J. London Math. Soc. (2) 70 (2004) 356–368 © 2004 London Mathematical Society
DOI: 10.1112/S0024610704005617

INJECTIVITY, PROJECTIVITY AND SUPERCUSPIDAL REPRESENTATIONS

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Abstract

Let G be a reductive p-adic group. Consider the category of smooth (complex) representations of G in which a (fixed) closed cocompact subgroup of the centre acts by a (fixed) character. It is well known that the supercuspidal representations in this category are both injective and projective. It is shown that, conversely, an admissible injective or projective object is necessarily supercuspidal.

0. *Introduction*

Let G be a unimodular locally profinite group and write $\mathfrak{R}(G)$ for the category of smooth complex representations of G. Let C be a closed cocompact subgroup of the centre Z of G and let $\chi : C \longrightarrow \mathbb{C}^\times$ be a smooth character (not necessarily unitary). Write $\mathfrak{R}_\chi(G)$ for the full subcategory of $\mathfrak{R}(G)$ whose objects are those (π, V) such that $\pi(c)v = \chi(c)v$ for all $c \in C,\ v \in V$.

Recall that a representation in $\mathfrak{R}(G)$ is *supercuspidal* if it is admissible and all its matrix coefficients have compact support modulo Z. It is an old observation of Casselman's that the supercuspidal representations in $\mathfrak{R}_\chi(G)$ are both injective and projective in this category [**4**, 5.4.1]. It is natural then to ask whether the converse holds, that is, whether an admissible injective or projective representation in $\mathfrak{R}_\chi(G)$ is necessarily supercuspidal.

Consider the projective case first. A straightforward argument shows that an irreducible admissible projective object in $\mathfrak{R}_\chi(G)$ is supercuspidal. (This follows, for example, from a trivial modification of part (b) of the proof of [**8**, 7.9].) Kutzko has proved, more generally, that a finitely generated admissible projective object in $\mathfrak{R}_\chi(G)$ is always supercuspidal [**5**]. (In fact, neither argument requires that G be unimodular. It is expected, however, that non-unimodular groups admit no non-zero supercuspidal representations.)

It is easy to see that the analogous statements, with 'injective' in place of 'projective', cannot hold in this degree of generality. Indeed, let F be a non-Archimedean local field (with finite residue field) and suppose that G is the group of F-points of a unipotent algebraic group defined over F. Let (π, V) be a smooth representation of G and put

$$V(G) = \langle \pi(g)v - v : g \in G,\ v \in V \rangle, \quad V_G = V/V(G).$$

Let $\mathbf{1}$ denote the trivial representation of G. Then

$$\mathrm{Hom}_G(\pi, \mathbf{1}) \cong \mathrm{Hom}_{\mathbb{C}}(V_G, \mathbb{C}).$$

Received 30 September 2003.

2000 *Mathematics Subject Classification* 22E50, 20G05.

The first author was partially supported by the National Security Agency (#MDA904-02-1-0020).

https://doi.org/10.1112/S0024610704005617 Published online by Cambridge University Press

The Jacquet functor $V \longmapsto V_G$ is exact, as G is the union of an increasing family of compact open subgroups. Hence $\mathbf{1}$ is injective in $\mathfrak{R}_1(G)$ (where the subscript denotes the trivial character of C). Clearly, however, $\mathbf{1}$ is not supercuspidal. In fact, Rodier has shown that any irreducible smooth representation of G is admissible and injective in the full smooth category $\mathfrak{R}(G)$ [**6**]. It follows that many of the categories $\mathfrak{R}_\chi(G)$ admit an abundance of admissible injective non-supercuspidal objects.

Suppose now that G is the group of F-points of a connected reductive algebraic group defined over F. Here one would expect that the full converse to Casselman's result holds. That this is indeed the case is the main result of this paper. That is, we prove the following.

THEOREM A. *Let G be the group of F-points of a connected reductive algebraic group defined over F. Let C be a closed cocompact subgroup of the centre of G and let $\chi : C \longrightarrow \mathbb{C}^\times$ be a smooth character. Let (π, V) be an admissible injective or projective object in the category $\mathfrak{R}_\chi(G)$ of smooth representations of G in which C acts via χ. Then (π, V) is supercuspidal.*

Thus the supercuspidal representations in $\mathfrak{R}_\chi(G)$ can be characterized as the admissible injective or admissible projective objects in this category. Note that, in general, an injective or projective object in $\mathfrak{R}_\chi(G)$ will not have the property that all of its matrix coefficients have compact support modulo Z. In other words, admissibility is an essential hypothesis in the theorem. (For example, suppose for simplicity that Z is trivial and G is not compact, and let K be a compact open subgroup of G. If we write 1 for the trivial character of K, then the full induced representation $\mathrm{Ind}_K^G 1$ is injective in $\mathfrak{R}(G)$ while the compactly induced representation $\mathrm{ind}_K^G 1$ is projective in $\mathfrak{R}(G)$, and both representations have non-compactly supported matrix coefficients.)

In the projective case, the theorem follows quickly from the result of Kutzko mentioned above. Indeed, let (π, V) be an admissible representation in $\mathfrak{R}_\chi(G)$. By [**1**, 2.10, 3.9], (π, V) is a direct sum of finitely generated admissible subrepresentations W, each of which clearly lies in $\mathfrak{R}_\chi(G)$. Of course, if (π, V) is projective, then each W is projective, and thus, by Kutzko's result, supercuspidal, whence their direct sum (π, V) is also supercuspidal.

The bulk of the paper is devoted to a proof of the theorem in the injective case. It amounts to an extended exercise in the application of certain fundamental ideas from [**1**, **3**]. After some preliminary reductions in Section 1, we first prove the result for semisimple groups in Section 2. We then use this and a Hecke algebra argument to treat general reductive groups in Section 3. (A key step hinges on the elementary lemma in Subsection 3.5. We learned this from lectures by Kottwitz, who credits the lemma to Tate.) Finally, in Section 4, we observe that, with trivial modifications, the arguments of Sections 1, 2 and 3 apply in the projective case also, and so we obtain a proof of the theorem in that case which is independent of [**5**].

