4 Technology beyond Trees

4.1 Loop Calculations

4.1.1 General View

Until now the calculations of Feynman diagrams were easy. Knowing the Feynman rules for vertices and propagators a simple multiplication of the corresponding expressions for them allowed to calculate the contribution of a given tree diagram to the decay amplitude we were interested in. However, generally more complicated diagrams, like the ones in Figure 6.9 have to be considered. If the interactions are sufficiently weak, we know that these *loop diagrams* involving higher powers of couplings will result in contributions that are smaller than the tree-level ones. Yet, we have to calculate them at least for three reasons:

- We have to check the accuracy of the tree-level predictions and whether the perturbation theory works at all.
- The true tests of a given quantum field theory start at one-loop level.
- Certain decays are absent at tree-level, and the leading contributions come from diagrams involving one loop.

We will soon see that there are other reasons for performing these, often tedious, calculations. But already these three reasons are sufficient motivations to learn how to efficiently calculate these diagrams.

The first obstacle one encounters in evaluating the diagrams in Figure 6.9 is not only the integration over the internal momentum in the loop but also the fact that the result for the left diagram is meaningless; it is divergent. This is bad news. The good news is that instead of trying to improve on this by studying the full diagram in Figure 6.9 or other diagrams with different loops, it is sufficient to look first at those subdiagrams (parts) of these diagrams, which are responsible for the divergences in question. The simplest of them are shown in Figure 4.1

Our goals for the first sections of this chapter are as follows.

- To learn the technology for loop calculations.
- To train Dirac algebra in $D \neq 4$ dimensions. This technology is crucial in addressing the divergences in loop diagrams.
- To train color algebra relevant for QCD calculations.
- To exhibit the calculations of the first two diagrams in Figure 4.1 in explicit terms and to indicate the structure of the calculation of the remaining diagrams.

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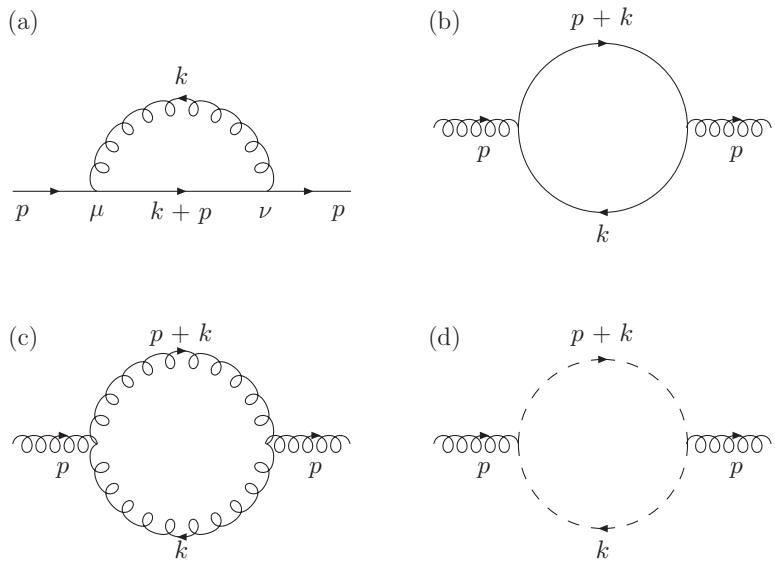


Figure 4.1 Self-energy diagrams.

- To exhibit differences between QED and QCD already at this stage by considering the exchanged gauge boson to be a photon and a gluon, respectively. These differences are important in the case of the first two diagrams but not as profound as the ones when the third and fourth diagram are considered: They are absent in QED, and in QCD they have important implications for the short-distance structure of the theory.

Once this is done we can address in subsequent sections the important topics of renormalization and renormalization group methods. While we will present all these topics in some detail, a very useful book in this context is the one by Muta [70].

At this stage the following comment should be made. Some of the younger readers may consider all these details unnecessary as these days several computer programs can perform complicated analytic calculations without entering the details presented next. Yet, such an approach prohibits a deep insight in what is going on and in my view takes away the pleasure of doing these calculations by hand. In fact, all one-loop calculations in the SM can be done easily by hand and the ones in QED and QCD also at the two-loop level. A different story are two-loop calculations of electroweak contributions and higher loop diagrams for which computer calculations are unavoidable.

4.1.2 Dimensional Regularization

Preliminaries

To deal with divergences that appear in loop diagrams we have to regularize the theory to have an explicit parametrization of the singularities. In this book we will employ *dimensional regularization* (DR), which is used basically by everybody these days except

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for lattice QCD experts, who in their numerical nonperturbative calculations use different regularization by discretizing the space. We will not discuss how these two regularizations are related, as the final results for physical quantities cannot depend on the regularization method used.

In DR Feynman diagrams are evaluated in $D = 4 - 2\,\varepsilon$ space-time dimensions, and singularities are extracted as poles for $\varepsilon \to 0$. Thus the results of one-loop and two-loop calculations have the following general structure:

$$\text{One-loop result} = \frac{a_1}{\varepsilon} + b_1\,, \tag{4.1}$$

$$\text{Two-loop result} = \frac{a_2}{\varepsilon^2} + \frac{b_2}{\varepsilon} + c_2\,, \tag{4.2}$$

where a_i, b_i, and c_2 are finite.

Several useful formulas for the evaluation of Feynman diagrams in $D = 4 - 2\varepsilon$ dimensions will be given soon. Here we only stress the following important point. Let us consider the second term in the second line in the QCD Lagrangian in (2.113). It describes the interaction of the gluon field A^a_μ with quarks q_i. The energy dimensions of q_i, A^a_μ and of $\mathcal{L}$ itself in D dimensions are $(D-1)/2$, $(D-2)/2$, and D, respectively. Consequently, the dimension of g_s in $D = 4 - 2\varepsilon$ dimensions is simply equal to ε. It is more useful to work with a dimensionless coupling constant in arbitrary D dimensions. To this end we make the replacement in (2.113):

$$\boxed{g_s \quad \to \quad g_s\,\mu^{\varepsilon}, \qquad \varepsilon = \frac{4-D}{2},} \tag{4.3}$$

where μ is an arbitrary parameter with the dimension of energy and g_s on the r.h.s. is dimensionless. The appearance of the scale μ has profound impact on this and subsequent sections.

D-Dimensional Integrals

In the calculations of one-loop Feynman diagrams we will encounter integrals over the internal four-momentum k_μ, and it is useful to have the results for them at hand. We will not derive these expressions as the derivations can be found in many field theory books. In particular we refer to pages 112–113 of Muta's book [70].

The three integrals we will need first are as follows

$$\boxed{\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - V + i\epsilon)^N} = i \frac{(-1)^N}{(4\pi)^{D/2}} \frac{\Gamma(N - D/2)}{\Gamma(N)} \frac{1}{(V - i\epsilon)^{N-D/2}},} \tag{4.4}$$

$$\boxed{\int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2 - V + i\epsilon)^N} = i \frac{(-1)^{N-1}}{(4\pi)^{D/2}} \frac{\Gamma(N - 1 - D/2)}{\Gamma(N)} \frac{D}{2} \frac{1}{(V - i\epsilon)^{N-1-D/2}},} \tag{4.5}$$

$$\boxed{\int \frac{d^D k}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2 - V + i\epsilon)^N} = i g_{\mu\nu} \frac{(-1)^{N-1}}{(4\pi)^{D/2}} \frac{\Gamma(N - 1 - D/2)}{\Gamma(N)} \frac{1}{2} \frac{1}{(V - i\epsilon)^{N-1-D/2}}.} \tag{4.6}$$

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Here N can be a complex number, and V is a function of masses of exchanged particles and momenta of external ones. $\Gamma(\alpha)$ is one of Euler functions for which useful expressions will be given soon. Also the following two relations are useful:

$$\boxed{\int \frac{d^D k}{(2\pi)^D} k_\mu k_\nu f(k^2) = \frac{1}{D} g_{\mu\nu} \int \frac{d^D k}{(2\pi)^D} k^2 f(k^2),} \tag{4.7}$$

$$\boxed{g_{\mu\nu} g^{\mu\nu} = D, \qquad \int \frac{d^D k}{(2\pi)^D} k_\mu f(k^2) = 0.} \tag{4.8}$$

Euler-Beta and Euler-Gamma Functions

While doing simple one-loop calculations, we will encounter integrals that can be expressed in terms of Euler-beta function $B(\alpha, \beta)$ and Euler-gamma function $\Gamma(\alpha)$, where α and β are complex numbers. We have

$$\boxed{\int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},} \tag{4.9}$$

with

$$\Gamma(1) = \Gamma(2) = 1, \qquad \Gamma(1+\alpha) = \alpha\Gamma(\alpha). \tag{4.10}$$

Of interest for us are various expansions of both functions in powers of the small parameter ε. In particular we have

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + O(\varepsilon), \qquad \gamma_E = 0.57721\ldots, \tag{4.11}$$

$$\Gamma(1-\varepsilon) = 1 + O(\varepsilon^2), \qquad \Gamma(2-\varepsilon) = 1-\varepsilon, \tag{4.12}$$

$$B(2-\varepsilon, 1-\varepsilon) = \frac{1}{2}[1+2\varepsilon] + O(\varepsilon^2), \tag{4.13}$$

$$B(2-\varepsilon, 2-\varepsilon) = \frac{1}{6}\left[1 + \frac{5}{3}\varepsilon\right] + O(\varepsilon^2), \tag{4.14}$$

$$B(3-\varepsilon, 1-\varepsilon) = \left(\frac{2-\varepsilon}{1-\varepsilon}\right) B(2-\varepsilon, 2-\varepsilon) + O(\varepsilon^2), \tag{4.15}$$

where γ_E is the Euler constant. Finally an important quantity is

$$\boxed{P_{div} \equiv \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} \left(\frac{\mu^2}{-p^2}\right)^\varepsilon = \frac{1}{16\pi^2}[\frac{1}{\varepsilon} + \ln 4\pi - \gamma_E + \ln\frac{\mu^2}{-p^2} + O(\varepsilon)],} \tag{4.16}$$

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where p (with $p^2 < 0$) is an external momentum, and we have performed the expansion around $\varepsilon = 0$. We will encounter P_{div} in the following one-loop calculations. The $1/\varepsilon$ is the divergence we mentioned earlier.

