cdot\frac{1}{4}, \dots, \frac{1}{2[\frac{N-2}{4}]}\right\}.$$

Notably, the min–max eigenvalues of the 1-Laplacian for P_N coincide with the multiway Cheeger constants [76]. For example, when $N = 6$, we have

$$\begin{aligned}
&h_1(P_6) = \lambda_1(P_6) = 0,\\
&h_2(P_6) = \lambda_2(P_6) = \frac{1}{5},\\
&h_3(P_6) = \lambda_3(P_6) = \frac{1}{2},\\
&h_4(P_6) = h_5(P_6) = h_6(P_6) = \lambda_4(P_6) = \lambda_5(P_6) = \lambda_6(P_6) = 1.
\end{aligned}$$

Here, we simply use $\lambda_k(\Gamma)$ or λ_k to denote the k-th min–max eigenvalue of the 1-Laplacian on Γ. Since the set of eigenvalues is

$$\left\{0, \frac{1}{5}, \frac{1}{3}, \frac{1}{2}, 1\right\},$$

this implies that $1/3$ is a nonvariational eigenvalue.

Theorem 7.2.3. *Let Γ be a connected graph, and let f be an eigenfunction corresponding to λ_2. Then,*

1. *Either $\{f \ge 0\}$ or $\{f < 0\}$ induces a connected subgraph of Γ;*
2. *Either $\{f \le 0\}$ or $\{f > 0\}$ induces a connected subgraph of Γ;*
3. *Either $\{f \le 0\}$ or $\{f \ge 0\}$ induces a connected subgraph of Γ.*

Proof We divide the proof into three parts.

1. We first prove that either $\{f \ge 0\}$ or $\{f < 0\}$ induces a connected subgraph of Γ. To do so, assume that $\{f \ge 0\}$ is not connected. Then, $\{f < 0\} \ne \emptyset$, and the set $\{f \ge 0\}$ can be expressed as the disjoint union of two nonadjacent, nonempty subsets V_1 and V_2. In particular, the sets V_1, V_2, $\{f < 0\} \cup V_1$ and

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$\{f < 0\} \cup V_2$ are all nonempty. Since f is an eigenfunction corresponding to λ_2, by Proposition 7.2.1, we have that $\mathbf{1}_{\{f<0\}}$ is also an eigenfunction corresponding to λ_2. Also, $\{f < 0\}$ is a Cheeger set with

$$\frac{|\partial\{f < 0\}|}{\mathrm{vol}\,(\{f < 0\})} = \frac{|\partial\{f < 0\}|}{\min\{\mathrm{vol}\,(\{f < 0\}), \mathrm{vol}\,(\{f \geq 0\})\}} = h.$$

Since $V_1 \cup V_2$ is the complement of $\{f < 0\}$ and the sets V_1 and V_2 are nonadjacent, we also have that

$$|\partial(V_1)| + |\partial(V_2)| = |\partial\{f < 0\}|.$$

Furthermore,

$$\begin{aligned}&\min\{\mathrm{vol}\,(\{f < 0\} \cup V_2), \mathrm{vol}\,(V_1)\} + \min\{\mathrm{vol}\,(\{f < 0\} \cup V_1), \mathrm{vol}\,(V_2)\}\\ &\geq \mathrm{vol}\,(\{f < 0\}).\end{aligned}$$

Accordingly,

$$\begin{aligned}h &= \frac{|\partial\{f < 0\}|}{\mathrm{vol}\,(\{f < 0\})}\\ &\geq \frac{|\partial(V_1)| + |\partial(V_2)|}{\min\{\mathrm{vol}\,(\{f < 0\} \cup V_2), \mathrm{vol}\,(V_1)\} + \min\{\mathrm{vol}\,(\{f < 0\} \cup V_1), \mathrm{vol}\,(V_2)\}}\\ &\geq \min\left\{\frac{|\partial(V_1)|}{\min\{\mathrm{vol}\,(V_1), \mathrm{vol}\,(\{f < 0\} \cup V_2)\}}, \frac{|\partial(V_2)|}{\min\{\mathrm{vol}\,(V_2), \mathrm{vol}\,(\{f < 0\} \cup V_1)\}}\right\}\\ &\geq h.\end{aligned}$$