0.1. *Notation*

We continue with the notation introduced above. In particular, F is a non-Archimedean local field (with finite residue field) and $G = \mathbb{G}(F)$, where $\mathbb{G}$ is a connected reductive algebraic group defined over F.

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Throughout the paper, we use the system of notation of [**3**, § 1], which we now briefly recall. Let L be an F-Levi subgroup of G (more properly, $L = \mathbb{L}(F)$, where $\mathbb{L}$ is a Levi subgroup defined over F of a parabolic subgroup of $\mathbb{G}$ defined over F). Let σ be an irreducible supercuspidal representation of L. Then $[L, \sigma]_G$ denotes the G-inertial equivalence class of (L, σ). (Thus $[L_1, \sigma_1]_G = [L_2, \sigma_2]_G$ if and only if there is a $g \in G$ such that ${}^gL_1 = L_2$ and ${}^g\sigma_1 \cong \sigma_2\nu$ for some unramified character ν of L_2.) We write $\mathfrak{B}(G)$ for the set of all G-inertial equivalence classes.

Each $\mathfrak{s} \in \mathfrak{B}(G)$ indexes a full subcategory $\mathfrak{R}^{\mathfrak{s}}(G)$ of $\mathfrak{R}(G)$. If $\mathfrak{s} = [L, \sigma]_G$, then $\mathfrak{R}^{\mathfrak{s}}(G)$ consists of those smooth representations of G each of whose irreducible subquotients has supercuspidal support represented by some unramified twist of σ. The full smooth category $\mathfrak{R}(G)$ is then the direct product of the subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$ as $\mathfrak{s}$ ranges through $\mathfrak{B}(G)$. This is the Bernstein decomposition of $\mathfrak{R}(G)$. The categories $\mathfrak{R}^{\mathfrak{s}}(G)$ are indecomposable and are therefore often called the components of $\mathfrak{R}(G)$.

If $\mathcal{A}$ is a ring with identity, then we write $\mathcal{A}$-$\mathfrak{mod}$ throughout for the category of (unital) left $\mathcal{A}$-modules. Other notation will be introduced as needed.

1. *Some reductions*

We begin with some straightforward preliminary reductions. We observe first that we may replace the closed cocompact subgroup C of Z by any smaller such subgroup.

PROPOSITION 1.1. *Let C and C_1 be closed cocompact subgroups of Z such that C contains C_1. Let $\chi : C \longrightarrow \mathbb{C}^\times$ be a smooth character and put $\chi_1 = \chi|_{C_1}$. Let (π, V) be a representation in $\mathfrak{R}_\chi(G)$. Then (π, V) is injective in $\mathfrak{R}_\chi(G)$ if and only if it is injective in $\mathfrak{R}_{\chi_1}(G)$.*

Proof. The 'if' direction is completely obvious (as any object in $\mathfrak{R}_\chi(G)$ is also in $\mathfrak{R}_{\chi_1}(G)$). For the other direction, suppose that (π, V) is injective in $\mathfrak{R}_\chi(G)$. Let (τ, U) be a representation in $\mathfrak{R}_{\chi_1}(G)$. For any smooth character $\theta : C \longrightarrow \mathbb{C}^\times$, we set

$$U^\theta = \{u \in U : \tau(c)u = \theta(c)u,\ \forall\, c \in C\}.$$

Since C/C_1 is compact and since χ_1 extends to C, it is immediate that

$$U = \bigoplus_\eta U^{\chi\eta},$$

where η ranges through the smooth characters of C/C_1. Then

$$\mathrm{Hom}_G(U, V) = \mathrm{Hom}_G(U^\chi, V).$$

As the functor $U \longmapsto U^\chi : \mathfrak{R}_{\chi_1}(G) \longrightarrow \mathfrak{R}_\chi(G)$ is exact, it follows that (π, V) is injective in $\mathfrak{R}_{\chi_1}(G)$. □

The intersection of two closed cocompact subgroups of Z is again a subgroup of the same form. Using this observation and Proposition 1.1, we see that it suffices to prove Theorem A for all closed cocompact subgroups of some fixed such subgroup. We now introduce a particular closed cocompact subgroup with which we find it convenient to work.

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Let $\mathbb{A}$ denote the maximal F-split torus in the centre $\mathbb{Z}$ of $\mathbb{G}$ and write $X_*(\mathbb{A})$ for the cocharacter lattice of $\mathbb{A}$. Fix a uniformizer ϖ in F and set

$$\Lambda = \{\lambda(\varpi) : \lambda \in X_*(\mathbb{A})\}.$$

The group $A = \mathbb{A}(F)$ is cocompact in $Z = \mathbb{Z}(F)$ and equals the direct product of Λ and the unique maximal compact open subgroup of A. It follows that Λ is a discrete cocompact subgroup of Z. In the remainder of the paper then, we will always take C to be a subgroup of Λ.

We next note that it suffices to prove Theorem A when the representation (π, V) in $\mathfrak{R}_\chi(G)$ belongs to a single component $\mathfrak{R}^{\mathfrak{s}}(G)$. Indeed, let (τ, U) be an object in $\mathfrak{R}_\chi(G)$ and write

$$U = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} U^{\mathfrak{s}}, \quad V = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} V^{\mathfrak{s}},$$

according to the Bernstein decomposition of $\mathfrak{R}(G)$. Then, for each $\mathfrak{s} \in \mathfrak{B}(G)$, the G-spaces $U^{\mathfrak{s}}$ and $V^{\mathfrak{s}}$ belong to $\mathfrak{R}_\chi(G)$. As any G-map from U to V takes each $U^{\mathfrak{s}}$ into $V^{\mathfrak{s}}$, there is an isomorphism

$$\mathrm{Hom}_G(U, V) \cong \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathrm{Hom}_G(U^{\mathfrak{s}}, V^{\mathfrak{s}})$$

which, in particular, is natural in U. It follows that (π, V) is injective in $\mathfrak{R}_\chi(G)$ if and only if each $V^{\mathfrak{s}}$ is injective in $\mathfrak{R}_\chi(G)$. Further, since (π, V) is admissible, each $V^{\mathfrak{s}}$ is admissible. Of course, to show that (π, V) is supercuspidal, it suffices to show that each $V^{\mathfrak{s}}$ is supercuspidal. From now on then, we assume, whenever it is convenient, that $V = V^{\mathfrak{s}}$ for some $\mathfrak{s} \in \mathfrak{B}(G)$.