Feynman Integrals

When writing the mathematical expression corresponding to a given diagram, we encounter the products of various propagators. To bring these products to the structure of the momentum integrals listed earlier we can use the following relations found by Feynman himself:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}, \qquad \frac{1}{A^\alpha B^\beta} = \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[xA + (1-x)B]^{\alpha+\beta}}, \tag{4.17}$$

$$\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \frac{y}{[xyA + y(1-x)B + (1-y)C]^3}. \tag{4.18}$$

Here A, B, and C represent the denominators in the propagators in loop diagrams. More general expressions can be found in Muta's book.

Dirac Algebra in D Dimensions

Here we give the most important relations between Dirac matrices γ_μ in D dimensions.

The three basic relations are

$$g_{\mu\nu}g^{\mu\nu} = D, \qquad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \qquad \gamma_\mu\gamma^\mu = D. \tag{4.19}$$

Next

$$\gamma_\nu\gamma_\mu\gamma^\nu = (2-D)\gamma_\mu = 2(\varepsilon - 1)\gamma_\mu, \tag{4.20}$$

$$\gamma_\lambda\gamma_\mu\gamma_\nu\gamma^\lambda = 4g_{\mu\nu} - 2\varepsilon\gamma_\mu\gamma_\nu, \tag{4.21}$$

$$\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho\gamma^\lambda = -2\gamma_\rho\gamma_\nu\gamma_\mu + 2\varepsilon\gamma_\mu\gamma_\nu\gamma_\rho, \tag{4.22}$$

and finally

$$\mathrm{Tr}[\hat{1}] = 4, \qquad \mathrm{Tr}[\gamma_\mu\gamma_\nu] = 4g_{\mu\nu}, \tag{4.23}$$

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho] = 4[g_{\mu\nu}g_{\lambda\rho} + g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}]. \tag{4.24}$$

Setting $\varepsilon = 0$ one recovers the usual relations for $D = 4$ known from any book on relativistic quantum mechanics or relativistic quantum field theory.

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Color Algebra

Here we will collect some useful formulas that we will need in our QCD calculations. The review in [12] is very useful in this context. In view of the large N approach to QCD, which we will discuss in Section 7.2, we will present these formulas for the group SU(N) and begin our presentation with SU(N) Gell-Mann matrices λ_a for which explicit expressions in the case $N = 3$ are given in (A.19)–(A.21) of Appendix A.2. But we will not need these expressions here as only the general relations listed by us will be relevant for our calculations.

There are $N^2 - 1$ matrices λ^a for the SU(N) group; they are Hermitian and traceless and normalized as

$$\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}, \qquad \delta_{ab}\delta_{ab} = N^2 - 1, \tag{4.25}$$

with δ_{ab} being Kronecker delta. The tensors f_{abc} and d_{abc} are defined by

$$\lambda_a \lambda_b = \frac{2}{N}\delta_{ab} + (if_{abc} + d_{abc})\lambda_c \tag{4.26}$$

and have the following properties

$$d_{aab} = 0, \qquad f_{aab} = 0, \qquad d_{acd}f_{bcd} = 0, \tag{4.27}$$

$$f_{acd}f_{bcd} = N\delta_{ab}, \qquad d_{acd}d_{bcd} = \left(N - \frac{4}{N}\right)\delta_{ab}, \tag{4.28}$$

$$(\lambda_a)_{\alpha\beta}(\lambda_a)_{\gamma\delta} = 2\left(\delta_{\alpha\delta}\delta_{\gamma\beta} - \frac{1}{N}\delta_{\alpha\beta}\delta_{\gamma\delta}\right), \tag{4.29}$$

$$\lambda_a\lambda_a = 4C_F\hat{1}, \qquad C_F = \frac{N^2-1}{2N}, \tag{4.30}$$

where $\hat{1}$ is the unit matrix in the color space. In all these formulas summation over repeated color indices is understood.

The color charges in the quark-gluon vertices are summarized by

$$t_a = \frac{\lambda_a}{2}, \qquad [t_a, t_b] = if_{abc}t_c. \tag{4.31}$$

Frequently used relations are

$$t_a t_a = C_F\hat{1}, \qquad t_a t_b t_a = -\frac{1}{2N}t_b, \qquad t_a t_b t_a t_b = -\frac{C_F}{2N}\hat{1}, \tag{4.32}$$

$$t_a t_b t_c t_a t_b = \frac{1}{4}\left(1 + \frac{1}{N^2}\right)t_c, \qquad t_a t_b t_c t_a = \frac{1}{4}\delta_{bc}\hat{1} - \frac{1}{2N}t_b t_c, \tag{4.33}$$

$$t_a t_b t_c t_b t_a = \frac{1}{4N^2}t_c. \tag{4.34}$$

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Fermion Self-Energy Diagram in QED

With this technology at hand we will now calculate explicitly the one-loop self-energy diagram (a) of Figure 4.1 in QED so that the exchanged gauge boson is photon and not gluon as shown there. Setting the fermion mass to zero, denoting the external fermion momentum by p (with $p^2 < 0$), and using QED Feynman rules, given in Appendix B, we find for this diagram

$$I_1 = N_1 \int \frac{d^D k}{(2\pi)^D} \frac{\gamma_\nu (\not{k} + \not{p}) \gamma_\mu}{(k+p)^2 k^2} g^{\mu\nu} = 2N_1(\varepsilon - 1) \int \frac{d^D k}{(2\pi)^D} \frac{(\not{k} + \not{p})}{(k+p)^2 k^2}, \tag{4.35}$$

where

$$N_1 = -e^2 Q_f^2 \mu^{2\varepsilon}, \qquad \not{p} \equiv p_\mu \gamma^\mu, \tag{4.36}$$

Q_f is the electric charge of the fermion, and e the QED coupling constant. It should be noted that we deal here with the so-called amputated Green function (see any book on field theory and Section 4.2) without external propagators.

We use next the first formula in (4.17) to obtain

$$\frac{1}{(k+p)^2 k^2} = \int_0^1 dx \frac{1}{[x(k+p)^2 + (1-x)k^2]^2} = \int_0^1 dx \frac{1}{[l^2 + p^2 x(1-x)]^2}, \tag{4.37}$$

where we made a shift in the internal momentum to bring the integral into the desired form so that we can calculate it in no time:

$$l = k + px, \qquad k + p = l + p(1-x). \tag{4.38}$$

Indeed, using (4.4) and (4.8), we obtain right away

$$I_1 = 2N_1(\varepsilon - 1)\ \not{p} \int_0^1 dx \int \frac{d^D l}{(2\pi)^D} \frac{(1-x)}{[l^2 + p^2 x(1-x)]^2}, \tag{4.39}$$

then

$$I_1 = 2N_1(\varepsilon - 1)\ \not{p} \int_0^1 dx (1-x) \left[i \frac{(-1)^2}{(4\pi)^{D/2}} \frac{\Gamma(2 - D/2)}{\Gamma(2)} \frac{1}{[-p^2 x(1-x)]^{2-D/2}} \right], \tag{4.40}$$

and subsequently

$$I_1 = i\ \not{p} Q_f^2 e^2 [2(1-\varepsilon)] P_{div}\, B(2-\varepsilon, 1-\varepsilon), \tag{4.41}$$

with P_{div} given in (4.16) and $B(2-\varepsilon, 1-\varepsilon)$ in (4.13). Expanding in ε we finally find

$$\boxed{I_a^{\text{em}} = i\ \not{p}\ Q_f^2 \frac{\alpha_{\text{em}}}{4\pi} \left[\frac{1}{\varepsilon} + \ln 4\pi - \gamma_E + \ln \frac{\mu^2}{-p^2} + 1 \right],} \tag{4.42}$$

where $O(\varepsilon)$ terms have been set to zero, and we added a superscript "em" to indicate that the calculation was done in QED. We have thus extracted the singularity as a $1/\varepsilon$ pole and have obtained a well-defined finite part. The appearance of the first four terms in the square bracket in (4.42), originating from P_{div} in (4.16), is characteristic for all divergent one-loop calculations. We will use this result in the process of renormalization of QED in the next section, but after this useful practice in one-loop calculations let us practice them a bit more.

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Quark Self-Energy Diagram in QCD

This is the diagram (a) of Figure 4.1 in which this time a gluon is exchanged. This calculation is identical to the one just performed except that $\alpha_{\rm em}$ should be replaced by α_s and Q_f^2 by the relevant color factor. This factor is calculated as follows

$$t^a_{\alpha\delta}t^b_{\delta\beta}\delta^{ab} = (t^a t^a)_{\alpha\beta} = C_F\delta_{\alpha\beta}, \qquad C_F = \frac{4}{3}, \tag{4.43}$$

where t^a come from quark-gluon vertices and δ^{ab} from gluon propagator. Making these changes in (4.42) we arrive at

$$\boxed{(I_a^{\rm QCD})_{\alpha\beta} = i\ \not{p}\ C_F\ \delta_{\alpha\beta}\frac{\alpha_s}{4\pi}[\frac{1}{\varepsilon} + \ln 4\pi - \gamma_E + \ln\frac{\mu^2}{-p^2} + 1].} \tag{4.44}$$

Again we will use this result in the next section where we will renormalize QCD but to be able to do this we have to work harder and to calculate first few additional diagrams.

