Check for updates

J. Appl. Probab. **62,** 1028–1043 (2025)
*doi:*10.1017/jpr.2024.112

MATRIX REPRESENTATIONS OF WIENER–HOPF FACTORISATIONS FOR LÉVY PROCESSES

SØREN ASMUSSEN,* *Aarhus University*
MOGENS BLADT ,** *University of Copenhagen*

Abstract

The Wiener–Hopf factors of a Lévy process are the maximum and the displacement from the maximum at an independent exponential time. The majority of explicit solutions assume the upward jumps to be either phase-type or to have a rational Laplace transform, in which case the traditional expressions are lengthy expansions in terms of roots located by means of Rouché's theorem. As an alternative, compact matrix formulas are derived, with parameters computable by iteration schemes.

Keywords: Phase-type distribution; matrix-exponential distribution; rational Laplace transform; iterative scheme; CGMY process; Padé approximation; barrier options; Poisson sampling

2010 Mathematics Subject Classification: Primary 60G51
Secondary 60-08

1. Introduction

Let X be a Lévy process with Lévy exponent $\kappa(\theta) = \log \mathbb{E}e^{\theta X(1)}$ and define $\overline{X}(t) = \sup\{X(s)\colon s \le t\}$, $\underline{X}(t) = \inf\{X(s)\colon s \le t\}$. Also let e_η be an independent exponentially distributed time, written as $e_\eta \sim \mathrm{Exp}(\eta)$. The standard form of the Wiener–Hopf (WH) factorisation of X then states that $\overline{X}(e_\eta)$ and the displacement $D(e_\eta) = \overline{X}(e_\eta) - X(e_\eta)$ of $X(e_\eta)$ from the maximum are independent, and that $D(e_\eta)$ has the same distribution as $-\underline{X}(e_\eta)$. In transform formulation, this gives the following factorisation of the moment generating function (m.g.f.) of $X(e_\eta) = \overline{X}(e_\eta) - (\overline{X}(e_\eta) - X(e_\eta))$:

$$\mathbb{E}e^{\theta X(e_\eta)} = \int_0^\infty e^{t\kappa(\theta)} \eta e^{-\eta t}\, dt = \frac{\eta}{\eta - \kappa(\theta)} = \mathbb{E}e^{\theta \overline{X}(e_\eta)} \cdot \mathbb{E}e^{\theta \underline{X}(e_\eta)}, \tag{1.1}$$

which is valid at least for θ purely imaginary. In order for these results to be non-trivial, assume in the following that X is not a subordinator, since in that case $\overline{X}(e_\eta) = X(e_\eta)$ and $\underline{X}(e_\eta) = 0$.

For the simplest case, where X is a Brownian motion $\mathrm{BM}(\mu, \sigma^2)$, $\overline{X}(e_\eta)$ and $-\underline{X}(e_\eta)$ are exponentially distributed with respective rates ρ_+ and ρ_-, where

$$\rho_\pm = \mp\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2\eta}{\sigma^2}}.$$

Received 3 November 2023; accepted 22 November 2024.
* Postal address: Department of Mathematics, Ny Munkegade, 8000 Aarhus C, Denmark. Email: asmus@math.au.dk
** Postal address: Department of Mathematical Sciences, Universitetsparken 5, 2100 Copenhagen Ø, Denmark. Email: bladt@math.ku.dk

© The Author(s), 2025. Published by Cambridge University Press on behalf of Applied Probability Trust.

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

The main case beyond BM that has been considered in the literature is when the jumps of X in one or both directions are phase-type (PH; see below) or, more generally, are matrix-exponential (ME), which is equivalent to the Laplace transform being rational (the ratio of polynomials). An often-cited paper in that vein is [25], but there are earlier references [7, 9, 26]. Among the many papers on WH factorisation in this setting, and covering related problems and extensions, we mention in particular [3, 8, 10, 17–19, 22, 23, 27, 28]. The solutions in the majority of these references are expressed in terms of the roots of the equation $\kappa^*(\theta) = \eta$, where κ^* is an approriate analytic continuation of the Lévy exponent. This equation involves polynomials and the location of the roots by means of Rouché's theorem. Beyond the BM or PH/ME setting, there are some specialised WH examples in [15, 20, 21, 23, 24].

The main result of the present paper is a different representation of the distribution of $\overline{X}(e_\eta)$. More precisely, we assume throughout that the jumps of X in the positive direction follow a compound Poisson (CP) process $Y(t)$ with Poisson rate λ and jumps which are PH or, more generally, ME. We can then write

$$X(t) = Y(t) + Z(t) = \sum_{i=1}^{N(t)} V_i + Z(t), \tag{1.2}$$

where Z is spectrally negative, with Lévy exponent say κ_Z, N is a Poisson(λ) process, $V_1, V_2, \ldots$ the upward jumps, and where obvious independence assumptions apply. The result is that $\overline{X}(e_\eta)$ is again PH/ME with computable parameters. This presents a structural property of $\overline{X}(e_\eta)$ and leads to matrix expressions like $f_{\overline{X}(e_\eta)}(x) = \boldsymbol{\gamma} \mathrm{e}^{\boldsymbol{G}x}\boldsymbol{g}$, $\mathbb{P}(\overline{X}(e_\eta) \le x) = 1 - \boldsymbol{\gamma}\mathrm{e}^{\boldsymbol{G}x}\mathbf{1}$, and $\mathbb{E}\mathrm{e}^{\theta\overline{X}(e_\eta)} = \boldsymbol{\gamma}(-\theta\boldsymbol{I} - \boldsymbol{G})^{-1}\boldsymbol{g}$ for the probability density function (p.d.f.), cumulative distribution function (c.d.f.), and m.g.f. of $\overline{X}(e_\eta)$ which are much more compact than the root expansions. Here, f_Y denotes the density of a random variable (r.v.) Y and $\mathbf{1}$ is the column vector with all components equal to 1. Further, $\boldsymbol{\gamma}$ is a row vector, $\boldsymbol{g}$ a column vector, and $\boldsymbol{G}$ a square matrix, all of which are computable by algorithms to be presented. In addition, the iterative form of the algorithms has proved less sensitive to the curse of dimension than the root-finding ones, as remarked, for example, in [17]. Note also that root-finding in high dimensions is often formulated as an eigenvalue problem, solved by iteration.

An argument in favour of considering only the PH setting is that our proofs are probabilistic rather than analytic/algebraic as in the ME case and that methods for fitting PH distributions [11, Chapter 13] are far better developed than for the ME case. In view of this, we start by considering the PH case in Section 4. A numerical example is presented in Section 5. In addition to illustrating the general WH results, the example also shows how PH approximations may be performed. The ME case is treated in Section 6; the extension from PH to ME can be motivated by the fact that the dimensions of matrices, vectors, etc. can be substantially reduced in some cases and that there are examples of distributions that are ME but not PH (nevertheless, both classes are dense). In both cases, it turns out that we need to distinguish between two cases according to whether $-Z$ is a subordinator or not (this distinction also occurs in the classical algorithms, where the number of roots is different in the two cases). This determines whether Z can attain strictly positive values, both for some $t > 0$ on the whole line and in a neighbourhood of $t = 0$. The subordinator case comprises the important case where both upward and downward jumps are CP/ME and no drift or Brownian parts are present. First, some preliminaries are given in Section 2.

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

2. Preliminaries

We start by recapitulating a few basic facts about PH distributions. The underlying object is a terminating Markov process $\{J(t)\}_{t<\zeta}$ on the finite phase space $E=\{1,\dots,p\}$ (or any other set with p elements), having transition rates t_{ij} for $i\neq j$, killing rate of $t_i\geq 0$ in state i, with t_{ii} then being defined by $t_{ii}=-\sum_{j\neq i}t_{ij}-t_i$, and initial probabilities $\alpha_i=\mathbb{P}(J(0)=i)$. We write $\boldsymbol{T}$ for the matrix $(t_{ij})_{i,j\in E}$, $\boldsymbol{t}$ for the column vector $(t_1\ \cdots\ t_p)^\top$, and $\boldsymbol{\alpha}$ for the row vector $(\alpha_1\ \cdots\ \alpha_p)$. Then $\boldsymbol{t}=-\boldsymbol{T}\boldsymbol{1}$, where $\boldsymbol{1}=(1\ \cdots\ 1)^\top$. The PH $(\boldsymbol{\alpha},\boldsymbol{T})$ distribution is then the distribution of the lifetime $\zeta=\inf\{t\geq 0\colon J(t)\notin E\}$. This allows the distribution to have an atom of size $1-\boldsymbol{\alpha}\boldsymbol{1}$ at $x=0$. When it is important to stress the dimension, we write $\mathrm{PH}_p\,(\boldsymbol{\alpha},\boldsymbol{T})$.

The c.d.f. is $\mathbb{P}(\zeta\leq x)=1-\boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{T}x}\boldsymbol{1}$, the p.d.f. is $\boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{T}x}\boldsymbol{t}$, and the m.g.f. is $\mathbb{E}\mathrm{e}^{\theta\zeta}=\boldsymbol{\alpha}(-\theta I-\boldsymbol{T})^{-1}\boldsymbol{t}$. PH distributions form a dense set in the set of all distributions on $[0,\infty)$, which, together with their underlying probabilistic structure and tractability, provides a strong motivation for their use. See further [11, Chapter 3].

