Math. Proc. Camb. Phil. Soc. (1977), **82**, 381
Printed in Great Britain

The 2-primary J-homomorphism

By VICTOR SNAITH
University of Western Ontario, London, Canada

(*Received* 13 *September* 1976, *revised* 9 *June* 1977)

§1. *Introduction*

In this paper every space will be 2-local, for example BO will mean the 2-localization of the space usually denoted BO.

The 2-primary J-homomorphism is a well-known map $J: O \to G$ for which the reader may consult the following references: (1); (8), p. 211; (16); (22); (23). It is an infinite loop map ((14), ch v, §4). In particular, it is an H-map. The solution of the Adams conjecture (3, 16, 22) implies that J can be factorized as a composite of the form $O \xrightarrow{w} JO \to G$ where JO is described in §2·5. By virtue of the following result no such factorizations exist by means of H-maps.

1·1. *Proposition* (Madsen (10)). Let $(JO)_0$ be the base-point component of JO. There exist no homotopy commutative diagrams of H-maps of the following form.

$$\begin{array}{ccc} SO & \xrightarrow{J} & SG \\ & {\scriptstyle w}\searrow \quad \nearrow{\scriptstyle \mu} & \\ & (JO)_0 & \end{array} \tag{1·2}$$

Here J and w are the natural H-maps.

Proof. In the notation of (11), (12) we have Bockstein spectral sequences with E^2-terms as follows:

$$E^2(SO) \cong \Lambda(u_1 u_2, u_3 u_4, \ldots),$$

$$E^2((JO)_0) \cong E^2(SO) \otimes P(b_2^2, b_4^2, \ldots),$$

where $H_*((JO)_0; Z/2) \cong \Lambda(u_1, u_2, \ldots) \otimes P(b_1, b_2, \ldots)$. Also $d_2(b_2^2) = u_1 u_2$. If SG is the identity component of G then $E^2(SG) \cong \Lambda(u_1 u_2) \otimes P(Q^2Q^2[1]_*[-3]) \otimes \ldots$ with $d_2(Q^2Q^2[1]_*[-3]) = u_1 u_2$. Hence μ induces μ_* in mod 2 homology which satisfies $\mu_*(b_2^2) = Q^2Q^2[1]_*[-3]$ which is indecomposable.

There have been other similar non-additivity results (6, 7, 12, 19) but Proposition 1·1 is the sharpest and for that reason I have included a proof.

Maps from JO to G arise, for example, from the solution of the Adams conjecture (2, 3, 22). Additive solutions would induce H-maps, μ, as in Proposition 1·1.

The aim of this paper is to study the non-additivity phenomenon in solutions of the Adams conjecture. This I will do for two quite separate approaches to the Adams conjecture. I will state all the results here (Theorems A, B and C) and give in §2 the proof of the first of these results. Theorems B and C will be proved in (21) because the proofs, while not difficult, require for their intelligibility all the technical background material which is established in (21), part I.

https://doi.org/10.1017/S0305004100054049 Published online by Cambridge University Press

The first study concerns the real analogue of the bundle-theoretic constructions given in (20) for complex vector bundles. Briefly the idea of (20) is to produce the complex analogue of diagram (1·2) by appealing to the bundle-theoretic description of J-theory given by Seymour in (17). The essential point was a technical modification of Seymour's bundle-theoretic J-theory of X, called $Ad_q^*(X)$, to a 'proper' Ad_q-theory which, while equivalent to Seymour's theory, admits a bundle-theoretic description of a homomorphism μ to spherical fibration theory which factors the complex J-homomorphism $U \to SG$. Now for the real analogue. In (17) Seymour constructed a functor $AdR^0(X)$; the construction is in terms of vector bundles. Details are given in §2. Seymour shows that the 2-localization of $AdR^0(X)$ is representable; we write JO for the essential part of the representing space. Seymour shows that JO participates in the following fibre sequence of infinite loop maps.

$$\dots O \xrightarrow{\psi^3-1} O \xrightarrow{w} JO \xrightarrow{k} BO \xrightarrow{\psi^3-1} BO \longrightarrow \dots . \tag{1·3}$$

A small change of Seymour's definition gives 'proper' AdR-theory, $PR(X)$, which also extends to a cohomology theory although we will not need that property. By analogy with (20) one would attempt to construct $\mu: PR(X) \to [X, G]$ thence to compare $PR(X)$ and $AdR^0(X)$ in order to establish the (impossible!) diagram (1·2). Our first result explains where this process fails. Let $TR(X) \subset PR(X)$ be the subgroup of $PR(X)$ represented by means of vector bundles admitting A_2-structures. For more on A_k-structures the reader is referred to (9).

