Econometric Theory, **19**, 2003, 225–228. Printed in the United States of America.
DOI: 10.1017/S0266466603001087

PROBLEMS AND SOLUTIONS

PROBLEMS

03.1.1. Deriving the Observed Information Matrix in Ordered Probit and Logit Models Using the Complete-Data Likelihood Function

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Louis (1982) presents a method for computing the observed information matrix and standard errors of maximum likelihood estimates obtained via the EM algorithm based on the complete-data log likelihood function. The problem illustrates the well-known method of Louis (1982) for a widely used qualitative response model in econometrics. The observed-data log likelihood function for the following model can, of course, be easily differentiated to obtain the observed information matrix; our objective is to illustrate the method and not to recommend its use for this model.

Consider the following ordered response model:

$$y_i^* = \beta' x_i + u_i, \qquad i = 1,2,\ldots,n, \tag{1}$$

where y^* is the underlying response variable, β is a $(k \times 1)$ vector of unknown parameters, x is a $(k \times 1)$ vector of known constants, and u is the residual. The term y^* is not observed, but we know which of the m categories it belongs to. It belongs to the jth category if $\alpha_{j-1} < y* < \alpha_j$ $(j = 1,2,\ldots,m)$, α's being known constants and $\alpha_0 = -\infty$, $\alpha_m = \infty$. Derive the observed information matrix for the model using the complete-data log likelihood function under the assumption that u_i are *i.i.d.* $N(0,1)$.

REFERENCE

Louis, T.A. (1982) Finding the observed information matrix when using the EM algorithm. *Journal of the Royal Statistical Society, Series B* 44, 226–233.

03.1.2. Redundancy of Lagged Regressors in a Conditionally Heteroskedastic Time Series Regression

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Consider the following stationary time series regression:

$$y_t = \beta x_t + e_t, \qquad E[e_t | x_t, x_{t-1}, \ldots] = 0,$$

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where all variables are scalars and the error e_t is conditionally heteroskedastic with the following form of heteroskedasticity:

$$E[e_t^2|x_t, x_{t-1}, \ldots] = \omega + \lambda(x_t - \mu)^2, \qquad \omega > 0, \qquad \lambda \geq 0.$$

The object of estimation is β. Usually, under conditional heteroskedasticity the use of lagged values of regressors as instruments increases the efficiency of generalized method of moments (GMM) estimation in comparison with ordinary least squares (OLS) estimation (see, e.g., Broze, Francq, and Zakoïan, 2001; West, 2001). This problem shows that it may not necessarily be so.

Assume that the regressor x_t can be represented as $x_t = \sum_{i=0}^{\infty} \varphi_i \eta_{t-i}$, where η_t's are independently and identically distributed standard normal. Also assume that the parameters are constrained so that all variables have finite fourth moments. Show that the OLS estimator is at least as efficient as any GMM estimator that uses an arbitrary fixed number of instruments from the list $\{x_t, x_{t-1}, x_{t-2}, \ldots\}$.

REFERENCES

Broze, L., C. Francq, & J.-M. Zakoïan (2001) Non-redundancy of high order moment conditions for efficient GMM estimation of weak AR processes. *Economics Letters* 71, 317–322.

West, K.D. (2001) On optimal instrumental variables estimation of stationary time series models. *International Economic Review* 42, 1043–1050.

SOLUTIONS

02.1.1. LS and BLUE Are Algebraically Identical—Solution[1]

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Let Z be an $n \times (n - q)$ matrix of rank $n - q$ whose columns span the space orthogonal to the space spanned by the columns of X and let $f = Z'y$. Then, on premultiplying by the $n \times n$ nonsingular matrix $[Z : X]$, we find that the conventional least squares problem:

$$\min e'e \quad \text{subject to } e = y - Xb \tag{1}$$

may be expressed as

$$\min e'e \quad \text{subject to } Z'e = Z'y \quad \text{and} \quad X'Xb = X'y - X'e \tag{2}$$

or, equivalently, as

$$\min e'e \quad \text{subject to } Z'e = f \tag{3}$$

and

$$b = (X'X)^{-1}X'(y - e), \tag{4}$$

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where equation (4) defines the optimal value of b in terms of the optimal value of e.