2. *Injectivity: the semisimple case*

In this section, we prove the theorem (for injective objects) under the assumption that $\mathbb{G}$ is semisimple. We will later deduce the general case from this special one.

Of course, as $\mathbb{G}$ is semisimple, the lattice Λ above is trivial. Our task then is to prove that $\mathfrak{R}(G)$ has no non-zero admissible injective objects that are not supercuspidal. By the final paragraph of Section 1, it suffices to show the following.

PROPOSITION 2.1. *Let $\mathfrak{R}^{\mathfrak{s}}(G)$ be a non-supercuspidal component of $\mathfrak{R}(G)$, that is, $\mathfrak{s} = [L, \sigma]_G$ with $L \neq G$. Then $\mathfrak{R}^{\mathfrak{s}}(G)$ has no non-zero admissible injective objects.*

Proof. Let $\mathcal{H} = C_c^\infty(G)$, the space of smooth compactly supported complex functions on G. This is a $\mathbb{C}$-algebra under convolution with respect to some fixed Haar measure on G. We view $\mathcal{H}$ as a smooth G-space under, say, the left regular representation. It then decomposes as

$$\mathcal{H} = \bigoplus_{\mathfrak{t}} \mathcal{H}^{\mathfrak{t}},$$

with $\mathcal{H}^{\mathfrak{t}}$ in $\mathfrak{R}^{\mathfrak{t}}(G)$ for all $\mathfrak{t}$. By [**3**, 3.1], the G-spaces $\mathcal{H}^{\mathfrak{t}}$ are two-sided ideals of $\mathcal{H}$ and hence, for $\mathfrak{t} \neq \mathfrak{t}'$,

$$\mathcal{H}^{\mathfrak{t}}\mathcal{H}^{\mathfrak{t}'} \subset \mathcal{H}^{\mathfrak{t}} \cap \mathcal{H}^{\mathfrak{t}'} = 0.$$

Further, by [**1**, 3.7–3.9], there is a compact open subgroup K of G such that $U^K \neq 0$ for every non-zero object (τ, U) in $\mathfrak{R}^{\mathfrak{s}}(G)$. Let $e_K \in \mathcal{H}$ denote the characteristic

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function of K times the inverse of the measure of K and write $e_K = \sum_{\mathfrak{t}} e_{\mathfrak{t}}$ with $e_{\mathfrak{t}} \in \mathcal{H}^{\mathfrak{t}}$ for all $\mathfrak{t}$. Since e_K is an idempotent, we see from the preceding paragraph that each $e_{\mathfrak{t}}$ is an idempotent. In particular, $e_{\mathfrak{s}}$ is an idempotent.

For any object (τ, U) in $\mathfrak{R}^{\mathfrak{s}}(G)$, we have $\mathcal{H}^{\mathfrak{t}}U = U^{\mathfrak{t}} = 0$, for all $\mathfrak{t} \neq \mathfrak{s}$ (again by [**3**, 3.1]). Therefore

$$e_{\mathfrak{s}}U = e_K U = U^K.$$

It then follows from [**3**, 3.12] that the functor

$$U \longmapsto e_{\mathfrak{s}}U : \mathfrak{R}^{\mathfrak{s}}(G) \longrightarrow e_{\mathfrak{s}}\mathcal{H}e_{\mathfrak{s}}\text{-}\mathfrak{mod}$$

is an equivalence of categories. To simplify the notation, we set $\mathcal{A}_{\mathfrak{s}} = e_{\mathfrak{s}}\mathcal{H}e_{\mathfrak{s}}$ and write $\mathcal{Z}_{\mathfrak{s}}$ for the centre of $\mathcal{A}_{\mathfrak{s}}$.

Suppose now that (τ, U) is an admissible non-zero object in $\mathfrak{R}^{\mathfrak{s}}(G)$. The $\mathcal{A}_{\mathfrak{s}}$-module $M = e_{\mathfrak{s}}U$ is then finite-dimensional and so has finite length. Let

$$M = M_1 \supset M_2 \supset \ldots \supset M_{n+1} = 0$$

be a composition series for M so that each M_i is an $A_{\mathfrak{s}}$-submodule and each M_i/M_{i+1} is simple. Of course $\mathcal{Z}_{\mathfrak{s}}$ is isomorphic to the centre of the category $\mathfrak{R}^{\mathfrak{s}}(G)$, whence by [**1**], $\mathcal{Z}_{\mathfrak{s}}$ is an integral domain with $\mathcal{Z}_{\mathfrak{s}} \neq \mathbb{C}$. Since $\mathcal{Z}_{\mathfrak{s}}$ acts by scalars on simple $\mathcal{A}_{\mathfrak{s}}$-modules, we see that there is a non-zero $z_i \in \mathcal{Z}_{\mathfrak{s}}$ with $z_i(M_i/M_{i+1}) = 0$ for $i = 1, \ldots, n$. Then $z = z_1 \ldots z_n \neq 0$ and $zM = 0$.