We will need a bit of functional calculus for matrices (see [11, Section 3.4), for example a matrix version of a cumulant function. In examples with exponential moments in at least one direction (as is sufficient for our purposes), we have $\kappa(\theta)=\sum_1^\infty\kappa_n\theta^n/n!$, where $\kappa_1,\kappa_2,\dots$ are the cumulants, and may then define $\kappa(\boldsymbol{A})$ as $\sum_1^\infty\kappa_n\boldsymbol{A}^n/n!$. Computationally, and for other purposes, it may sometimes be more convenient to represent $\kappa(\boldsymbol{A})$, and other analytic functions on the form $\varphi(\boldsymbol{A})$ (e.g. in Theorem 4.1), as the Cauchy integral

$$\varphi(\boldsymbol{A})=\frac{1}{2\pi\mathrm{i}}\oint_C\varphi(z)(z\boldsymbol{I}-\boldsymbol{A})^{-1}\,\mathrm{d}z, \tag{2.1}$$

where all eigenvalues of $\boldsymbol{A}$ are inside the simple closed curve C.

Lemma 2.1. *For an r.v. V with cumulant function $\ell(\theta)=\log\mathbb{E}\mathrm{e}^{\theta V}$, $\mathbb{E}\mathrm{e}^{\boldsymbol{A}V}=\mathrm{e}^{\ell(\boldsymbol{A})}$ provided any eigenvalue ρ of $\boldsymbol{A}$ is in the interior of the domain of ℓ. In particular, for a Lévy process X with Lévy exponent κ, $\mathbb{E}\mathrm{e}^{\boldsymbol{A}X(t)}=\mathrm{e}^{t\kappa(\boldsymbol{A})}$.*

Proof. Since a Laplace transform is analytic wherever it is defined, (2.1) gives

$$\mathbb{E}\mathrm{e}^{\boldsymbol{A}V}=\frac{1}{2\pi\mathrm{i}}\oint_C\mathbb{E}\mathrm{e}^{zV}(z\boldsymbol{I}-\boldsymbol{A})^{-1}\,\mathrm{d}z=\frac{1}{2\pi\mathrm{i}}\oint_C\mathrm{e}^{\ell(z)}(z\boldsymbol{I}-\boldsymbol{A})^{-1}\mathrm{d}z=\mathrm{e}^{\ell(\boldsymbol{A})}.$$ □

The following lemma will serve as a check of the accuracy of our numerical computations in Section 5.

Lemma 2.2. *Assume that $X\overset{\mathrm{d}}{=}-X$. Then $\mathbb{V}\mathrm{ar}\overline{X}(e_\eta)=\mathbb{V}\mathrm{ar}X(1)/2\eta$.*

Proof. Let $Y=X(e_\eta)$ and $Z=\overline{X}(e_\eta)$, with corresponding cumulant functions κ_Y, κ_Z. Taking logarithms in (1.1) and using the symmetry property of X then gives $\kappa_Y(\theta)=\kappa_Z(\theta)+\kappa_Z(-\theta)$. Differentiating twice and letting $\theta=0$ gives $\mathbb{V}\mathrm{ar}Y=\kappa_Y''(0)=2\kappa_Z''(0)=2\mathbb{V}\mathrm{ar}Z$, but

$$\begin{aligned}\mathbb{V}\mathrm{ar}Y&=\mathbb{E}[\mathbb{V}\mathrm{ar}(X(e_\eta)\mid e_\eta)]+\mathbb{V}\mathrm{ar}[\mathbb{E}(X(e_\eta)\mid e_\eta)]\\&=\mathbb{E}[e_\eta\cdot\mathbb{V}\mathrm{ar}X(1)]+\mathbb{V}\mathrm{ar}(e_\eta\cdot\mathbb{E}X(1))=\mathbb{V}\mathrm{ar}X(1)/\eta+0.\end{aligned}$$ □

In the following, ω_0 will denote the time of the first jump of the CP/PH part and $\omega=\omega_0\wedge e_\eta$.

3. The PH case with Z being a subordinator

Theorem 3.1. *Assume in 1.2 that the $V_i\sim\mathrm{PH}_p\,(\boldsymbol{\alpha},\boldsymbol{T})$, and that $-Z$ is a subordinator. Then the distribution of $\overline{X}(e_\eta)$ consists of an atom at 0 of size $1-\boldsymbol{\beta}\boldsymbol{1}$, and an absolute continuous*

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

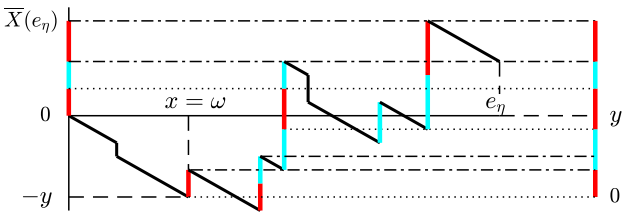


FIGURE 1. The case where $-Z$ is a subordinator.

part which is $\mathrm{PH}_p(\boldsymbol{\beta}, \boldsymbol{G}(\boldsymbol{\beta}))$ *distributed. Here,* $\boldsymbol{G}(\boldsymbol{\beta}) = \boldsymbol{T} + \boldsymbol{t}\boldsymbol{\beta}$*, where* $\boldsymbol{\beta}$ *solves* $\boldsymbol{\beta} = \psi(\boldsymbol{\beta})$ *with* $\psi(\boldsymbol{\beta}) = \lambda\boldsymbol{\alpha}((\lambda+\eta)\boldsymbol{I} - \kappa_Z(-\boldsymbol{G}(\boldsymbol{\beta})))^{-1}$*. The row vector* $\boldsymbol{\beta}$ *can be computed by iteration,* $\boldsymbol{\beta}^{(n)} \uparrow \boldsymbol{\beta}$*, where* $\boldsymbol{\beta}^{(n)} = \psi\left(\boldsymbol{\beta}^{(n-1)}\right)$ *and* $\boldsymbol{\beta}^{(0)} = \boldsymbol{0}$*.*

Note that the eigenvalue assumptions of Lemma 2.1 are fulfilled to make $\kappa_Z(-\boldsymbol{G}(\boldsymbol{\beta}))$ well-defined: $\boldsymbol{G}(\boldsymbol{\beta})$ will be seen to be the generator of a terminating Markov process, hence all eigenvalues have strictly negative real part, and $\kappa_Z(\theta)$ is well-defined whenever $\Re(\theta) \geq 0$.

Proof. The intuition underlying our analysis is illustrated in Figure 1. Here, Z is taken as a sum of a negative drift and CP negative jumps, and the phase space E is given by $E = \{\text{RED}, \text{CYAN}\}$. We then construct a terminating Markov process ξ on the vertical axis with the same state space. The process initiates at 0 when the first jump (if at all) upcrosses the horizontal axis, by projecting the underlying Markov process, which generates the jump onto the left vertical axis. Upon termination of the jump, the construction is repeated relative to the new running maximum: the next time a jump upcrosses the current maximum level, the Markov jump process underlying Markov process from the overshoot is projected onto the (left) vertical axis, and thereby concatenated with the part relating to the first overshoot. In the figure, this happens when the vertical Markov process from a jump crosses a dotted line in $(0, \infty)$.

The process ξ has two types of transition $i \to j$. The first ones correspond to jumps in the CP process occurring at rate t_{ij} and are marked with a dotted horizontal line. In Figure 1 there is just one of this type, the first from RED to CYAN. The second ones, marked with a dash-dotted horizontal line, occur at the top of a jump (rate t_i) and are reset according to a certain probability $\beta_j < 1$. In Figure 1, the first is the transition CYAN to RED, while there is no resetting on the top of the jump leading to $\overline{X}(e_\eta)$. Therefore, the total generator is $\boldsymbol{T} + \boldsymbol{t}\boldsymbol{\beta}$, where $\boldsymbol{\beta} = (\beta_1 \ \cdots \ \beta_p)$ is also the initial vector for ξ. The remaining problem is to compute $\boldsymbol{\beta}$.

To prove that $\boldsymbol{\beta} = \psi(\boldsymbol{\beta})$, we condition on the value $-y$ of Z just before the first upward jump of Y (in the RED phase in the figure), occurring before e_η at time $x = \omega$ at rate $\lambda \mathrm{e}^{-(\lambda+\eta)x}$; cf. Lemma 4.1(i). We now construct a second terminating Markov jump process, ξ', by shifting the origin (0,0) to $(x, -y)$, cf. the dashed horizontal line. Next, we project the underlying Markov processes from the jumps onto the vertical axis in exactly the same way as for ξ. This axis containing ξ' is shown on the right of the figure and has a zero marked at the original level $-y$, and we have used the same convention for dotted and dash-dotted lines as for ξ.

The right vertical process ξ' is then again Markov with generator $\boldsymbol{G}(\boldsymbol{\beta})$, though its initial distribution is $\boldsymbol{\alpha}$ rather than $\boldsymbol{\beta}$. Then $\boldsymbol{\beta}$, the initial distribution for ξ, is precisely its state vector $\boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{G}(\boldsymbol{\beta})y}$, the distribution of ξ' at time y. It follows from Lemma 2.1 that

$$\boldsymbol{\beta} = \boldsymbol{\alpha} \int_0^\infty \lambda \mathrm{e}^{-(\lambda+\eta)x} \mathbb{E}\mathrm{e}^{\boldsymbol{G}(\boldsymbol{\beta})(-Z(x))} \, \mathrm{d}x$$

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

$$= \boldsymbol{\alpha} \int_0^\infty \lambda \exp\{-(\lambda+\eta)\boldsymbol{I}x + \kappa_Z(-\boldsymbol{G}(\boldsymbol{\beta}))x\}\,\mathrm{d}x = \psi(\boldsymbol{\beta}). \tag{3.1}$$

The argument for $\boldsymbol{\beta}^{(n)} \uparrow \boldsymbol{\beta}$ is essentially the same as in [1], [2, pp 241–242], or [11, pp. 373–375]. We add here that the monotonicity of the iteration follows from (3.1) since $-Z(x) \geq 0$ and $\mathrm{e}^{\boldsymbol{G}(\boldsymbol{\beta})x}$ is non-decreasing in $\boldsymbol{\beta}$. This last fact is not obvious from the series representation of the matrix exponential function since $\boldsymbol{G}(\boldsymbol{\beta})$, and hence possibly the $\boldsymbol{G}(\boldsymbol{\beta})^n$, contains negative elements. However, it may be seen from the uniformity representation, cf. [11, pp. 51ff], since the matrix $\boldsymbol{I} + \boldsymbol{G}(\boldsymbol{\beta})/a$, and hence its powers, only contain non-negative elements for sufficiently large a. □

