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A new generating function for Jacobi polynomials

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1. *Introduction.* In this paper we give a new generating function for the Jacobi polynomials. The result obtained is of general character and includes as particular cases some of the results given earlier by Carlitz (2), Salam (1), Manocha and Sharma (3, 4).

2. We require the following results (8) and ((7), p. 257) in our investigation

$$F_k[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3,; x, y, z]$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_{n+p} (\beta_1)_{m+p} (\beta_2)_n}{m!\, n!\, p!\, (\gamma_1)_m (\gamma_2)_n (\gamma_3)_p} x^m y^n z^p. \quad (1)$$

$$D^n[(1-x)^{n+\alpha}(1+x)^{n+\beta}] = \frac{2^n n!}{(-1)^n (1-x)^{-\alpha}(1+x)^{-\beta}} P_n^{(\alpha,\beta)}(x), \quad (2)$$

where
$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n (1+\beta)_n}{n!} {}_2F_1\left[-n, 1+\alpha+\beta+n; 1+\beta; \frac{1+x}{2}\right], \quad (3)$$

is the Jacobi polynomial of degree n. It is interesting to note that Saran's function F_k (8) and Lauricella's function F_D (5) are connected by the following relation

$$F_k[\lambda, \rho+\mu, \rho+\mu, \lambda+\alpha, \rho+\beta, \lambda+\alpha; \lambda+\alpha, \rho+\beta+\mu, \rho+\mu; x, y, z]$$
$$= (1-y)^{\lambda+\alpha-\rho}(1-y-z)^{-\alpha}(1+xy-x-y-z)^{-\lambda}$$
$$\times F_D\left[\mu; \lambda, \alpha, \beta; \rho+\beta+\mu; \frac{-yz}{1+xy-x-y-z}, \frac{-yz}{1-y-z}, y\right]. \quad (4)$$

3. The main result to be proved is

$$\sum_{n=0}^{\infty} \frac{(m+n)!\,(n+p)!}{n!\,(\lambda)_n} P_{n+m}^{(\alpha-n-m,\beta)}(x)\, P_{n+p}^{(\gamma-n-p,\delta)}(y)\, t^n$$
$$= (-1)^{m+p}(1+\beta)_m (1+\delta)_p \left[\frac{1-x}{2}\right]^{m-\alpha} \left[\frac{1-y}{2}\right]^{p-\gamma}$$
$$\times F_k\left[-\alpha, 1+p+\delta, 1+p+\delta, 1+\beta+m, -\gamma, 1+\beta+m; 1+\beta, 1+\delta, \lambda;\right.$$
$$\left.\times \frac{1+x}{2}, \frac{1+y}{2}, \frac{(1-x)(1-y)t}{4}\right]. \quad (5)$$

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Proof. In the proof θ and ϕ will denote d/dx and d/dy respectively. We consider the expression

$$\theta^m\phi^p\,{}_0F_1[;\lambda;t\theta\phi(1+x)(1+y)][(1+x)^{\beta+m}(1-x)^{\alpha}(1+y)^{\delta+p}(1-\gamma)^{\gamma}]$$
$$=\sum_{n=0}^{\infty}\frac{t^n}{n!\,(\lambda)_n}\theta^{m+n}[(1-x)^{\alpha}(1+x)^{\beta+m+n}]\phi^{n+p}[(1-y)^{\gamma}(1+y)^{\delta+n+p}]$$
$$=(-2)^{m+p}(1-x)^{\alpha-m}(1+x)^{\beta}(1+y)^{\delta}(1-y)^{\gamma-p}$$
$$\times\sum_{n=0}^{\infty}\frac{(m+n)!\,(n+p)!}{n!\,(\lambda)_n}P_{n+m}^{(\alpha-n-m,\beta)}(x)\,P_{n+p}^{(\gamma-n-p,\delta)}(y)\left[\frac{4t}{(1-x)(1-y)}\right]^n \text{ by (2).} \quad (6)$$

Again we consider the same expression.