THEOREM A. *Let X be a finite CW complex. Then*

(*a*) *there is a commutative diagram of natural homomorphisms*

$$\begin{array}{ccccc} KO^{-1}(X) & \xrightarrow{w'} & TR(X) & \subset & PR(X) \\ & \searrow^{J} & \downarrow & & \\ & & [X, G] & & \end{array}$$

(*b*) *if $\widetilde{KO}^0(X) \xrightarrow{\eta} KO^{-1}(X) \xrightarrow{J} [X, G]$ is zero then $TR(X) = PR(X)$. Here η is multiplication by $0 \neq \eta \in KO^{-1}(*)$.*

Theorem A gives us one interpretation of the obstruction to the existence of diagram (1·2), namely there is some finite subcomplex, X, of JO for which $J \circ \eta$ is non-zero. In fact JO may be written $RP^\infty \times RP^\infty \times J'$ and it is easy to see that such X exist inside the $RP^\infty \times RP^\infty$ factor.

Now let us turn to our second study which concerns the Becker–Gottlieb 'solution of the Adams conjecture' (3) as presented in (2). This is a 2-local map $\alpha: BO \to G/O$ such that the following diagram homotopy commutes.

$$\begin{array}{ccc} BO & \xrightarrow{\alpha} & G/O \\ & \searrow^{\psi^3-1} & \downarrow \\ & & BO \end{array} \tag{1·4}$$

Here $G/O \to BO$ is the natural map from the J-homomorphism fibre sequence and α is constructed as follows. Set $H(n)$ equal to the wreath product $\Sigma_n \int O(2) \subset O(2n)$

https://doi.org/10.1017/S0305004100054049 Published online by Cambridge University Press

and put $X_n = O(4n)/H(2n)$. Also write $P(2n)$ for $O(n) \times O(n)$. Adams constructed a map $\alpha_1 : BO(2) \to G/O$ ((1); (5), §6). The transfer maps associated with the fibrings $BH(n) \xrightarrow{\pi_n} BO(2n)$ fit together ((2); (21), part I, §§2, 3) to give a map $\tau : BO \to QBO(2)$ where $QW = \varinjlim_n \Omega^n \Sigma^n W$. Then α is defined as the composite

$$BO \xrightarrow{\tau} QBO(2) \xrightarrow{Q(\alpha_1)} QG/O \xrightarrow{d} G/O, \tag{1·5}$$

where d is the structure map of the infinite loopspace G/O ((15), §5). In (1·5) d and $Q(\alpha_1)$ are infinite loop maps. In particular they are additive. (1·4) induces a map $JO \to G$, factorizing the J-homomorphism, which would be an H-map if α were additive. In order to understand the deviation from additivity of α it suffices to study the deviation from additivity of τ. This deviation is a map

$$\gamma : BO \times BO \to QBO(2)$$

defined by a homotopy commutative diagram

$$\begin{array}{ccc} BO \times BO & \xrightarrow{\Delta} & BO \times BO \times BO \times BO \\ \downarrow{\scriptstyle m_1} & & \downarrow{\scriptstyle \tau \times \tau \times \gamma} \\ BO & & QBO(2) \times QBO(2) \times QBO(2) \\ & {\scriptstyle \tau}\searrow \quad \swarrow{\scriptstyle m_2} & \\ & QBO(2) & \end{array} \tag{1·6}$$

Here m_1 and m_2 come from the H-space multiplication and $\Delta(x, y) = (x, y, x, y)$. The map γ may be explicitly described in terms of the geometry of the orthogonal groups. The result takes the following form.

Set $Y_n = E \times_{P(4n)} X_n$, where E is a contractible, free $O(4n)$-space.