Example 3.1. If Z is CP with Poisson rate λ_- and generic jump $\zeta_- < 0$, then $\kappa_Z(\theta) = \lambda_-(\mathbb{E}\mathrm{e}^{\theta\zeta_-} - 1)$ and $\kappa_Z(-\boldsymbol{G}) = \lambda_-(\mathbb{E}\mathrm{e}^{-\boldsymbol{G}\zeta_-} - \boldsymbol{I})$. If $-\zeta_- \sim \mathrm{PH}(\boldsymbol{\alpha}_-, \boldsymbol{T}_-)$, this last expression may be evaluated via

$$\begin{aligned}
\mathbb{E}\mathrm{e}^{-\boldsymbol{G}\zeta_-} &= \int_0^\infty \boldsymbol{\alpha}_- \mathrm{e}^{\boldsymbol{T}_- x}\boldsymbol{t}_- \cdot \mathrm{e}^{\boldsymbol{G}x}\,\mathrm{d}x \\
&= \int_0^\infty \boldsymbol{\alpha}_- \mathrm{e}^{\boldsymbol{T}_- x}\boldsymbol{t}_- \cdot \boldsymbol{I}\mathrm{e}^{\boldsymbol{G}x}\boldsymbol{I}\,\mathrm{d}x \\
&= \int_0^\infty (\boldsymbol{\alpha}_- \otimes \boldsymbol{I})\mathrm{e}^{(\boldsymbol{T}_- \oplus \boldsymbol{G})x}(\boldsymbol{t}_- \otimes \boldsymbol{I})\,\mathrm{d}x = -(\boldsymbol{\alpha}_- \otimes \boldsymbol{I})(\boldsymbol{T}_- \oplus \boldsymbol{G})^{-1}(\boldsymbol{t}_- \otimes \boldsymbol{I}),
\end{aligned}$$

cf. [11, Section A.4] for the Kronecker calculus. If $-\zeta_-$ is exponential μ, we simply have $\mathbb{E}\mathrm{e}^{-\boldsymbol{G}\zeta_-} = (\boldsymbol{I} - \boldsymbol{G}/\mu)^{-1}$; this generalises straightforwardly to a mixture of exponentials or Erlangs.

4. The PH case with Z not a subordinator

We first remark that with positive values of Z occurring with positive probability, we are always in one of the situations where Z either (i) contains a Brownian component, and/or (ii) contains an infinite-variation jump component, or (iii) has the form $Z(t) = at + J(t)$ where $a > 0$ and J is a finite-variation jump process with the Lévy measure concentrated on $(-\infty, 0)$.

The situation is illustrated in Figure 2, where the blue segments of the path of X between upward PH jumps are governed by Z. Downward jumps of Z are shown in green.

Since the first upward jump occurs at some strictly positive time, the path of X starts by such a segment attaining positive values immediately. Thus the state space of ξ needs to be augmented with an extra state BLUE, denoted 0 in the following. The initial state of ξ is always BLUE and is possibly followed by a sequence of the states in E coming from the PH jumps. Upon exit at the top of a jump, ξ returns to the BLUE state.

Formally, we let $E_0 = \{0\} \cup E = \{0, 1, \ldots, p\}$. For a $1 \times (p+1)$ row vector $\boldsymbol{\beta} = (\beta_0\ \beta_1\ \cdots\ \beta_p)$, we write $\boldsymbol{\beta}_{1:p} = (\beta_1\ \cdots\ \beta_p)$. We also write $\mathbf{1}_0 = (1\,0\cdots 0)$ for the $(p+1)$-dimensional unit row vector.

The following lemma contains the key steps in our derivation of the distribution of $\overline{X}(e_\eta)$. In part (i), $\widetilde{X}_{0:T}(t) = X(t \wedge T)$ denotes the process X being absorbed at time T at its running value $X(T)$.

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press



FIGURE 2. The case of Z taking positive values.

Lemma 4.1.

(i) *Let* $e(\lambda_1) \sim \mathrm{Exp}\,(\lambda_1)$ *and* $e(\lambda_2) \sim \mathrm{Exp}\,(\lambda_2)$ *be mutually independent and independent of* X, *and write* $e(\lambda_1 + \lambda_2) = e(\lambda_1) \wedge e(\lambda_2)$. *Then*

$$\begin{aligned}\mathbb{P}\big(\widetilde{X}_{0:e(\lambda_1)} \in A,\ e(\lambda_1) < e(\lambda_2)\big) &= \mathbb{P}\big(\widetilde{X}_{0:e(\lambda_1+\lambda_2)} \in A\big) \cdot \mathbb{P}(e(\lambda_1) < e(\lambda_2)) \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \mathbb{P}\big(\widetilde{X}_{0:e(\lambda_1+\lambda_2)} \in A\big)\end{aligned}$$

for all measurable subsets A *of the Skorokhod space* $D[0, \infty)$.

(ii) $\kappa_Z(\theta) = \log \mathbb{E}\mathrm{e}^{\theta Z(1)}$ *is well-defined for all* θ *with real parts* $\Re(\theta) \in [0, \infty)$ *and increases strictly monotonically from* $\min_{\theta \geq 0} \kappa_Z(\theta) \leq 0$ *to* $+\infty$ *as* $\theta \uparrow \infty$.

(iii) *Conditionally on* $\omega < \mathrm{e}_\eta$, $\overline{X}(\omega-) = \overline{Z}(\omega) \sim \mathrm{Exp}\,(\delta)$, *where* δ *is the unique positive solution of* $\kappa_Z(\delta) = \lambda + \eta$.

(iv) *Define* $\varphi(\theta) = \mathbb{E}\mathrm{e}^{\theta D(\omega)}$ *as the m.g.f. of the displacement* $D(\omega) = \overline{X}(\omega) - X(\omega)$ *of* $X(\omega)$ *from* $\overline{X}(\omega)$ *given* $\omega < \mathrm{e}_\eta$. *Then* $\varphi(\theta)$ *is well-defined for all* θ *with* $\Re(\theta) \in (-\infty, 0]$ *and equals* $(1 + \theta/\delta)/(1 - \kappa_Z(-\theta)/(\lambda + \eta))$.

Proof. For (i), we have $\mathbb{P}\big(\widetilde{X}_{0:e(\lambda)} \in A\big) = \int_0^\infty \mathbb{P}\big(\widetilde{X}_{0:x} \in A\big)\lambda \mathrm{e}^{-\lambda x}\,\mathrm{d}x$. Similarly,

$$\begin{aligned}\mathbb{P}\big(\widetilde{X}_{0:e(\lambda_1)} \in A,\ e(\lambda_1) < e(\lambda_2)\big) &= \int_0^\infty \mathbb{P}\big(\widetilde{X}_{0:x} \in A,\ x < e(\lambda_2)\big)\lambda_1 \mathrm{e}^{-\lambda_1 x}\,\mathrm{d}x \\ &= \int_0^\infty \mathbb{P}\big(\widetilde{X}_{0:x} \in A\big)\mathrm{e}^{-\lambda_2 x}\lambda_1 \mathrm{e}^{-\lambda_1 x}\,\mathrm{d}x \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \int_0^\infty \mathbb{P}\big(\widetilde{X}_{0:x} \in A\big)(\lambda_1 + \lambda_2)\mathrm{e}^{-(\lambda_1+\lambda_2)x}\,\mathrm{d}x \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \mathbb{P}\big(\widetilde{X}_{0:e(\lambda_1+\lambda_2)} \in A\big).\end{aligned}$$

Part (ii) is just a well-known fact about spectrally negative Lévy processes, where $\psi(\theta) \to \infty$ as $\theta \uparrow \infty$ follows from the assumption of $-Z$ not being a subordinator so that $\mathbb{P}(Z(1) > 0) > 0$. Also, (iii) is well known, but the proof is short, so we include it for

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

completeness. First, it is probabilistically clear from the upward skip-free property of Z that the distribution of $\overline{X}(\omega)$ is memoryless, hence exponential for some δ. By (i), we treat ω as Exp $(\lambda+\eta)$-distributed, and (1.1) then gives

$$\frac{\lambda+\eta}{\lambda+\eta-\kappa_Z(\theta)} = \frac{\delta}{\delta-\theta}\mathbb{E}\mathrm{e}^{\theta\underline{Z}(\omega)}. \tag{4.1}$$

But $\mathbb{E}\mathrm{e}^{\theta\underline{Z}(\omega)}$ is finite and non-zero for all $\theta>0$, so we must have $\kappa_Z(\delta)=\lambda+\eta$.

Finally, in (iii), $D(\omega)>0$ implies $\varphi(\theta)$ is well-defined and analytic for all θ with $\Re(\theta)\in(-\infty,0]$. The expression for $\varphi(\theta)$ in the region $\Re(\theta)\in(-\delta,0]$ then follows from (1.1), (4.1), and $D(\omega)=-\underline{Z}(\omega)$. At $\theta=-\delta$, both the numerator and denominator are zero but have non-zero derivatives, and for $\theta<-\mu$, both are negative. The validity of the expression for $\varphi(\theta)$ for all θ with $\Re(\theta)\in(-\infty,0]$ thus follows by analytic continuation.