Vacuum Polarization in QED

We calculate next the second diagram (b) in Figure 4.1 in QED. We find first

$$(I_b^{\rm em})_{\mu\nu} = N_2\int\frac{d^Dk}{(2\pi)^D}\frac{{\rm Tr}(\gamma_\nu(\not{k}+\not{p})\gamma_\mu\not{k})}{(k+p)^2k^2} = N_2\int\frac{d^Dk}{(2\pi)^D}\frac{(k+p)_{\sigma_1}k_{\sigma_2}}{(k+p)^2k^2}{\rm T}^{\sigma_1\sigma_2}_{\nu\mu}, \tag{4.45}$$

where

$$N_2 = -e^2Q_f^2\mu^{2\varepsilon}, \qquad {\rm T}^{\sigma_1\sigma_2}_{\nu\mu} = {\rm Tr}(\gamma_\nu\gamma^{\sigma_1}\gamma_\mu\gamma^{\sigma_2}). \tag{4.46}$$

The additional minus sign comes from the closed fermion loop. The structure of the denominator is the same as in the self-energy diagram calculations, and consequently the same shift in momenta as given in (4.38) should be made to bring the four-momentum integral in the desired form. We find then

$$(I_b^{\rm em})_{\mu\nu} = N_2{\rm T}^{\sigma_1\sigma_2}_{\mu\nu}\left[G_1^{\sigma_1\sigma_2} + G_2^{\sigma_1\sigma_2}\right], \tag{4.47}$$

where

$$G_1^{\sigma_1\sigma_2} = \int_0^1 dx\int\frac{d^Dl}{(2\pi)^D}\frac{l_{\sigma_1}l_{\sigma_2}}{[l^2+p^2x(1-x)]^2}, \tag{4.48}$$

$$G_2^{\sigma_1\sigma_2} = -p_{\sigma_1}p_{\sigma_2}\int_0^1 dx\int\frac{d^Dl}{(2\pi)^D}\frac{x(1-x)}{[l^2+p^2x(1-x)]^2}. \tag{4.49}$$

With the list of integrals given a few pages earlier, these integrals can be straightforwardly evaluated to find

$$\mu^{2\varepsilon}G_1^{\sigma_1\sigma_2} = \frac{i}{2}g_{\sigma_1\sigma_2}\frac{(-p^2)}{(1-\varepsilon)}P_{div}B(2-\varepsilon,2-\varepsilon), \tag{4.50}$$

$$\mu^{2\varepsilon}G_2^{\sigma_1\sigma_2} = -ip_{\sigma_1}p_{\sigma_2}P_{div}B(2-\varepsilon,2-\varepsilon). \tag{4.51}$$

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Next the products of the tensor $T^{\sigma_1\sigma_2}_{\nu\mu}$ and $g_{\sigma_1\sigma_2}$ and $p_{\sigma_1}p_{\sigma_2}$ have to be evaluated. We find

$$T^{\sigma_1\sigma_2}_{\nu\mu}g_{\sigma_1\sigma_2} = \mathrm{Tr}(\gamma_\nu\gamma^{\sigma_1}\gamma_\mu\gamma_{\sigma_1}) = 2(\varepsilon-1)\mathrm{Tr}(\gamma_\nu\gamma_\mu) = 8(\varepsilon-1)g_{\mu\nu}, \tag{4.52}$$

$$T^{\sigma_1\sigma_2}_{\nu\mu}p_{\sigma_1}p_{\sigma_2} = \mathrm{Tr}(\gamma_\nu \not{p}\gamma_\mu \not{p}) = 4(2p_\mu p_\nu - p^2 g_{\mu\nu}) = 8p_\mu p_\nu - 4p^2 g_{\mu\nu}. \tag{4.53}$$

What remains to be done is to insert the results of all these intermediate calculations into (4.47) and expand in powers of ε. We obtain first

$$(I_b^{\mathrm{em}})_{\mu\nu} = -ie^2Q_f^2 8[p^2 g_{\mu\nu} - p_\mu p_\nu]P_{div}B(2-\varepsilon, 2-\varepsilon), \tag{4.54}$$

and expanding in powers of ε and dropping $O(\varepsilon)$ terms, we find

$$\boxed{(I_b^{\mathrm{em}})_{\mu\nu} = -iQ_f^2\frac{\alpha_{\mathrm{em}}}{4\pi}\frac{4}{3}[p^2 g_{\mu\nu} - p_\mu p_\nu]\left[\frac{1}{\varepsilon} + \ln 4\pi - \gamma_E + \ln\frac{\mu^2}{-p^2} + \frac{5}{3}\right].} \tag{4.55}$$

From gauge invariance we know that the following relation has to be satisfied

$$p^\mu(I_b^{\mathrm{em}})_{\mu\nu} = 0. \tag{4.56}$$

Fortunately our result satisfies this relation.

Vacuum Polarization in QCD

Again the calculation is identical to the one just performed except that α_{em} should be replaced by α_s and Q_f^2 by the relevant color factor. This factor is this time

$$t^a_{\beta\alpha}t^b_{\alpha\beta} = \mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}, \tag{4.57}$$

and using (4.54) we can first write

$$(I_b^{\mathrm{QCD}})^{ab}_{\mu\nu} = -ig_s^2\delta^{ab}n_F 4[p^2 g_{\mu\nu} - p_\mu p_\nu]P_{div}B(2-\varepsilon, 2-\varepsilon), \tag{4.58}$$

where we included the contribution of n_F massless quarks. Finally we find

$$\boxed{(I_b^{\mathrm{QCD}})^{ab}_{\mu\nu} = -in_F\delta^{ab}\frac{\alpha_s}{4\pi}\frac{2}{3}[p^2 g_{\mu\nu} - p_\mu p_\nu]\left[\frac{1}{\varepsilon} + \ln 4\pi - \gamma_E + \ln\frac{\mu^2}{-p^2} + \frac{5}{3}\right].} \tag{4.59}$$

Yet, whereas the result in (4.55) was the final result in QED, the one just obtained is only one contribution to vacuum polarization in QCD. We have to include additional diagrams (c) and (d) shown in Figure 4.1, which are typical for a nonabelian gauge theory. The one involving the so-called tadpole, in which two fermion lines and two gauge boson lines meet at one point, vanishes in dimensional regularization as demonstrated on page 172 in Muta's book [70]. But the other two have to be calculated. In particular the calculation of the one involving triple gluon vertices is rather tough, and we will not be able to present it in all details here. But let us show at least the most important steps.

The integral to be evaluated is this time

$$(I_c^{\mathrm{QCD}})^{ab}_{\mu\nu} = N_3\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k+p)^2k^2}V_{\mu\rho\sigma}(p,k,-k-p)V_\nu^{\sigma\rho}(-p,p+k,-k), \tag{4.60}$$

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where

$$N_3 = (-i)^2 g_s^2 f_{bcd} f_{adc} \frac{1}{2} \mu^{2\varepsilon} = g_s^2 \frac{N}{2} \mu^{2\varepsilon} \delta_{ab} \tag{4.61}$$

with 1/2 being a symmetry factor explained on page 91 of Muta's book. Now the general formula for the triple gluon vertex involves, in addition to $g_s f_{abc}$, the function

$$V_{\mu\nu\lambda}(k_1, k_2, k_3) = (k_1 - k_2)_\lambda g_{\mu\nu} + (k_2 - k_3)_\mu g_{\nu\lambda} + (k_3 - k_1)_\nu g_{\mu\lambda}, \tag{4.62}$$

where all momenta are incoming. The crucial thing in the next steps is not to get lost in indices. While a computer program like *Mathematica* can help us here, it is useful once in your life to do such calculation by hand. We find then

$$V_{\mu\rho\sigma}(p, k, -k - p) = (p - k)_\sigma g_{\mu\rho} + (2k + p)_\mu g_{\rho\sigma} + (-2p - k)_\rho g_{\mu\sigma}, \tag{4.63}$$

$$V_{\nu\sigma\rho}(-p, p + k, -k) = (-2p - k)_\rho g_{\nu\sigma} + (2k + p)_\nu g_{\rho\sigma} + (p - k)_\sigma g_{\nu\rho}. \tag{4.64}$$

Subsequently the product of these two expressions and contraction over repeated ρ and σ indices gives

$$V_{\mu\rho\sigma} V_\nu^{\sigma\rho} = (2k^2 + 2pk + 5p^2) g_{\mu\nu} + (4D - 6) k_\mu k_\nu + (D - 6) p_\mu p_\nu + (2D - 3)(k_\mu p_\nu + k_\nu p_\mu). \tag{4.65}$$

The suppressed arguments in the vertex functions are as in the previous two expressions. Inserting this result into (4.60), we are faced with a number of integrals that can be easily performed using the technology developed by us before. We quote here the final result

$$\boxed{(I_c^{\mathrm{QCD}})_{\mu\nu}^{ab} = i\delta^{ab} g_s^2 N [g_{\mu\nu} p^2 (19 - 12\varepsilon) - 2(11 - 7\varepsilon) p_\mu p_\nu] P_{div} \frac{B(2 - \varepsilon, 2 - \varepsilon)}{2(1 - \varepsilon)}.} \tag{4.66}$$

The striking feature of this result is that it does not satisfy the condition (4.56).

Yet, the diagram (d) in Figure 4.1, in which a ghost is exchanged, removes this problem. Its contribution is

$$(I_d^{\mathrm{QCD}})_{\mu\nu}^{ab} = N_4 \int \frac{d^D k}{(2\pi)^D} \frac{(k + p)_\mu k_\nu}{(k + p)^2 k^2}, \tag{4.67}$$

where

$$N_4 = (-1)(-i)^2 g_s^2 \mu^{2\varepsilon} f_{bdc} f_{acd} = -g_s^2 \mu^{2\varepsilon} N \delta_{ab}. \tag{4.68}$$

The first minus sign comes from the ghost loop. The next steps are similar to the ones we already made in the process of the calculation of the fermion loop (integral I_2), and we will not repeat them here. We find then

$$\boxed{(I_d^{\mathrm{QCD}})_{\mu\nu}^{ab} = i\delta^{ab} g_s^2 N [g_{\mu\nu} p^2 + 2(1 - \varepsilon) p_\mu p_\nu] P_{div} \frac{B(2 - \varepsilon, 2 - \varepsilon)}{2(1 - \varepsilon)}.} \tag{4.69}$$

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This result does not satisfy the gauge invariance condition in (4.56) either. But the gluon and ghost loops together satisfy it:

$$\boxed{(I_c^{\rm QCD}+I_d^{\rm QCD})^{ab}_{\mu\nu}=i\delta^{ab}g_s^2N[g_{\mu\nu}p^2-p_\mu p_\nu]P_{div}\frac{B(2-\varepsilon,2-\varepsilon)}{(1-\varepsilon)}(10-6\varepsilon).} \tag{4.70}$$

Adding the quark loop contribution in (4.59), we complete the calculation of the vacuum polarization in QCD ($K_{\mu\nu}=[g_{\mu\nu}p^2-p_\mu p_\nu]$)

$$\boxed{(I_b^{\rm QCD}+I_c^{\rm QCD}+I_d^{\rm QCD})^{ab}_{\mu\nu}=i\delta^{ab}g_s^2K_{\mu\nu}P_{div}\frac{B(2-\varepsilon,2-\varepsilon)}{(1-\varepsilon)}[N(10-6\varepsilon)-4n_F(1-\varepsilon)].} \tag{4.71}$$

This result will play an important role in the next sections. But let us first address an important issue present in the dimensional regularization.