THEOREM B. *Let $\gamma : BO \times BO \to QBO(2)$ be the deviation from additivity of the transfer $\tau : BO \to QBO(2)$. Then, associated with the canonical fibring*

$$X_n \to Y_n \to BP(4n)$$

there is a transfer-like map, described in (21), *$\phi_n : BP(4n) \to QY_n$ such that γ restricted to $BP(4n) = BO(2n) \times BO(2n)$ is homotopic to a composite*

$$BP(4n) \xrightarrow{\phi_n} QY_n \xrightarrow{h} QBO(2).$$

Here $P(4n)$, X_n and Y_n are the spaces introduced above and h is the map described in (21).

In Theorem B it would be preferable to have a description of γ in terms of genuine transfer maps. In principle this is possible from the description of ϕ_n given in (21). This would be accomplished using the technique of (5) as extended and explained in ((21), part I, §2). However, this would involve a lengthy analysis of the double-coset spaces of the symmetric groups (cf. (21)) which would be out of place here.

Here is the answer when $n = 1$.

THEOREM C. *Let γ be as in Theorem B. Then the restriction of γ to*

$$BP(4) = BO(2) \times BO(2)$$

https://doi.org/10.1017/S0305004100054049 Published online by Cambridge University Press

is given by a composition of the following form, whose ingredients are described in (21),

$$BO(2)^2 \xrightarrow{\bar{\tau}^2} (QB\Sigma_2 \int \Sigma_2)^2 \xrightarrow{\theta} QBO(2).$$

Here $\Sigma_2 \int \Sigma_2$ *is a subgroup of* $O(2)$ *and* $\bar{\tau}$ *is the transfer associated with* $B\Sigma_2 \int \Sigma_2 \to BO(2)$.

The description of results is now complete. The material is arranged as follows. §2 contains the material on $AdR^0(X)$, $PR(X)$, $TR(X)$ and the proof of Theorem A. Since much of this is analogous to results in (20) I have, for brevity, assumed that the reader is familiar with ((20), §4) and confined myself to describing the modifications necessary in the real case. The material on the Becker–Gottlieb transfer will be proved in (21), part I. This is because the proofs require to be presented alongside the technical results of (21), part I, §2 and because, since my first draft of this paper, I have learned that J. Tornehave (24) has constructed a (possibly different) solution to the Adams conjecture, $BO \to G/O$, following Quillen's method (16), for which he has an explicit formula for the deviation from additivity.

Finally, I am indebted to several persons. This work was greatly influenced by Ib Madsen. Firstly because of his result (Proposition 1·1), which I wanted to understand conceptually. Secondly, because in conversations at the University of Chicago in August 1975 he convinced me that the deviation from additivity of α was geometrically interesting. Also I would like to express my gratitude to the mathematicians of the University of Chicago, especially Peter May, for their hospitality during August 1975, where I started the paper, and to those of the Centro de Investigacion, Mexico City, for their hospitality during May–June 1976, which gave me an opportunity to finish the first draft.

§2 *Proper AdR-theory and the J-homomorphism*

2·1. *Definitions* (cf. (20), §4). Let X be a compact Hausdorff space. A *Real* ψ^3*-vector bundle* over X is a triple (E, E', Θ). Here E and E' are real vector bundles over X, but we need some notations in order to explain what Θ is. Set

$$E_c = E \otimes_{\mathbb{R}} \mathbb{C} \quad \text{and} \quad E'_c = E' \otimes_{\mathbb{R}} \mathbb{C}.$$

Let N be the regular representation of $Z/3$ on $\mathbb{C}^3$. Endow $Z/3$ with the involution

$$\tau(g) = g^{-1} \quad (g \in Z/3)$$

and give N the involution

$$\tau(z_1, z_2, z_3) = (\bar{z}_1, \bar{z}_3, \bar{z}_2) \quad (z_i \in \mathbb{C}).$$