Therefore, we must have equality everywhere in the above inequalities, and this implies that

$$\mathrm{vol}\,(\{f < 0\}) = \mathrm{vol}\,(V_1 \cup V_2). \tag{7.2.8}$$

Now, our goal is to prove that $\{f < 0\}$ induces a connected subgraph of Γ. Assume, by contradiction, that is not true. Then, $\{f < 0\}$ can be written as the disjoint union of two nonadjacent, nonempty subsets V_3 and V_4. By (7.2.8), we then have $\mathrm{vol}\,(V_3 \cup V_4) = \mathrm{vol}\,(V_1 \cup V_2)$. Also, by the same reasoning as above,

$$\partial(V_1 \cup V_3) + \partial(V_1 \cup V_4) = \partial(V_1 \cup V_2)$$

and

$$\begin{aligned}&\min\{\mathrm{vol}\,(V_1 \cup V_3), \mathrm{vol}\,(V_2 \cup V_4)\} + \min\{\mathrm{vol}\,(V_1 \cup V_4), \mathrm{vol}\,(V_2 \cup V_3)\}\\ &> \mathrm{vol}\,(V_1 \cup V_2).\end{aligned}$$

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Therefore,

$$\begin{aligned} h &= \frac{|\partial(V_1 \cup V_2)|}{\mathrm{vol}\,(V_1 \cup V_2)} \\ &> \frac{|\partial(V_1 \cup V_3)| + |\partial(V_1 \cup V_4)|}{\min\{\mathrm{vol}\,(V_1 \cup V_3), \mathrm{vol}\,(V_2 \cup V_4)\} + \min\{\mathrm{vol}\,(V_1 \cup V_4), \mathrm{vol}\,(V_2 \cup V_3)\}} \\ &\geq \min\left\{ \frac{|\partial(V_1 \cup V_3)|}{\min\{\mathrm{vol}\,(V_1 \cup V_3), \mathrm{vol}\,(V_2 \cup V_4)\}}, \frac{|\partial(V_1 \cup V_4)|}{\min\{\mathrm{vol}\,(V_1 \cup V_4), \mathrm{vol}\,(V_2 \cup V_3)\}} \right\} \\ &> 0, \end{aligned}$$

and this contradicts the definition of the Cheeger constant. This proves the first claim.

2. The proof is analogous to that of the first claim.
3. Finally, we want to show that either $\{f \geq 0\}$ or $\{f \leq 0\}$ induces a connected subgraph on Γ. Suppose the contrary. Then, by the same reasoning as above, we obtain

$$\mathrm{vol}\,(\{f \geq 0\}) = \mathrm{vol}\,(\{f < 0\}) \text{ and } \mathrm{vol}\,(\{f \leq 0\}) = \mathrm{vol}\,(\{f > 0\})$$

implying that

$$\{f = 0\} = \emptyset,\ \{f > 0\} = \{f \geq 0\} \text{ and } \mathrm{vol}\,(\{f < 0\}) = \mathrm{vol}\,(\{f > 0\}),$$

but, this contradicts the first statement of the theorem. □

As a consequence of the above theorem, for any Cheeger cut (A, A^c) of a connected graph, either A or A^c induces a connected subgraph[2]. As we shall see, a more interesting consequence is a nodal domain property presented in Proposition 9.2.2 in Chapter 9.

A graph Γ is *perfect* [60] if for every induced subgraph Γ' of Γ, the chromatic number of Γ' equals the size of the largest clique of Γ'.

Proposition 7.2.2. *Fix a graph Γ and let α be its independence number. Then,*

1. *The multiplicity of the eigenvalue 1 for Δ_1 is at least α and at most 2α.*
2. *If Γ is a triangle-free perfect graph, then the multiplicity of the eigenvalue 1 for Δ_1 is α.*

Proof 1. Let t denote the multiplicity of the eigenvalue 1 for Δ_1. To prove that $t \geq \alpha$, we need to show that $\lambda_k = 1$ for all $k \geq N - \alpha + 1$.

[2] This property is folklore, and such formulation presented here was first proposed by Zhang in [131] with a proof in [53]. The proof of Theorem 7.2.3 in this book provides new insights into this folklore property.

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Note that, if $N-k+1 \leq \alpha$, then there exist at least $N-k+1$ pairwise nonadjacent vertices in Γ. We denote them by $v_1,\ldots,v_{N-k+1}$. Let $\mathbf{1}_v \in C(V)$ be the indicator function on v, that is, $\mathbf{1}_v(v)=1$ and $\mathbf{1}_v(v')=0$ for all $v' \neq v$. Let also

$$X_{N-k+1} := \operatorname{span}(\mathbf{1}_{v_1},\ldots,\mathbf{1}_{v_{N-k+1}}).$$

Note that, for each f in X_{N-k+1}, there exists $(l_1,\ldots,l_{N-k+1}) \neq \mathbf{0}$ such that

$$f = l_1\mathbf{1}_{v_1} + \ldots + l_{n-k+1}\mathbf{1}_{v_{N-k+1}}.$$

Also, if $f(v)f(v') \neq 0$, then $v \nsim v'$. Thus, each nodal domain of f must be a singleton set, which implies that $R_1(f)=1$. Therefore, the intersection property of the genus derived in Lemma 2.6.3 implies that

$$A \cap X_{N-k+1} = A \cap \operatorname{span}(\mathbf{1}_{v_1},\ldots,\mathbf{1}_{v_{N-k+1}}) \neq \emptyset$$

for any A with $\operatorname{genus}(A) \geq k$. Hence,

$$\sup_{f\in A} R_1(f) \geq \inf_{f\in X_{N+1-k}} R_1(f) = 1,$$

implying that $\lambda_k = 1$. This shows that $t \geq \alpha$.