Theorem 4.1. *Assume that the V_i in (1.2) are* $\mathrm{PH}_p(\boldsymbol{\alpha},\boldsymbol{T})$ *and that $-Z$ is not a subordinator. Then* $\overline{X}(e_\eta)\sim\mathrm{PH}_{p+1}(\mathbf{1}_0,\boldsymbol{G}(\boldsymbol{\beta}))$ *on* $(0,\infty)$ *with phase space E_0, initial vector $\mathbf{1}_0$, and phase generator*

$$\boldsymbol{G}(\boldsymbol{\beta})=\begin{pmatrix}-\delta(1-\beta_0) & \delta\boldsymbol{\beta}_{1:p}\\ \boldsymbol{t} & \boldsymbol{T}\end{pmatrix}.$$

Here, δ is the positive solution of $\kappa_Z(\delta)=\lambda+\eta$, and $\boldsymbol{\beta}:1\times(p+1)$ solves $\boldsymbol{\beta}=\psi(\boldsymbol{\beta})$ where

$$\psi(\boldsymbol{\beta})=\frac{\lambda}{\lambda+\eta}(0\ \boldsymbol{\alpha})\varphi(\boldsymbol{G}(\boldsymbol{\beta})),$$

with φ as in Lemma 4.1(iv).

The row vector $\boldsymbol{\beta}$ can be computed by iteration, $\boldsymbol{\beta}^{(n)}\uparrow\boldsymbol{\beta}$ where $\boldsymbol{\beta}^{(n)}=\psi\big(\boldsymbol{\beta}^{(n-1)}\big)$ and $\boldsymbol{\beta}^{(0)}=\mathbf{0}$.

Proof. Let ξ' be the ξ-process on the right in Figure 2 and define $\boldsymbol{\beta}$ as the vector of state probabilities of ξ' at its upcrossing of $\overline{X}(\omega)$. We now use Lemma 4.1 with $e(\lambda_1)=\omega$, $e(\lambda_2)=e_\eta$. The event $\mathrm{e}_\eta<\omega$ gives no contribution to $\boldsymbol{\beta}$, so we have to treat ω as Exp$(\lambda+\eta)$. Thus by Lemma 4.1(iii), $\overline{X}(\omega)\sim\mathrm{Exp}\,(\delta)$, where δ is as stated. However, the ξ-process on the left may return immediately to the BLUE state 0, which occurs with probability β_0 (an example of such a backcoupling is in the last BLUE segment of ξ in Figure 2). Thus, the exit rate of ξ from state 0 is $\delta(1-\beta_0)$, which gives the upper left entry of the $\boldsymbol{G}(\boldsymbol{\beta})$ matrix. The forms of the remaining entries are evident.

The process ξ' has initial distribution $(0\quad\boldsymbol{\alpha})$ and has to traverse the distance $D=D(\omega)$ to upcross $\overline{X}(\omega)$, but only starts if $\omega<e_\eta$. Thus

$$\boldsymbol{\beta}=\mathbb{P}(\omega<e_\eta)(0\quad\boldsymbol{\alpha})\mathbb{E}\mathrm{e}^{\xi'(D(\omega))}=\frac{\lambda}{\lambda+\eta}(0\quad\boldsymbol{\alpha})\varphi(\boldsymbol{G}(\boldsymbol{\beta})).$$

That $\boldsymbol{\beta}^{(n)}\uparrow\boldsymbol{\beta}$ follows by similar reasoning to the proof of Theorem 3.1. □

5. Numerical example

The traditional way to approximate the positive jumps would be to truncate the density $n(x)$ of the Lévy measure ν to (ε,∞) for some small ε and use the CP model with $\lambda=\int_\varepsilon^\infty n(x)\,\mathrm{d}x$ and density $h(x)=n(x)/\lambda$, $x>\varepsilon$, of the jumps (possibly, this would be supplemented by some

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

approximation of the contribution from jumps in $(0, \varepsilon)$). In the PH setting, we would then use some PH fit of h. This has, however, the disadvantage that distributions with sharp jumps, such as that of $h(x)$ from $h(\varepsilon-) = 0$ to $h(\varepsilon) = n(\varepsilon)/\lambda > 0$, require a huge number of phases to give a reasonable fit.

Instead, we use the completely monotone structure present in most of the standard examples [16] where

$$n(x) = \int_0^\infty \mathrm{e}^{-tx} v(t)\,\mathrm{d}t, \quad x > 0, \tag{5.1}$$

for some function $v \geq 0$ on $(0, \infty)$ (and similarly for $n(x)$, $x < 0$). That is, $n(x)$ is a mixture of $\mathrm{Exp}\,(t)$ distributions, with respective weights $v(t)/t$. In [16], this structure is used to approximate the upward jumps via Padé approximations. This leads to a scheme with the special structure of the PH jumps being hyperexponential (mixtures of exponentials); this is, of course, natural in view of (5.1).

Specifically, we considered a symmetric ($G = M$) Carr–Geman–Madan–Yor (CGMY) (tempered stable) process with Lévy density $C\mathrm{e}^{-M|x|}/|x|^{Y+1}$, $-\infty < x < \infty$, [13]. Such a process can be written as $X = X^{(u)} - X^{(d)}$, where $X^{(u)}$ and $X^{(d)}$ are independent spectrally positive CGMY processes both having Lévy densities $C\mathrm{e}^{-Mx}/x^{Y+1}$, $x > 0$, and Lévy exponents

$$\kappa^{(\mathrm{u})}(\theta) = \kappa^{(\mathrm{d})}(\theta) = C\Gamma(-Y)((M-\theta)^Y - M^Y).$$

We took the variance of $X(1)$ to be 1 and the kurtosis to be $K = 4$. According to [4], such a process exists for any $Y \in (0, 2)$, and we took $Y = 3/2$ corresponding to the infinite-variation case. Then the formulas of [4] give

$$M = \frac{1}{K^{1/2}}\sqrt{(2-Y)(3-Y)} = 0.433, \qquad C = \frac{1}{2\Gamma(2-Y)}M^{2-Y} = 0.186.$$

It is an easy consequence of the integral representation of the gamma function that the Lévy density $C\mathrm{e}^{-Mx}/x^{Y+1}$ of the positive jumps is completely monotone corresponding to $v(t) = C(t-M)^Y/\,\Gamma(Y+1)$, $t > M$. The relevant Padé approximations from [16] are the $[n+1/1]$ and $[n+2/2]$ ones in Proposition 2 there. We use the $[n+1/1]$ one, where the approximation of $\kappa^{(\mathrm{u})}\theta)$ is given as

$$\widehat{\kappa}_n^{(\mathrm{u})}(\theta) = \frac{P(\theta)}{Q(\theta)} = a\theta + \sum_{i=1}^n \lambda_i\left(\frac{\mu_i}{\mu_i - \theta} - 1\right) \tag{5.2}$$

for certain polynomials P,Q of degree $n+1$ (respectively n), and certain a, $\lambda_1, \ldots, \lambda_n > 0$, $\mu_1, \ldots, \mu_n > 0$. The last expression is a slight rewriting of [16, (25)], but the key observation leading to the equivalence is that

$$\int_0^\infty (\mathrm{e}^{\theta x} - 1 - \theta x)\mu\mathrm{e}^{-\mu x}\,\mathrm{d}x = \frac{\mu}{\mu-\theta} - 1 - \frac{\theta}{\mu} = \frac{\theta^2/\mu}{\mu-\theta}.$$

The last expression in (5.2) is the Lévy exponent of a hyperexponential Lévy process with jumps arriving at rate $\lambda = \sum_1^n \lambda_i$ and being distributed as the mixture of n exponential distributions, with rate μ_i and weight λ_i/λ for the ith. To this is added a linear drift at rate a. Computationally, we used the linear equations for the coefficients of Q (e.g. the Appendix in [16] or [29]) to get $\mu_1, \ldots, \mu_n$ as the roots of Q. Since the $[n+1/n]$ Padé approximation preserves the first $2n+2$ cumulants, we could then first get $\lambda_1, \ldots, \lambda_n$ as solutions of the linear equations

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

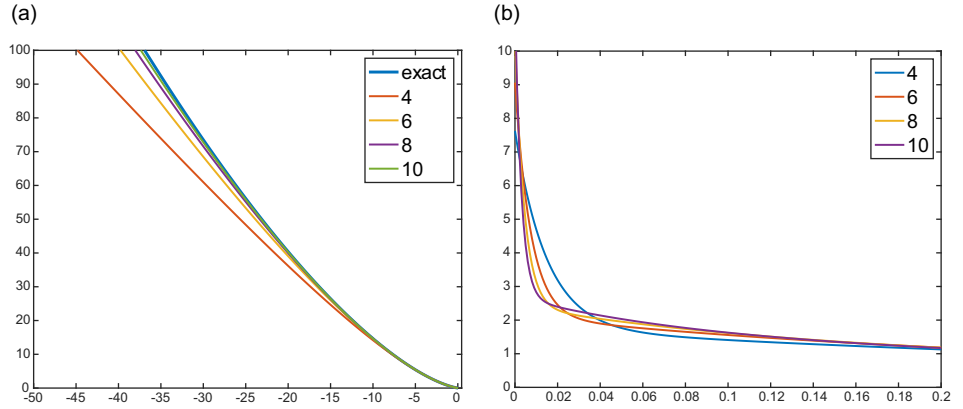


FIGURE 3. (a) Exact Lévy exponent $\kappa^{(\mathrm{u})}(\theta)$ and Padé approximations $\widehat{\kappa}_n^{(\mathrm{u})}(\theta)$. (b) Approximated density of $\overline{X}(e_\eta)$ computed via the hyperexponential Padé scheme and Theorem 4.1.

$$\int_0^\infty x^k n(x)\,\mathrm{d}x = C\Gamma(-Y)Y\cdots(Y-k+1)M^{Y-k} = \sum_{i=1}^{n}\frac{k!}{\mu_i^k}\lambda_i, \quad k=2,\ldots,n+1,$$

and next compute a as $C\Gamma(-Y)YM^{Y-1} - \sum_1^n \lambda_i/\mu_i$. The Z process becomes $Z(t) = at - X^{(d)}(t)$.