Then N becomes a real representation of the real group $Z/3$. That is, the conjugate-linear involution on N satisfies

$$\tau(gw) = \tau(g)\tau(w) \quad (g \in Z/3; w \in N).$$

A real $Z/3$-vector bundle structure on $E_c \oplus (E_c \otimes N)$ is given by

$$g(e \oplus (e' \otimes n)) = e \oplus (e' \otimes gn) \quad (g \in Z/3, e \in E_c, e' \in E'_c, n \in N),$$

$$\tau(e \oplus (e' \otimes n)) = \tau(e) \oplus (\tau(e') \otimes (n)).$$

https://doi.org/10.1017/S0305004100054049 Published online by Cambridge University Press

Here τ is complex conjugation on E_c and E'_c. The tensor product is taken over the complex numbers. A real $Z/3$-vector bundle structure on $E_c^{\otimes 3}$ is given by cyclic permutation of the factors together with the involution

$$\tau(e_1 \otimes e_2 \otimes e_3) = \tau e_1 \otimes \tau e_3 \otimes \tau e_2.$$

Then Θ is an isomorphism of real $Z/3$-vector bundles over X of the form

$$\Theta: E_c^{\otimes 3} \to E_c \oplus (E'_c \otimes N).$$

For our purposes X has no involution. We refer to Θ as a *Real* Ψ^3*-isomorphism*. A triple (E, E', Θ) is called *proper* if $\Theta(e^{\otimes 3})$ has a non-zero E_c-component whenever

$$e \in (E_c)_x (x \in X)$$

is non-zero. When (E, E', Θ) is proper we call Θ a *proper real* Ψ^3*-isomorphism*.

Two real Ψ^3-vector bundles (E, E', Θ) and (F, F', ϕ) over X are equivalent if there exist Real vector bundle isomorphisms (i.e. respecting involutions)

$$\alpha: E_c \to F_c \quad \text{and} \quad \alpha': E'_c \to F'_c$$

such that $\phi \circ \alpha^{\otimes 3}$ is homotopic through real Ψ^3-isomorphisms to $\alpha \oplus (\alpha' \otimes id) \circ \theta$ where 'id' is the identity map of N.

Two proper Real Ψ^3-vector bundles are *properly equivalent* if they are equivalent through a homotopy of proper Real Ψ^3-isomorphisms.

2·2. *Example.* We have a Real ψ^3-vector bundle over a point, $n = (\mathbb{R}^n, \mathbb{R}^d, \Theta_n)$ $(3d = n^3 - n)$, described as follows. Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{R}^n_c = \mathbb{C}^n$. Then

$$(\mathbb{C}^n)^{\otimes 3} \cong \mathbb{C}^n \oplus (\mathbb{C}^d \otimes N),$$

where the inclusion of $\mathbb{C}^n$ into $(\mathbb{C}^n)^{\otimes 3}$ corresponds to sending e_i to $e_i^{\otimes 3}$. $\mathbb{C}^d \otimes N$ is isomorphic ((17), §1) to the vector space spanned by the monomials

$$e_{i_1} \otimes e_{i_2} \otimes e_{i_3}$$

in which not all the e_{i_j} are equal. This is a real Ψ^3-isomorphism. n is a proper triple because

$$\Theta_n \left[\left(\sum_{i=1}^{n} z_i e_i \right)^{\otimes 3} \right]$$

has $\mathbb{C}^n$-component equal to $(z_1^3, \ldots, z_n^3)$ $(Z_i \in \mathbb{C})$.

2·3. $AdR^0(X)$, $PR(X)$ and $TR(X)$. The set of equivalence classes of real Ψ^3-vector bundles over X form an abelian monoid. The addition is defined as follows. If E and F are real vector bundles over X there is a vector bundle

$$U(E, F) = 3((E^{\otimes 2} \underset{\mathbb{R}}{\otimes} F) \oplus (E \underset{\mathbb{R}}{\otimes} F^{\otimes 2}))$$

such that $U(E, F)_c$ has a Real $Z/3$-vector bundle structure ((17), proposition 1·2 and §4) and there is a natural Real $Z/3$-vector bundle isomorphism

$$E_c^{\otimes 3} \oplus F_c^{\otimes 3} \oplus U(E, F)_c \cong (E_c \oplus F_c)^{\otimes 3}.$$

As a Real $Z/3$-vector bundle $U(E, F)_c \cong U'(E, F)_c \otimes N$, although we will not need to know the precise isomorphism ((17), proposition 1·2).

