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Global solution to a three-dimensional spherical piston problem for the relativistic Euler equations

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The study of spherically symmetric motion is important for the theory of explosion waves. In this paper, we consider a 'spherical piston' problem for the relativistic Euler equations, which describes the wave motion produced by a sphere expanding into an infinite surrounding medium. We use the reflected characteristics method to construct a global piecewise smooth solution with a single shock of this spherical piston problem, provided that the speed of the sphere is a small perturbation of a constant speed.

Key words: Relativistic Euler equations, spherical piston problem, shock wave, characteristic

2020 Mathematics Subject Classification: Primary: 35L65. Secondary: 35L60, 35L67

1 Introduction

In a frame of reference where the fluid moves with 3-velocity $\mathbf{u} = (u_1, u_2, u_3)$, the 4-velocity of the fluid is given by $\bar{\mathbf{u}} = (u^0, u^1, u^2, u^3) = \chi(1, u_1, u_2, u_3)$, where the speed of light is 1, $\chi = \frac{1}{\sqrt{1-u^2}}$ is the Lorentz factor and $u = \sqrt{u_1^2 + u_2^2 + u_3^2}$ is required to be less than 1.

The thermodynamic quantities n, p, e, T, S, $\rho = n + e$ and $i = \rho + p$ denote the average rest-mass density, the pressure, the internal energy per unit volume, the absolute temperature, the entropy per particle, the total mass-energy per unit volume and the enthalpy per unit volume, respectively.

The motion of a relativistic fluid in the Minkowski space-time is governed by the conservation laws for energy and momentum:

$$(T^{\alpha\beta})_{x_\beta} = 0, \quad \alpha, \beta = 0, 1, 2, 3, \tag{1.1}$$

and the conservation law for the number of particles:

$$(nu^\alpha)_{x_\alpha} = 0, \tag{1.2}$$

where repeated indices imply summation,

$$T^{\alpha\beta} = (p + \rho)u^\alpha u^\beta - pg^{\alpha\beta} \tag{1.3}$$

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denotes the stress-energy tensor, and

$$g=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}; \tag{1.4}$$

see Taub [24].

There are many results about Cauchy problems for the one-dimensional relativistic Euler equations. Smoller and Temple [22] considered the system of conservation laws in energy and momentum in special relativity:

$$\begin{cases}\left(\dfrac{iu^2}{1-u^2}+\rho\right)_t+\left(\dfrac{iu}{1-u^2}\right)_x=0,\\ \left(\dfrac{iu}{1-u^2}\right)_t+\left(\dfrac{iu^2}{1-u^2}+p\right)_x=0\end{cases} \tag{1.5}$$

with the equation of state $p=\sigma^2\rho$, where σ is a positive constant smaller than 1. They solved the Riemann problem and the Cauchy problem for the system. Martí and Müller [20] also studied the Riemann problem for (1.5) with the equation of state $p=\sigma^2\rho$. Chen [4], Hsu et al. [13] and Li et al. [2, 17] extended the result of Smoller and Temple [22] to more general equations of state $p=p(\rho)$. For more results about one-dimensional relativistic Euler equations, the readers can see [3, 5, 6, 7, 18, 26].

We are concerned with spherically symmetric motion. The conservation laws in energy and momentum in special relativity with spherical symmetry have the form:

$$\begin{cases}\left(\dfrac{iu^2}{1-u^2}+\rho\right)_t+\left(\dfrac{iu}{1-u^2}\right)_r+\dfrac{2iu}{(1-u^2)r}=0,\\ \left(\dfrac{iu}{1-u^2}\right)_t+\left(\dfrac{iu^2}{1-u^2}+p\right)_r+\dfrac{2iu^2}{(1-u^2)r}=0,\end{cases} \tag{1.6}$$

where $t>0$ an $r>0$. We take the equation of state $p=\sigma^2\rho$.

We consider the 'spherical piston' problem which describes the wave motion produced by a sphere expanding into a quiet gas; see Courant and Friedrichs [8]. The spherical piston problem for the compressible Euler equations was first worked out by Taylor [25]. He constructed a self-similar solution with a single shock of the problem. Chen et al. [1] constructed a global weak solution of the spherical piston problem for the compressible Euler system. The spherical piston problem for the relativistic Euler system was first studied by Ding and Li [11, 12]. They studied the local existence of shock wave solutions of the spherical piston problem for the relativistic Euler system (1.6). In a recent study, the author [15] studied self-similar solutions of the radially symmetric relativistic Euler equations (1.6).

In this paper, we consider system (1.6) with the initial and boundary conditions:

$$(u,\rho)(r,0)=(0,\rho_0),\quad u(r,\alpha(r))=\frac{1}{\alpha'(r)},\quad r>0. \tag{1.7}$$

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We assume that $\alpha(r) \in C^2[0, +\infty)$ satisfies the following assumptions:

(A1): $\alpha(r) = \frac{r}{u_p}$ as $r \in [0, r_0]$, where $u_p \in (0, 1)$;

(A2): $|\alpha'(r) - \frac{1}{u_p}| < \frac{\varepsilon}{r^\delta}$, and $|\alpha''(r)| < \frac{\varepsilon}{r^{1+\delta}}$ as $r \in [r_0, +\infty)$.

We also assume $\delta \in \left(0, \frac{3}{17}\right)$,

$$\frac{1-4\sigma}{(1-\frac{2}{3}\sigma)^\delta} + \frac{2}{\delta}\left[\frac{1}{(1-\frac{2}{3}\sigma)^\delta} - 1\right] + \frac{17\sigma}{12(1-\frac{1}{3}\sigma)^\delta} < 1-\sigma, \tag{1.8}$$

and

$$\frac{1-\frac{10}{3}\sigma}{(1-\frac{2\sigma}{3})^{1+\delta}} + 2\left(\frac{1}{(1-\frac{2\sigma}{3})^{1+\delta}} - 1\right) < 1-\sigma. \tag{1.9}$$

Remark 1.1 *By a direct computation, we know that there exists a $\sigma_1 > 0$, such that when $0 < \sigma < \sigma_1$ there hold*

$$\lim_{\delta\to 0}\left\{\frac{1-4\sigma}{(1-\frac{2}{3}\sigma)^\delta} + \frac{2}{\delta}\left[\frac{1}{(1-\frac{2}{3}\sigma)^\delta} - 1\right] + \frac{17\sigma}{12(1-\frac{1}{3}\sigma)^\delta}\right\}$$
$$= 1 - 4\sigma - 2\ln\left(1-\frac{2\sigma}{3}\right) + \frac{17\sigma}{12} < 1-\sigma$$

and

$$\lim_{\delta\to 0}\left\{\frac{1-\frac{10}{3}\sigma}{(1-\frac{2\sigma}{3})^{1+\delta}} + 2\left(\frac{1}{(1-\frac{2\sigma}{3})^{1+\delta}} - 1\right)\right\} = \frac{1-\frac{6\sigma}{3}}{1-\frac{2\sigma}{3}} < 1-\sigma.$$

Therefore, when $0 < \sigma < \sigma_1$ there exists a sufficiently small $\delta > 0$ such that (1.8) and (1.9) can be satisfied.

Remark 1.2 *Assumptions (1.8) and (1.9) are actually 'dissipative' conditions for the problem.*

We use the reflected characteristics method to solve the (1.6), (1.7). Such a method has been used in [10, 9, 19, 27] to study the stability of shock waves of the compressible Euler equations. The main result of the paper can be stated as the following theorem:

Theorem 1.1 (Main theorem) *There exists a $\sigma_2 \in (0, \sigma_1)$ which depends only on u_p, such that for any $\sigma \in (0, \sigma_2)$, when ε is sufficiently small, the problem (1.6), (1.7) admit a global piecewise smooth entropy solution with a single shock.*

The rest of the paper is organised as follows.

In Section 2, we consider the system (1.6) with the initial and boundary conditions:

$$(u, \rho)(r, 0) = (0, \rho_0), \quad u\left(r, \frac{r}{u_p}\right) = u_p, \quad r > 0. \tag{1.10}$$

We construct a self-similar solution $(u, \rho) = (\hat{u}, \hat{\rho})(\frac{r}{t})$ to the problem (1.6), (1.10) with a single shock located at some point $\frac{r}{t} = \frac{1}{s_0}$; see Figure 1(left). We also establish some useful estimates for the self-similar solution. These estimates will be used to establish uniform a priori estimates for the perturbed solution.