To prove that $t \leq 2\alpha$, it suffices to show that $\lambda_k < 1$ for all $k \leq N-2\alpha$. Given $k \leq N-2\alpha$, let $g_i \in C(V)$ be defined by $g_i(v_\nu) := \nu^{i-1}$ for $\nu \in \{1,\ldots,N\}$ and $i \in \{1,2,\ldots,k\}$. Let also $X_k := \operatorname{span}(g_1,\ldots,g_k)$. Then, X_k is a closed symmetric set with $\operatorname{genus}(X_k)=k$. Now, for any $f \neq \mathbf{0}$ in X_k, there exists $(l_1,l_2,\ldots,l_k) \neq \mathbf{0}$ such that $f = l_1g_1 + \ldots + l_kg_k$. By the properties of the Vandermonde matrix, not every element of the form l_ig_i, for $i=1,\ldots,k$, can vanish on k vertices simultaneously. This implies that the number of zeros of f is at most $(k-1)$. Therefore, the support of f has cardinality at least $N+1-k$. By the pigeonhole principle, it follows that at least $\lceil \frac{N+1-k}{2} \rceil$ vertices with a nonzero f-value have the same sign, where $\lceil \frac{N+1-k}{2} \rceil$ is the smallest integer larger than or equal to $\frac{N+1-k}{2}$. Since $\alpha < \lceil \frac{N+1-k}{2} \rceil$, there exist two adjacent vertices whose values for f have the same sign, which implies that $R_1(f) < 1$. Note that R_1 is continuous on the compact set $X_k \cap \mathbb{S}^{N-1}$, and we have proved that $R_1(f) < 1$ for any $f \in X_k$. Since every continuous function on a compact set attains its maximum, this implies that

$$\lambda_k \leq \sup_{f\in X_k} R_1(f) = \max_{f\in X_k\cap\mathbb{S}^{N-1}} R_1(f) < 1.$$

2. By [278], any triangle-free graph is such that

$$\alpha \leq t \leq \kappa,$$

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where κ denotes the clique covering number of the graph, that is, the smallest possible number of cliques that can cover Γ. Moreover, if Γ is a perfect graph, then $\alpha = \kappa$. By putting everything together, we obtain that $\alpha = t = \kappa$, if Γ is both triangle-free and perfect.

□

We refer to [278] for more results on the multiplicity of the eigenvalue 1 for Δ_1.

Proposition 7.2.3. *For a simple graph $\Gamma = (V, E)$ on N nodes, the 1-Laplacian Δ_1 has at most $\mathrm{vol}\,(V)^2/4 + 1$ distinct eigenvalues. Hence, in particular, Δ_1 has at most $(N^2(N-1)^2)/4 + 1$ distinct eigenvalues, and counting multiplicity, the number of the eigenvalues of Δ_1 does not exceed $O(N^5)$.*

Proof Let λ and $\tilde{\lambda}$ be two distinct eigenvalues. By Proposition 7.2.1 and Corollary 7.2.1, there exist $A, B \subset V$ with $\mathrm{vol}\,(A), \mathrm{vol}\,(B) \leq \mathrm{vol}\,(V)/2$, such that

$$\lambda = \frac{|\partial A|}{\mathrm{vol}\,(A)} \quad \text{and} \quad \tilde{\lambda} = \frac{|\partial B|}{\mathrm{vol}\,(B)}.$$

We have that

$$\begin{aligned} |\lambda - \tilde{\lambda}| &= \left| \frac{|\partial A|}{\mathrm{vol}\,(A)} - \frac{|\partial B|}{\mathrm{vol}\,(B)} \right| \\ &= \frac{||\partial A| \mathrm{vol}\,(B) - |\partial B| \mathrm{vol}\,(A)|}{\mathrm{vol}\,(A)\mathrm{vol}\,(B)} \\ &\geq \frac{1}{\mathrm{vol}\,(A)\mathrm{vol}\,(B)} \\ &\geq \frac{4}{\mathrm{vol}\,(V)^2}, \end{aligned}$$

which implies that the distance between two distinct eigenvalues is at least $4/\mathrm{vol}\,(V)^2$. Since the eigenvalues lie in $[0,1]$, this implies that there are at most $\mathrm{vol}\,(V)^2/4 + 1$ distinct eigenvalues of Δ_1. This proves the first claim. The second claim follows from the fact that $\mathrm{vol}\,(V) \leq N(N-1)$ and that the multiplicity of any eigenvalue does not exceed N. □