4.1.3 The Issue of γ_5 in D Dimensions

Preliminaries

The dimensional regularization is the favorite regularization in gauge theories as it preserves all symmetries of the theory. Possible problems are connected with the treatment of γ_5 in $D\neq 4$ dimensions. This issue did not play any role in QED and QCD, in which the fermion-gauge boson couplings are vectorlike (γ_μ), but it is important in the study of weak interactions, where γ_5 plays a prominent role. Let us then discuss this issue now. Even if the material in this section is rather technical and could be skipped, motivated readers should at least look briefly at the following pages. We follow here [71].

Let us describe the three distinct sets of computational rules, for the manipulation of covariants and Dirac matrices, most commonly used in perturbative calculations in the SM. These schemes all employ the method of dimensional regularization of the Feynman integrals used already earlier. We will not discuss other regularization schemes such as BPHZ and lattice. These work directly in four dimensions and hence don't have algebraic consistency problems with respect to γ_5, but their use introduces other subtleties, and already two-loop calculations therewith are extremely tedious.

Naive Dimensional Regularization

The most commonly used set of rules is one we shall call "naive dimensional regularization" (NDR). This is the one we used already earlier, but now our presentation will be more formal and will be generalized to include γ_5. Only the D-dimensional metric tensor g is introduced satisfying,

$$g_{\mu\nu}=g_{\nu\mu},\qquad g_{\mu\rho}g^{\rho}_{\nu}=g_{\mu\nu},\qquad g^{\mu}_{\mu}=D, \tag{4.72}$$

and the Dirac matrices γ_μ and γ_5 obey

$$\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu},\qquad \{\gamma_\mu,\gamma_5\}=0. \tag{4.73}$$

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It is standard (but inessential) to set the trace of the unit matrix to equal 4; we shall adopt this convention in this and other schemes discussed by us. When γ_5 appears in the Feynman vertices the manipulation rule adopted in this scheme is, as given here, that it anticommutes with the Dirac matrices.

It has repeatedly been emphasized in the literature [72–76] that the latter rule leads to obvious algebraic inconsistencies. Nevertheless this scheme has been most widely employed for most calculations because of its ease to incorporate standard software in computer programs. It is known to lead to incorrect results in certain cases, e.g., the axial anomaly is not reproduced. On the other hand, in many cases it does reproduce the correct results. A necessary condition for this seems to be that the calculated amplitude does not involve the evaluation of a closed odd parity fermion loop. Indeed, with the NDR rules one does not know how to unambiguously handle the expression $\mathrm{Tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda)$.

Fortunately beginning with the work in [71] it has been demonstrated in many explicit calculations that the NDR scheme gives correct results, consistent with the schemes without the γ_5 problems, provided one can avoid the calculations of traces like the one just mentioned. In fact, all the higher-order QCD calculations for weak decays performed in the NDR scheme in more than twenty-five years and reviewed in [77] could avoid the direct calculation of such traces.

Dimensional Reduction

A second set of manipulation rules initially introduced in [78] for the renormalization of supersymmetric theories goes under the name of dimensional reduction (DRED). Here the Dirac matrices $\tilde{\gamma}$ are taken to be in four dimensions, thus

$$\{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\tilde{g}_{\mu\nu}, \tag{4.74}$$

where $\tilde{g}$ is the four dimensional metric tensor,

$$\tilde{g}_{\mu\nu} = \tilde{g}_{\nu\mu}, \qquad \tilde{g}_{\mu\rho}\tilde{g}^{\rho}_{\nu} = \tilde{g}_{\mu\nu}, \qquad \tilde{g}^{\mu}_{\mu} = 4. \tag{4.75}$$

When evaluating the Feynman integrals the D-dimensional, $g_{\mu\nu}$ inevitably makes its appearance, and it is necessary to supplement the rules with one that stipulates the result of contraction of the four- and D-dimensional metric tensors. To preserve gauge invariance and in apparent concord with the reduction to $D < 4$ dimensions the rule employed is

$$\tilde{g}_{\mu\rho}g^{\rho}_{\nu} = g_{\mu\nu}. \tag{4.76}$$

The advantage of this scheme is that the four-dimensional Dirac algebra can be used to reduce the algebraic complexity of the amplitudes. However, there is a price to be paid, which involves a number of field theoretical subtleties, some of which are already present in the pure QCD part of the dynamics. These are discussed in [79]. Again, this scheme has been criticized [72–76] because it leads to similar difficulties as the NDR described earlier. In particular, it implies that identities homogeneous in the metric tensor in four dimensions are also satisfied in generic D dimensions, which is manifestly algebraically inconsistent. Although the axial anomaly can be reproduced [80, 81], and although there is to our knowledge as yet no known explicit calculation using DRED that gives the wrong

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result, it has not yet been established as a consistent scheme and thus maintains at present merely the status of a prescription.

In the field of weak decays the DRED scheme has been used for the first time in [79] for the calculation of higher-order QCD corrections to weak decays. This result has been confirmed in [71] and shown to be compatible with the NDR scheme as well as with the 't Hooft–Veltman (HV) scheme discussed next. Similarly the initial problem of calculating higher-order QCD corrections to the $B \to X_s\gamma$ in the DRED scheme [82] has been resolved by Misiak [83]. These days the DRED scheme is less popular, and most calculations of QCD corrections are performed in the NDR scheme and sometimes in the 't Hooft–Veltman scheme, to which we turn our attention now.

The 't Hooft–Veltman Rules

The third set of rules is the one originally proposed by 't Hooft and Veltman [84] and by Akyeampong and Delbourgo [85–87] and systematized by Breitenlohner and Maison [72–74]. The latter authors showed that this is a consistent formulation of dimensional regularization even when γ_5 couplings are present.

To write down the rules it is convenient to introduce, in addition to the D- and four-dimensional metric tensors g and $\tilde{g}$ satisfying (4.72) and (4.75), respectively, the -2ε- dimensional tensor $\hat{g}$ satisfying,

$$\hat{g}_{\mu\nu} = \hat{g}_{\nu\mu}, \qquad \hat{g}_{\mu\rho}\hat{g}^{\rho}_{\nu} = \hat{g}_{\mu\nu}, \qquad \hat{g}^{\mu}_{\mu} = -2\varepsilon. \tag{4.77}$$

The important difference with respect to DRED is that instead of the rule (4.76) for contracting the different metric tensors one imposes

$$\tilde{g}_{\mu\rho}g^{\rho}_{\nu} = \tilde{g}_{\mu\nu}, \tag{4.78}$$

which does not lead to manifest algebraic inconsistencies. In addition to (4.78), one has

$$\hat{g}_{\mu\rho}g^{\rho}_{\nu} = \hat{g}_{\mu\nu}, \qquad \hat{g}_{\mu\rho}\tilde{g}^{\rho}_{\nu} = 0. \tag{4.79}$$

The D-dimensional Dirac matrix is now split into a 4- and -2ε-dimensional part,

$$\gamma_\mu = \tilde{\gamma}_\mu + \hat{\gamma}_\mu, \tag{4.80}$$

with γ and $\tilde{\gamma}$ obeying the anticommutation relations (4.73) and (4.74), respectively. $\hat{\gamma}$ on the other hand satisfies

$$\{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} = 2\hat{g}_{\mu\nu}, \tag{4.81}$$

and it anticommutes with $\tilde{\gamma}$

$$\{\hat{\gamma}_\mu, \tilde{\gamma}_\nu\} = 0. \tag{4.82}$$

Note also by virtue of (4.79) that it follows

$$\hat{\gamma}_\mu\tilde{\gamma}^\mu = 0, \qquad \hat{g}^{\nu}_{\mu}\tilde{\gamma}_\nu = 0, \qquad \tilde{g}^{\nu}_{\mu}\hat{\gamma}_\nu = 0. \tag{4.83}$$

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In [72–74] it is shown that a γ_5 can be introduced, which anticommutes with $\tilde{\gamma}$ but commutes with $\hat{\gamma}$,

$$\gamma_5^2 = 1, \qquad \{\gamma_5, \tilde{\gamma}_\nu\} = 0, \qquad [\gamma_5, \hat{\gamma}_\nu] = 0. \tag{4.84}$$

Because γ_5 does not have simple commutation properties with γ_μ it is important to consistently define the coupling to chiral fields in a model such as the SM; e.g., for couplings to left-handed fields the symmetrically defined vertex

$$\frac{1}{2}(1+\gamma_5)\gamma_\mu(1-\gamma_5) = \tilde{\gamma}_\mu(1-\gamma_5), \tag{4.85}$$

should be used [88].

This scheme has admittedly some rather unattractive features. In particular, it is more inconvenient to implement in algebraic computer programs than the NDR scheme. Nevertheless it must be stressed again that it is to date the only known scheme (within the framework of dimensional regularization) that has been demonstrated to be consistent [72–76], and thus its inconvenience must be tolerated. For this reason a computer package for Dirac algebra manipulation in the HV and NDR schemes called TRACER has been developed by Jamin and Lautenbacher [89]. Using this program one can appreciate the simplicity of the NDR scheme compared with the HV scheme for which the computer calculations turned out in the 1990s to be really time consuming. With much improved computers this is not the case now. But as we mentioned earlier, most researchers use the NDR scheme making sure that no mathematical inconsistencies appear. In particular, even two-loop calculations are doable by hand in the NDR scheme, whereas in the HV scheme already one-loop calculations are rather involved.