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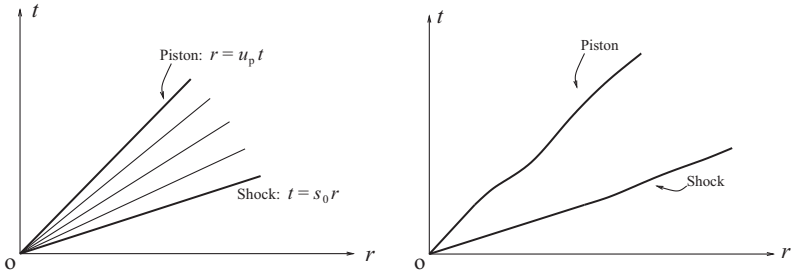


FIGURE 1. Left: self-similar solution; right: perturbed solution.

We aim to construct a global piecewise smooth solution containing a single shock to the problem (1.6), (1.7); see Figure 1(right). To this aim, we reduce the problem (1.6), (1.7) to a free boundary problem for (1.6) in Section 3.1. It is obvious that $(\hat{u}, \hat{\rho})(\frac{r}{t})$ $(u_p < \frac{r}{t} < \frac{1}{s_0})$ is the solution to the free boundary problem for $\alpha(r) = \frac{r}{u_p}$ $(r > 0)$. Then the problem can be seen as a small perturbation problem for the self-similar solution with a single shock. In Section 3.2, we introduce the Riemann invariants:

$$R_1 = \frac{1}{2}\ln\frac{1+u}{1-u} - \frac{\sigma}{1+\sigma^2}\ln\rho, \quad R_2 = \frac{1}{2}\ln\frac{1+u}{1-u} + \frac{\sigma}{1+\sigma^2}\ln\rho.$$

We give a reformulation of the free boundary problem (see (3.10), (3.21)–(3.22) and (3.26)) using the perturbations of the self-similar solution:

$$W_1(r,t) = R_1(r,t) - \hat{R}_1\left(\frac{r}{t}\right) \quad \text{and} \quad W_2(r,t) = R_2(r,t) - \hat{R}_2\left(\frac{r}{t}\right),$$

where

$$\hat{R}_1 = \frac{1}{2}\ln\frac{1+\hat{u}}{1-\hat{u}} - \frac{\sigma}{1+\sigma^2}\ln\hat{\rho} \quad \text{and} \quad \hat{R}_2 = \frac{1}{2}\ln\frac{1+\hat{u}}{1-\hat{u}} + \frac{\sigma}{1+\sigma^2}\ln\hat{\rho}.$$

In Sections 3.3 and 3.4, the free boundary condition (3.21) and the slip boundary condition (3.26) in terms of the variables W_1 and W_2 are formed by the sum of a linear part plus a small perturbation. We also derive some useful estimates on the coefficients of the reformulated nonlinear system and its boundary conditions in this section.

In order to establish the a priori estimates for the derivatives of W_1 and W_2, we derive characteristic equations and boundary conditions for the derivatives of the perturbations in Section 4. In Section 4.1, we derive the characteristic equations for $\widetilde{W}_1$ and $\widetilde{W}_2$, where

$$\widetilde{W}_i = \partial_r W_i + \frac{\partial_t W_i}{u_p}, \quad i = 1, 2;$$

see (4.3). In Sections 4.2 and 4.3, the boundary conditions for the derivatives $\widetilde{W}_1$ and $\widetilde{W}_2$ are derived and reformulated by the sum of a linear part plus a small perturbation; see (4.15) and (4.17).

In Section 5, we use the reflected characteristics method to establish the required uniform a priori C^1 norm estimates for the classical solution $(W_1, W_2)(r,t)$ for the reformulated free boundary problem; see Lemma 5.1. Using the uniform a priori C^1 norm estimates, one can extend the local classical solution to a global classical solution; see Theorem 5.1.

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There are also some other related works about the spherically symmetric relativistic Euler system. Hsu et al. [14] and Mizohata [21] constructed spherically symmetric weak solutions of the relativistic Euler system in the exterior of a hard sphere.

2 Self-similar solution of the problem (1.6), (1.10) and some useful estimates

2.1 Ordinary differential equations

By self-similar transformation, system (1.6) can be written as:

$$\begin{cases} -\xi\dfrac{\mathrm{d}}{\mathrm{d}\xi}\Big(\dfrac{i}{1-u^2}\Big)+\xi\dfrac{\mathrm{d}p}{\mathrm{d}\xi}+\dfrac{\mathrm{d}}{\mathrm{d}\xi}\Big(\dfrac{iu}{1-u^2}\Big)+\dfrac{2iu}{(1-u^2)\xi}=0,\\ -\xi\dfrac{\mathrm{d}}{\mathrm{d}\xi}\Big(\dfrac{iu}{1-u^2}\Big)+\dfrac{\mathrm{d}}{\mathrm{d}\xi}\Big(\dfrac{iu^2}{1-u^2}\Big)+\dfrac{\mathrm{d}p}{\mathrm{d}\xi}+\dfrac{2iu^2}{(1-u^2)\xi}=0, \end{cases}$$

where $\xi>0$. Hence, we get

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}\xi}=-\dfrac{2iu(u-\xi)}{\xi\Big\{\Big[\xi(1-u^2)+(1+\frac{1}{\sigma^2})(u-\xi)\Big](u-\xi)+(1+u^2-2u\xi)(u\xi-1)\Big\}},\\ \dfrac{\mathrm{d}u}{\mathrm{d}\xi}=-\dfrac{2(u\xi-1)u(1-u^2)}{\xi\Big\{\Big[\xi(1-u^2)+(1+\frac{1}{\sigma^2})(u-\xi)\Big](u-\xi)+(1+u^2-2u\xi)(u\xi-1)\Big\}}. \end{cases} \tag{2.1}$$

Let $s=1/\xi$. Then, we have

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}s}=\dfrac{2(u-s)u(1-u^2)}{f(u,s)},\\ \dfrac{\mathrm{d}p}{\mathrm{d}s}=\dfrac{2(1+\frac{1}{\sigma^2})up(us-1)}{f(u,s)}, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} f(u,s) &= \Big[(1-u^2)+(1+\frac{1}{\sigma^2})(us-1)\Big](us-1)+\Big[(1+u^2)s-2u\Big](u-s)\\ &= \frac{(us-1)^2}{\sigma^2}-(u-s)^2. \end{aligned}$$

2.2 Rankine–Hugoniot conditions for shock waves

Assume that there is a shock wave with the speed $\xi\in(0,1)$. We make the Lorentz transformation:

$$\breve{r}=\frac{r-\xi t}{\sqrt{1-\xi^2}},\quad \breve{t}=\frac{t-\xi r}{\sqrt{1-\xi^2}}. \tag{2.3}$$

Then in the coordinates $(\breve{r},\breve{t})$, the shock is stationary. Thus, the Rankine–Hugoniot conditions for the shock are

$$\begin{cases} \dfrac{\rho_1\breve{u}_1}{1-\breve{u}_1^2}=\dfrac{\rho_2\breve{u}_2}{1-\breve{u}_2^2},\\ (1+\sigma^2)\dfrac{\rho_1\breve{u}_1^2}{1-\breve{u}_1^2}+p_1=(1+\sigma^2)\dfrac{\rho_2\breve{u}_2^2}{1-\breve{u}_2^2}+p_2; \end{cases} \tag{2.4}$$

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see [16, 23]. Here, the subscript '1' denotes the fluid in front of the shock, the subscript '2' denotes the fluid behind the shock and

$$\breve{u}_k = \frac{u_k - \xi}{1 - \xi u_k} \quad (k = 1, 2) \tag{2.5}$$

denotes the velocity of the fluid relatively to the shock.

In addition to the jump conditions (2.4), the admissible forward shock waves must satisfy the entropy condition:

$$\breve{u}_1 < \breve{u}_2 < 0. \tag{2.6}$$

From (2.4), we immediately have

$$\breve{u}_1\breve{u}_2(\breve{u}_1 - \breve{u}_2) = \sigma^2(\breve{u}_1 - \breve{u}_2).$$

Hence, for shock waves, we have

$$\breve{u}_1\breve{u}_2 = \sigma^2; \tag{2.7}$$

see for example [23].

2.3 Self-similar solution of the problem (1.6), (1.10)

In what follows we are going to show that for any $u_p \in (0, 1)$, the problem (1.6), (1.10) admits a self-similar solution with a single shock.