The bound $O(N^5)$ from Proposition 7.2.3 may seem excessive, but the graph 1-Laplacian can indeed have many eigenvalues. For example, let $\Gamma = (V, E)$ be the graph displayed in Figure 7.1 with vertex set $V = \{v_1 \ldots, v_6\}$: Then, as shown in [278], the spectrum of Δ_1 is

$$\left\{0, \frac{2}{5}, \frac{5}{9}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, 1\right\}.$$

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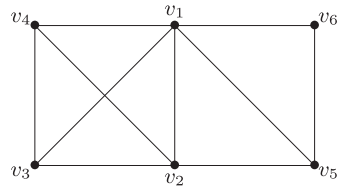


Figure 7.1 A graph with six vertices and nine distinct 1-Laplacian eigenvalues.

Hence, the graph has six nodes, and its 1-Laplacian has nine distinct eigenvalues.

7.3 The Graph ∞-Laplacian

In this section, we consider the ∞-Laplacian on graphs, which has been studied, for instance, in [35, 40, 77, 157, 186]. As we shall see, the spectrum of this operator is related to the maximal radius that allows us to inscribe a fixed number of disjoint balls in the graph.

As the 1-Laplacian, also the ∞-Laplacian Δ_∞ on a graph $\Gamma = (V, E)$ is the subgradient of a functional. (Recall (6.1.6), (6.1.7) for the definition of the subgradient.) Here, such functional is $f \mapsto \sum_{\{v,v'\}\in E} |f(v) - f(v')|$, that is,

$$\Delta_\infty f := \nabla_f \max_{\{v,v'\}\in E} |f(v) - f(v')|. \tag{7.3.1}$$

In this case, we are interested in the nonlinear eigenvalue problem

$$0 \in \Delta_\infty f - \lambda \nabla \|f\|_\infty, \tag{7.3.2}$$

and we say that (λ, f) is an eigenpair of the ∞-Laplacian Δ_∞ on Γ if (7.3.2) holds. Thus, (λ, f) is an eigenpair of Δ_∞ if there exist $z_{vv'}$ and z_v with

$$\sum_{v'\in V\,:\,v'\sim v} z_{vv'} = \lambda z_v, \quad \forall v \in V,$$

$$\begin{cases} z_{vv'}(f(v) - f(v')) \geq 0, z_{v'v} = -z_{vv'} \\ \sum_{vv'\in E} |z_{vv'}| = 1 \\ z_{vv'} = 0 \text{ if } |f(v) - f(v')| < \max\limits_{\hat{v}\tilde{v}\in E} |f(\hat{v}) - f(\tilde{v})| \end{cases} \quad \text{and} \quad \begin{cases} z_v f(v) \geq 0 \\ \sum_{v\in V} |z_v| = 1 \\ z_v = 0 \text{ if } |f(v)| \\ < \|f\|_\infty. \end{cases} \tag{7.3.3}$$

The following characterization of the first nonzero eigenvalue of Δ_∞ has been given in [186].

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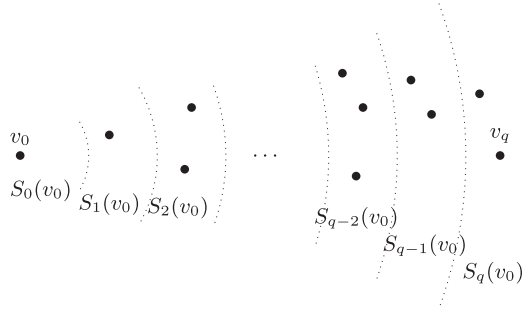


Figure 7.2 A sequence of concentric discrete spheres, used in the proof of Proposition 7.3.1.

Proposition 7.3.1. *For any connected graph* Γ*,*

$$\lambda_2(\Delta_\infty) = \frac{2}{\operatorname{diam}(\Gamma)}.$$

Proof Similarly to the proofs of (6.4.1) and Theorem 6.3.1 or by applying Theorem 2.1 in [160] to the functionals $f \mapsto \sum_{\{v,v'\}\in E} |f(v) - f(v')|$ and $f \mapsto \|f\|_\infty$ directly, the second eigenvalue $\lambda_2(\Delta_\infty)$ can be written as

$$\min_{f \text{ nonconstant}} \frac{\max_{\{v,v'\}\in E} |f(v) - f(v')|}{\min_{t\in\mathbb{R}} \max_{v\in V} |f(v) - t|} = \min_{\substack{\max f = 1 \\ \min f = -1}} \max_{\{v,v'\}\in E} |f(v) - f(v')|. \tag{7.3.4}$$