We recall the following key fact: the ring $e_K\mathcal{H}e_K$ is torsion-free over its centre [**1**, 3.14 (ii)]. Since $e_K\mathcal{H}e_K = \bigoplus_{\mathfrak{t}} e_{\mathfrak{t}}\mathcal{H}e_{\mathfrak{t}}$, it follows that $\mathcal{A}_{\mathfrak{s}}$ is a torsion-free $\mathcal{Z}_{\mathfrak{s}}$-module. Therefore $z\mathcal{A}_{\mathfrak{s}} \cong \mathcal{A}_{\mathfrak{s}}$ as left $\mathcal{A}_{\mathfrak{s}}$-modules and thus the space $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{s}}}(z\mathcal{A}_{\mathfrak{s}}, M)$ of $\mathcal{A}_{\mathfrak{s}}$-module homomorphisms is non-zero. On the other hand, for any $\mathcal{A}_{\mathfrak{s}}$-module homomorphism $f : \mathcal{A}_{\mathfrak{s}} \longrightarrow M$,

$$f(z\mathcal{A}_{\mathfrak{s}}) = zf(\mathcal{A}_{\mathfrak{s}}) \subset zM = 0.$$

This says that no non-zero element of $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{s}}}(z\mathcal{A}_{\mathfrak{s}}, M)$ extends to $\mathcal{A}_{\mathfrak{s}}$. Hence M is not an injective $\mathcal{A}_{\mathfrak{s}}$-module and so (τ, U) is not injective in $\mathfrak{R}^{\mathfrak{s}}(G)$. □

3. *Injectivity: the general case*

We turn now to the general case. Thus $G = \mathbb{G}(F)$ with $\mathbb{G}$ reductive but not (necessarily) semisimple. As explained in Section 1, we may and do assume that $C = \Lambda_1$ is a subgroup of Λ. We have fixed a character $\chi : \Lambda_1 \longrightarrow \mathbb{C}^\times$. As also explained in Section 1, we may and do assume that our injective admissible object (π, V) in $\mathfrak{R}_\chi(G)$ belongs to a fixed component $\mathfrak{R}^{\mathfrak{s}}(G)$.

3.1.

Let G' denote the group of F-points of the derived group of $\mathbb{G}$. By [**7**], the restriction to G' of an irreducible smooth representation of G has finite length. Of course, it is then immediate that the restriction to G' of a smooth finite length representation of G again has finite length. Since admissible objects in $\mathfrak{R}^{\mathfrak{s}}(G)$ have finite length, we see, in particular, that $V|_{G'}$ has finite length.

Let M denote a minimal F-Levi subgroup of G and let K_0 denote a corresponding special maximal compact subgroup of G. Thus K_0 is the G-stabilizer of a special vertex in the apartment corresponding to M of the extended affine building of G. Then $M' = M \cap G'$ is the group of F-points of a minimal F-Levi subgroup of the

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derived group of $\mathbb{G}$ with $K_0' = K_0 \cap G'$ as a corresponding special maximal compact subgroup of G'.

We recall from [**2**, 1.2] that there is a neighbourhood basis of the identity in G consisting of open normal subgroups K of K_0 which are totally decomposed with respect to M (see [**2**, 1.1] for this notion). The groups $K' = K \cap G'$ then form a neighbourhood basis of the identity in G' each of which is open and normal in K_0' and totally decomposed with respect to M'.

There is a compact open subgroup K as in the preceding paragraph, which we fix for the remainder of this section, such that V is generated over G' by its space $V^{K'}$ of K'-fixed vectors (see [**2**, 1.5]). Further (again see [**2**, 1.5]), the trivial representation of K (respectively of K') is a type in G (respectively in G'). To state this more explicitly, let $\mathfrak{R}_K(G)$ denote the full subcategory of $\mathfrak{R}(G)$ whose objects are those (τ, U) such that U is generated over G by U^K, and write $\mathcal{H}(G, K)$ for the convolution algebra (with respect to some Haar measure on G) of compactly supported K-bi-invariant complex functions on G. Let $\mathfrak{R}_{K'}(G')$ and $\mathcal{H}(G', K')$ denote the analogous objects for G'. Then the functors

$$\begin{aligned} U &\longmapsto U^K : \mathfrak{R}_K(G) \longrightarrow \mathcal{H}(G, K)\text{-}\mathfrak{mod} \\ W &\longmapsto W^{K'} : \mathfrak{R}_{K'}(G') \longrightarrow \mathcal{H}(G', K')\text{-}\mathfrak{mod} \end{aligned} \tag{3.1}$$

are equivalences of categories.

We will show by a Hecke algebra argument that $V|_{G'}$ is injective in $\mathfrak{R}(G')$. Of course it will then follow from Section 2 that $V|_{G'}$ is supercuspidal, and hence that (π, V) is supercuspidal. (This last step follows, for example, from Jacquet module considerations, as the parabolic subgroups of G and G' share the same sets of unipotent radicals.)

3.2.

Let $G_1 = \Lambda_1 G' K$. The compact-induction functor $\mathrm{ind} = \mathrm{ind}_{G_1}^G$ takes objects in $\mathfrak{R}_\chi(G_1)$ to objects in $\mathfrak{R}_\chi(G)$. Further, since G_1 is open, there are natural isomorphisms

$$\mathrm{Hom}_{G_1}(\sigma, \pi|_{G_1}) \cong \mathrm{Hom}_G(\mathrm{ind}\,\sigma, \pi)$$

for any object σ in $\mathfrak{R}_\chi(G_1)$. Since ind is exact and π is injective, it follows that $\pi|_{G_1}$ is injective in $\mathfrak{R}_\chi(G_1)$.

3.3.

We next describe the space $V^{K'}$ of K'-fixed vectors in V in terms of the K-isotypic decomposition of V. Let ρ be a smooth irreducible representation of K such that $\rho|_{K'}$ contains the trivial representation of K'. Let $J = \mathrm{Ker}\,\rho$. Then ρ is effectively a representation of the finite group K/J which contains the trivial representation of JK'/J. As JK'/J is normal in K/J, Clifford theory implies that ρ is trivial on JK'. Hence

$$V^{K'} = \bigoplus_\eta V^\eta, \tag{3.2}$$

where the sum is over certain smooth characters $\eta : K \longrightarrow \mathbb{C}^\times$ such that $\eta|_{K'} = 1$.