https://doi.org/10.1017/S0305004100054049 Published online by Cambridge University Press

The sum of (E, E', Θ) and (F, F', ϕ) is $(E \oplus F, E' \oplus F' \oplus U'(E,F), \Theta + \phi)$, where $\Theta + \phi$ is the composite (setting $U'(E,F) = U'$)

$$\begin{array}{ccc} (E_c \oplus F_c)^{\otimes 3} & \xrightarrow{\cong} & E_c^{\otimes 3} \oplus F_c^{\otimes 3} \otimes (U'_c \otimes N) \\ & & \downarrow \lambda \\ & & E_c \oplus F_c \oplus ((E'_c \oplus F'_c \oplus U'_c) \otimes N) \end{array}$$

in which λ equals $\Theta \oplus \phi \oplus 1$ followed by shuffling of summands.

Notice the addition respects propriety. The n-fold sum of 1 is n.

$AdR^0(X)$ is the Grothendieck group associated to the abelian monoid of equivalence classes of real Ψ^3-vector bundles over X. Similarly $PR(X)$ is the Grothendieck group of proper equivalence classes of proper real Ψ^3-vector bundles over X.

Every element in $PR(X)$ or $AdR^0(X)$ has a representative of the form $(E, E', \Theta) - n$ (the proof is similar to (17), §2·5). Define $TR(X)$ to be the subgroup of $PR(X)$ generated by elements of the form $(E, E, \Theta) - n$ $(n = \dim E)$ for which the real vector bundle E admits an A_2-structure (9). E is defined to have an A_2-structure if the map from the Stiefel bundle of orthonormal 2-frames in E to the sphere bundle of E (given by taking the first vector) has a section.

2·4. *Proof of Theorem* A. Define $w' : KO^{-1}(X) \to TR(X)$ as follows. Since X is finite we may represent $x \in KO^{-1}(X)$ by $\alpha : X \to O(n)$. If n is even then $\mathbb{R}^n$ admits an A_2-structure ((9), §1·9). Set $w'(x)$ equal to the element

$$(\mathbb{R}^n, \mathbb{R}^d, (\alpha_c \oplus \mathrm{id}).\Theta_n) - n,$$

where $3d = (n^3 - n)$, 'id' is the identity map of $\mathbb{C}^d \otimes N$, α_c is the complexification of α and $n = (\mathbb{R}^n, \mathbb{R}^d, \Theta_n)$ is the element defined in §2·2.

To define μ we proceed as follows. Let $x \in TR(X)$ be represented by $(E, E', \theta) - n$, where E admits an A_2-structure. Form the proper map

$$\bar{\mu} : E_c \to E_c$$

given by $\bar{\mu}(e) = \pi_1(\theta(e^{\otimes 3}))$, where $\pi_1 : E_c \oplus (E'_c \otimes N) \to E_c$ is the projection. Now choose a complementary bundle F for E and an isomorphism $E \oplus F \cong X \times \mathbb{R}^M$. Let

$$\mathrm{Map}_{Z/2}((\mathbb{C}^n)^+)$$

be the space of $Z/2$-self-maps of $(\mathbb{C}^n)^+$ which send infinity to itself. Here $(\mathbb{C}^n)^+$ is the one-point compactification of $\mathbb{C}^n$ on which $Z/2$ acts by conjugation. Set

$$V(Z/2) = \varinjlim_n \mathrm{Map}_{Z/2}((\mathbb{C}^n)^+).$$

The join product makes $V(Z/2)$ into an H-space. As in ((20), §5) if we form $\bar{\mu} \oplus 1_F$ (1_F = identity on F) and stabilize we obtain a map $\mu' : X \to V(Z/2)$. Restriction to the fixed point set is an H-map

$$\lambda : V(Z/2) \longrightarrow QS^0 = \varinjlim_n \Omega^n S^n.$$

Set $\mu(x) = \lambda \circ \mu'$. This gives a well-defined homomorphism, μ, as required. For example, suppose that we were to choose a second complementary bundle F', so there

https://doi.org/10.1017/S0305004100054049 Published online by Cambridge University Press