Assume the speed of the single shock is $\xi = 1/s$. Then by (2.5)–(2.7), we have

$$-1 < \breve{u}_1 = -\frac{1}{s} < -\sigma,$$

and hence

$$1 < s < \frac{1}{\sigma}.$$

Using (2.4), (2.5) and (2.7), we know that if the front side state of the shock is $(0, \rho_0)$, then the backside state of the shock is

$$\begin{cases} \rho = \bar{\rho}(s) := \dfrac{\rho_0(1 - s^2\sigma^4)}{(s^2 - 1)\sigma^2}, \\ u = \bar{u}(s) := \dfrac{1 - s^2\sigma^2}{(1 - \sigma^2)s}. \end{cases} \tag{2.8}$$

Eliminating s from (2.8), we get the shock wave curve through $(0, \rho_0)$:

$$H(u, \rho) = \frac{u(1 - \sigma^2)}{\sigma}\sqrt{\frac{\rho_0 + \rho\sigma^2}{\rho + \rho_0\sigma^2}} + \frac{\rho_0 + \rho\sigma^2}{\rho + \rho_0\sigma^2} - 1 = 0, \quad \rho > \rho_0. \tag{2.9}$$

Thus, in order to construct a shock wave solution to the problem (1.6), (1.10), we consider the problem:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}s} = \dfrac{2(u - s)u(1 - u^2)}{f(u, s)}, \quad s < \dfrac{1}{u_p}, \\ u\left(\frac{1}{u_p}\right) = u_p. \end{cases} \tag{2.10}$$

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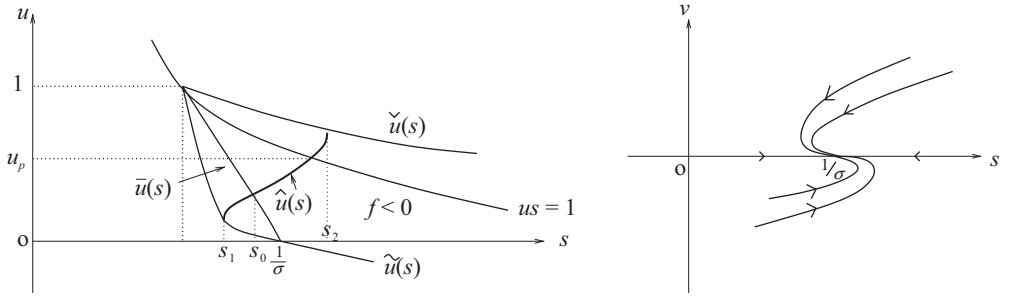


FIGURE 2. Left: the integral curve of the problem (2.10); right: integral curves of $\frac{\mathrm{d}v}{\mathrm{d}s}=\frac{2(v-s)v(1-v^2)}{f(v,s)}$ near the point $(s,v)=(\frac{1}{\sigma},0)$.

The problem (2.10) is classically well defined which has an unique local solution $u=\hat{u}(s)$.

Lemma 2.1 *For any $u_p\in(0,1)$, there exists a $s_0\in(1,\frac{1}{\sigma})$ such that the solution $u=\hat{u}(s)$ of the problem (2.10) satisfies $\hat{u}(s_0)=\bar{u}(s_0)$. Moreover, we have*

$$\lim_{\sigma\to 0} s_0=\frac{1}{u_p}\quad \textit{and}\quad \lim_{\sigma\to 0}\left\|\frac{\mathrm{d}\hat{u}}{\mathrm{d}s}-2u_p^2\right\|_{C^0[s_0,\frac{1}{u_p}]}=0. \tag{2.11}$$

Proof Let

$$\tilde{u}(s)=\frac{1-\sigma s}{s-\sigma}\quad\text{and}\quad \check{u}(s)=\frac{1+\sigma s}{s+\sigma}. \tag{2.12}$$

Then by computation, we have

(C1): $\tilde{u}(s)<\frac{1}{s}<\check{u}(s)$ as $s>1$;
(C2): $\tilde{u}(s)<\bar{u}(s)<\frac{1}{s}$ as $1<s<\frac{1}{\sigma}$;
(C3): $f(u,s)<0$ as $(u,s)\in\{(u,s)\mid s>1,\tilde{u}(s)<u<\check{u}(s)\}$;

see Figure 2(left).

We first prove that there exists a $s_0\in(1,\frac{1}{\sigma})$ such that the solution $u=\hat{u}(s)$ of the problem (2.10) satisfies $\hat{u}(s_0)=\bar{u}(s_0)$. We divide the discussions into two cases: (1) $u_p\geq\sigma$ and (2) $u_p<\sigma$.

If $u_p\geq\sigma$, then by (C1)–(C3), we can see that there exists a $s_0\in(1,\frac{1}{\sigma})$ such that $\hat{u}(s_0)=\bar{u}(s_0)$.

We now consider the case of $u_p<\sigma$. We first show that $\hat{u}(s)>0$ as $\frac{1}{\sigma}<s<\frac{1}{u_p}$. If there exists a $s_*\in(\frac{1}{\sigma},\frac{1}{u_p})$ such that $\hat{u}(s)>0$ as $s\in(s_*,\frac{1}{u_p})$ and $\hat{u}(s_*)=0$, then we have

$$\int_{u_p}^{0}\frac{1}{\hat{u}}\,\mathrm{d}\hat{u}=\int_{\frac{1}{u_p}}^{s_*}\frac{2(\hat{u}(s)-s)(1-\hat{u}^2(s))}{f(\hat{u}(s),s)}\,\mathrm{d}s, \tag{2.13}$$

which leads to a contradiction, since the right part of (2.13) is finite. Thus, we have $\hat{u}(s)>0$ as $\frac{1}{\sigma}<s<\frac{1}{u_p}$.

Next, we shall show $\hat{u}(\frac{1}{\sigma})>0$. Let us consider the system of ordinary differential equations:

$$\begin{cases}\dfrac{\mathrm{d}v}{\mathrm{d}t}=2(v-s)v(1-v^2),\\[2mm] \dfrac{\mathrm{d}s}{\mathrm{d}t}=\dfrac{1}{\sigma^2}(vs-1)^2-(v-s)^2.\end{cases} \tag{2.14}$$

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At the point $(v, s) = (0, \frac{1}{\sigma})$, we find that the linear part of the right-hand side of (2.14) is given by $M(v, s - \frac{1}{\sigma})^T$ where

$$M = \begin{pmatrix} -\frac{2}{\sigma} & 0 \\ \frac{2}{\sigma} - \frac{2}{\sigma^3} & -\frac{2}{\sigma} \end{pmatrix}.$$

The matrix has eigenvalues $\lambda_1 = \lambda_2 = -\frac{2}{\sigma}$. Since $\frac{2}{\sigma} - \frac{2}{\sigma^3} < 0$, along the integral curves of (2.14), we have $\frac{ds}{dv} \to -\infty$ as $(v, s) \to (0, \frac{1}{\sigma})$ and $v \neq 0$; see Figure 2 (right). Thus, we have $\hat{u}(\frac{1}{\sigma}) > 0$.

Therefore, there exists a $s_0 \in (1, \frac{1}{\sigma})$ such that $\hat{u}(s_0) = \bar{u}(s_0)$.

From $\lim_{\sigma \to 0} \|\bar{u}(s) - \frac{1}{s}\|_{C^0[1, \frac{1}{u_p}]} = 0$ and $\hat{u}'(s) > 0$ as $s_0 < s < \frac{1}{u_p}$, we have $\lim_{\sigma \to 0} s_0 = \frac{1}{u_p}$.

When $\sigma < u_p$, we have

$$\frac{(\hat{u}(s)s - 1)^2}{\sigma^2} < \frac{1}{\sigma^2}\left(\frac{1 - s_0^2\sigma^2}{1 - \sigma^2} - 1\right)^2 = \frac{(1 - s_0^2)^2\sigma^2}{(1 - \sigma^2)^2} < \frac{(1 - \frac{1}{u_p^2})^2\sigma^2}{(1 - \sigma^2)^2}$$

as $s_0 < s < \frac{1}{u_p}$, since $0 < \bar{u}(s_0)s_0 = \hat{u}(s_0)s_0 < \hat{u}(s)s < 1$ as $1 < s_0 < s < \frac{1}{u_p}$. Hence, by a direct computation, we can get $\lim_{\sigma \to 0} \left\| \frac{d\hat{u}}{ds} - 2u_p^2 \right\|_{C^0[s_0, \frac{1}{u_p}]} = 0$.

We then complete the proof of Lemma 2.1. □

Let $\rho = \hat{\rho}(s)$ $(s_0 < s < \frac{1}{u_p})$ be the solution of the problem:

$$\begin{cases} \dfrac{d\rho}{ds} = \dfrac{2(1 + \frac{1}{\sigma^2})\rho\hat{u}(\hat{u}s - 1)}{f(\hat{u}, s)}, \quad s_0 < s < \dfrac{1}{u_p}, \\ \rho(s_0) = \bar{\rho}(s_0). \end{cases} \tag{2.15}$$

Therefore, the self-similar solution of the problem (1.6), (1.10) has the form:

$$(u, \rho)(t, r) = \begin{cases} (\hat{u}, \hat{\rho})(\frac{t}{r}), \ s_0 < \frac{t}{r} < \frac{1}{u_p}; \\ (0, \rho_0), \quad 0 < \frac{t}{r} < s_0. \end{cases}$$

Remark 2.1 *Actually, there exist s_1 and s_2 with $s_1 < s_0 < \frac{1}{u_p} < s_2$, such that the problem (2.10) admits a solution in (s_1, s_2), and $f(\hat{u}(s_1), s_1) = f(\hat{u}(s_2), s_2) = 0$; see Figure 2(left).*

2.4 Some useful estimates about the self-similar solution $(\hat{u}, \hat{p})(s)$

System (1.6) has two distinct eigenvalues:

$$\lambda_1 = \frac{u - \sigma}{1 - u\sigma} \quad \text{and} \quad \lambda_2 = \frac{u + \sigma}{1 + u\sigma}. \tag{2.16}$$

So, we can define the C_i $(i = 1, 2)$ characteristic curves as the integral curves $\frac{dr}{dt} = \lambda_i(u(r, t))$.