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Note that each such η extends to a smooth character of G. Indeed, we may view K/K' as a subgroup of G/G', and so, as $\mathbb{C}^\times$ is divisible, each η extends to G/G' (and each extended character is smooth since the original characters were smooth.)

We now show that the direct sum (3.2) contains just a single term. Indeed, suppose that $V^{\eta_1} \neq 0$ and $V^{\eta_2} \neq 0$. Since the trivial character of K is a type and since each η_i extends to G, we see easily that each η_i is a type (for $i = 1, 2$). It follows that each irreducible representation in $\mathfrak{R}^{\mathfrak{s}}(G)$ contains η_1 and η_2. In particular, η_1 and η_2 must intertwine in G, that is, there is an $x \in G$ such that

$$ {}^x\eta_1|_{{}^xK\cap K} = \eta_2|_{{}^xK\cap K}. $$

By the preceding paragraph, η_1 extends to G. As any such extension is invariant under conjugation, ${}^x\eta_1|_{{}^xK\cap K} = \eta_1|_{{}^xK\cap K}$. We have $G = K_0MK_0$, and so $x = k_1mk_2$ with $k_1, k_2 \in K_0$ and $m \in M$. Then, since K is normal in K_0,

$$ {}^xK \cap K = {}^{k_1m}K \cap K. $$

It follows that ${}^{k_1^{-1}}\eta_1$ and ${}^{k_1^{-1}}\eta_2$ coincide on ${}^mK \cap K$. However, ${}^{k_i^{-1}}\eta_i = \eta_i$, as each η_i extends to G, and hence $\eta_1 = \eta_2$ on ${}^mK \cap K$. Since M normalizes $K_M = K \cap M$, we see that

$$ \eta_1|_{K_M} = \eta_2|_{K_M}, $$

and thus $\eta_1 = \eta_2$ on K (as $K = K_MK'$). Therefore

$$ V^{K'} = V^{\eta} \tag{3.3} $$

for a unique smooth character $\eta : K \longrightarrow \mathbb{C}^\times$ such that $\eta|_{K'} = 1$.

3.4.

Let $\mathcal{H}_1 = C_c^\infty(G_1)$. This is a $\mathbb{C}$-algebra under convolution with respect to a fixed Haar measure dg on G_1. To simplify some later expressions, we normalize this measure so that the volume of K is 1 and take counting measure as our Haar measure on the discrete abelian group Λ_1. There is then an induced Haar measure $d\dot{g}$ on G_1/Λ_1 such that, for any $f \in \mathcal{H}_1$,

$$ \int_{G_1} f(g)\, dg = \int_{G_1/\Lambda_1} \left(\sum_{x\in\Lambda_1} f(xg) \right) d\dot{g}. \tag{3.4} $$

We write $\mathcal{H}(G_1, \eta)$ for the set of all $f \in \mathcal{H}_1$ such that

$$ f(kgk') = \eta(k)^{-1} f(g) \eta(k')^{-1}, $$

for $k, k' \in K$ and $g \in G_1$. This is a subalgebra of $\mathcal{H}_1$. It has an identity element e_η which is supported on K and satisfies

$$ e_\eta(k) = \eta(k)^{-1}, \qquad k \in K. $$

We have $\mathcal{H}(G_1, \eta) = e_\eta \mathcal{H}_1 e_\eta$.

Let $\mathcal{H}_{1,\chi}$ denote the set of smooth functions $f : G_1 \longrightarrow \mathbb{C}$ of compact support modulo Λ_1 such that

$$ f(xg) = \chi(x)^{-1} f(g), \qquad x \in \Lambda_1,\ g \in G_1. $$

This is a $\mathbb{C}$-algebra under convolution with respect to our Haar measure $d\dot{g}$ on G_1/Λ_1. Each object (τ, U) in $\mathfrak{R}_\chi(G_1)$ is naturally a non-degenerate $\mathcal{H}_{1,\chi}$-module

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and this process induces, in the usual manner, an equivalence of categories between $\mathfrak{R}_\chi(G_1)$ and the category of non-degenerate $\mathcal{H}_{1,\chi}$-modules.

The character η of K clearly extends to the character $\chi\eta$ of $\Lambda_1 K$ given by $\chi\eta(xk) = \chi(x)\eta(k)$ for $x \in \Lambda_1$, $k \in K$ (which is well defined since $\Lambda_1 \cap K = 1$). We write $\mathcal{H}_\chi(G_1, \eta)$ for the subset of $\mathcal{H}_{1,\chi}$ which consists of all $f \in \mathcal{H}_{1,\chi}$ such that

$$f(ygy') = \chi\eta(y)^{-1} f(g) \chi\eta(y')^{-1}, \qquad y, y' \in \Lambda_1 K,\ g \in G.$$

Again, this is a subalgebra of $\mathcal{H}_{1,\chi}$ with an identity element $e_{\chi\eta}$ which is supported on $\Lambda_1 K$ and satisfies

$$e_{\chi\eta}(y) = \chi\eta(y)^{-1}, \qquad y \in \Lambda_1 K.$$

We have $\mathcal{H}_\chi(G_1, \eta) = e_{\chi\eta} \mathcal{H}_{1,\chi} e_{\chi\eta}$.

There is a surjective $\mathbb{C}$-algebra homomorphism $p_1 : \mathcal{H}_1 \longrightarrow \mathcal{H}_{1,\chi}$ given by

$$p_1(\phi)(g) = \sum_{x \in \Lambda_1} \chi(x) f(xg), \qquad g \in G_1,\ \phi \in \mathcal{H}_1. \tag{3.5}$$

Note that $p_1(e_\eta) = e_{\chi\eta}$ so that $p_1(\mathcal{H}(G_1, \eta)) = \mathcal{H}_\chi(G_1, \eta)$.

Let (τ, U) be an object in $\mathfrak{R}_\chi(G_1)$. Then

$$U^\eta = U^{\chi\eta} = e_{\chi\eta} U$$

is, of course, a left $\mathcal{H}_\chi(G_1, \eta)$-module. We need the following observation.