Further, let u be an $n \times 1$ matrix satisfying $X'u = c$ and let $d = (Z'Z)^{-1}Z'(a - u)$ be an $(n - q) \times 1$ function of a. Then the conventional minimum variance unbiased linear estimation problem

$$\min a'a \quad \text{subject to } X'a = c \tag{5}$$

may be expressed as

$$\min a'a \quad \text{subject to } X'a = X'u - X'Zd \quad \text{and} \quad Z'a = Z'u - Z'Zd, \tag{6}$$

and, on premultiplying by the inverse of the $n \times n$ nonsingular matrix $[Z : X]$, we have

$$\min a'a \quad \text{subject to } a = u - Zd. \tag{7}$$

Clearly, problems (1) and (3) are algebraically identical to problems (7) and (5) when X, y, b, f, and e are replaced by Z, u, d, c, and a, respectively.

NOTE

1. Two excellent solutions were independently proposed by H. Neudecker and S. Puntanen, G. Styan, and H. Werner.

02.1.2. A Particular Symmetric Idempotent Matrix—Solution[1]

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It is easy to see that Problem 02.1.2 holds more generally in that the result is valid for B Hermitian complex (rather than "symmetric real"), for $m = 0,1,\ldots,q$ (rather than just for "$m < q$"), for $r \neq 0$ (rather than just for "integer $r > 1$"), and for $\mathrm{tr}(B^k) = \mathrm{tr}(C^k)$, $k = 1,2,3,4$ (rather than for "$k = 1,2,\ldots$"), as shown by Shanbhag (1970).

To see this let $A = (1/r)B$, $D = (1/r)C$, and $\alpha_1, \alpha_2, \ldots, \alpha_q$ denote the eigenvalues of A. Then

$$m = \sum_{i=1}^{q} \alpha_i = \sum_{i=1}^{q} \alpha_i^2 = \sum_{i=1}^{q} \alpha_i^3 = \sum_{i=1}^{q} \alpha_i^4,$$

because we know that if $\lambda_1,\ldots,\lambda_q$ are the (not necessarily distinct) eigenvalues of $M \in \mathbb{C}^{q\times q}$, then $\lambda_1^n,\ldots,\lambda_q^n$ are the eigenvalues of M^n, irrespective of $n \in \mathbb{N}$ (see, e.g., Lancaster, 1969, Theorem 2.5.2). A fundamental result on characteristic polynomials tells us that $\mathrm{tr}(M) = \sum_1^q \lambda_i$ (cf. Lancaster, 1969, p. 55). Hence

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$$\sum_{i=1}^{q} \alpha_i^2(1-\alpha_i)^2 = 0,$$

and so the $\alpha_i = 0$ or 1. Because $\sum_{i=1}^{q} \alpha_i = m$ it follows that precisely m of the α_i are equal to 1 and $q - m$ are equal to 0. A Hermitian matrix is unitarily similar to the diagonal matrix of its eigenvalues, and the eigenvalues are all real (cf. e.g., Lancaster, 1969, Theorem 2.9.5 and Theorem 2.9.1). It follows then, as is well known, that a Hermitian complex matrix with all eigenvalues equal to 0 or 1 is idempotent. Our proof is complete.

One might be tempted to believe that a complex square matrix M, not Hermitian, with all eigenvalues equal to 0 and/or 1 is always idempotent. This, however, is erroneous, as the nilpotent matrix

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

shows. Although this matrix fails to be idempotent, it has the eigenvalue 0 with (algebraic) multiplicity 2.

When A and B are both Hermitian nonnegative definite, then the eigenvalues $\alpha_i \geq 0$, and, as Shanbhag (1970) points out, the condition $\mathrm{tr}(B^k) = \mathrm{tr}(C^k)$, $k = 1,2,3,4$ may be replaced by $\mathrm{tr}(B^k) = \mathrm{tr}(C^k)$, $k = 1,2,3$. This leads to $\sum_{i=1}^{q} \alpha_i(1 - \alpha_i)^2 = 0$, and because the α_i are now all nonnegative we see immediately that $\alpha_i = 0$ or 1.

NOTE

1. Excellent solutions were independently proposed by G. Dhaene, J. Hill, S. Lawford and C. Dehon, P. Omtzigt, G. Trenkler, M. Van de Velden, D. Wiens, and H. Neudecker (the poser of the problem).

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