In this book we will exclusively work in the NDR scheme, but we will occasionally quote results obtained in the HV scheme. Of course physical results cannot depend on the scheme used, but one should always remember that in field theory there are many intermediate objects that are scheme and also gauge dependent, and only after these objects are combined into a physical quantity that can be measured in experiments these nonphysical dependences drop out. Therefore it is often useful to perform calculations in different schemes and different gauges to verify the correctness of obtained results.

4.2 Renormalization

4.2.1 General Remarks

We have seen in the previous section that the simplest one-loop diagrams are divergent, and these divergences are then also present in Green functions[1] and finally in decay amplitudes. The process of renormalization allows to remove these divergences and to obtain finite decay amplitudes and consequently finite predictions for the observables as functions of

[1] The Green functions are just propagators, vertices and more complicated diagrams and are discussed formally in various books. We will define them later.

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the parameters of a given theory. There are many books discussing renormalization of gauge theories, among them [14, 16, 18, 90], but for this book I found most useful the one of Muta [70], where further details can be found.

The idea of renormalization is actually rather simple. A given theory depends on

- Fields, as φ_0, ψ_0, A_0^μ etc.
- Parameters, as couplings g_0, masses m_0 etc.

The index "0" indicates that these are the so-called bare fields and parameters. Working with the bare fields and parameters, as we did until now, suppressing the index "0," results in divergent Green functions. The renormalization program is just a set of rules for replacing the bare fields and bare parameters by the so-called renormalized fields and renormalized parameters so that the Green functions and decay amplitudes written in terms of them are finite and can be compared with experiment.

The renormalized fields φ, ψ, and A^μ and renormalized parameters g, m, etc. are related to the bare ones as follows

$$\begin{aligned} A^a_{0\mu} &= Z_3^{1/2} A^a_\mu, \qquad & q_0 &= Z_q^{1/2} q, \\ g_{0,s} &= Z_g g_s\, \mu^\varepsilon, \qquad & m_0 &= Z_m\, m, \end{aligned} \tag{4.86}$$

where we have just shown the most important examples. In particular g_s is the renormalized QCD coupling and m the renormalized quark mass.

The factors Z_i are the renormalization constants. They are divergent quantities, chosen in such a manner that the divergences disappear once the Greens functions have been expressed in terms of renormalized quantities only and a particular rescaling of them, exhibited soon, has been made. We have gauge field renormalization (Z_3), quark field renormalization (Z_q), coupling renormalization (Z_g), and mass renormalization (Z_m). There will be other quantities that we will have to renormalize later.

Within DR the renormalization constants have the following structure

$$Z_i = 1 + \frac{\alpha_s}{4\pi}\left(\frac{a_i}{\varepsilon} + b_i\right) + O(\alpha_s^2) \tag{4.87}$$

with the following properties:

- a_i are characteristic for a given theory, here QCD, and are independent of the renormalization scheme used.
- b_i are not characteristic for a given theory but depend on the renormalization scheme.

It should be stressed that the unrenormalized parameters $g_{0,s}$ and m_0 are independent of the scale μ. This implies, in particular, that g_s must be μ-dependent. Because Z_i have a perturbative expansion in g_s they must also depend on μ. Consequently, also the renormalized mass m is μ-dependent.

Before entering the details let us stress that not all theories can be renormalized in this manner. Generally one encounters:

- *Super renormalizable theories*: Only a finite number of Feynman diagrams diverge in all orders of perturbation theory.

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- *Renormalizable theories*: Only a finite number of Feynman subdiagrams diverge, but they occur at each order of perturbation theory.
- *Nonrenormalizable theories*: All amplitudes are divergent at a sufficiently high order of perturbation theory.

The SM is a renormalizable theory, but as we will see at various places in this book, in particular in Chapter 14, also effective theories that look nonrenormalizable can provide useful results if properly interpreted [91].

4.2.2 MS and $\overline{\text{MS}}$ Renormalization Schemes

The simplest renormalization scheme is the *Minimal Subtraction scheme* MS [92] in which only divergences are subtracted. In this scheme, the renormalization constants are given by

$$Z_i = 1 + \frac{\alpha_s}{4\pi}\frac{a_{1i}}{\varepsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{a_{2i}}{\varepsilon^2} + \frac{b_{2i}}{\varepsilon}\right) + O(\alpha_s^3), \tag{4.88}$$

where a_{ji} and b_{ji} are μ-independent constants. The fact that in this scheme the renormalization constants do not have any explicit μ-dependence and depend on μ only through g_s is an important virtue of this scheme. This property is discussed in the context of renormalization group equations in the next section. Similarly the renormalization constants Z_i do not depend on masses. Therefore the MS-scheme and the schemes discussed next belong to the class of mass independent renormalization schemes [93].

Now, starting with the MS scheme, one can construct a whole class of subtraction schemes that differ from MS by a different continuation of the renormalized coupling constant to D dimensions. For these MS-like schemes we have

$$g_{0,s} = Z_g^k g_s^k \mu_k^\varepsilon, \qquad \mu_k = \mu f_k, \tag{4.89}$$

where f_k is an arbitrary number that defines the particular scheme "k." Because different schemes in this class differ from the MS scheme only by a shift in μ, the renormalization constants for these schemes can be obtained from (4.88) by replacing α_s by α_s^k characteristic for a given scheme. The constants a_{ji} and b_{ji}, being μ-independent, remain unchanged.

Of particular interest is the so-called $\overline{\text{MS}}$ scheme [94] in which

$$\mu_{\overline{MS}} = \mu e^{\gamma_E/2}(4\pi)^{-1/2} \tag{4.90}$$

and P_{div} in (4.16) is replaced by

$$\bar{P}_{div} \equiv \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}\left(\frac{\mu_{\overline{MS}}^2}{-p^2}\right)^\varepsilon = \frac{1}{16\pi^2}\left[\frac{1}{\varepsilon} + \ln\frac{\mu^2}{-p^2} + O(\varepsilon)\right]. \tag{4.91}$$

We observe that in this scheme the terms $\ln 4\pi - \gamma_E$, the artifacts of the dimensional regularization, are absent! In summary, then:

$$\{\text{MS} \to \overline{\text{MS}}\} \equiv \{\mu \to \mu_{\overline{MS}}\} \tag{4.92}$$

$$\{Z_i^{MS} \to Z_i^{\overline{MS}}\} \equiv \{\alpha_s^{MS} \to \alpha_s^{\overline{MS}}\}. \tag{4.93}$$

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In this book, we will exclusively work with the $\overline{\text{MS}}$ scheme. To simplify the notation we will denote $\mu_{\overline{MS}}$ simply by μ and simultaneously drop the $\ln 4\pi - \gamma_E$ terms in any finite contribution. Similarly α_s in this book will always stand for $\alpha_s^{\overline{\text{MS}}}$.

In later subsections of this section we will use the results of the previous section to calculate explicitly Z_q, Z_m, Z_3, and Z_{g_s} in the $\overline{\text{MS}}$ scheme, but it is useful to see the result already now. We have in the case of QCD

$$Z_q = 1 - \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon} + O(\alpha_s^2), \tag{4.94}$$

$$Z_m = 1 - \frac{\alpha_s}{4\pi} 3C_F \frac{1}{\varepsilon} + O(\alpha_s^2), \tag{4.95}$$

$$Z_3 = 1 - \frac{\alpha_s}{4\pi} \left[\frac{2}{3} f - \frac{5}{3} N \right] \frac{1}{\varepsilon} + O(\alpha_s^2), \tag{4.96}$$

$$Z_{g_s} = 1 - \frac{\alpha_s}{4\pi} \left[\frac{11}{6} N - \frac{2}{6} f \right] \frac{1}{\varepsilon} + O(\alpha_s^2), \tag{4.97}$$

where N denotes the number of colors ($N = 3$ in QCD), C_F is given in (4.30), and f stands for the number of quark flavors. Z_q and Z_3 are gauge dependent and are given here in the Feynman gauge ($\xi = 1$). But this gauge dependence is cancelled by other contributions to physical amplitudes.

There are two routes to find the Z_i. One uses the relation between renormalized and unrenormalized Green functions, and the second one is the so-called counter method. We will first introduce these methods and subsequently use them to calculate Z_i in QCD and QED.

4.2.3 Renormalization of Green Functions

To discuss all this in explicit terms let us denote by

$$G^{(n_F,n_G)}(p_j, g_s, m, \mu, \varepsilon) \equiv \langle 0|T(q_1, \ldots q_{n_F}, A_1^\mu, \ldots A_{n_G}^\mu)|0\rangle \tag{4.98}$$

a *connected renormalized* Green function with n_F quark and n_G gluon external legs carrying momenta p_j. Here m indicates general dependence on masses. The corresponding *amputated renormalized one-particle irreducible* Green function is given by

$$\Gamma^{(n_F,n_G)} = \frac{G^{(n_F,n_G)}}{\prod^{n_F} G^{(2,0)} \prod^{n_G} G^{(0,2)}} \,. \tag{4.99}$$

The division by the products of propagators of external fields amputates the Green function. Similar expressions exist for the unrenormalized Green functions $G_0^{(n_F,n_G)}$ and $\Gamma_0^{(n_F,n_G)}$ with all renormalized parameters and fields replaced by the corresponding bare quantities. With (4.86), $\Gamma^{(n_F,n_G)}$ and $\Gamma_0^{(n_F,n_G)}$ are related to each other by

$$\Gamma^{(n_F,n_G)}(p_j, g_s, m, \mu, \varepsilon) = Z_q^{n_F/2} Z_3^{n_G/2} \Gamma_0^{(n_F,n_G)}(p_j, g_{0,s}, m_0, \varepsilon) \,. \tag{4.100}$$

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The factors in front of $\Gamma_0^{(n_F,n_G)}$ result from the rescaling of the fields in (4.86). The renormalization then means that when $g_{0,s}$ and m_0 on the r.h.s. of (4.100) are expressed through g and m according to (4.86), $\Gamma^{(n_F,n_G)}$ are finite and the limit

$$\lim_{\varepsilon\to 0}\Gamma^{(n_F,n_G)}(p_j,g_s,m,\mu,\varepsilon)=\Gamma^{(n_F,n_G)}(p_j,g_s,m,\mu) \tag{4.101}$$

exists.