PROPOSITION 3.1. *With notation as above, $V^\eta = V^{\chi\eta}$ is an injective $\mathcal{H}_\chi(G_1, \eta)$-module.*

We will deduce the proposition from the general result stated in the next subsection.

3.5.

Let $\mathcal{H}$ be an idempotented $\mathbb{C}$-algebra and let $\mathcal{H}$-$\mathfrak{mod}$ denote the category of non-degenerate $\mathcal{H}$-modules (see, for example, [**1**, 1.1]). Let e be an idempotent in $\mathcal{H}$. For any object W in $\mathcal{H}$-$\mathfrak{mod}$, we write $W(e)$ for the sum of all $\mathcal{H}$-submodules of W annihilated by e. Thus

$$W(e) = \{w \in W : e\mathcal{H}w = 0\}.$$

Put $W_e = W/W(e)$ and let $\mathcal{H}_e$-$\mathfrak{mod}$ denote the full subcategory of $\mathcal{H}$-$\mathfrak{mod}$ whose objects are all (non-degenerate) $\mathcal{H}$-modules W such that

$$W = \mathcal{H}eW \quad \text{and} \quad W(e) = 0.$$

LEMMA 3.2. *The functor*

$$W \longmapsto eW : \mathcal{H}_e\text{-}\mathfrak{mod} \longrightarrow e\mathcal{H}e\text{-}\mathfrak{mod}$$

is an equivalence of categories.

The inverse (up to equivalence) is given by

$$M \longmapsto (\mathcal{H} \bigotimes_{e\mathcal{H}e} M)_e : e\mathcal{H}e\text{-}\mathfrak{mod} \longrightarrow \mathcal{H}_e\text{-}\mathfrak{mod}.$$

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While there are several related results in the literature (see, for example, [**3**, Proposition 3.3]), we do not know a reference for precisely this statement, and so will include the (straightforward) proof. We first saw the lemma during a course of lectures by Kottwitz on automorphic forms. He credits the observation to Tate.

Proof of Lemma 3.2. First note that for any object W in $\mathcal{H}$-$\mathfrak{mod}$, there is a natural isomorphism $eW_e \cong eW$ (since $eW(e) = 0$). Hence, for each $e\mathcal{H}e$-module M,

$$\begin{aligned} e(\mathcal{H} \bigotimes_{e\mathcal{H}e} M)_e &\cong e(\mathcal{H} \bigotimes_{e\mathcal{H}e} M) \\ &= e\mathcal{H} \bigotimes_{e\mathcal{H}e} M = e\mathcal{H}e \bigotimes_{e\mathcal{H}e} M \cong M. \end{aligned}$$

Next let W be an object in $\mathcal{H}_e$-$\mathfrak{mod}$ and let $\mathcal{K}$ denote the kernel of the surjective map

$$h \otimes ew \longmapsto hew : \mathcal{H} \bigotimes_{e\mathcal{H}e} eW \longrightarrow \mathcal{H}eW = W. \tag{3.6}$$

Of course we then have $(\mathcal{H} \bigotimes_{e\mathcal{H}e} eW)/\mathcal{K} \cong W$. To complete the proof, we verify that $\mathcal{K} = (\mathcal{H} \bigotimes_{e\mathcal{H}e} eW)(e)$ so that there is a natural isomorphism

$$(\mathcal{H} \bigotimes_{e\mathcal{H}e} eW)_e \cong W.$$

Let $\sum_i h_i \otimes ew_i \in \mathcal{K}$. Then $\sum_i h_i ew_i = 0$ and so, for all $h \in \mathcal{H}$,

$$\begin{aligned} eh\left(\sum_i h_i \otimes ew_i\right) &= \sum_i ehh_i \otimes ew_i = \sum_i ehh_i e \otimes ew_i \\ &= e \otimes \sum_i ehh_i ew_i = e \otimes eh\left(\sum_i h_i ew_i\right) = 0. \end{aligned}$$

That is, $\mathcal{K} \subset (\mathcal{H} \bigotimes_{e\mathcal{H}e} eW)(e)$.

For the reverse inclusion, let $\sum_i h_i \otimes ew_i \in (\mathcal{H} \bigotimes_{e\mathcal{H}e} eW)(e)$. Then, for all $h \in \mathcal{H}$,

$$eh\left(\sum_i h_i \otimes ew_i\right) = 0.$$

As above, we can rewrite this as $e \otimes \sum_i ehh_i ew_i = 0$. Applying the map (3.6), we see that

$$\sum_i ehh_i ew_i = 0$$

for all $h \in \mathcal{H}$, which states exactly that $\sum_i h_i ew_i \in W(e)$. Since $W(e) = 0$, we have $\sum_i h_i ew_i = 0$, and so $(\mathcal{H} \bigotimes_{e\mathcal{H}e} eW)(e) \subset \mathcal{K}$. □

3.6.

We now use Lemma 3.2 to prove Proposition 3.1. We identify the categories $\mathfrak{R}_\chi(G_1)$ and $\mathcal{H}_{1,\chi}$-$\mathfrak{mod}$ and write $\mathcal{H} = \mathcal{H}_{1,\chi}$, $e = e_{\chi\eta}$ so that $e\mathcal{H}e = \mathcal{H}_\chi(G_1, \eta)$.

We claim that $V|_{G_1}$ belongs to $\mathcal{H}_e$-$\mathfrak{mod}$. Once this is established, then the injective object $V|_{G_1}$ in $\mathcal{H}$-$\mathfrak{mod}$ is clearly also injective in the full subcategory $\mathcal{H}_e$-$\mathfrak{mod}$. Lemma 3.2 then implies that $eV = V^{\chi\eta} = V^\eta$ is an injective $\mathcal{H}_\chi(G_1, \eta)$-module.