We took $n = 4$, 6, 8, 10 and computed $\varphi(\boldsymbol{G})$ via diagonalisation. The iteration was stopped at $\overline{m} = \inf\{m\colon \|\boldsymbol{\beta}^{(m)} - \boldsymbol{\beta}^{(m-1)}\| < 10^{-8}\}$, which required $\overline{m} = 43$, 69, 97, and 127 in the four cases. The program ran in just a few seconds on a 2022 MacBook Air. Graphical illustrations are given in Figure 3.

As a check, we used Lemma 2.2 and the computed variances of the approximated PH distribution of $\overline{X}(e_\eta)$, which were 0.500 34, 0.500 07, 0.500 02, and 0.500 01, to be compared with the exact value of 0.500 00. All of the above confirms the expected increase in accuracy as n increases. As an example of the numerical values of the parameters, the computations for $n = 8$ gave

$$\delta = 207.2, \qquad a = -5.18,$$

$$\lambda_1,\ldots,\lambda_8 = 234.9,\ 26.68,\ 0.290,\ 0.057,\ 0.136,\ 0.0032,\ 6.2\times10^{-4},\ 6.2\times10^{-5},$$

$$\boldsymbol{\beta} = \begin{pmatrix}0.165 & 0.740 & 0.036 & 0.0067 & 0.0017 & 4.7\times10^{-4} & 1.2\times10^{-4} & 2.4\times10^{-5} & 2.5\times10^{-6}\end{pmatrix},$$

$$\boldsymbol{G} = \begin{pmatrix}
-172.9 & 153.2 & 7.52 & 3.96 & 3.53 & 0.096 & 0.025 & 0.0050 & 0.0005\\
56.65 & -56.65 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
6.42 & 0 & -6.42 & 0 & 0 & 0 & 0 & 0 & 0\\
2.41 & 0 & 0 & -2.41 & 0 & 0 & 0 & 0 & 0\\
1.31 & 0 & 0 & 0 & -1.31 & 0 & 0 & 0 & 0\\
0.861 & 0 & 0 & 0 & 0 & -0.861 & 0 & 0 & 0\\
0.641 & 0 & 0 & 0 & 0 & 0 & -0.641 & 0 & 0\\
0.524 & 0 & 0 & 0 & 0 & 0 & 0 & -0.524 & 0\\
0.461 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.461
\end{pmatrix}$$

(where, obviously, $\mu_1,\ldots,\mu_8$ are the last eight elements of the first column of $\boldsymbol{G}$).

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

Altogether, we take the results of this example as strong support for the usefulness of both our WH scheme and the Padé approximation.

Remark 5.1. Note that our implementation of the Padé approximation is fairly primitive, and more sophisticated methods involving double or multiple precision would be required to get much beyond $n = 10$.

An alternative allowing much higher n is a truncation scheme where the integral in (5.1) is cut off at some $A < \infty$ and the reduced part $\int_0^A \mathrm{e}^{-tx} \nu(t)\,\mathrm{d}t$ of $n(x)$ coming from this is scaled to be a distribution. This is then fitted to a general PH distribution using maximum likelihood (which is typically implemented via the EM algorithm, [11, Chapter 13]). The idea behind this is somewhat similar to that in the ε scheme above, in the sense of disregarding 'small' jumps in the setting of (5.1), meaning those coming from a large t. We then approximate $X^{(u)}$ by the contribution $X^{(u)}_{0:A}$ from jumps coming from some $t < A$ in (5.1), to which we add some approximation $\widehat{X}^{(u)}_{A:\infty}$ to the contribution $X^{(u)}_{A:\infty}$ from jumps coming from some $t \geq A$. This scheme has been considered for simulation purposes in [5]. The simulation aspect is not relevant here, but we could use the observation from [5] that taking $\widehat{X}^{(u)}_{A:\infty}$ as BM can be justified by a limit theorem (actually, taking $\widehat{X}^{(u)}_{A:\infty}$ as a gamma process was observed to be somewhat more precise but is not relevant here since it would add some non-PH upward jumps). In view of the success of the Padé approximation in our example, we have not implemented this scheme.

6. Extension to matrix-exponentially distributed jumps

We now consider the same model as in Theorem 4.1, but we replace the distributional assumption of the upward jumps being of PH type by assuming that they follow the more general matrix-exponential (ME) distribution, written as $\mathrm{ME}(\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{t})$ (if we need to stress the dimension of the matrices and vectors, we write $\mathrm{ME}_p(\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{t})$). This means that the upward jumps have a density of the form $f(x) = \boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{T}x}\boldsymbol{t}$, whereas for the PH type case we may take $\boldsymbol{t} = -\boldsymbol{T}\mathbf{1}$. Here, the matrix $\boldsymbol{T}$ does not need to have the form of a sub-intensity matrix, and $\boldsymbol{\alpha}$ is not a (sub-)probability vector (e.g. it may contain negative values). In the ME terminology, $\boldsymbol{\alpha}$ is referred to as the starting vector, $\boldsymbol{T}$ as the generator, and $\boldsymbol{t}$ as the closing vector. However, we can always find a representation for which $\boldsymbol{t} = -\boldsymbol{T}\mathbf{1}$, in which case the mass of the distribution is given by $\boldsymbol{\alpha}\mathbf{1}$, whereas before $\mathbf{1}$ is the vector of ones.

In the PH case, $\boldsymbol{\alpha}$ is always chosen as the vector of initial probabilities for the underlying Markov process and $\boldsymbol{T}$ as the rate matrix. In the ME case, there is no such obvious choice, and many representations $(\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{t})$ of the same distribution are possible, sometimes creating a need for regularity assumptions. In particular, dimensionality becomes an issue, and the assumption we shall need is condition (R) from [11, pp. 215ff]. For a representation $\mathrm{ME}_q(\boldsymbol{\gamma}, \boldsymbol{G}, \boldsymbol{g})$, it states that $\boldsymbol{g}, \boldsymbol{G}\boldsymbol{g}, \ldots, \boldsymbol{G}^{q-1}\boldsymbol{g}$ are linearly independent. It implies that if $\mathrm{ME}_q(\boldsymbol{\gamma}^*, \boldsymbol{G}, \boldsymbol{g})$ is also a representation, then $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}$. Any ME has some representation satisfying (R). For these and other facts on ME distributions used without detailed reference in the following, see [11, Chapter 4].

We shall repeatedly use the representation

$$(\boldsymbol{\alpha}_1 \ \mathbf{0}), \qquad \begin{pmatrix} \boldsymbol{S}_1 & \boldsymbol{s}_1\boldsymbol{\alpha}_2 \\ \mathbf{0} & \boldsymbol{S}_2 \end{pmatrix}, \qquad \begin{pmatrix} (1 - \boldsymbol{\alpha}_2\mathbf{1})\boldsymbol{s}_1 \\ \boldsymbol{s}_2 \end{pmatrix}$$

for the sum $U_1 + U_2$ of two independent (possibly defective) r.v.s $U_1 \sim \mathrm{ME}_{p_1}(\boldsymbol{\alpha}_1, \boldsymbol{S}_1, \boldsymbol{s}_1)$, $U_2 \sim \mathrm{ME}_{p_2}(\boldsymbol{\alpha}_2, \boldsymbol{S}_2, \boldsymbol{s}_2)$ (slighly adapted from [11, p. 228]).

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

We shall give a detailed derivation of the following analogue of Theorem 4.1 and only at the end state the simpler parallel of Theorem 3.1.

Theorem 6.1. *Assume that $X(t) = Y(t) + Z(t)$, where $-Z$ is not a subordinator and Y is* $\mathrm{CP}(\lambda, H)$ *with H being* $\mathrm{ME}_p(\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{t})$ *with $\boldsymbol{t} = -\boldsymbol{T}\mathbf{1}$ and satisfying (R). Then $\overline{X}(e_\eta)$ is* $\mathrm{ME}_{p+1}(\mathbf{1}_0, \boldsymbol{G}(\boldsymbol{\beta}), \boldsymbol{g}(\boldsymbol{\beta}))$ *on $(0, \infty)$ with starting vector $\mathbf{1}_0 = (1 \quad \mathbf{0}_{1:p})$, and generator $\boldsymbol{G}(\boldsymbol{\beta})$ and closing vector $\boldsymbol{g}(\boldsymbol{\beta})$ given by*

$$\boldsymbol{G}(\boldsymbol{\beta}) = \begin{pmatrix} -\delta(1-\beta_0) & \delta\boldsymbol{\beta}_{1:p} \\ \boldsymbol{t} & \boldsymbol{T} \end{pmatrix}, \qquad \boldsymbol{g}(\boldsymbol{\beta}) = -\boldsymbol{G}(\boldsymbol{\beta})\mathbf{1} = \begin{pmatrix} \delta(1 - \boldsymbol{\beta}\mathbf{1}) \\ \mathbf{0} \end{pmatrix}.$$

Here, δ is the positive solution of $\kappa_Z(\delta) = \lambda + \eta$, and $\boldsymbol{\beta} : 1 \times (p+1)$ solves $\boldsymbol{\beta} = \psi(\boldsymbol{\beta})$ where

$$\psi(\boldsymbol{\beta}) = \frac{\lambda}{\lambda + \eta}(0 \quad \boldsymbol{\alpha})\varphi(\boldsymbol{G}(\boldsymbol{\beta})),$$

with φ as in Lemma 4.1(iv).

The row vector $\boldsymbol{\beta}$ can be computed by iteration, $\boldsymbol{\beta}^{(n)} \to \boldsymbol{\beta}$ as $n \to \infty$, where $\boldsymbol{\beta}^{(n)} = \psi\left(\boldsymbol{\beta}^{(n-1)}\right)$ with $\boldsymbol{\beta}^{(0)} = \mathbf{0}$.

Note that in contrast to the PH case, the convergence $\boldsymbol{\beta}^{(n)} \to \boldsymbol{\beta}$ need not be monotone, which is one of the reasons that the proof is substantially more involved in the ME case.