Now, let q be the diameter of Γ, let v_0 and v_q be a pair of vertices that have distance q, and let $v_0, v_1, \ldots, v_q$ form a shortest path between v_0 and v_q. Let also $f\colon V \to \mathbb{R}$ be defined by

$$f(v) \coloneqq 1 - \frac{2}{q} \cdot \operatorname{dist}(v_0, v), \quad \forall v \in V, \tag{7.3.5}$$

where $\operatorname{dist}(v_0, v)$ denotes the distance between v_0 and v. Then, the maximum value of f is 1, and the minimum value of f is -1. For $k \geq 0$, we also let

$$S_k(v_0) \coloneqq \{v \in V\colon \operatorname{dist}(v_0, v) = k\} \tag{7.3.6}$$

be the spheres that are centered at v_0, and we observe that any edge $\{v, v'\} \in E$ intersects at most two adjacent spheres $S_k(v_0)$ and $S_{k+1}(v_0)$, for some $k \geq 0$. This is illustrated in Figure 7.2.

Since f is constant on each sphere and $|f(v) - f(v')| = 2/q$ whenever $v \in S_k(v_0)$ and $v' \in S_{k+1}(v_0)$ for some $k \geq 0$, we can infer that, for any $\{v, v'\} \in E$,

$$|f(v) - f(v')| \leq \frac{2}{q},$$

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which implies that

$$\lambda_2(\Delta_\infty) \le \frac{2}{q}.$$

Now, fix f such that the maximum value of f is 1, the minimum value of f is -1, and $\|f\|_\infty = 1$. Let also $v_0, v_1, \dots, v_l$ be a shortest path satisfying $f(v_0) = 1$ and $f(v_l) = -1$. Then, $l \le q$ and

$$\begin{aligned}
\max_{\{v,v'\}\in E} |f(v) - f(v')| &\ge \max_{i=0,\dots,l-1} |f(v_i) - f(v_{i+1})| \\
&\ge \frac{1}{l} \sum_{i=0}^{l-1} |f(v_i) - f(v_{i+1})| \\
&\ge \frac{1}{l} |f(v_0) - f(v_l)| \\
&= \frac{2}{l} \\
&\ge \frac{2}{q},
\end{aligned}$$

which implies that $\lambda_2(\Delta_\infty) \ge 2/q$. □

7.4 p-Laplacians on Oriented Hypergraphs

We now extend the previous concepts and considerations to oriented hypergraphs. This section covers the entire range $1 \le p \le \infty$, that is, also includes the limiting cases $p = 1$ and $p = \infty$ that had been giving special treatments above. While some aspects are straightforward, also some new phenomena and difficulties arise. The first issue is the definition of the p-Laplacian, and in fact, several notions of p-Laplacians on hypergraphs have been proposed and investigated in the literature. For instance, the hypergraph p-Laplacians which are introduced in [133] are related to the Lovász extension. The corresponding functionals for the p-Laplacians in [133] are of the form

$$f \mapsto \sum_{e\in E} \max_{v,v'\in e} |f(v) - f(v')|^p,$$

and they are nonsmooth in general, even for $p = 2$.

In this section, we consider the p-Laplacian on oriented hypergraphs proposed in [157], which is a natural extension of the hypergraph Laplacian defined in Section 2.5.4, the case of $p = 2$.

Let $\Gamma = (V, E, \varphi)$ be an oriented hypergraph. Recall that, from Definition 2.4.7, if $\varphi(v, e) = 1$ (respectively $\varphi(v, e) = -1$), then v is said to be an *output* (respectively *input*) for e. Also, if $v \ne v'$ and

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$$\varphi(v,e) = \varphi(v',e) \neq 0 \quad (\text{respectively } \varphi(v,e) = -\varphi(v',e) \neq 0),$$

the vertices v and v' are *co-oriented* (respectively *antioriented*) in e. We also let

$$o_e(v,v') \coloneqq \begin{cases} -1, & \text{if } v \text{ and } v' \text{ are co-oriented in } e \\ 1, & \text{if } v \text{ and } v' \text{ are antioriented in } e \\ 0, & \text{otherwise.} \end{cases}$$

For $p \in \mathbb{R}_{>1}$, the *p-Laplacian* $\Delta_p : C(V) \to C(V)$ is defined by

$$\Delta_p f(v) \coloneqq \frac{1}{\deg v} \sum_{e \ni v} |\delta f(e)|^{p-2} \left(\sum_{v' \in e \colon o_e(v,v')=-1} f(v') - \sum_{v' \in e \colon o_e(v,v')=1} f(v') \right),$$

where, recalling (2.5.37),

$$\delta f(e) = \sum_{v \colon \varphi(v,e)=1} f(v) - \sum_{v \colon \varphi(v,e)=-1} f(v).$$

We define its *eigenvalue problem* as

$$\Delta_p f = \lambda |f|^{p-2} f. \tag{7.4.1}$$

A nonzero function f and a real number λ satisfying (7.4.1) are called an *eigenfunction* and the corresponding *eigenvalue* for Δ_p.