To verify our claim, note first that V is generated over G' by $V^{K'} = V^\eta$, and thus *a fortiori* V is generated over G_1 by $V^\eta = V^{\chi\eta} = eV$, that is, $V = \mathcal{H}eV$. Next, let W be a G_1-subrepresentation of V. Since the trivial representation of K' is a type,

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we know that W is generated over G' by $W^{K'}$. By the arguments of Subsection 3.3,

$$W^{K'} = \bigoplus_{\lambda} W^{\lambda},$$

where λ varies through certain smooth characters of K/K'. Of course, if $W^{\lambda} \neq 0$, then $V^{\lambda} \neq 0$. As η is the unique smooth character of K/K' such that $V^{\eta} \neq 0$ (see (3.3)), we see that $W^{K'} = W^{\eta}$. Therefore W is generated by W^{η} over G_1. In other words, every $\mathcal{H}$-submodule W of V is generated by eW, and thus $V(e) = 0$. This establishes our claim and so completes the proof of Proposition 3.1.

3.7.

We next show that the Hecke algebras $\mathcal{H}(G_1, \eta)$ and $\mathcal{H}_{\chi}(G_1, \eta)$ are isomorphic to certain spherical Hecke algebras on $\Lambda_1 G'$ and G', respectively.

PROPOSITION 3.3. *The restriction map*

$$\phi \overset{r}{\longmapsto} \phi|_{\Lambda_1 G'} : \mathcal{H}(G_1, \eta) \longrightarrow \mathcal{H}(\Lambda_1 G', K')$$

is a $\mathbb{C}$*-algebra isomorphism.*

Proof. We first verify that r is a $\mathbb{C}$-algebra homomorphism. It is clearly a map of $\mathbb{C}$-vector spaces. To show that it preserves multiplication, we use the following identity. For $f \in \mathcal{H}_1 = C_c^{\infty}(\Lambda_1 G' K)$,

$$\int_{\Lambda_1 G' K} f(g)\, dg = \int_{\Lambda_1 G'} \int_{K} f(yk)\, dk\, dy.$$

Here dg and dk denote the Haar measures on G_1 and K for which K has measure 1, and dy denotes the Haar measure on $\Lambda_1 G'$ for which K' has measure 1. Let $h \in \Lambda_1 G'$. Then

$$\begin{aligned}(\phi_1\phi_2)(h) &= \int_{\Lambda_1 G' K} \phi_1(g)\phi_2(g^{-1}h)\, dg \\ &= \int_{\Lambda_1 G'} \int_{K} \phi_1(yk)\phi_2(k^{-1}y^{-1}h)\, dk\, dy \\ &= \int_{\Lambda_1 G'} \phi_1(y)\phi_2(y^{-1}h)\, dy \\ &= (r(\phi_1)r(\phi_2))(h).\end{aligned}$$

The map r is obviously injective. To complete the proof, we now show that it is also surjective. By the Cartan decomposition,

$$G' = \bigcup_{m \in M'} K'_0 m\, K'_0.$$

Hence, as m varies through M' and x through Λ_1, and as k_1, k_2 vary through K'_0,

$$\Lambda_1 G' = \bigcup K' x k_1 m k_2 K'.$$

Surjectivity of r is clearly equivalent to the statement that, for each $x \in \Lambda_1$, $m \in M'$ and $k_1, k_2 \in K'_0$,

$$K x k_1 m k_2 K \cap \Lambda_1 G' = K' x k_1 m k_2 K'. \tag{3.7}$$

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Let G_0 denote the intersection of the kernels of the unramified characters of G (equivalently, the subgroup of G generated by the compact subgroups of G). Note that $w \longmapsto wG_0 : \Lambda \longrightarrow G/G_0$ is injective and $Kk_1mk_2K \subset G_0$, $G' \subset G_0$. Hence

$$Kxk_1mk_2K \cap \Lambda_1 G' = xKk_1mk_2K \cap xG' = x(Kk_1mk_2K \cap G').$$

Of course, since $k_1, k_2 \in G'$,

$$x(Kk_1mk_2K \cap G') = x(k_1KmKk_2 \cap G') = xk_1(KmK \cap G')k_2.$$

We have $K = (K \cap M)K'$ and $K \cap M$ is normal in M, and thus $KmK = KmK'$. Therefore, since $m \in G'$,

$$KmK \cap G' = KmK' \cap G' = K'm\,K'.$$

The relation (3.7) now follows. □

The character χ of Λ_1 clearly extends to a character of $\Lambda_1 K'$ which is trivial on K' and which we again denote by χ. As above, we write $\mathcal{H}_\chi(\Lambda_1 G', K')$ for the space of functions $\phi : \Lambda_1 G' \longrightarrow \mathbb{C}$ of compact support modulo Λ_1 such that

$$\phi(ygy') = \chi(y)^{-1}\,\phi(g)\chi(y')^{-1}$$

for $y, y' \in \Lambda_1 K'$, $g \in \Lambda_1 G'$. This is again a $\mathbb{C}$-algebra with respect to convolution on $\Lambda_1 G'/\Lambda_1 = G'$, and the map $p : \mathcal{H}(\Lambda_1 G', K') \longrightarrow \mathcal{H}_\chi(\Lambda_1 G', K')$ given by

$$p(\phi)(g) = \sum_{x \in \Lambda_1} \chi(x)\phi(xg) \tag{3.8}$$

for $\phi \in \mathcal{H}(\Lambda_1 G', K')$, $g \in \Lambda_1 G'$, is a surjective $\mathbb{C}$-algebra homomorphism.

COROLLARY 3.4. *The restriction map*

$$\phi \overset{r_\chi}{\longmapsto} \phi|_{\Lambda_1 G'} : \mathcal{H}_\chi(G_1, \eta) \longrightarrow \mathcal{H}_\chi(\Lambda_1 G', K')$$

is a $\mathbb{C}$-algebra isomorphism.