Lemma 2.2 *Assume the C_2 characteristic curve passing through $(r_1, s_0 r_1)$ intersects with $r = u_p t$ at some point $(r_2, r_2/u_p)$; see Figure 3(left). Then, for any given small $\epsilon > 0$, when σ is sufficiently small we have*

$$1 \geq \frac{r_2}{r_1} \geq 1 - \frac{\sigma}{(3 - 2\epsilon)u_p}, \quad \forall r_1 > 0.$$

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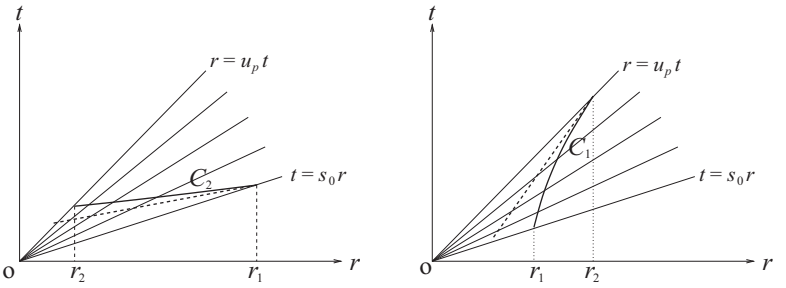


FIGURE 3. C_1 and C_2 characteristic curves.

Proof Let

$$u_2 := \bar{u}(s_0) = \frac{1 - s_0^2\sigma^2}{(1-\sigma^2)s_0}. \tag{2.17}$$

By the entropy condition, we have

$$\lambda_2(u_2) > \frac{1}{s_0}.$$

By a direct computation, we know that the straight line $r = r_1 + \lambda_2(u_2)(t - s_0 r_1)$ intersects with the line $r = u_p t$ at the point:

$$\left(u_p r_1 \frac{\lambda_2(u_2)s_0 - 1}{\lambda_2(u_2) - u_p},\ r_1 \frac{\lambda_2(u_2)s_0 - 1}{\lambda_2(u_2) - u_p} \right).$$

Since $\hat{u}(s) \geq u_2$ as $s_0 < s < \frac{1}{u_p}$, we have

$$\lambda_2(\hat{u}(s)) \geq \lambda_2(u_2) \quad \text{as } s_0 < s < \frac{1}{u_p}. \tag{2.18}$$

Consequently, we have

$$\frac{r_2}{r_1} \geq u_p s_0 \left(\frac{\lambda_2(u_2) - \frac{1}{s_0}}{\lambda_2(u_2) - u_p} \right).$$

By (2.11), we know that for any small $\epsilon > 0$, if σ is sufficiently small then,

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}s} > (2-\epsilon)u_p^2 \quad \text{as} \quad s_0 < s < \frac{1}{u_p}. \tag{2.19}$$

Let $s_\epsilon(\sigma)$ be determined by

$$\begin{cases} (u - u_p) = (2-\epsilon)u_p^2 \left(s_\epsilon - \frac{1}{u_p} \right), \\ u = \dfrac{1 - s_\epsilon^2\sigma^2}{s_\epsilon(1-\sigma^2)}. \end{cases} \tag{2.20}$$

Thus, we have

$$1 - s_\epsilon^2\sigma^2 = \left(u_p(\epsilon - 1) + (2-\epsilon)u_p^2 s_\epsilon \right) s_\epsilon (1 - \sigma^2).$$

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By computation, we also have

$$s_\epsilon(0)=\frac{1}{u_p},\quad s'_\epsilon(0)=0,\quad \text{and}\quad s''_\epsilon(0)=\frac{2}{3-\epsilon}\frac{u_p^2-1}{u_p^3}.$$

Hence, by Taylor's formula, we have

$$\frac{1}{s_\epsilon}=u_p+\frac{1-u_p^2}{(3-\epsilon)u_p}\sigma^2+o(\sigma^2). \tag{2.21}$$

From (2.19) and (2.20), we have $s_\epsilon < s_0$. Hence, we have

$$u_p s_0\left(\frac{\lambda_2(u_2)-\frac{1}{s_0}}{\lambda_2(u_2)-u_p}\right) > u_p s_\epsilon\left(\frac{\lambda_2(u_2)-\frac{1}{s_\epsilon}}{\lambda_2(u_2)-u_p}\right). \tag{2.22}$$

Since $s_0 < \frac{1}{u_p}$, we have

$$u_2 > \bar{u}\left(\frac{1}{u_p}\right) = \frac{u_p^2-\sigma^2}{u_p(1-\sigma^2)} = u_p+\left(u_p-\frac{1}{u_p}\right)\sigma^2+o(\sigma^2), \tag{2.23}$$

and hence

$$\lambda_2(u_2)>\lambda_2\Big(\frac{u_p^2-\sigma^2}{u_p(1-\sigma^2)}\Big)=\frac{u_p+\sigma+\gamma\sigma^2+o(\sigma^2)}{1+(u_p+\gamma\sigma^2+o(\sigma^2))\sigma},$$

where $\gamma := u_p-\frac{1}{u_p}$.

Moreover, by $\frac{1}{s_\epsilon} > u_p$, we have

$$\frac{\lambda_2(u_2)-\frac{1}{s_\epsilon}}{\lambda_2(u_2)-u_p} > \frac{\lambda_2\Big(\frac{u_p^2-\sigma^2}{u_p(1-\sigma^2)}\Big)-\frac{1}{s_\epsilon}}{\lambda_2\Big(\frac{u_p^2-\sigma^2}{u_p(1-\sigma^2)}\Big)-u_p}. \tag{2.24}$$

Combining with (2.21), (2.22), (2.23) and (2.24), we get

$$\begin{aligned}
&u_p s_0\left(\frac{\lambda_2(u_2)-\frac{1}{s_0}}{\lambda_2(u_2)-u_p}\right) > u_p s_\epsilon\left(\frac{\lambda_2(u_2)-\frac{1}{s_\epsilon}}{\lambda_2(u_2)-u_p}\right) > u_p s_\epsilon\left(\frac{\lambda_2\Big(\frac{u_p^2-\sigma^2}{u_p(1-\sigma^2)}\Big)-\frac{1}{s_\epsilon}}{\lambda_2\Big(\frac{u_p^2-\sigma^2}{u_p(1-\sigma^2)}\Big)-u_p}\right)\\
&=u_p\cdot\left(\frac{1}{u_p}+\frac{1}{3-\epsilon}\frac{u_p^2-1}{u_p^3}\sigma^2+o(\sigma^2)\right)\\
&\quad\cdot\frac{\big(u_p+\sigma+\gamma\sigma^2+o(\sigma^2)\big)-\left(u_p+\frac{1-u_p^2}{(3-\epsilon)u_p}\sigma^2+o(\sigma^2)\right)\big(1+(u_p+\gamma\sigma^2+o(\sigma^2))\sigma\big)}{\big(u_p+\sigma+\gamma\sigma^2+o(\sigma^2)\big)-u_p\big(1+(u_p+\gamma\sigma^2+o(\sigma^2))\sigma\big)}\\
&=\big(1+o(\sigma)\big)\cdot\frac{(1-u_p^2)\sigma+\left(\gamma-\frac{1-u_p^2}{(3-\epsilon)u_p}\right)\sigma^2+o(\sigma^2)}{(1-u_p^2)\sigma+\gamma\sigma^2+o(\sigma^2)}\\
&=1-\frac{1}{(3-\epsilon)u_p}\sigma+o(\sigma).
\end{aligned}$$

We then complete the proof of Lemma 2.2. □

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Lemma 2.3 *Assume that the C_1 characteristic curve passing through $(r_2, r_2/u_p)$ intersects with $t = s_0 r$ at some point $(r_1, s_0 r_1)$. Then, for any given small $\epsilon > 0$, when σ is sufficiently small, we have*

$$1 \geq \frac{r_1}{r_2} \geq 1 - \frac{\sigma}{(3-2\epsilon)u_p}, \quad \forall r_2 > 0.$$