The 1-*Laplacian* is the set-valued operator which, for $f \in C(V)$, is defined by

$$\Delta_1 f \coloneqq \left\{ \sum_{v \in V} \frac{1}{\deg v} \sum_{e \ni v} z_{ve} \mathbf{1}_v \;\middle|\; \begin{array}{l} z_{ve} \in \operatorname{Sgn}\left(\sum_{v' \in e \colon o_e(v,v')=-1} f(v') - \sum_{v' \in e \colon o_e(v,v')=1} f(v') \right) \\ z_{ve} = -o_e(v,v') z_{v'e} \end{array} \right\}$$

where $\mathbf{1}_1, \dots, \mathbf{1}_N$ form the orthonormal basis of $\mathbb{R}^N$ and

$$\operatorname{Sgn}(t) \coloneqq \begin{cases} \{1\} & \text{if } t > 0, \\ [-1,1] & \text{if } t = 0, \\ \{-1\} & \text{if } t < 0. \end{cases}$$

The *eigenvalue problem* of Δ_1 is to find the eigenpairs (λ, f) for which there exist

$$z_{ve} \in \operatorname{Sgn}\left(\sum_{v' \in e \colon o_e(v,v')=-1} f(v') - \sum_{v' \in e \colon o_e(v,v')=1} f(v') \right)$$

with $z_{ve} = -o_e(v,v') z_{v'e}$ for $v, v' \in e$, and $z_v \in \operatorname{Sgn}(f(v))$ satisfying

$$\sum_{e \ni v} z_{ve} = \lambda \deg v \cdot z_v, \quad \forall v \in V.$$

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A shorter coordinate form of the eigenvalue problem for the 1-Laplacian is

$$\exists z_v \in \mathrm{Sgn}(f(v)) \text{ and } z_e \in \mathrm{Sgn}\left(\sum_{v' :\, \varphi(v',e)=1} f(v') - \sum_{v' :\, \varphi(v',e)=-1} f(v')\right)$$
$$\text{such that } \sum_{e :\, \varphi(v,e)=1} z_e - \sum_{e :\, \varphi(v,e)=-1} z_e = \lambda \deg v \cdot z_v, \quad \forall v \in V. \tag{7.4.2}$$

Moreover, we can rewrite (7.4.2) as

$$\exists z_e \in \mathrm{Sgn}\left(\sum_{v' :\, \varphi(v',e)=1} f(v') - \sum_{v' :\, \varphi(v',e)=-1} f(v')\right) \text{ such that}$$
$$\sum_{e :\, \varphi(v,e)=1} z_e - \sum_{e :\, \varphi(v,e)=-1} z_e \in \lambda \deg v \cdot \mathrm{Sgn}(f(v)), \quad \forall v \in V. \tag{7.4.3}$$

Similar to the graph ∞-Laplacian presented in Section 7.3, we also define the ∞-Laplacian Δ_∞ on an oriented hypergraph as the subgradient of the functional $f \mapsto \|\delta f\|_\infty$. The kth min–max eigenvalue $\lambda_k(\Delta_\infty)$ is similarly defined as

$$\inf_{S \in \mathrm{Gen}_k} \sup_{f \in S \setminus \{\mathbf{0}\}} \frac{\|\delta f\|_\infty}{\|f\|_\infty}.$$

Then, we have

Proposition 7.4.1. *For any $k \geq 1$,*

$$\lim_{p \to +\infty} (\lambda_k(\Delta_p))^{\frac{1}{p}} = \lambda_k(\Delta_\infty).$$

Proof We have that

$$\frac{|E|^{\frac{1}{p}} \max_{e \in E} |\delta f(e)|}{\max_{v \in V} |f(v)|} \geq \frac{(\sum_{e \in E} |\delta f(e)|^p)^{\frac{1}{p}}}{(\sum_{v \in V} \deg v\, |f(v)|^p)^{\frac{1}{p}}} \geq \frac{\max_{e \in E} |\delta f(e)|}{\mathrm{vol}\,(V)^{\frac{1}{p}} \max_{v \in V} |f(v)|},$$

which implies

$$|E|^{\frac{1}{p}} \frac{\|\delta f\|_\infty}{\|f\|_\infty} \geq (\mathrm{RQ}_p(f))^{\frac{1}{p}} \geq \mathrm{vol}\,(V)^{-\frac{1}{p}} \frac{\|\delta f\|_\infty}{\|f\|_\infty}.$$

By definition of min–max eigenvalues, we then obtain

$$|E|^{\frac{1}{p}} \lambda_k(\Delta_\infty) \geq (\lambda_k(\Delta_p))^{\frac{1}{p}} \geq \mathrm{vol}\,(V)^{-\frac{1}{p}} \lambda_k(\Delta_\infty).$$