4.2.4 The Counter-Term Method

A straightforward way to implement renormalization is provided by the counter-term method. Thereby parameters and fields in the original Lagrangian, considered as unrenormalized (bare) quantities, are reexpressed through renormalized ones by means of (4.86). Thus

$$\mathcal{L}^0_{QCD}=\mathcal{L}_{QCD}+\mathcal{L}_C, \tag{4.102}$$

where $\mathcal{L}_{QCD}$ is given in (2.113). $\mathcal{L}^0_{QCD}$ is also given by (2.113) but with q replaced by q_0 and similarly for A^a_μ, g_s, and m. $\mathcal{L}_C$ is the *counter-term* Lagrangian. It is simply defined by (4.102). For instance:

$$\mathcal{L}_q=\bar{q}_0 i\not{\partial}q_0-m_0\bar{q}_0q_0\equiv\bar{q}i\not{\partial}q-m\bar{q}q+(Z_q-1)\bar{q}i\not{\partial}q-(Z_qZ_m-1)m\bar{q}q. \tag{4.103}$$

$\mathcal{L}_{QCD}$ given entirely in terms of renormalized quantities leads to the usual Feynman rules of Appendix B. The counter-terms ($\sim(Z-1)$) can be formally treated as new interaction terms that contribute to Green functions calculated in perturbation theory. For these new interactions also Feynman rules can be derived. For instance, the Feynman rule for the counter-terms in (4.103) reads (p is the quark momentum)

$$i\delta_{\alpha\beta}[(Z_q-1)\not{p}-(Z_qZ_m-1)m]\,. \tag{4.104}$$

The constants Z_i are determined such that the contributions from these new interactions cancel the divergences in the Green functions resulting from the calculations based on $\mathcal{L}_{QCD}$ in (4.102) only.

There is some arbitrariness how this can be done because a given renormalization prescription can in general subtract not only the divergences but also finite parts. The subtractions of finite parts is, however, not uniquely defined, which results in the *renormalization scheme dependence* of Z_i and of the renormalized fields and parameters as we already announced on previous pages.

4.2.5 Explicit Calculation of Renormalization Constants (QCD)

Let us first calculate Z_q and Z_m. To this end we repeat the calculation of the self-energy diagram of Figure 4.1, this time keeping the quark mass m. Dropping finite terms, which are of no concern for finding Z_i in the $\overline{\text{MS}}$ scheme, we find

$$(i\Sigma_{\alpha\beta})_{div}=iC_F\delta_{\alpha\beta}\frac{\alpha_s}{4\pi}(\not{p}-4m)\frac{1}{\varepsilon}+O(\alpha_s^2). \tag{4.105}$$

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The l.h.s. is just fancy notation for $(I_1^{\rm QCD})_{\alpha\beta}$ in (4.44). Note that for $m = 0$ (4.105) gives the divergent part in (4.44). Adding to this result the counter-term (4.104) and requiring the final result to be zero, we readily find Z_q and Z_m in (4.94) and (4.95), respectively.

As an exercise we can check whether Z_q in (4.94) indeed renormalizes the two-point Green function, which for $m = 0$ is simply given by $(I_1^{\rm QCD})_{\alpha\beta}$ in (4.44). In the notation of (4.100) its divergent part added to the "tree level" propagator is given by

$$\Gamma_0^{(2,0)} = iC_F\delta_{\alpha\beta}\ \not{p}\left(1 + \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}\right). \tag{4.106}$$

The corresponding renormalized two-point function is given by

$$\Gamma^{(2,0)} = Z_q\Gamma_0^{(2,0)}, \tag{4.107}$$

which with (4.94) is indeed finite. In this case at $O(\alpha_s)$ only quark field renormalization is needed to obtain a finite result. Coupling renormalization is necessary first at $O(\alpha_s^2)$.

Let us next calculate Z_3 in QCD. The result in (4.71) is just the $O(\alpha_s)$ contribution to the unrenormalized Green function $\Gamma_0^{(0,2)}$. Adding the tree-level part and expanding the expression in (4.71) in powers of ε and keeping only the divergent part we find

$$\Gamma_0^{(0,2)} = i\delta^{ab}[p_\mu p_\nu - g_{\mu\nu}p^2]\left(1 + \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}r\right), \qquad r = \frac{2}{3}f - \frac{5}{3}N. \tag{4.108}$$

But the corresponding renormalized Green function $\Gamma^{0,2}$ is given by

$$\Gamma^{(0,2)} = Z_3\Gamma_0^{(0,2)}. \tag{4.109}$$

Requiring it to be finite gives Z_3 in (4.96).

The calculation of Z_g is much harder as it involves either the diagrams in Figure 4.2 or the diagrams in Figure 4.3. There is no space for doing these calculations in detail here,

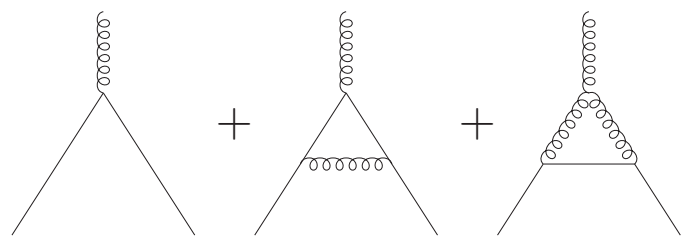


Figure 4.2 The diagrams contributing to Z_g.

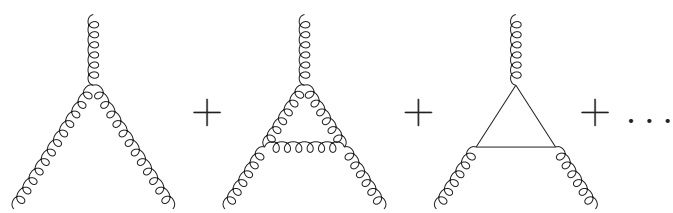


Figure 4.3 Different route to Z_g.

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and it will suffice to quote only the results for them. Keeping only the divergent parts one finds first

$$\Gamma_0^{(2,1)} = ig_s Z_g \gamma_\mu t^a (Z_1^F)^{-1}, \qquad Z_1^F = 1 - \frac{\alpha_s}{4\pi}\frac{(N + C_F)}{\varepsilon}, \tag{4.110}$$

$$\Gamma_0^{(0,3)} \propto ig_s Z_g (Z_1^G)^{-1}, \qquad Z_1^G = 1 - \frac{\alpha_s}{4\pi}\frac{2}{3\varepsilon}(f - N). \tag{4.111}$$

In writing these expressions we replaced the bare coupling constant in the tree-level terms by the renormalized coupling. In the correction terms this replacement is immaterial at this order in perturbation theory. Moreover, we introduced Z_1 familiar from QED. Due to the complicated Lorenz structure of the triple gluon vertex, we suppressed it as it does not play any role in finding Z_g.

But the corresponding renormalized Green functions are given by

$$\Gamma^{(2,1)} = Z_2 Z_3^{1/2} \Gamma_0^{(2,1)}, \qquad \Gamma^{(0,3)} = Z_3^{3/2} \Gamma_0^{(0,3)}. \tag{4.112}$$

Using (4.110) and (4.111) and requiring these Green functions to be finite implies, respectively,

$$Z_g = Z_q^{-1} Z_3^{-1/2} Z_1^F, \qquad Z_g = Z_3^{-3/2} Z_1^G. \tag{4.113}$$

Inserting the already known expressions for Z_q, Z_3, Z_1^F, and Z_1^G and keeping only the $O(\alpha_s)$ terms we find from both expressions the same Z_g given in (4.97).

The fact that we have obtained the same Z_g from two different Green functions follows from gauge invariance of QCD as expressed by the Slavnov–Taylor identity:

$$\frac{Z_1^F}{Z_2} = \frac{Z_1^G}{Z_3}. \tag{4.114}$$

4.2.6 Renormalization Constants in QED

After this practice of calculating the renormalization constants in QCD, it is straightforward to find corresponding results in QED. We leave it as an exercise and just list the results that are analogous to the ones in (4.94)–(4.97),

$$Z_f = 1 - \frac{\alpha_{\text{em}}}{4\pi} Q_f^2 \frac{1}{\varepsilon} + O(\alpha_{\text{em}}^2), \tag{4.115}$$

$$Z_m = 1 - \frac{\alpha_{\text{em}}}{4\pi} 3Q_f^2 \frac{1}{\varepsilon} + O(\alpha_{\text{em}}^2), \tag{4.116}$$

$$Z_3 = 1 - \frac{\alpha_{\text{em}}}{4\pi}\left[\frac{4}{3}\sum_f Q_f^2\right]\frac{1}{\varepsilon} + O(\alpha_{\text{em}}^2), \tag{4.117}$$

$$Z_e = 1 - \frac{\alpha_{\text{em}}}{4\pi}\left[\frac{2}{3}\sum_f Q_f^2\right]\frac{1}{\varepsilon} + O(\alpha_{\text{em}}^2). \tag{4.118}$$

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We note that for fermion and mass renormalization, in going from QCD to QED one just replaces C_F by Q_f^2. Also in QED one has relations resulting from gauge invariance. These are the so-called Ward identities:

$$Z_1^F = Z_f, \qquad Z_e = Z_3^{-1/2}. \tag{4.119}$$

On the other hand, the change in coupling renormalization when going from QCD to QED is very profound. To appreciate it we will in the next section look at the dependence of QCD and QED couplings on the scale μ, which is governed by renormalization group equations.