Proof By a direct computation, we know that the straight line $r = r_2 + \lambda_1(u_p)(t - \frac{r_2}{u_p})$ intersects with the line $t = s_0 r$ at the point:

$$\left(\frac{r_2(1 - \frac{\lambda_1(u_p)}{u_p})}{1 - \lambda_1(u_p)s_0}, \quad \frac{s_0 r_2(1 - \frac{\lambda_1(u_p)}{u_p})}{1 - \lambda_1(u_p)s_0} \right).$$

Since $\hat{u}(s) \leq u_p$ as $s_0 < s < \frac{1}{u_p}$, we have

$$\lambda_1(\hat{u}) \leq \lambda_1(u_p) = \frac{u_p - \sigma}{1 - u_p\sigma} \quad \text{as} \quad s_0 < s < \frac{1}{u_p}. \tag{2.25}$$

Consequently, by $\lambda_1(u_p) < u_p < \frac{1}{s_0}$ and $s_\epsilon < s_0$, we have

$$\begin{aligned}
\frac{r_1}{r_2} &\geq \frac{1 - \frac{1}{u_p}\frac{u_p - \sigma}{1 - u_p\sigma}}{1 - s_0\frac{u_p - \sigma}{1 - u_p\sigma}} \geq \frac{1 - \frac{1}{u_p}\frac{u_p - \sigma}{1 - u_p\sigma}}{1 - s_\epsilon\frac{u_p - \sigma}{1 - u_p\sigma}} = \frac{1}{u_p s_\epsilon} \cdot \frac{u_p - \frac{u_p - \sigma}{1 - u_p\sigma}}{\frac{1}{s_\epsilon} - \frac{u_p - \sigma}{1 - u_p\sigma}} \\
&= \frac{1}{u_p s_\epsilon} \cdot \frac{(1 - u_p^2)\sigma}{(1 - u_p^2)\sigma + \frac{1 - u_p^2}{(3-\epsilon)u_p}\sigma^2 + o(\sigma^2)} \\
&= 1 - \frac{1}{(3-\epsilon)u_p}\sigma + o(\sigma),
\end{aligned}$$

where s_ϵ is defined by (2.20).

If σ is sufficiently small, then by (2.11), we also have $\lambda_1(u) \geq 0$, and hence $\frac{r_1}{r_2} \leq 1$. We then complete the proof of Lemma 2.3. □

Lemma 2.4 *For any given small $\epsilon > 0$, when σ is sufficiently small, we have*

$$\left| u_p - \lambda_j\left(\hat{u}\left(\frac{t}{r}\right)\right) \right| > (1 - u_p^2 - \epsilon)\sigma \quad \text{and} \quad \left| \frac{\frac{1}{u_p} - \frac{t}{r}}{1 - \frac{\lambda_j(\hat{u}(\frac{t}{r}))}{u_p}} \right| \leq \frac{\sigma}{(3-2\epsilon)u_p^2} \tag{2.26}$$

as $(r, t) \in \Delta := \{(r, t) \mid r > 0, s_0 r \leq t \leq \frac{r}{u_p}\}$, where $j = 1, 2$.

Proof From (2.18) and (2.23), we have

$$\begin{aligned}
\left| u_p - \lambda_2\left(\hat{u}\left(\frac{t}{r}\right)\right) \right| > \left| u_p - \lambda_2(u_2) \right| &= \left| u_p - \frac{u_p + \sigma + \left(u_p - \frac{1}{u_p}\right)\sigma^2 + o(\sigma^2)}{1 + \left(u_p + \left(u_p - \frac{1}{u_p}\right)\sigma^2 + o(\sigma^2)\right)\sigma} \right| \\
&= (1 - u_p^2)\sigma + o(\sigma).
\end{aligned}$$

From (2.25), we have

$$\left| u_p - \lambda_1\left(\hat{u}\left(\frac{t}{r}\right)\right) \right| > \left| u_p - \lambda_1(u_p) \right| = (1 - u_p^2)\sigma + o(\sigma).$$

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Let s_ϵ be defined by (2.20). Then, we have

$$\left|\frac{\frac{1}{u_p}-\frac{t}{r}}{1-\frac{\lambda_1(\hat{u}(\frac{t}{r}))}{u_p}}\right| \le \left|\frac{\frac{1}{u_p}-s_\epsilon}{1-\frac{\lambda_1(u_p)}{u_p}}\right| \le \frac{\sigma}{(3-\epsilon)u_p^2}+o(\sigma)$$

and

$$\left|\frac{\frac{1}{u_p}-\frac{t}{r}}{1-\frac{\lambda_2(\hat{u}(\frac{t}{r}))}{u_p}}\right| \le \left|\frac{\frac{1}{u_p}-s_\epsilon}{1-\frac{\lambda_2(u_2)}{u_p}}\right| \le \frac{\sigma}{(3-\epsilon)u_p^2}+o(\sigma).$$

We then have (2.26) as σ is sufficiently small.

We then complete the proof of Lemma 2.4. □

3 Reformulation of the problem (1.6), (1.7)

3.1 Free boundary problem

We intend to look for a piecewise smooth solution with a single shock of the problem (1.6), (1.7). So, we consider system (1.6) with the fixed boundary condition:

$$u(t,r)=\frac{1}{\alpha'(r)} \quad \text{on } \Gamma_p, \tag{3.1}$$

where $\Gamma_p := \{(r,t) \mid r>0, t=\alpha(r)\}$, and the free boundary condition

$$H(u,\rho)=0 \quad \text{and} \quad \beta'(r)=\bar{s}(u) \quad \text{on } \Gamma_s, \tag{3.2}$$

where $\Gamma_s := \{(r,t) \mid r>0, t=\beta(r)\}$ is the shock free boundary and

$$\bar{s}(u)=\frac{-u(1-\sigma^2)+\sqrt{u^2(1-\sigma^2)^2+4\sigma^2}}{2\sigma^2}=\frac{2}{u(1-\sigma^2)+\sqrt{u^2(1-\sigma^2)^2+4\sigma^2}} \tag{3.3}$$

is the inverse function of $u=\bar{u}(s)$ $(1\le s\le \frac{1}{\sigma})$.

Remark 3.1 *It is easy to see that for any* $0<a<1$,

$$\lim_{\sigma\to 0}\left\|\bar{s}(u)-\frac{1}{u}\right\|_{C^0[0,a]}=0 \quad \textit{and} \quad \lim_{\sigma\to 0}\left\|\bar{s}'(u)+\frac{1}{u^2}\right\|_{C^0[0,a]}=0. \tag{3.4}$$

3.2 Riemann invariants and characteristic equations

We introduce the Riemann invariables:

$$R_1=\frac{1}{2}\ln\frac{1+u}{1-u}-\frac{\sigma}{1+\sigma^2}\ln\rho, \quad R_2=\frac{1}{2}\ln\frac{1+u}{1-u}+\frac{\sigma}{1+\sigma^2}\ln\rho. \tag{3.5}$$

By computation, we have

$$u=\frac{e^{R_1+R_2}-1}{e^{R_1+R_2}+1},$$

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$$\begin{pmatrix} \dfrac{\partial R_1}{\partial u} & \dfrac{\partial R_1}{\partial \rho} \\ \dfrac{\partial R_2}{\partial u} & \dfrac{\partial R_2}{\partial \rho} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{1-u^2} & -\dfrac{\sigma}{(1+\sigma^2)\rho} \\ \dfrac{1}{1-u^2} & \dfrac{\sigma}{(1+\sigma^2)\rho} \end{pmatrix}, \tag{3.6}$$

and

$$\begin{pmatrix} \dfrac{\partial u}{\partial R_1} & \dfrac{\partial u}{\partial R_2} \\ \dfrac{\partial \rho}{\partial R_1} & \dfrac{\partial \rho}{\partial R_2} \end{pmatrix} = \begin{pmatrix} \dfrac{1-u^2}{2} & \dfrac{1-u^2}{2} \\ -\dfrac{(1+\sigma^2)\rho}{2\sigma} & \dfrac{(1+\sigma^2)\rho}{2\sigma} \end{pmatrix}. \tag{3.7}$$

In terms of the Riemann invariables, system (1.6) can be written as:

$$\begin{cases} \partial_t R_1 + \lambda_1(u)\partial_r R_1 = \dfrac{F_1(u)}{r}, \\ \partial_t R_2 + \lambda_2(u)\partial_r R_2 = -\dfrac{F_2(u)}{r}, \end{cases} \tag{3.8}$$

where

$$F_1(u) = \frac{2u\sigma}{1-u\sigma} \quad \text{and} \quad F_2(u) = \frac{2u\sigma}{1+u\sigma}.$$