$$\theta^m\phi^p\,{}_0F_1[;\lambda;t\theta\phi(1+x)(1+y)][(1-x)^{\alpha}(1+x)^{\beta+m}(1-y)^{\gamma}(1+y)^{\delta+p}]$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-\alpha)_r(-\gamma)_s 2^{\alpha+\gamma-r-s}t^n}{n!\,r!\,s!\,(\lambda)_n}\theta^{n+m}[1+x]^{\beta+m+n+r}\phi^{n+p}[1+\gamma]^{\delta+n+p+s}$$
$$=(1+x)^{\beta}(1+y)^{\delta}(1+\beta)_m(1+\delta)_p 2^{\alpha+\gamma}$$
$$\times\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-\alpha)_r(-\gamma)_s(1+\beta+m)_{n+r}(1+\delta+p)_{n+s}t^n}{n!\,r!\,s!\,(1+\beta)_r(1+\delta)_s(\lambda)_n}\left(\frac{1+x}{2}\right)^r\left(\frac{1+y}{2}\right)^s$$
$$=(1+x)^{\beta}(1+y)^{\delta}(1+\beta)_m(1+\delta)_p 2^{\alpha+\gamma}$$
$$\times F_k\left[-\alpha,1+p+\delta,1+p+\delta,1+\beta+m,-\gamma,1+\beta+m;1+\beta,1+\delta,\lambda;\right.$$
$$\left.\times\frac{1+x}{2},\frac{1+y}{2},t\right], \quad (7)$$

by virtue of (1).

(5) follows immediately from (6) and (7) after a little simplification. This completes the proof of (5).

If we make use of the relation [(9), p. 63]

$$P_n^{(\alpha-n,-\alpha-\gamma-n)}(x)=\left(\frac{1+x}{2}\right)^n P_n^{(\alpha-n,\gamma-1)}\left[\frac{3-x}{1+x}\right] \quad (8)$$

and the formula (6)

$$F_k[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z]=(1-x)^{-\beta_1}(1-y)^{-\alpha_2},$$
$$\times F_k\left[\gamma_1-\alpha_1,\alpha_2,\alpha_2,\beta_1,\gamma_2-\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;\frac{x}{x-1},\frac{y}{y-1},\frac{z}{(1-x)(1-y)}\right] \quad (9)$$

in (5), we find that

$$\sum_{n=0}^{\infty}\frac{(n+m)!\,(n+p)!}{n!\,(\lambda)_n}P_{n+m}^{(\alpha-n-m,\beta-n-m)}(x)\,P_{n+p}^{(\gamma-n-p,\delta-n-p)}(y)\,t^n$$
$$=(-1)^{m+p}(-\alpha-\beta)_m(-\gamma-\delta)_p\left(\frac{x-1}{x+1}\right)^{\beta}\left(\frac{y-1}{y+1}\right)^{\delta},$$
$$\times F_k\left[-\beta,p-\gamma-\delta,p-\gamma-\delta,m-\alpha-\beta,-\delta,m-\alpha-\beta;-\alpha-\beta,-\gamma-\delta,\lambda;\right.$$
$$\left.\times\frac{2}{1-x},\frac{2}{1-y},\frac{(x+1)(y+1)}{4}t\right]. \quad (10)$$

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4. *Particular cases.* We shall mention some interesting particular cases of the general result (5).

(*a*) Assuming $p = 0$, $\lambda = 1+\beta$ and using the formulae (3) and (4) in (5), it reduces to a known result due to Manocha and Sharma ((3), p. 75, eq. (12)).

(*b*) Taking $y = -1$ and using the formula ((7), p. 254)

$$P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n(1+\beta)_n}{n!} \tag{11}$$

in (5), we find that

$$\sum_{n=0}^{\infty} \frac{(m+n)!\,(\delta)_n}{n!\,(\lambda)_n} P_{n+m}^{(\beta,\alpha-m-n)}(x)\,t^n = (1+\beta)_m \left(\frac{1+x}{2}\right)^{m-\alpha},$$
$$\times F_2\left[1+\beta+m, -\alpha, \delta; 1+\beta, \lambda; \frac{1-x}{2}, \frac{1+x}{2}t\right]. \tag{12}$$

In case $m = 0$, (12) reduces to a result due to Salam (1). In case $\lambda = \delta$ and using the formula (5)

$$F_2(\alpha,\beta,\beta';\beta,\gamma';x,y) = (1-x)^{-\alpha}\,{}_2F_1\left(\alpha,\beta';\gamma';\frac{y}{1-x}\right) \tag{13}$$

in (12), it reduces to a result due to Manocha and Sharma (4). In case $\delta = 1$, $\lambda = m+1$ in (12), we find that

$$\sum_{n=0}^{\infty} P_{n+m}^{(\beta,\alpha-m-n)}(x)\,t^n = \frac{(1+\beta)_m}{m!}\left(\frac{1+x}{2}\right)^{m-\alpha},$$
$$\times F_2\left[1+\beta+m, -\alpha, 1; 1+\beta, m+1; \frac{1-x}{2}, \frac{1+x}{2}t\right]. \tag{14}$$

If we take $m = 0$ and using the formula (8) in (14), it reduces to a result due to Carlitz ((2), p. 2).

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