Therefore, $(\lambda_k(\Delta_p))^{\frac{1}{p}}$ tends to $\lambda_k(\Delta_\infty)$, as $p \to +\infty$. □

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The *independence number* of a hypergraph Γ is defined in the same manner as that for a graph in Definition 7.1.1. Thus, a subset $U \subseteq V$ is *independent* if $\#(U \cap e) \leq 1$ for all $e \in E$, and the independence number is

$$\alpha := \max\{\#U \colon U \subseteq V \text{ independent}\}.$$

Theorem 7.4.1. *Let $p \in [1,\infty]$. Then,*

$$\alpha \leq \min\{\#\{i \colon \lambda_i(\Delta_p) \leq 1\}, \#\{i \colon \lambda_i(\Delta_p) \geq 1\}\}, \tag{7.4.4}$$

and the inequality is sharp.

Proof We first consider the case where $p < \infty$. Let $U \subseteq V$ be an independent set such that $\#U = \alpha$. Then, for any $f \in C(V)$ with $\operatorname{supp}(f) \subset U$,

$$\sum_{e \in E} |\delta f(e)|^p = \sum_{e \in E} \sum_{v \in e} |f(v)|^p = \sum_{v \in V} \sum_{e \in E, e \ni v} |f(v)|^p = \sum_{v \in V} \deg v \cdot |f(v)|^p,$$

which implies that $\mathcal{R}_p(f) = 1$. Now, the linear subspace

$$X := \{f \in C(V) \colon \operatorname{supp}(f) \subset U\}$$

has dimension α, and for any $f \in X$, $\mathcal{R}_p(f) = 1$. Thus,

$$\lambda_1(\Delta_p) \leq \ldots \leq \lambda_\alpha(\Delta_p) = \inf_{X' \in \mathrm{Gen}_\alpha} \sup_{f \in X'} \mathcal{R}_p(f) \leq \sup_{f \in X} \mathcal{R}_p(f) = 1.$$

Similarly, by the intersection property of the $\mathbb{Z}_2$-genus (see Lemma 2.6.3), for any $X' \in \mathrm{Gen}_{N-\alpha+1}$, we have that $X' \cap X \neq \emptyset$. Hence,

$$\lambda_N(\Delta_p) \geq \ldots \geq \lambda_{N-\alpha+1}(\Delta_p) = \inf_{X' \in \mathrm{Gen}_{N-\alpha+1}} \sup_{f \in X'} \mathcal{R}_p(f) \geq \inf_{f \in X} \mathcal{R}_p(f) = 1.$$

Hence, there are at least α eigenvalues which are smaller (respectively, larger) than 1. This proves the first claim. To see that (7.4.4) is sharp, assume that Γ is such that $V = \{v_1, \ldots, v_N\}$ and $E = \{\{v_1\}, \ldots, \{v_N\}\}$. Then, clearly, $\alpha = N$ and all eigenvalues are equal to 1. This proves the claim for $p < \infty$.

In the case when $p = \infty$, assume, without loss of generality, that

$$\begin{aligned}\lambda_1(\Delta_\infty) \leq \ldots \leq \lambda_k(\Delta_\infty) < 1 = \lambda_{k+1}(\Delta_\infty) = \ldots \\ = \lambda_{k+r}(\Delta_\infty) < \lambda_{k+r+1}(\Delta_\infty) \leq \ldots\end{aligned}$$

By Proposition 7.4.1, $\lambda_i(\Delta_p)^{\frac{1}{p}} \to \lambda_i(\Delta_\infty)$ as p tends to $+\infty$, which implies that

$$\lambda_1(\Delta_p) \leq \ldots \leq \lambda_k(\Delta_p) < 1 < \lambda_{k+r+1}(\Delta_p) \leq \ldots \leq \lambda_N(\Delta_p)$$

for a sufficiently large p. Therefore, by applying (7.4.4) to a sufficiently large $p < \infty$, it follows that

$$\alpha \leq \min\{\#\{i \colon \lambda_i(\Delta_p) \leq 1\}, \#\{i \colon \lambda_i(\Delta_p) \geq 1\}\}$$

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$$\leq \min\{k + r, N - k\}$$
$$= \min\{\#\{i\colon \lambda_i(\Delta_\infty) \leq 1\}, \#\{i\colon \lambda_i(\Delta_\infty) \geq 1\}\}.$$

□

A *clique* of an oriented hypergraph Γ is a set of vertices $U \subseteq V$ such that, for any $\{v, v'\} \subset U$, there exists a hyperedge $e \supset \{v, v'\}$. The *clique covering number* of Γ is

$$\kappa := \min\{k\colon \exists k \text{ cliques } C_1, \ldots, C_k \text{ s.t. } V = \cup_{i=1}^{k} C_i\}.$$

Theorem 7.4.2 is a sandwich theorem for hypergraphs with only inputs which involves the independence number, the clique covering number, and the multiplicity of the eigenvalue 1 for Δ_1.