Let

$$\begin{gathered} \hat{R}_1(r,t) = \frac{1}{2}\ln\left(\frac{1+\hat{u}(s)}{1-\hat{u}(s)}\right) - \frac{\sigma \ln \hat{\rho}(s)}{1+\sigma^2}, \quad \hat{R}_2(r,t) = \frac{1}{2}\ln\left(\frac{1+\hat{u}(s)}{1-\hat{u}(s)}\right) + \frac{\sigma \ln \hat{\rho}(s)}{1+\sigma^2}, \\ W_1(r,t) = R_1(r,t) - \hat{R}_1(r,t), \quad W_2(r,t) = R_2(r,t) - \hat{R}_2(r,t), \end{gathered} \tag{3.9}$$

where $s = \frac{t}{r}$, $\hat{u}(s)$ is the solution of the problem (2.10) and $\hat{\rho}(s)$ is the solution of the problem (2.15). Then, in view of $\partial_t \hat{R}_j + \lambda_j(\hat{u})\partial_r \hat{R}_j = (-1)^{j+1}\frac{F_j(\hat{u})}{r}$ $(j = 1, 2)$, we have

$$\begin{cases} \partial_t W_1 + \lambda_1(u)\partial_r W_1 &= \dfrac{F_1(u) - F_1(\hat{u})}{r} + \big(\lambda_1(\hat{u}) - \lambda_1(u)\big)\partial_r \hat{R}_1, \\ \partial_t W_2 + \lambda_2(u)\partial_r W_2 &= -\dfrac{F_2(u) - F_2(\hat{u})}{r} + \big(\lambda_2(\hat{u}) - \lambda_2(u)\big)\partial_r \hat{R}_2. \end{cases} \tag{3.10}$$

By Taylor's formula, we know that there exist $\theta_j \in (0, 1)$ $(j = 1, 2, 3, 4)$, such that

$$F_1(u) - F_1(\hat{u}) = \frac{4\sigma e^{\hat{R}+\theta_1 W} W}{\left[1 - \left(\frac{e^{\hat{R}+\theta_1 W}-1}{e^{\hat{R}+\theta_1 W}+1}\right)\sigma\right]^2 \left(1 + e^{\hat{R}+\theta_1 W}\right)^2}, \tag{3.11}$$

$$F_2(u) - F_2(\hat{u}) = \frac{4\sigma e^{\hat{R}+\theta_2 W} W}{\left[1 - \left(\frac{e^{\hat{R}+\theta_2 W}-1}{e^{\hat{R}+\theta_2 W}+1}\right)\sigma\right]^2 \left(1 + e^{\hat{R}+\theta_2 W}\right)^2}, \tag{3.12}$$

$$\lambda_1(u) - \lambda_1(\hat{u}) = \frac{2(1-\sigma^2)e^{\hat{R}+\theta_3 W} W}{\left[1 - \left(\frac{e^{\hat{R}+\theta_3 W}-1}{e^{\hat{R}+\theta_3 W}+1}\right)\sigma\right]^2 \left(1 + e^{\hat{R}+\theta_3 W}\right)^2}, \tag{3.13}$$

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$$\lambda_2(u)-\lambda_2(\hat{u})=\frac{2(1-\sigma^2)e^{\hat{R}+\theta_4 W}W}{\left[1+\left(\frac{e^{\hat{R}+\theta_4 W}-1}{e^{\hat{R}+\theta_4 W}+1}\right)\sigma\right]^2\left(1+e^{\hat{R}+\theta_4 W}\right)^2}, \tag{3.14}$$

where $W=W_1+W_2$ and $\hat{R}=\hat{R}_1+\hat{R}_2$.

Lemma 3.1 *We have*

$$\lim_{\sigma\to 0}\left\|r\partial_r\hat{R}_j-\frac{2u_p}{1-u_p^2}\right\|_{C^0(\Delta)}=0 \quad \textit{and} \quad \lim_{\sigma\to 0}\left\|\frac{2e^{\hat{R}}}{(1+e^{\hat{R}})^2}-\frac{1-u_p^2}{2}\right\|_{C^0(\Delta)}=0. \tag{3.15}$$

Proof By computation, we have

$$\partial_r\hat{R}_1=-\frac{1}{r}\left(\frac{s}{1-\hat{u}^2}\frac{\mathrm{d}\hat{u}}{\mathrm{d}s}-\frac{\sigma s}{(1+\sigma^2)\hat{\rho}}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}s}\right) \tag{3.16}$$

and

$$\partial_r\hat{R}_2=-\frac{1}{r}\left(\frac{s}{1-\hat{u}^2}\frac{\mathrm{d}\hat{u}}{\mathrm{d}s}+\frac{\sigma s}{(1+\sigma^2)\hat{\rho}}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}s}\right). \tag{3.17}$$

Thus, using Lemma 2.1 and $\lim_{\sigma\to 0}\|\hat{u}-u_p\|_{C^0[s_0,\frac{1}{u_p}]}=0$, we can get Lemma 3.1. □

3.3 Free boundary condition in terms of W_1 and W_2

Next, we reformulate the free boundary condition.

The shock wave curve (2.9) can be written as:

$$H\left[u(W_1+\hat{R}_1,W_2+\hat{R}_2),\rho(W_1+\hat{R}_1,W_2+\hat{R}_2)\right]=0. \tag{3.18}$$

Since $\hat{R}_j$ $(j=1,2)$ are given functions of $\frac{t}{r}$, equation (3.18) can be written as:

$$\widetilde{H}\left(W_1,W_2,\frac{t}{r}\right)=0. \tag{3.19}$$

Moreover, we have

$$\widetilde{H}(0,0,s_0)=0. \tag{3.20}$$

Let

$$\eta(r)=\frac{\beta(r)}{r}-s_0, \quad r>0.$$

Then, using Taylor's formula and implicit function theorem, the free boundary condition (3.2) can be reformulated as:

$$W_1(t,r)=A_1W_2(t,r)+A_2\eta(r)+I(W_2(t,r),\eta(r)) \tag{3.21}$$

and

$$(r\eta)'=\bar{s}(u(r,\beta(r)))-s_0 \tag{3.22}$$

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on Γ_s, where

$$A_1=-\frac{m_2}{m_1},\quad A_2=-\frac{\hat{R}_1'(s_0)m_1+\hat{R}_2'(s_0)m_2}{m_1},$$
$$m_1=\frac{(1-u_2)^2}{2}\cdot\frac{\partial H}{\partial u}\Big|_{(u_2,\rho_2)}-\frac{(1+\sigma^2)\rho_2}{2\sigma}\cdot\frac{\partial H}{\partial \rho}\Big|_{(u_2,\rho_2)},$$
$$m_2=\frac{(1-u_2)^2}{2}\cdot\frac{\partial H}{\partial u}\Big|_{(u_2,\rho_2)}+\frac{(1+\sigma^2)\rho_2}{2\sigma}\cdot\frac{\partial H}{\partial \rho}\Big|_{(u_2,\rho_2)},$$

$u_2=\bar{u}(s_0)$, $\rho_2=\bar{\rho}(s_0)$ and $I(0,0)=I_{W_2}(0,0)=I_\eta(0,0)=0$. Moreover, for the function $I(W_2,\eta)$, there exists a positive constant $\mathcal{C}$ independent of $|W_2|$ and $|\eta|$, such that when $|W_2|$ and $|\eta|$ are sufficiently small:

$$|I(W_2,\eta)|<\mathcal{C}\big(|W_2|^2+|\eta|^2\big)\quad\text{and}\quad|\partial_{W_2}I(W_2,\eta)|<\mathcal{C}\big(|W_2|+|\eta|\big).\tag{3.23}$$

There exist $\theta_j\in(0,1)$ $(j=5,6,7,8)$, such that the condition (3.22) on Γ_s can be written as:

$$\begin{aligned}(r\eta)'&=\bar{s}(u(r,\beta(r)))-s_0\\&=\bar{s}(u(r,\beta(r)))-\bar{s}\left(\hat{u}\left(\frac{\beta(r)}{r}\right)\right)+\bar{s}\left(\hat{u}\left(\frac{\beta(r)}{r}\right)\right)-\bar{s}\big(\hat{u}(s_0)\big)\\&=\bar{s}'\left(\hat{u}\left(\frac{\beta(r)}{r}\right)+\theta_5\left(u(r,\beta(r))-\hat{u}\left(\frac{\beta(r)}{r}\right)\right)\right)\left(u(r,\beta(r))-\hat{u}\left(\frac{\beta(r)}{r}\right)\right)\\&\quad+\bar{s}'\left(\hat{u}(s_0)+\theta_6\left(\hat{u}\left(\frac{\beta(r)}{r}\right)-\hat{u}(s_0)\right)\right)\left(\hat{u}\left(\frac{\beta(r)}{r}\right)-\hat{u}(s_0)\right)\\&=\bar{s}'\left(\hat{u}\big(s_0+\eta\big)+\frac{2\theta_6e^{\hat{R}+\theta_7W}W}{\big(1+e^{\hat{R}+\theta_7W}\big)^2}\right)\frac{2e^{\hat{R}+\theta_7W}W}{\big(1+e^{\hat{R}+\theta_7W}\big)^2}\\&\quad+\bar{s}'\big(\hat{u}(s_0)+\theta_6\hat{u}'(s_0+\theta_8\eta)\eta\big)\hat{u}'(s_0+\theta_8\eta)\eta,\quad r>0.\end{aligned}\tag{3.24}$$