4.3 Renormalization Group Equations

4.3.1 The Basic Equations

In the process of renormalization we have introduced an arbitrary mass parameter μ. The μ-dependence of the renormalized coupling constant g_s and of the renormalized quark mass m is governed by the renormalization group equations. These equations are derived from the definitions (4.86) using the fact that bare quantities are μ-independent. One finds ($g \equiv g_s$):

$$\frac{dg(\mu)}{d\ln\mu} = \beta(g(\mu), \varepsilon), \tag{4.120}$$

$$\frac{dm(\mu)}{d\ln\mu} = -\gamma_m(g(\mu))m(\mu), \tag{4.121}$$

where

$$\beta(g, \varepsilon) = -\varepsilon g + \beta(g), \tag{4.122}$$

$$\beta(g) = -g\frac{1}{Z_g}\frac{dZ_g}{d\ln\mu}, \qquad \gamma_m(g) = \frac{1}{Z_m}\frac{dZ_m}{d\ln\mu}. \tag{4.123}$$

Equation (4.122) is valid in arbitrary dimensions. In four dimensions $\beta(g, \varepsilon)$ reduces to $\beta(g)$. Let us prove (4.122). Using (4.86) we have

$$\begin{aligned}\beta(g, \varepsilon) &= g_0\mu\frac{d}{d\mu}[\mu^{-\varepsilon}Z_g^{-1}] = g_0\left[-\varepsilon\mu^{-\varepsilon}Z_g^{-1} + \mu^{-\varepsilon+1}\frac{dZ_g^{-1}}{d\mu}\right] \\ &= -\varepsilon g - g_0\mu^{-\varepsilon+1}\frac{1}{Z_g^2}\frac{dZ_g}{d\mu} = -\varepsilon g - g\mu\frac{dZ_g}{d\mu}\frac{1}{Z_g}.\end{aligned} \tag{4.124}$$

Similarly, one can derive the expression for γ_m in (4.123) by inserting $m = m_0/Z_m$ into (4.121).

$\beta(g)$ and $\gamma(g)$ are called *renormalization group functions.* $\beta(g)$ governs the μ-dependence of $g(\mu)$. γ_m, the *anomalous dimension* of the mass operator, governs

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the μ-dependence of $m(\mu)$. In the MS ($\overline{\text{MS}}$)-scheme they depend only on g. In particular they carry no explicit μ-dependence and are independent of masses. Writing

$$Z_i = 1 + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} Z_{i,k}(g), \tag{4.125}$$

and using (4.122) and (4.123) one finds

$$\beta(g) = 2g^3 \frac{dZ_{g,1}(g)}{dg^2}, \tag{4.126}$$

$$\gamma_m(g) = -2g^2 \frac{dZ_{m,1}(g)}{dg^2}. \tag{4.127}$$

Thus $\beta(g)$ and $\gamma_m(g)$ can be directly obtained from the $1/\varepsilon$-pole parts of the renormalization constants Z_g and Z_m, respectively. This is a very useful property of the MS-like schemes. Let us demonstrate that (4.126) is indeed true. We follow Muta [70] and write

$$\beta(g,\varepsilon) = -\varepsilon g - g f(g), \qquad f(g) = \frac{\mu}{Z_g}\frac{dZ_g}{d\mu}. \tag{4.128}$$

Specializing the expansion (4.125) to Z_g and inserting it into the formula for $f(g)$ gives

$$f(g)\left(1 + \frac{Z_{g,1}}{\varepsilon} + \frac{Z_{g,2}}{\varepsilon^2} + \cdots\right) = \frac{1}{\varepsilon}\beta(g,\varepsilon)\left(\frac{dZ_{g,1}}{dg} + \frac{1}{\varepsilon}\frac{dZ_{g,2}}{dg} + \cdots\right). \tag{4.129}$$

Now finiteness of $\beta(g)$ implies finiteness of $f(g)$. Consequently, the equality (4.129) should hold for each coefficient of the power $1/\varepsilon$. In particular the nonsingular terms give

$$f(g) = -g\frac{dZ_{g,1}}{dg}, \tag{4.130}$$

which with $\beta(g) = -g f(g)$ gives (4.126). The proof of (4.127) can be done in an analogous manner using the finiteness of γ_m. It is left as a homework problem.

With Z_g and Z_m in (4.97) and (4.95), respectively, the formulas (4.126) and (4.127) give immediately the leading terms for $\beta(g)$ and $\gamma_m(g)$:

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}N - \frac{2}{3}f\right], \tag{4.131}$$

$$\gamma_m(g) = \frac{g^2}{16\pi^2} 6\, C_F. \tag{4.132}$$

With this technique it is also easy to show that the anomalous dimensions of the quark field (γ_q) and the qluon field (γ_G) defined by

$$\gamma_q(g) = \frac{1}{2}\frac{1}{Z_q}\frac{dZ_q}{d\ln\mu}, \qquad \gamma_G(g) = \frac{1}{2}\frac{1}{Z_3}\frac{dZ_3}{d\ln\mu}, \tag{4.133}$$

are given by

$$\gamma_i(g) = -g^2\frac{dZ_{i,1}(g)}{dg^2}, \qquad (i = q, G). \tag{4.134}$$

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4.3.2 Compendium of Useful Results

It will be useful to have a collection of results for $\beta(g)$, $\gamma(\alpha_s)$, and $Z_{q,1}(\alpha_s)$, including also two-loop contributions. They are

$$\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2}, \tag{4.135}$$

$$\gamma_m(\alpha_s) = \gamma_m^{(0)} \frac{\alpha_s}{4\pi} + \gamma_m^{(1)} \left(\frac{\alpha_s}{4\pi}\right)^2, \tag{4.136}$$

$$Z_{q,1}(\alpha_s) = a_1 \frac{\alpha_s}{4\pi} + a_2 \left(\frac{\alpha_s}{4\pi}\right)^2, \tag{4.137}$$

where we dropped higher-order terms in $g = g_s$ and

$$\beta_0 = \frac{11N - 2f}{3}, \qquad \beta_1 = \frac{34}{3}N^2 - \frac{10}{3}Nf - 2C_F f, \tag{4.138}$$

$$\gamma_m^{(0)} = 6C_F, \qquad \gamma_m^{(1)} = C_F \left(3C_F + \frac{97}{3}N - \frac{10}{3}f\right), \tag{4.139}$$

$$a_1 = -C_F, \qquad a_2 = C_F \left(\frac{3}{4}C_F - \frac{17}{4}N + \frac{1}{2}f\right), \tag{4.140}$$

$$C_F = \frac{N^2 - 1}{2N}. \tag{4.141}$$

These results are valid in the MS ($\overline{\text{MS}}$) scheme. N is the number of colors and f the number of quark flavors. Whereas β_0, β_1, $\gamma_m^{(0)}$, and $\gamma_m^{(1)}$ are gauge independent, a_1 and a_2 given here are valid only in the $\xi = 1$ gauge. The two-loop β_1 has been calculated in [95], the two-loop $\gamma_m^{(1)}$ in [96] and the three-loop $\gamma_m^{(2)}$ in [1336].

4.3.3 Running Coupling Constant

With the expansion (4.135), the renormalization group equation (4.120) for $g(\mu)$ can be written as follows:

$$\frac{d\alpha_s}{d\ln\mu} = -2\beta_0 \frac{\alpha_s^2}{4\pi} - 2\beta_1 \frac{\alpha_s^3}{(4\pi)^2}. \tag{4.142}$$

Solving it, one finds [94]:

$$\frac{\alpha_s(\mu)}{4\pi} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2_{\overline{MS}})} - \frac{\beta_1}{\beta_0^3} \frac{\ln[\ln(\mu^2/\Lambda^2_{\overline{MS}})]}{\ln^2(\mu^2/\Lambda^2_{\overline{MS}})}. \tag{4.143}$$

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Let us make a few comments:

- $\Lambda_{\overline{MS}}$ is a QCD scale characteristic for the $\overline{\text{MS}}$ scheme. It can be determined by measuring $\alpha_s(\mu)$ at a single value of μ. To this end the quantity used to determine $\alpha_s(\mu)$ has to be calculated in the $\overline{\text{MS}}$ scheme. Strictly speaking $\alpha_s(\mu)$ should really read $\alpha_{s,\overline{\text{MS}}}$, but we will work exclusively in the $\overline{\text{MS}}$ scheme, and this complication of the notation is unnecessary. Yet it is useful to quote the relation to the MS scheme. Inserting (see (4.90))

$$\mu \equiv \mu_{\overline{MS}} = \mu_{MS} e^{\gamma_E/2} (4\pi)^{-1/2} \tag{4.144}$$

into (4.143), one finds the relation between α_s in the MS and $\overline{\text{MS}}$ schemes:

$$\alpha_{s,MS} = \alpha_{s,\overline{MS}} \left(1 + \beta_0 (\gamma_E - \ln 4\pi) \frac{\alpha_{s,\overline{MS}}}{4\pi} \right) \tag{4.145}$$

or

$$\Lambda^2_{\overline{MS}} = 4\pi e^{-\gamma_E} \Lambda^2_{MS}. \tag{4.146}$$

- $\Lambda_{\overline{MS}}$ and $\alpha_s(\mu)$ depend on f, the number of "effective" flavors present in β_0 and β_1. These effective flavors are defined by

$$\begin{cases} f = 6 & \mu \geq m_{\text{t}}, \\ f = 5 & m_{\text{b}} \leq \mu \leq m_{\text{t}}, \\ f = 4 & m_{\text{c}} \leq \mu \leq m_{\text{b}}, \\ f = 3 & \mu \leq m_{\text{c}}. \end{cases} \tag{4.147}$$

Denoting by $\alpha_s^{(f)}$ the effective coupling constant for a theory with f effective flavors and by $\Lambda^{(f)}_{\overline{MS}}$ the corresponding QCD scale parameter, we have the following boundary conditions that follow from the continuity of α_s:

$$\alpha_s^{(6)}(m_{\text{t}}) = \alpha_s^{(5)}(m_{\text{t}}), \qquad \alpha_s^{(5)}(m_{\text{b}}) = \alpha_s^{(4)}(m_{\text{b}}), \qquad \alpha_s^{(4)}(m_{\text{c}}) = \alpha_s^{(3)}(m_{\text{c}}). \tag{4.148}$$

The preceding continuity conditions allow to find values of $\Lambda^{(f)}_{\overline{MS}}$ for different f once one particular $\Lambda^{(f)}_{\overline{MS}}$ is known. In Table 4.1 we show different $\alpha_s^{(f)}(\mu)$ and $\Lambda^{(f)}_{\overline{MS}}$ corresponding to a much larger range than the present world average

$$\alpha_s^{(5)}(M_{\text{Z}}) = 0.1181 \pm 0.0006, \tag{4.149}$$

extracted from different processes, partly to allow the reader to reproduce this table and to underline the big progress in the determination of $\alpha_s^{(5)}(M_{\text{Z}})$ in the last twenty years. But our central value is in fact very close to the present world average.