The proof is based on random walk theory in the presence of ME components. We start with a number of auxiliary results.

Lemma 6.1. *If, for all $x \geq 0$, $f_\beta(x) = \boldsymbol{\beta}\mathrm{e}^{Sx}\boldsymbol{s} \leq \boldsymbol{\gamma}\mathrm{e}^{Sx}\boldsymbol{s} = f_\gamma(x)$ are defective ME densities, then also $\boldsymbol{\alpha}\mathrm{e}^{(S+\boldsymbol{s}\boldsymbol{\beta})t}\mathrm{e}^{Sx}\boldsymbol{s} \leq \boldsymbol{\alpha}\mathrm{e}^{(S+\boldsymbol{s}\boldsymbol{\gamma})t}\mathrm{e}^{Sx}\boldsymbol{s}$ for all $x \geq 0$.*

Proof. An induction argument immediately implies that $f_\beta^{*n}(x) \leq f_\gamma^{*n}(x)$ for all $x \geq 0$ and all n. Hence, the corresponding renewal densities also satisfy

$$u_\beta(x) = \left(f_0 * \sum_{n=1}^{\infty} f_\beta^{*n}\right)(x) \leq \left(f_0 * \sum_{n=1}^{\infty} f_\gamma^{*n}(x)\right) = u_\gamma(x),$$

where $f_0(x) = \boldsymbol{\alpha}\mathrm{e}^{Sx}\boldsymbol{s}$ denotes the delay density. Given a time t, denote the residual lifetimes $R_t^\beta = S^\beta_{N^\beta(t)} - t$ and $R_t^\gamma = S^\gamma_{N^\gamma(t)} - t$, where N^β and N^γ denote the corresponding delayed and terminating (defective) renewal processes with respective interarrival densities f_β and f_γ and delay density f_0. Then

$$\begin{aligned} f_{R_t^\beta}(x) &= \boldsymbol{\alpha}\mathrm{e}^{S(t+x)}\boldsymbol{s} + \int_0^t f_\beta(x+t-z)u_\beta(z)\,\mathrm{d}z \\ &\leq \boldsymbol{\alpha}\mathrm{e}^{S(t+x)}\boldsymbol{s} + \int_0^t f_\gamma(x+t-z)u_\gamma(z)\,\mathrm{d}z = f_{R_t^\gamma}(x) \end{aligned}$$

for all $x \geq 0$. But then, by [11, Theorem 5.7.5], $\boldsymbol{\alpha}\mathrm{e}^{(S+\boldsymbol{s}\boldsymbol{\beta})t}\mathrm{e}^{Sx}\boldsymbol{s} \leq \boldsymbol{\alpha}\mathrm{e}^{(S+\boldsymbol{s}\boldsymbol{\gamma})t}\mathrm{e}^{Sx}\boldsymbol{s}$. □

Proposition 6.1. *Consider a stopped random walk $\{S_n\}_{n=0,1,\ldots,N-1} = \{U_1 - T_1 + \cdots + U_n - T_n\}_{n=0,1,\ldots,N-1}$, where all the r.v.s are independent, the $T_i \geq 0$ have distribution $A(\mathrm{d}t)$, $U_i \sim \mathrm{ME}(\boldsymbol{\gamma}, S, \boldsymbol{s})$ satisfying condition* (R)*, and N is* Geometric (π) *with $\mathbb{P}(N \geq n) = \pi^{n-1}$, $n = 1, 2, \ldots$ Define $\tau_+ = \inf\{n = 0, \ldots, N-1 : S_n > 0\}$. Then the overshoot*

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

S_{τ_+} over 0 is $\mathrm{ME}(\boldsymbol{\beta}, S, s)$*, where* $\boldsymbol{\beta}$ *solves the fixed point problem* $\boldsymbol{\beta} = \upsilon(\boldsymbol{\beta})$ *with* $\upsilon(\boldsymbol{\beta}) = \pi\boldsymbol{\beta}\int_0^\infty \mathrm{e}^{(S+s\boldsymbol{\beta})t}A(\mathrm{d}t)$*. The row vector* $\boldsymbol{\beta}$ *can be computed by iteration,* $\boldsymbol{\beta}^{(n)} \to \boldsymbol{\beta}$ *where* $\boldsymbol{\beta}^{(n)} = \upsilon\big(\boldsymbol{\beta}^{(n-1)}\big)$ *and* $\boldsymbol{\beta}^{(0)} = \mathbf{0}$*.*

Proof. The argument is essentially the same as the proof of [11, Theorem 6.2.8, p. 373]. The extra factor π in $\upsilon(\boldsymbol{\beta})$ is the probability that there is a first step in the random walk. The fact that $S_{\tau_+} \sim \mathrm{ME}(\boldsymbol{\beta}, \boldsymbol{T}, \boldsymbol{t})$ for some $\boldsymbol{\beta}$, the derivation of $\boldsymbol{\beta} = \upsilon(\boldsymbol{\beta})$, and the proof of convergence of the iteration is then very similar.

However, the proof in [11, pp. 374–375] requires a small correction in the paragraph starting 'Let S_i^β and S_i^γ ...': it is shown there that for two terminating renewal processes with interarrival densities $\boldsymbol{\alpha}\mathrm{e}^{Sx}s \le \boldsymbol{\beta}\mathrm{e}^{Sx}s$, the overshoot distributions are stochastically ordered. This needs to be changed to the conclusion and proof of Lemma 6.1.

Lemma 6.2. *Consider the convolution of the* $\mathrm{Exp}(\delta)$ *distribution and the* $\mathrm{ME}_p(\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{t})$ *distribution in Theorem 6.1 in the representation*

$$(0\ \boldsymbol{\alpha}), \qquad \boldsymbol{S}_0 = \begin{pmatrix} -\delta & 0 \\ \boldsymbol{t} & \boldsymbol{T} \end{pmatrix}, \qquad \boldsymbol{s}_0 = -\boldsymbol{S}_0\mathbf{1} = \begin{pmatrix} \delta \\ \mathbf{0} \end{pmatrix}. \tag{6.1}$$

Then $(\boldsymbol{S}_0, \boldsymbol{s}_0)$ *has property* (R).

Proof. It is easy to see that

$$\boldsymbol{S}_0^k\boldsymbol{s}_0 = \begin{pmatrix} (-1)^k\delta^{k+1} \\ \delta\boldsymbol{T}^{k-1}\boldsymbol{t} - \delta^2\boldsymbol{T}^{k-2}\boldsymbol{t} + \cdots + (-1)^{k-1}\delta^k\boldsymbol{t} \end{pmatrix},$$

so that the determinant $\Delta = |\boldsymbol{s}_0\boldsymbol{S}_0\boldsymbol{s}_0 \cdots \boldsymbol{S}_0^p\boldsymbol{s}_0|$ equals

$$\begin{vmatrix} \delta & -\delta^2 & \delta^3 & \cdots & (-1)^{p-1}\delta^p \\ 0 & \delta\boldsymbol{t} & \delta\boldsymbol{T}\boldsymbol{t} - \delta^2\boldsymbol{t} & \cdots & \delta\boldsymbol{T}^{p-1}\boldsymbol{t} - \delta^2\boldsymbol{T}^{p-2}\boldsymbol{t} + \cdots + (-1)^{p-1}\delta^p\boldsymbol{t} \end{vmatrix}.$$

Multiplying column i by δ and adding to column $i+1$ then gives

$$\begin{aligned} \boldsymbol{\Delta} &= \delta^{-p}\begin{vmatrix} \delta & 0 & 0 & \cdots & 0 \\ 0 & \delta\boldsymbol{t} & \delta\boldsymbol{T}\boldsymbol{t} & \cdots & \delta\boldsymbol{T}^{p-1}\boldsymbol{t} \end{vmatrix} \\ &= \delta^{-p}\delta|\delta\boldsymbol{t}, \delta\boldsymbol{T}\boldsymbol{t}, \ldots, \delta\boldsymbol{T}^{p-1}\boldsymbol{t}| = \delta|\boldsymbol{t}, \boldsymbol{T}\boldsymbol{t}, \ldots, \boldsymbol{T}^{p-1}\boldsymbol{t}| \neq 0 \end{aligned}$$

by property (R) of $(\boldsymbol{T}, \boldsymbol{t})$. Hence, $(\boldsymbol{S}_0, \boldsymbol{s}_0)$ has property (R) as well.

Corollary 6.1. *In the setting of Theorem 6.1, define* $\sigma = \max\{t\colon X(t) = \overline{X}(\omega-),\ \tau = \inf\{t > \sigma\colon X(t) > \overline{X}(\omega-)\}$*. Then the overshoot* $X(\tau) - \overline{X}(\omega-)$ *is* $\mathrm{ME}_{p+1}(\boldsymbol{\beta}, \boldsymbol{S}_0, \boldsymbol{s}_0)$*, where* $\boldsymbol{\beta}$ *is as in Proposition 6.1, with A equal to the distribution of the displacement D of* $X(\omega)$ *from* $\overline{X}(\omega-)$ *and* U_i *distributed as the independent sum of an* $\mathrm{Exp}(\delta)$ *r.v. and an* $\mathrm{ME}_p(\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{t})$ *one, where we take (6.1) as the representation. Further,* $\boldsymbol{\beta}$ *satisfies the same fixed-point problem with* υ *replaced by* ψ *and the same iteration converges.*

Proof. The argument is illustrated schematically in Figure 4. It uses the alternative dashed coordinate system where the dotted red vertical lines divide the process into segments starting with a downward move with distribution sized as the displacement $D(\omega)$ of Z from $\overline{X}(\omega)$ and is followed by an independent upward move with the stated distribution of U_i. The overshoot over

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

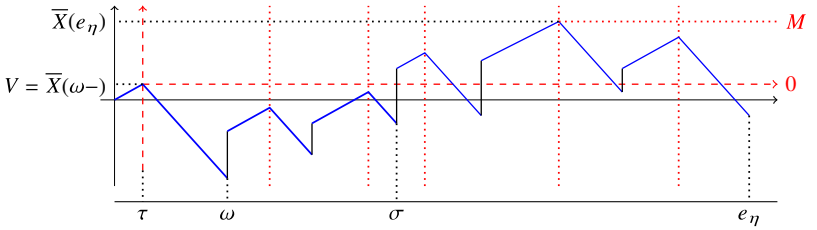


FIGURE 4. The random walk connection.