Lemma 3.2 *For the constants A_1 and A_2, we have*

$$|A_1|=1-\frac{4}{u_p}\sigma+o(\sigma)\quad\textit{and}\quad|A_2|=\frac{4u_p\sigma}{1-u_p^2}+o(\sigma).\tag{3.25}$$

Proof By computation, we have

$$\frac{m_2}{m_1}=\frac{\left(\frac{1-u_2^2}{2}\right)\left(\frac{1-\sigma^2}{\sigma}\right)\sqrt{\frac{\rho_0+\rho_2\sigma^2}{\rho_2+\rho_0\sigma^2}}-\left(\frac{u_2(1-\sigma^2)}{2\sigma}\sqrt{\frac{\rho_2+\rho_0\sigma^2}{\rho_0+\rho_2\sigma^2}}+1\right)\left(\frac{1+\sigma^2}{2\sigma}\right)\frac{\rho_0\rho_2(1-\sigma^4)}{(\rho_2+\rho_0\sigma^2)^2}}{\left(\frac{1-u_2^2}{2}\right)\left(\frac{1-\sigma^2}{\sigma}\right)\sqrt{\frac{\rho_0+\rho_2\sigma^2}{\rho_2+\rho_0\sigma^2}}+\left(\frac{u_2(1-\sigma^2)}{2\sigma}\sqrt{\frac{\rho_2+\rho_0\sigma^2}{\rho_0+\rho_2\sigma^2}}+1\right)\left(\frac{1+\sigma^2}{2\sigma}\right)\frac{\rho_0\rho_2(1-\sigma^4)}{(\rho_2+\rho_0\sigma^2)^2}}.$$

From $\rho_2=\frac{\rho_0(1-s_0^2\sigma^4)}{(s_0^2-1)\sigma^2}$, we have

$$\frac{\rho_0+\rho_2\sigma^2}{\rho_2+\rho_0\sigma^2}=s_0^2\sigma^2$$

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and

$$\frac{\rho_0\rho_2(1-\sigma^4)}{(\rho_2+\rho_0\sigma^2)^2} = \frac{\rho_0^2(1-\sigma^4)(1-s_0^2\sigma^4)}{(s_0^2-1)\sigma^2}\cdot\frac{1}{\rho_0^2\big(\sigma^2+\frac{1-s_0^2\sigma^4}{(s_0^2-1)\sigma^2}\big)^2}$$
$$= \frac{(1-s_0^2\sigma^4)(s_0^2-1)\sigma^2}{1-\sigma^4}.$$

Hence, we get

$$\frac{m_2}{m_1} = \frac{s_0(1-\sigma^2)\frac{1-u_2^2}{2} - \left(\frac{u_2(1-\sigma^2)}{2s_0\sigma^2}+1\right)\left(\frac{1+\sigma^2}{2\sigma}\right)\frac{(1-s_0^2\sigma^4)(s_0^2-1)\sigma^2}{1-\sigma^4}}{s_0(1-\sigma^2)\frac{1-u_2^2}{2} + \left(\frac{u_2(1-\sigma^2)}{2s_0\sigma^2}+1\right)\left(\frac{1+\sigma^2}{2\sigma}\right)\frac{(1-s_0^2\sigma^4)(s_0^2-1)\sigma^2}{1-\sigma^4}}$$

$$= \frac{2s_0^2\sigma(1-\sigma^2)^2(1-u_2^2) - u_2(s_0^2-1)(1-s_0^2\sigma^4)(1-\sigma^2) - 2(s_0^2-1)(1-s_0^2\sigma^4)s_0\sigma^2}{2s_0^2\sigma(1-\sigma^2)^2(1-u_2^2) + u_2(s_0^2-1)(1-s_0^2\sigma^4)(1-\sigma^2) + 2(s_0^2-1)(1-s_0^2\sigma^4)s_0\sigma^2}$$

$$= \frac{-u_2(s_0^2-1)+2s_0^2(1-u_2^2)\sigma+o(\sigma)}{u_2(s_0^2-1)+2s_0^2(1-u_2^2)\sigma+o(\sigma)}$$

$$= -1+\frac{4s_0^2(1-u_2^2)}{u_2(s_0^2-1)}\sigma + \frac{o(\sigma)}{u_2(s_0^2-1)+2s_0^2(1-u_2^2)\sigma+o(\sigma)}.$$

Thus, by Lemma 2.1, we can get $|A_1| = 1-\frac{4}{u_p}\sigma+o(\sigma)$.

From Lemma 2.1, we have

$$\frac{\sigma\hat\rho'(s_0)}{(1+\sigma^2)\hat\rho(s_0)} = \frac{\frac{2u_2(s_0u_2-1)}{\sigma}}{\frac{(s_0u_2-1)^2}{\sigma^2}-(u_2-s_0)^2} = \frac{\frac{2u_2(1-s_0^2)\sigma}{1-\sigma^2}}{\frac{(1-s_0^2)^2\sigma^2}{(1-\sigma^2)^2}-(u_2-s_0)^2} = \frac{2u_p\sigma}{1-u_p^2}+o(\sigma)$$

and

$$\frac{\hat u'(s_0)}{1-\hat u^2(s_0)} = \frac{2u_p^2}{1-u_p^2}+o(1).$$

Hence, we have

$$A_2 = \frac{\sigma\hat\rho'(s_0)}{(1+\sigma^2)\hat\rho(s_0)} - \frac{\hat u'(s_0)}{1-\hat u^2(s_0)} - \left(\frac{\hat u'(s_0)}{1-\hat u^2(s_0)}+\frac{\sigma\hat\rho'(s_0)}{(1+\sigma^2)\hat\rho(s_0)}\right)\left(-1+\frac{4}{u_p}\sigma+o(\sigma)\right)$$
$$= \frac{2\sigma\hat\rho'(s_0)}{(1+\sigma^2)\hat\rho(s_0)} - \left(\frac{\hat u'(s_0)}{1-\hat u^2(s_0)}+\frac{\sigma\hat\rho'(s_0)}{(1+\sigma^2)\hat\rho(s_0)}\right)\left(\frac{4}{u_p}\sigma+o(\sigma)\right)$$
$$= \frac{-4u_p\sigma}{1-u_p^2}+o(\sigma).$$

We then complete the proof of Lemma 3.2. □

3.4 Fixed boundary condition in terms of W_1 and W_2

The fixed boundary condition (3.1) can be written as:

$$W_2(r,t) = -W_1(r,t)+G_{11}(r)+G_{12}(r,t) \quad \text{on } \Gamma_p, \tag{3.26}$$

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where

$$G_{11}(r)=\ln\Big(\frac{\alpha'(r)+1}{\alpha'(r)-1}\Big)-\ln\Big(\frac{1+u_p}{1-u_p}\Big),\quad G_{12}(r,t)=\ln\Big(\frac{1+u_p}{1-u_p}\Big)-(\hat{R}_1+\hat{R}_2)\Big(\frac{t}{r}\Big).$$

Lemma 3.3 *Under assumptions (A1)–(A2), when ε is sufficiently small we have*

$$\begin{cases} |G_{11}(r)|<\dfrac{4u_p^2}{1-u_p^2}\cdot\dfrac{\varepsilon}{r^\delta},\quad r>r_0; \\ |G_{12}(r,t)|<\dfrac{8u_p^2}{1-u_p^2}\cdot\dfrac{\varepsilon}{r^\delta},\ (r,t)\in\Gamma_p. \end{cases} \tag{3.27}$$

Proof By computation, we have

$$G_{11}(r)=\frac{2(\alpha'(r)-\frac{1}{u_p})}{1-\Big[\frac{1}{u_p}+\theta\Big(\alpha'(r)-\frac{1}{u_p}\Big)\Big]^2},$$

where $\theta\in(0,1)$. Thus, by assumptions (A1)–(A2), we have that when ε is sufficiently small $|G_{11}(r)|<\frac{4u_p^2}{1-u_p^2}\cdot\frac{\varepsilon}{r^\delta}$ as $r>r_0$.