Theorem 7.4.2. *Let Γ be an oriented hypergraph with only inputs, and let η denote the multiplicity of the eigenvalue* 1 *for the* 1*-Laplacian. Then,*

$$\alpha \leq \eta \leq \kappa.$$

Proof By Theorem 7.4.1 and since 1 is the largest eigenvalue of Δ_1,

$$\alpha \leq \#\{i\colon \lambda_i(\Delta_1) \geq 1\} = \#\{i\colon \lambda_i(\Delta_1) = 1\} \leq \eta.$$

This proves the first inequality. Now, given a clique U, let $f \in C(V)$ with $\operatorname{supp}(f) \subset U$. Since every hyperedge has only inputs,

$$R_1(f) = 1 \iff f(v)f(v') \geq 0 \ \forall v, v' \in U \iff \text{either } f \geq 0 \text{ or } f \leq 0.$$

Now, one can check that the map

$$\{f \in C(V) \setminus \{0\} \text{ with } \operatorname{supp}(f) \subset U\colon f \geq 0 \text{ or } f \leq 0\} \to \{-1, 1\}$$

defined by

$$f \mapsto \begin{cases} 1, & \text{if } f \geq 0 \\ -1, & \text{if } f \leq 0 \end{cases}$$

is an odd continuous map. Therefore, the $\mathbb{Z}_2$-genus of the set

$$\{f \in C(V) \setminus \{0\}\colon \operatorname{supp}(f) \subset U,\ R_1(f) = 1\}$$

is 1. Now, let $U_1, \ldots, U_\kappa$ be pairwise disjoint cliques with $\cup_{i=1}^{\kappa} U_i = V$, and let f be such that $R_1(f) = 1$. Then, clearly $f = \sum_{j=1}^{\kappa} f|_{U_j}$. Also, from

$$TV_1(f) \leq \sum_{j=1}^{\kappa} TV_1(f|_{U_j}) \quad \text{and} \quad \sum_{v \in V} \deg v \cdot |f(v)| = \sum_{j=1}^{\kappa} \sum_{v \in V} \deg v \cdot |f|_{U_j}(v)|,$$

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it follows that

$$1 = R_1(f) \leq \frac{\sum_{j=1}^{\kappa} TV_1(f|_{U_j})}{\sum_{j=1}^{\kappa} \sum_{v \in V} \deg v \cdot |f|_{U_j}(v)|} \leq \max_{j=1,\dots,\kappa} R_1(f|_{U_j}) \leq 1.$$

Hence, $R_1(f|_{U_j}) = 1$, for all $j = 1, \dots, \kappa$. Together with the fact that R_1 is zero-homogeneous, this implies that $R_1(N \cdot f|_{U_i}) = R_1(f|_{U_i}) = 1$, for $i = 1, \dots, \kappa$. Thus, by the fact that

$$f = \sum_{i=1}^{\kappa} f|_{U_i} = \frac{1}{N} \sum_{i=1}^{\kappa} N \cdot f|_{U_i}$$

and

$$N \cdot f|_{U_i} \in \{f \neq 0 \colon \operatorname{supp}(f) \subset U_i, R_1(f) = 1\},$$

we have

$$\{f \neq 0 \colon R_1(f) = 1\} \subset \bigbowtie_{i=1}^{\kappa} \{f \neq 0 \colon \operatorname{supp}(f) \subset U_i, R_1(f) = 1\},$$

where $\bowtie$ is the simplicial join operator defined by

$$\bigbowtie_{i=1}^{\kappa} S_i := \left\{ t_1 s_1 + \ldots + t_\kappa s_\kappa \colon s_i \in S_i, t_i \geq 0, \sum_{i=1}^{\kappa} t_i = 1 \right\},$$

for subsets $S_1, \dots, S_\kappa$ of a linear space.

The $\mathbb{Z}_2$-genus is monotonic, as established in Lemma 2.6.3, and subadditive with respect to the join operator, which follows from the fact that $S^{i_1-1} \times \cdots \times S^{i_k-1}$ naturally injects into $S^{i_1+\ldots+i_k-1}$. Therefore, we have

$$\begin{aligned}
\eta &= \operatorname{genus}\{f \neq 0 \colon R_1(f) = 1\} \\
&\leq \operatorname{genus}\left(\bigbowtie_{i=1}^{\kappa} \{f \neq 0 \colon \operatorname{supp}(f) \subset U_i, R_1(f) = 1\} \right) \\
&\leq \sum_{i=1}^{\kappa} \operatorname{genus}\{f \neq 0 \colon \operatorname{supp}(f) \subset U_i, R_1(f) = 1\} \\
&= \kappa.
\end{aligned}$$

□

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