is an isomorphism $E \oplus F \oplus E \oplus F' \cong X \times (\mathbb{R}^M \oplus \mathbb{R}^N)$. The two definitions of μ', stabilized to be considered as maps

$$X \to \mathrm{Map}_{Z/2}((\mathbb{C}^M \oplus \mathbb{C}^N)^+),$$

differ by composition with the map which interchanges the two copies of E in the bundle $E \oplus F \oplus E \oplus F'$. However, if E has an A_2-structure, the antipodal map in each fibre is homotopic as a bundle map to the identity map ((9), p. 362) and $\mu(x)$ is independent of the choice of complementary bundle. Showing that $\mu(x)$ is independent of other choices and that $\mu \circ w' = J$ is entirely analogous to the complex case (20).

Finally if $J \circ \eta = 0$ then every stable vector bundle over X admits an A_2-structure ((9), theorem 1·9).

REFERENCES

(1) Adams, J. F. On the groups $J(X)$. I and IV. *Topology* **2** (1963), 181–195; *Topology* **5** (1966), 21–80.

(2) Becker, J. C. Characteristic classes and K-theory; *Lecture Notes in Mathematics*, no. 428 (1975), 132–143.

(3) Becker, J. C. and Gottlieb, D. H. The transfer map and fibre bundles. *Topology* **14** (1975), 1–12.

(4) Becker, J. C. and Gottlieb, D. H. Transfer maps for fibrations and duality. *Compositio Math.* (in the Press).

(5) Brumfiel, G. and Madsen, I. Evaluation of the transfer and the universal surgery classes. *Inventiones Math.* **32** (1976), 133–169.

(6) Clough, R. R. The map $SJ \to SF$ does not deloop mod 2. *Canad. J. Math.* **17** (1975), 737–745.

(7) Clough, R. R. and Stasheff, J. D. BSJ does not map correctly into BSF mod 2. *Manuscripta Math.* **7** (1972), 205–214.

(8) Husemoller, D. *Fibre bundles* (McGraw-Hill, 1966).

(9) James, I. M. Bundles with special structure: I. *Ann. of Math.* **89** (1969), 359–390.

(10) Madsen, I. Private communication.

(11) Madsen, I. Higher torsion in SG and BSG. *Math. Z.* **143** (1975), 55–80.

(12) Madsen, I. On the action of the Dyer–Lashof algebra on $H_*(G)$. *Pacific J. Math.* **60** (1975), 235–275.

(13) Madsen, I., Snaith, V. and Tornehave, J. Infinite loop maps in geometric topology. (Aarhus Universitet Preprint Series 31 (1974/5)) *Math. Proc. Cambridge Philos Soc.* **81** (1977), 399–430.

(14) May, J. P. (with contributions by F. Quinn and N. Ray): *E_∞ ring spaces and E_∞ ring spectra* (to appear).

(15) May, J. P. The geometry of iterated loop spaces. *Lecture Notes in Mathematics*, no. 271 (1972).

(16) Quillen, D. G. The Adams conjecture. *Topology* **10** (1971), 67–80.

(17) Seymour, R. M. Vector bundles invariant under the Adams operations. *Quart. J. Math. Oxford Ser.* (2) **25** (1975), 395–414.

(18) Seymour, R. M. The infinite loop Adams conjecture. *University College preprint* (1976).

(19) Snaith, V. P. Dyer–Lashof operations in K-theory, *Lecture Notes in Mathematics*, no. 496 (1976), 104–294.

(20) Snaith, V. P. The complex J-homomorphism. I. *Proc. London Math. Soc.* **34** (1977), 269-302.

(21) Snaith, V. P. Algebraic cobordism and K-theory. (Submitted to *Mem. Amer. Math. Soc.*)

(22) Sullivan, D. Genetics of homotopy theory and the Adams conjecture. *Ann. of Math.* **100** (1974), 1–79.

(23) Toda, H. Composition methods in homotopy groups of spheres. *Ann. of Math.*, study no. **49** (1962).

(24) Tornehave, J. Deviation from additivity of a solution of the Adams conjecture (lecture given at Am. Math. Soc. Summer Inst., Stanford, 1976).

https://doi.org/10.1017/S0305004100054049 Published online by Cambridge University Press