Proof. The map r_χ is clearly an injective homomorphism of $\mathbb{C}$-vector spaces. It is easy to check that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}(G_1, \eta) & \xrightarrow{\ r\ } & \mathcal{H}(\Lambda_1 G', K') \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{H}_\chi(G_1, \eta) & \xrightarrow{\ r_\chi\ } & \mathcal{H}_\chi(\Lambda_1 G', K'), \end{array}$$

where p_1 and p are given by (3.5) and (3.8), respectively. Since p_1 is a surjective algebra homomorphism and r and p are algebra homomorphisms, we deduce that r_χ preserves multiplication and so is an algebra homomorphism. Further, as p and r are surjective, r_χ is surjective. □

Finally, as $\Lambda_1 G'$ is the product of Λ_1 and G', it is clear that the restriction map

$$\phi \longmapsto \phi|_{G'} : \mathcal{H}_\chi(\Lambda_1 G', K') \longrightarrow \mathcal{H}(G', K')$$

is again a $\mathbb{C}$-algebra isomorphism.

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3.8.

We can now prove Theorem A (in the injective case). From the preceding subsection, we have the algebra homomorphisms

$$\mathcal{H}(G_1,\eta) \xrightarrow{p_1} \mathcal{H}_\chi(G_1,\eta) \cong \mathcal{H}_\chi(\Lambda_1 G', K') \cong \mathcal{H}(G', K').$$

Note that the composition of these maps is simply the restriction

$$\phi \longmapsto \phi|_{G'} : \mathcal{H}(G_1,\eta) \longrightarrow \mathcal{H}(G', K').$$

The $\mathcal{H}_\chi(G_1,\eta)$-module $V^{\chi\eta} = V^{K'}$ acquires an $\mathcal{H}(G', K')$-module structure via the isomorphism $\mathcal{H}_\chi(G_1,\eta) \cong \mathcal{H}(G', K')$. We first need to verify that this coincides with the natural $\mathcal{H}(G', K')$-module structure on $V^{K'}$. Let $\phi \in \mathcal{H}(G_1,\eta)$ and $v \in V^\eta = V^{\chi\eta}$. By (3.4), the $\mathcal{H}(G_1,\eta)$- and $\mathcal{H}_\chi(G_1,\eta)$-module structures on $V^\eta = V^{\chi\eta}$ are related by $\phi\, v = p_1(\phi)v$. Further,

$$\phi\, v = \int_{G_1} \phi(g)\pi(g)v\, dg = \int_{\Lambda_1 G'} \int_K \phi(yk)\pi(yk)v\, dk\, dy.$$

Since $\pi(k)v = \eta(k)v$ and $\phi(yk) = \phi(y)\eta(k)^{-1}$ (for $k \in K$, $y \in \Lambda_1 G'$) and since K has measure 1, this equals

$$\begin{aligned}\int_{\Lambda_1 G'} \phi(y)\pi(y)v\, dy &= \int_{G'} \left(\sum_{x \in \Lambda_1} \phi(gx)\pi(gx)v \right) dg \\ &= \int_{G'} \phi(g)\pi(g)v\, dg \\ &= \phi|_{G'}\, v.\end{aligned}$$

That is, the isomorphism $\mathcal{H}_\chi(G_1,\eta) \cong \mathcal{H}(G', K')$ gives $V^{K'} = V^\eta$ its natural $\mathcal{H}(G', K')$-module structure, as required.

Proposition 3.1 now implies that $V^{K'}$ is an injective $\mathcal{H}(G', K')$-module. Since the trivial representation of K' is a type (see (3.1)), $V|_{G'}$ is injective in $\mathfrak{R}_{K'}(G')$. Further, as $\mathfrak{R}_{K'}(G')$ is a (finite) sum of components of the Bernstein decomposition of G', we see that $V|_{G'}$ is injective in $\mathfrak{R}(G')$. Hence, by Section 2, $V|_{G'}$ is supercuspidal. It now follows, for example from the final paragraph of Subsection 3.1, that (π, V) is supercuspidal.

4. *The projective case*

Finally we observe that the above arguments, with trivial modifications, also establish the projective case of Theorem A.

Let (π, V) be an admissible projective object in $\mathfrak{R}_\chi(G)$. First, one can check easily that the reductions of Section 1 all go through. We assume then that $C = \Lambda_1$ is a subgroup of Λ and that $V = V^{\mathfrak{s}}$ for some $\mathfrak{s} \in \mathfrak{B}(G)$.

Suppose first that $\mathbb{G}$ is semisimple. To prove the result in this case, we note that the analogue of Proposition 2.1 holds, that is, $\mathfrak{R}^{\mathfrak{s}}(G)$ has no non-zero admissible projective objects when $\mathfrak{s} = [L, \sigma]_G$ with $L \neq G$. Indeed, suppose that (τ, U) is admissible in $\mathfrak{R}^{\mathfrak{s}}(G)$. In the notation of Section 2, $M = e_{\mathfrak{s}}U$ has finite length and so there is a non-zero $z \in \mathcal{Z}_{\mathfrak{s}}$ with $zM = 0$. Since $A_{\mathfrak{s}}$ is torsion-free over $\mathcal{Z}_{\mathfrak{s}}$, each projective $\mathcal{A}_{\mathfrak{s}}$-module is torsion-free over $\mathcal{Z}_{\mathfrak{s}}$. Hence M cannot be projective over $\mathcal{A}_{\mathfrak{s}}$, and (τ, U) is not projective in $\mathfrak{R}^{\mathfrak{s}}(G)$.

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In the general reductive case, we again choose a compact open subgroup K as in Subsection 3.1 and set $G_1 = \Lambda_1 G' K$. Since the restriction functor from $\mathfrak{R}_\chi(G)$ to $\mathfrak{R}_\chi(G_1)$ has an exact right adjoint (given by ordinary smooth induction), we see that $\pi|_{G_1}$ is projective. Therefore $V^\eta = V^{\chi\eta}$ is a projective $\mathcal{H}_\chi(G_1, \eta)$-module (by the proof of Proposition 3.1). The Hecke algebra arguments of Section 3 then imply that $V|_{G'}$ is projective, and hence supercuspidal, and so we conclude that (π, V) is supercuspidal.

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