To this end we have set $m_{\text{c}} = 1.3$ GeV, $m_{\text{b}} = 4.4$ GeV, and $m_{\text{t}} = 170$ GeV. We observe that for $\mu \geq m_{\text{c}}$ the values of $\alpha_s(\mu)$ are sufficiently small that the effects of strong interactions can be treated in perturbation theory. When one moves to low-energy scales, α_s increases and at $\mu \approx O(1\ \text{GeV})$ and high values of $\Lambda^{(3)}_{\overline{MS}}$ one finds $\alpha_s^{(3)}(\mu) > 0.5$. This signals breakdown of perturbation theory for scales lower than 1 GeV. Yet it is gratifying that strong interaction contributions to weak decays coming from scales higher than 1 GeV can be treated by perturbative methods.

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Table 4.1 Values of $\alpha_s^{(f)}(\mu)$ and $\Lambda_{\overline{MS}}^{(f)}$ corresponding to given values of $\alpha_s^{(5)}(M_Z)$.

$\alpha_s^{(6)}(m_t)$	0.1037	0.1054	0.1079	0.1104	0.1120
$\Lambda_{\overline{MS}}^{(6)}$[MeV]	66	76	92	110	123
$\alpha_s^{(5)}(M_Z)$	0.113	0.115	0.118	0.121	0.123
$\Lambda_{\overline{MS}}^{(5)}$[MeV]	169	190	226	267	296
$\alpha_s^{(5)}(m_b)$	0.204	0.211	0.222	0.233	0.241
$\Lambda_{\overline{MS}}^{(4)}$[MeV]	251	278	325	376	413
$\alpha_s^{(4)}(m_c)$	0.336	0.357	0.396	0.443	0.482
$\Lambda_{\overline{MS}}^{(3)}$[MeV]	297	325	372	421	457
$\alpha_s^{(3)}$(1 GeV)	0.409	0.444	0.514	0.605	0.690

Finally, we would like to give an equivalent expression for α_s, which allows to calculate $\alpha_s(\mu)$ directly from the experimental value given in (4.149):

$$\boxed{\alpha_s(\mu) = \frac{\alpha_s(M_Z)}{v(\mu)}\left[1 - \frac{\beta_1}{\beta_0}\frac{\alpha_s(M_Z)}{4\pi}\frac{\ln v(\mu)}{v(\mu)}\right], \qquad v(\mu) = 1 - \beta_0\frac{\alpha_s(M_Z)}{2\pi}\ln\left(\frac{M_Z}{\mu}\right).} \tag{4.150}$$

Strictly speaking, (4.150) is valid for the $f = 5$ theory. To find $\alpha_s(\mu)$ for $f \neq 5$ one has to proceed as in (4.147) and (4.148).

We have discussed here only two-loop contributions to the QCD β function. By now also three-loop [97, 98], four-loop [99, 100], and five-loop [101, 102] contributions are known.

4.3.4 Running Quark Mass

Let us next find the μ-dependence of $m(\mu)$. With $dg/d\ln\mu = \beta(g)$ the solution of

$$\frac{dm(\mu)}{d\ln\mu} = -\gamma_m(g)m(\mu) \tag{4.151}$$

is obviously

$$m(\mu) = m(\mu_0)\exp\left[-\int_{g(\mu_0)}^{g(\mu)} dg'\frac{\gamma_m(g')}{\beta(g')}\right]. \tag{4.152}$$

Here $m(\mu_0)$ is the value of the running mass at the scale μ_0. For instance: $m_s(2\ \text{GeV})$. Inserting the expansions for $\gamma_m(g)$ and $\beta(g)$ into (4.152) and expanding in α_s gives:

$$\boxed{m(\mu) = m(\mu_0)\left[\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]^{\frac{\gamma_m^{(0)}}{2\beta_0}}\left[1 + \left(\frac{\gamma_m^{(1)}}{2\beta_0} - \frac{\beta_1\gamma_m^{(0)}}{2\beta_0^2}\right)\frac{\alpha_s(\mu) - \alpha_s(\mu_0)}{4\pi}\right].} \tag{4.153}$$

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In the literature the running quark mass is often denoted by $\overline{m}(\mu)$, but in this book we will use simply $m(\mu)$.

Because formulas similar to (4.151)–(4.153) will often appear in subsequent sections, it is useful to derive at least the leading term in (4.153). Keeping the leading terms in $\gamma_m(g)$ and $\beta(g)$, we have

$$-\int_{g(\mu_0)}^{g(\mu)} dg' \frac{\gamma_m(g')}{\beta(g')} = \int_{g(\mu_0)}^{g(\mu)} dg' \frac{\gamma_m^{(0)}}{\beta_0} \frac{1}{g'} = \frac{1}{2} \frac{\gamma_m^{(0)}}{\beta_0} \ln \frac{g^2(\mu)}{g^2(\mu_0)}, \tag{4.154}$$

which inserted into (4.152) gives the leading term in (4.153). Keeping also the NLO terms in $\gamma_m(g)$ and $\beta(g)$ and proceeding in a similar manner, one readily finds the NLO term in (4.153).

Because the power $\gamma_m^{(0)}/2\beta_0$ is positive, $m(\mu)$ similar to α_s decreases with increasing μ. For low μ this dependence is sizable so that between 1 GeV and 2 GeV the light quark masses decrease typically by 25%. On the other hand the μ dependence of the top quark mass $m_{\rm t}(\mu_t)$ is much weaker. In the range $100\,\text{GeV} \le \mu_t \le 300\,\text{GeV}$ the top quark mass decreases only by 8%.

By now also three-loop [103], four-loop [104, 105], and five loop [106] contributions to $\gamma_m(\alpha_s)$ are known.

The most difficult part in this analysis is the determination of the mass $m(\mu_0)$ for a given quark from the data. For the top quark, which is not confined, this can be done through high-energy processes, but in the case of the remaining five quarks, which are confined in hadrons, nonperturbative methods are required to find $m(\mu_0)$. The most accurate values are presently obtained from lattice simulations. One finds then in FLAG and PDG reports

$$m_u(2\,\text{GeV}) = (2.16 \pm 0.11)\,\text{MeV}, \qquad m_d(2\,\text{GeV}) = (4.68 \pm 0.15)\,\text{MeV}, \tag{4.155}$$

$$m_c(m_c) = (1.279 \pm 0.013)\,\text{GeV}, \qquad m_s(2\,\text{GeV}) = (93.8 \pm 2.4)\,\text{MeV}, \tag{4.156}$$

$$m_b(m_b) = 4.19^{+0.18}_{-0.06}\,\text{GeV}, \qquad m_t(m_t) = 163(1)\,\text{GeV}, \tag{4.157}$$

where we also list the value of $m_t(m_t)$, which is found from collider experiments. In Table 4.2 we show the values of m_i for different scales μ using as input the values in (4.155)–(4.157).[2]

4.3.5 RG Improved Perturbation Theory

The structure of (4.150) and (4.153) makes it clear that the RG approach goes beyond the usual perturbation theory. To see what is going on, let us consider the leading term in (4.150):

$$\alpha_s(\mu) = \frac{\alpha_s(M_Z)}{1 - \beta_0 \frac{\alpha_s(M_Z)}{2\pi} \ln\left(\frac{M_Z}{\mu}\right)}\,. \tag{4.158}$$

[2] The author thanks Jason Aebischer and Christoph Bobeth for providing this table.

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Table 4.2 Central values of quark masses at different scales.[a]

μ [TeV]	α_s	m_u [MeV]	m_d [MeV]	m_s [MeV]	m_c [GeV]	m_b [GeV]	m_t [GeV]
1	0.0886	1.05	2.27	45.5	0.535	2.41	143.8
5	0.0762	0.96	2.07	41.5	0.488	2.20	131.2
10	0.0719	0.92	2.00	40.1	0.471	2.13	126.6
20	0.0680	0.89	1.94	38.8	0.456	2.06	122.5
50	0.0635	0.86	1.86	37.2	0.437	1.97	117.5
100	0.0605	0.83	1.80	36.2	0.425	1.92	114.1
200	0.0578	0.81	1.75	35.2	0.413	1.86	111.0

[a] The input values are given in (4.155)–(4.157). The second column gives the corresponding values of α_s.

Expanding it in $\alpha_s(M_Z)$, we find:

$$\alpha_s(\mu) = \alpha_s(M_Z)\left[1 + \sum_{n=1}^{\infty}\left(\beta_0 \frac{\alpha_s(M_Z)}{2\pi}\ln\left(\frac{M_Z}{\mu}\right)\right)^n\right]. \tag{4.159}$$

We conclude that the solution of the renormalization group equations sums automatically large logarithms $\log(M_Z/\mu)$, which appear for $\mu \ll M_Z$. More generally

$$\text{LO}: \quad \text{Summation of} \quad \left[\alpha_s(M_Z)\ln\left(\frac{M_Z}{\mu}\right)\right]^n, \tag{4.160}$$

$$\text{NLO}: \quad \text{Summation of} \quad \alpha_s(M_Z)^n\left[\ln\left(\frac{M_Z}{\mu}\right)\right]^{n-1}. \tag{4.161}$$

In particular we note that the expansion (4.153) in terms of $\alpha_s(\mu)$ does not involve large logarithms, and a few terms suffice to obtain reliable results. Equation (4.153) is an example of a *renormalization group improved perturbative expansion*. We will encounter similar expansions for other quantities in the later chapters.

In this chapter we have collected certain information about QCD and tools like RG methods that allow us to sum large logarithms. We have also discussed the μ dependences of the running QCD coupling and the running quark masses. Yet all these nice and powerful tools are still insufficient to discuss properly weak decays. Yes, what we still need is *the operator product expansion* and more generally the technology of *effective field theories.*

THE REAL FUN BEGINS ONLY NOW!

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