$\overline{X}(\omega)$ is simply the overshoot over 0 of the RW with increments distributed as the increments over a section. Since U_i has property (R) by Lemma 6.2, the overshoot can be calculated according to Proposition 6.1, which also immediately gives $\upsilon = \psi$ and the convergence. □

Lemma 6.3 *The sets* $(\boldsymbol{\alpha}_i, \boldsymbol{S}_i, \boldsymbol{s}_i)$, $i = 1, 2, 3$, *given by*

$$\boldsymbol{\alpha}_1 = (1\ 0\ \mathbf{0}_{1:p}\ 0), \quad \boldsymbol{S}_1 = \begin{pmatrix} -\delta & \delta\boldsymbol{\beta} & \delta\boldsymbol{\beta}_{1:p} \\ 0 & \delta(1-\beta_0) & \delta\boldsymbol{\beta}_{1:p} \\ \mathbf{0}_{1:p}^\top & \boldsymbol{t} & \boldsymbol{T} \end{pmatrix}, \quad \boldsymbol{s}_1 = \begin{pmatrix} \delta(1-\boldsymbol{\beta}\mathbf{1}) \\ \delta(1-\boldsymbol{\beta}\mathbf{1}) \\ \mathbf{0} \end{pmatrix},$$

$$\boldsymbol{\alpha}_2 = (0\ 1\ \mathbf{0}_{1:p}\ \ 0), \quad \boldsymbol{S}_2 = \begin{pmatrix} 0 & 0 & \mathbf{0}_{1:p} \\ 0 & -\delta(1-\beta_0) & \delta\boldsymbol{\beta}_{1:p} \\ \mathbf{0}_{1:p}^\top & \boldsymbol{t} & \boldsymbol{T} \end{pmatrix}, \quad \boldsymbol{s}_2 = \begin{pmatrix} 0 \\ \delta(1-\boldsymbol{\beta}\mathbf{1}) \\ \mathbf{0} \end{pmatrix},$$

$$\boldsymbol{\alpha}_3 = (1\ \mathbf{0}_{1:p}\ 0), \quad \boldsymbol{S}_3 = \begin{pmatrix} -\delta(1-\beta_0) & \delta\boldsymbol{\beta}_{1:p} \\ \boldsymbol{t} & \boldsymbol{T} \end{pmatrix}, \quad \boldsymbol{s}_3 = \begin{pmatrix} \delta(1-\boldsymbol{\beta}\mathbf{1}) \\ \mathbf{0} \end{pmatrix}$$

all represent the same ME distribution.

Proof. Consider the movement of a particle with marks a,b,c,d, where marks a and b indicate an Exp(δ) part of the movement, mark c an ME($\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{t}$) one, and d that the movement has stopped. The $i = 1$ representation then models the total length L travelled if the particle starts with mark a and after the Exp(δ) time takes mark b with probability β_0, mark d with probability $1 - \beta_0$, and otherwise mark c. The behaviour with mark b is exactly the same, and after a mark c time, the particle takes mark b. But this does not differ from the movement starting with mark b, which is the one modeled by the $i = 2$ representation. Since the evolvement of marks does not influence L, representations 1 and 2 therefore give the same distribution of L. That representation 3 also does follows from

$$\mathrm{e}^{\boldsymbol{S}_2 x} = \begin{pmatrix} 1 & \mathbf{0}_{1:p} \\ \mathbf{0}_{1:p}^\top & \mathrm{e}^{\boldsymbol{S}_3 x} \end{pmatrix}.$$

Alternatively, we could prove equality of the Laplace transforms $\boldsymbol{\alpha}_i(\theta\boldsymbol{I} - \boldsymbol{S}_i)^{-1}\boldsymbol{s}_i$ by using the formula in [11, p. 711] for the inverse of a block partitioned matrix. □

Proof of Theorem 6.1. We use that $\overline{X}(e_\eta) = V + M$, where M is the maximum of the random walk in Corollary 6.1 and its proof, cf. Figure 4, and an independent Exp(δ) r.v. The ladder height distribution of this random walk was computed in Corollary 6.1, and we can then appeal to [11, Theorem 6.2.6, p. 372] to conclude that $M \sim \mathrm{ME}_{p+1}(\boldsymbol{\beta}, \boldsymbol{S}_0 + \boldsymbol{s}_0\boldsymbol{\beta}, (1 - \boldsymbol{\beta}\mathbf{1})\boldsymbol{s}_0)$ with $\boldsymbol{S}_0$ and $\boldsymbol{s}_0$ as in (6.1). This gives the $i = 1$ representation of the distribution of $\overline{X}(e_\eta) = V + M$ in

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

Lemma 6.3. Now just notice that the $i = 3$ representation is the one stated in Theorem 6.1. That $\boldsymbol{\beta}$ solves the fixed point equation and that the iteration converges was established in Corollary 6.1. □

Similarly to the proof of Theorem 6.1 but easier, we have the following analogue of Theorem 3.1.

Theorem 6.2 *Assume in (1.2) that $-Z$ is a subordinator and Y is* $\mathrm{CP}(\lambda, H)$*, with H being* $\mathrm{ME}_p(\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{t})$ *with $\boldsymbol{t} = -\boldsymbol{T}\mathbf{1}$ and satisfying* (R)*. Then $\overline{X}(e_\eta)$ restricted to $(0, \infty)$ is* $\mathrm{ME}_p(\boldsymbol{\beta}, \boldsymbol{G}(\boldsymbol{\beta}), \boldsymbol{g}(\boldsymbol{\beta}))$*, where $\boldsymbol{G}(\boldsymbol{\beta}) = \boldsymbol{T} + \boldsymbol{t}\boldsymbol{\beta}$, $\boldsymbol{g}(\boldsymbol{\beta}) = -\boldsymbol{G}(\boldsymbol{\beta})\mathbf{1}$. Here, $\boldsymbol{\beta}$ solves $\boldsymbol{\beta} = \psi(\boldsymbol{\beta})$ where*

$$\psi(\boldsymbol{\beta}) = \lambda\boldsymbol{\alpha}((\lambda + \eta)\boldsymbol{I} - \kappa_Z(-\boldsymbol{G}(\boldsymbol{\beta})))^{-1}.$$

Further, $\overline{X}(e_\eta)$ has an atom of size $1 - \boldsymbol{\beta}\mathbf{1}$ at 0.

The row vector $\boldsymbol{\beta}$ can be computed by iteration, $\boldsymbol{\beta}^{(n)} \to \boldsymbol{\beta}$, where $\boldsymbol{\beta}^{(n)} = \psi\left(\boldsymbol{\beta}^{(n-1)}\right)$ and $\boldsymbol{\beta}^{(0)} = \mathbf{0}$.

7. Remarks

In exponential Lévy models in finance, the price of a barrier option with fixed maturity T is an integral in the joint distribution of either $\overline{X}(T), X(T)$ or $\underline{X}(T), X(T)$. In the case of, say, $\overline{X}(T), X(T)$, this is obviously given by the joint distribution of $\overline{X}(T), D(T)$ (note that $X(T) = \overline{X}(T) - D(T)$). However, Theorems (3.1) and (4.1) determine the bivariate Laplace transform, characteristic function, etc. with respect to T of $\overline{X}(T), D(T)$. The price can thus be computed by transform inversion. This or closely related procedures have been implemented, e.g., in [18]. For an alternative, using an extension of the present paper to an Erlangian rather than exponential horizon, see [6].

Similar transform inversion approaches have been used e.g. in [14] and references therein for European options. For a call option with strike $K > S_0$ where S_0 is the spot price, this involves only the distribution of the positive part $X(T)^+$ of $X(T)$.

Corollary 7.1. *In the model of the present paper, the distribution of $X(e_\eta)^+$ is defective PH with generator $\boldsymbol{G}$ and initial vector $\boldsymbol{\beta}\mathbb{E}\mathrm{e}^{\boldsymbol{G}D(e_\eta)}$ when Z is a subordinator. Here,*

$$\mathbb{E}\mathrm{e}^{\boldsymbol{G}D(e_\eta)} = -\eta(\eta\boldsymbol{I} - \kappa(-\boldsymbol{G}))^{-1}((\boldsymbol{\beta} \otimes \boldsymbol{I})(\boldsymbol{G} \otimes \boldsymbol{I} - \boldsymbol{I} \otimes \boldsymbol{G})^{-1}(\boldsymbol{g} \otimes \boldsymbol{I}))^{-1}.$$

If Z is not a subordinator, the same holds with $\boldsymbol{\beta}$ replaced by e_0. The formulas also hold in the ME case.

Proof. For $x > 0$,

$$\begin{aligned}\mathbb{P}(X^+(e_\eta) \in \mathrm{d}x) &= \int_0^\infty \mathbb{P}(D(e_\eta) \in \mathrm{d}y)\mathbb{P}(\overline{X}(\omega) \in \mathrm{d}(x+y)\\ &= \int_0^\infty \boldsymbol{\beta}\mathbb{P}(D(e_\eta) \in \mathrm{d}y)\mathrm{e}^{\boldsymbol{G}(x+y)}\boldsymbol{g} = \boldsymbol{\beta}\mathbb{E}\mathrm{e}^{\boldsymbol{G}D(e_\eta)}\mathrm{e}^{\boldsymbol{G}x}\boldsymbol{g}.\end{aligned}$$

For the expression for $\mathbb{E}\mathrm{e}^{\boldsymbol{G}D(e_\eta)}$, just use the WH identity $\mathbb{E}\mathrm{e}^{-\boldsymbol{G}X(e_\eta)} = \mathbb{E}\mathrm{e}^{-\boldsymbol{G}\overline{X}(e_\eta)}\mathbb{E}\mathrm{e}^{\boldsymbol{G}D(e_\eta)}$ together with

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

$$\mathbb{E}\mathrm{e}^{-\boldsymbol{G}\overline{X}(\omega)} = \int_0^\infty \boldsymbol{\beta}\mathrm{e}^{\boldsymbol{G}x}\boldsymbol{g}\cdot\mathrm{e}^{-\boldsymbol{G}x}\,\mathrm{d}x$$
$$= \int_0^\infty \boldsymbol{\beta}\mathrm{e}^{\boldsymbol{G}x}\boldsymbol{g}\cdot\boldsymbol{I}\mathrm{e}^{-\boldsymbol{G}x}\boldsymbol{I}\,\mathrm{d}x = -(\boldsymbol{\beta}\otimes\boldsymbol{I})(\boldsymbol{G}\otimes\boldsymbol{I}-\boldsymbol{I}\otimes\boldsymbol{G})^{-1}(\boldsymbol{g}\otimes\boldsymbol{I}). \qquad \square$$

In the barrier option example, transform inversion then applies in the same manner.

Another application of our results is the Poisson sampling setting of [12] dealing with $\overline{X}_{\eta,\gamma}$, the maximum of X sampled at Poisson(γ) epochs before e_η. It is shown in [12] that $\overline{X}(e_\eta)$ has the same distribution as the independent sum $\overline{X}_{\eta,\gamma}+\overline{X}(e_{\eta+\gamma})$. This is used to derive the transform of $\overline{X}_{\eta,\gamma}$ in the spectrally positive case. It is further shown that $X_{\eta,\gamma}$ is defective exponential in the spectrally negative cases. However, our approach extends this result for the spectrally negative case to a PH distribution result in the presence of positive PH/ME jumps.

Corollary 7.2. *In the model of the present paper, the distribution of $\overline{X}_{e_{\eta,\gamma}}$ is defective PH with generator $\boldsymbol{G}^\gamma+\boldsymbol{g}^\gamma\boldsymbol{v}$ and initial vector $\boldsymbol{v}$. Here, $\boldsymbol{G}^\gamma$ is as in Corollary 7.2 with η replaced by γ, $\boldsymbol{g}^\gamma=-\boldsymbol{G}^\gamma\mathbf{1}$ and $\boldsymbol{v}=\lim_{n\to\infty}\boldsymbol{v}^{(n)}$, where*

$$\boldsymbol{v}^{(n)}=\mathbf{0},\qquad \boldsymbol{v}^{(\mathbf{n+1})}=\frac{\gamma}{\gamma+\eta}\boldsymbol{\beta}^\gamma\mathbb{E}\exp\{\boldsymbol{G}^\gamma+\boldsymbol{g}^\gamma\boldsymbol{v}^{(\mathbf{n})})\mathbf{D}(\mathbf{e}_\gamma)\},$$

when Z is a subordinator and $\boldsymbol{v}^{(n)}$ is given by the same recursion with $\boldsymbol{\beta}^\gamma$ replaced by e_0 otherwise.

Proof. Clearly, $\overline{X}_{\eta,\gamma}$ is distributed as the maximum of a terminating random walk with increments Y_k distributed as $X(e_\gamma)=\overline{X}(e_\gamma)-D(e_\gamma)$ (the killing probability is $\mathbb{P}(e_\gamma>e_\eta)=\gamma/(\gamma+\eta)$). Here, $\overline{X}(e_\gamma)$ is PH with generator $\boldsymbol{G}^\gamma$. Combining with [1, Theorem 2.2] for the PH case and [11, Theorems 6.2.6 and 6.2.8] for the ME case gives the result. $\square$

We finally remark that in the case where Z is a sum of BM and downward PH jumps, we can avoid using the somewhat complicated form of $\kappa_Z(-\boldsymbol{G})$ by using a variant of the present approach, where ω is redefined as the first upward or downward jump instant before e_η. This variant is the case $q=1$ of [6].

Funding information

There are no funding bodies to thank relating to the creation of this article.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

References

[1] ASMUSSEN, S. (1992). Phase-type representations in random walk and queueing problems. *Ann. Prob.* **20**, 772–789.

[2] ASMUSSEN, S. (2003). *Applied Probability and Queues*, 2nd ed. Springer, New York.

[3] ASMUSSEN, S. (2014). Lévy processes, phase-type distributions and martingales. *Stoch. Models* **30**, 443–468.

[4] ASMUSSEN, S. (2022). On the role of skewness and kurtosis in tempered stable (CGMY) Lévy models in finance. *Finance Stoch.* **26**, 383–416.

[5] ASMUSSEN, S. (2022). Remarks on Lévy process simulation. In *Advances in Modeling and Simulation. Festschrift for Pierre L'Ecuyer*, eds Z. Botev, A. Keller, C. Lemieux, B. Tuffin, Springer, New York, pp. 21–40.

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press

[6] ASMUSSEN, S. (2024). Erlangization/Canadization of phase-type jump diffusions, with applications to barrier options. *J. Indian Soc. Statist. Prob.* **25**, 575–596.
[7] ASMUSSEN, S., AVRAM, F. AND PISTORIUS, M. R. (2004). Russian and American put options under exponential phase-type Lévy models. *Stoch. Process. Appl.* **109**, 79–111.
[8] ASMUSSEN, S. AND IVANOVS, J. (2018). A factorization identity for a Lévy process over a phase-type horizon. *Stoch. Models* **34**, 397–408.
[9] AVRAM, F., PISTORIUS, M. R. AND USABEL, M. (2003). The two barriers ruin problem via a Wiener–Hopf decomposition approach. *Ann. Univ. Craiova Math. Comp. Sci. Ser.* **30**, 38–44.
[10] BLADT, M. AND IVANOVS, J. (2021). Fluctuation theory for one-sided Lévy processes with a matrix-exponential time horizon. *Stoch. Process. Appl.* **142**, 105–123.
[11] BLADT, M. AND NIELSEN, B. F. (2017). *Matrix-Exponential Distributions in Applied Probability*. Springer, New York.
[12] BOXMA, O. AND MANDJES, M. (2023). A decomposition for Lévy processes sampled at Poisson moments. *J. Appl. Prob.* **60**, 557–569.
[13] CARR, P., GEMAN, H., MADAN, D. B. AND YOR, M. (2002). The fine structure of asset returns: An empirical investigation. *J. Business* **75**, 305–332.
[14] DEELSTRA, G. AND SIMON, M. (2017). Multivariate European option pricing in a Markov-modulated Lévy framework. *J. Comput. Appl. Math.* **317**, 171–187.
[15] HACKMANN, D. (2022). Wiener–Hopf factorization for the normal inverse Gaussian process. Preprint, arXiv:1902.08804v2.
[16] HACKMANN, D. AND KUZNETSOV, A. (2016). Approximating Lévy processes with completely monotone jumps. *Ann. Appl. Prob.* **26**, 328–359.
[17] IVANOVS, J. (2021). On scale functions for Lévy processes with negative phase-type jumps. *Queueing Systems* **98**, 3–19.
[18] JEANNIN, M. AND PISTORIUS, M. (2010). A transform approach to compute prices and Greeks of barrier options driven by a class of Lévy processes. *Quant. Finance* **10**, 629–644.
[19] JIANG, Z. AND PISTORIUS, M. R. (2008). On perpetual American put valuation and first passage in a regime-switching model with jumps. *Finance Stoch.* **12**, 331–355.
[20] KUZNETSOV, A. (2010). Wiener–Hopf factorization and distribution of extrema for a family of Lévy processes. *Ann. Appl. Prob.* **20**, 1801–1830.
[21] KUZNETSOV, A. (2010). Wiener–Hopf factorization for a family of Lévy processes related to theta functions. *J. Appl. Prob.* **47**, 1023–1033.
[22] KUZNETSOV, A. (2012). On the distribution of exponential functionals for Lévy processes with jumps of rational transform. *Stoch. Process. Appl.* **122**, 654–663.
[23] KUZNETSOV, A., KYPRIANOU, A. E., PARDO, J.-C. AND VAN SCHAIK, K. (2011). A Wiener–Hopf Monte Carlo simulation technique for Lévy processes. *Ann. Appl. Prob.* **21**, 2171–2190.
[24] KUZNETSOV, A. AND PENG, X. (2012). On the Wiener–Hopf factorization for Lévy processes with bounded positive jumps. *Stoch. Process. Appl.* **122**, 2610–2638.
[25] LEWIS, A. L. AND MORDECKI, E. (2008). Wiener–Hopf for Lévy processes having positive jumps with rational transform. *J. Appl. Prob.* **45**, 118–134.
[26] MORDECKI, E. (2002). The distribution of the maximum of a Lévy process with positive jumps of phase-type. In *Proc. Conf. Dedicated to the 90th Anniversary of Boris Vladimirovich Gnedenko*, Kyiv, pp. 309–316.
[27] MORDECKI, E. (2003). Ruin probabilities for Lévy processes with mixed-exponential negative jumps. *Teor. Veroyat Primen.* **48**, 188–194.
[28] PISTORIUS, M. (2006). On maxima and ladder processes for a dense class of Lévy processes. *J. Appl. Prob.* **43**, 208–220.
[29] WEISSTEIN, E. W. Padé approximant. From *MathWorld* – A Wolfram web resource. https://mathworld.wolfram.com/PadeApproximant.html.

https://doi.org/10.1017/jpr.2024.112 Published online by Cambridge University Press