By computation, we have

$$\begin{aligned} G_{12}(r,t)&=\ln\left(\frac{1+u_p}{1-u_p}\right)-(\hat{R}_1+\hat{R}_2)\left(\frac{t}{r}\right)\\ &=(\hat{R}_1+\hat{R}_2)\left(\frac{1}{u_p}\right)-(\hat{R}_1+\hat{R}_2)\left(\frac{t}{r}\right)\\ &=-(\hat{R}_1+\hat{R}_2)'\left(\frac{1}{u_p}+\theta\left(\frac{t}{r}-\frac{1}{u_p}\right)\right)\cdot\left(\frac{t}{r}-\frac{1}{u_p}\right)\\ &=\frac{2\hat{u}'\left(\frac{1}{u_p}+\theta\left(\frac{t}{r}-\frac{1}{u_p}\right)\right)}{1-\hat{u}^2\left(\frac{1}{u_p}+\theta\left(\frac{t}{r}-\frac{1}{u_p}\right)\right)}\cdot\left(\frac{t}{r}-\frac{1}{u_p}\right), \end{aligned} \tag{3.28}$$

where $\theta\in(0,1)$. By assumptions (A1)–(A2), we have

$$\Big|\alpha(r)-\frac{r}{u_p}\Big|<\int_{r_0}^{r}\Big|\alpha'(r)-\frac{1}{u_p}\Big|\mathrm{d}r<\frac{\varepsilon}{1-\delta}r^{1-\delta},\quad r>r_0. \tag{3.29}$$

Thus, using Lemma 2.1 and Remark 2.1, we know that when ε is sufficiently small $|G_{12}(r,t)|<\frac{8u_p^2}{1-u_p^2}\cdot\frac{\varepsilon}{r^\delta}$ as $t=\alpha(r)$ and $r>r_0$.

We then complete the proof of Lemma 3.3. □

4 Characteristic equations and boundary conditions for the derivatives of W

Let

$$\partial_z=\partial_r+\frac{1}{u_p}\partial_t, \tag{4.1}$$

and set $\widetilde{W}_i=\partial_z W_i$.

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4.1 Characteristic equations about $\widetilde{W}_1$ and $\widetilde{W}_2$

By computation, we have

$$\partial_r = \frac{\partial_j - u_p\partial_z}{\lambda_j - u_p} \quad \text{and} \quad \partial_t = \frac{\partial_j - \lambda_j\partial_z}{1 - \frac{\lambda_j}{u_p}}, \tag{4.2}$$

where $\partial_j = \partial_t + \lambda_j\partial_r$ $(j = 1, 2)$. Then by (3.10), we obtain

$$\begin{cases} \partial_t\widetilde{W}_1 + \lambda_1(u)\partial_r\widetilde{W}_1 = H_1, \\ \partial_t\widetilde{W}_2 + \lambda_2(u)\partial_r\widetilde{W}_2 = H_2, \end{cases} \tag{4.3}$$

where

$$\begin{aligned} H_1 = {} & \frac{1}{r}\partial_z\big[F_1(u) - F_1(\hat{u})\big] + \partial_z\big[\lambda_1(\hat{u}) - \lambda_1(u)\big](\partial_r\hat{R}_1 - \partial_r W_1) \\ & + \big[\lambda_1(\hat{u}) - \lambda_1(u)\big]\partial_z\partial_r\hat{R}_1 + \partial_z\lambda_1(\hat{u})\partial_r W_1 - \frac{F(u) - F(\hat{u})}{r^2}, \\ H_2 = {} & -\frac{1}{r}\partial_z\big[F_2(u) - F_2(\hat{u})\big] + \partial_z\big[\lambda_2(\hat{u}) - \lambda_2(u)\big](\partial_r\hat{R}_2 - \partial_r W_2) \\ & + \big[\lambda_2(\hat{u}) - \lambda_2(u)\big]\partial_z\partial_r\hat{R}_2 + \partial_z\lambda_2(\hat{u})\partial_r W_2 + \frac{F(u) - F(\hat{u})}{r^2}. \end{aligned}$$

By computations, we have

$$\partial_z\big[F_1(u) - F_1(\hat{u})\big] = F_1'(u)(\partial_z u - \partial_z\hat{u}) + \big[F_1'(u) - F_1'(\hat{u})\big]\hat{u}'\left(\frac{t}{r}\right)\left(\frac{1}{u_p} - \frac{t}{r}\right)\frac{1}{r}, \tag{4.4}$$

$$\partial_z\big[F_2(u) - F_2(\hat{u})\big] = F_2'(u)(\partial_z u - \partial_z\hat{u}) + \big[F_2'(u) - F_2'(\hat{u})\big]\hat{u}'\left(\frac{t}{r}\right)\left(\frac{1}{u_p} - \frac{t}{r}\right)\frac{1}{r}, \tag{4.5}$$

$$\partial_z\lambda_j(\hat{u})\partial_r W_j = \lambda_j'(\hat{u})\hat{u}'\left(\frac{t}{r}\right)\left\{\frac{\frac{1}{u_p} - \frac{t}{r}}{1 - \frac{\lambda_j}{u_p}}\widetilde{W}_j - \frac{\frac{1}{u_p} - \frac{t}{r}}{u_p - \lambda_j}\partial_j W_j\right\}\frac{1}{r}, \tag{4.6}$$

$$\partial_z\big[\lambda_j(u) - \lambda_j(\hat{u})\big]\partial_r\hat{R}_j = \Big(\big[\lambda_j'(u) - \lambda_j'(\hat{u})\big]\partial_z\hat{u} + \lambda_j'(u)\partial_z(u - \hat{u})\Big)\partial_r\hat{R}_j, \tag{4.7}$$

$$\begin{aligned} \partial_z\big[\lambda_j(u) - \lambda_j(\hat{u})\big]\partial_r W_j &= \Big(\big[\lambda_j'(u) - \lambda_j'(\hat{u})\big]\partial_z\hat{u} + \lambda_j'(u)\partial_z(u - \hat{u})\Big)\partial_r W_j \\ &= \big[\lambda_j'(u) - \lambda_j'(\hat{u})\big]\hat{u}'\left(\frac{t}{r}\right)\left\{\frac{\frac{1}{u_p} - \frac{t}{r}}{1 - \frac{\lambda_j}{u_p}}\widetilde{W}_j - \frac{\frac{1}{u_p} - \frac{t}{r}}{u_p - \lambda_j}\partial_j W_j\right\}\frac{1}{r} \\ &\quad + \lambda_j'(u)\partial_z(u - \hat{u})\left\{\frac{u_p}{u_p - \lambda_j}\widetilde{W}_j - \frac{1}{u_p - \lambda_j}\partial_j W_j\right\}, \end{aligned} \tag{4.8}$$

$$\partial_z\partial_r\hat{R}_j = \frac{1}{r^2}\left\{\Big(s^2 - \frac{s}{u_p}\Big)\hat{R}_j''\left(\frac{t}{r}\right) + \Big(2s - \frac{1}{u_p}\Big)\hat{R}_j'\left(\frac{t}{r}\right)\right\}. \tag{4.9}$$

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By Taylor's formula, there exist $\theta_j \in (0,1)$ $(j=9,10,11,12,13)$, such that

$$\begin{aligned}\partial_z u - \partial_z \hat{u} &= \frac{2e^R}{(1+e^R)^2}(\partial_z R - \partial_z \hat{R}) + \left(\frac{2e^R}{(1+e^R)^2} - \frac{2e^{\hat{R}}}{(1+e^{\hat{R}})^2}\right)\partial_z \hat{R} \\ &= \frac{2e^{\hat{R}+W}\widetilde{W}}{(1+e^{\hat{R}+W})^2} + \frac{2\left(\frac{1}{u_p} - \frac{t}{r}\right)(e^{\hat{R}+\theta_9 W} - e^{3(\hat{R}+\theta_9 W)})\hat{R}' W}{r(1+e^{\hat{R}+\theta_9 W})^4},\end{aligned} \tag{4.10}$$

$$F_1'(u) - F_1'(\hat{u}) = \frac{8\sigma^2 e^{\hat{R}+\theta_{10}W} W}{\left[1 - \left(\frac{e^{\hat{R}+\theta_{10}W}-1}{e^{\hat{R}+\theta_{10}W}+1}\right)\sigma\right]^3 \left(1+e^{\hat{R}+\theta_{10}W}\right)^2}, \tag{4.11}$$

$$F_2'(u) - F_2'(\hat{u}) = \frac{8\sigma^2 e^{\hat{R}+\theta_{11}W} W}{\left[1 + \left(\frac{e^{\hat{R}+\theta_{11}W}-1}{e^{\hat{R}+\theta_{11}W}+1}\right)\sigma\right]^3 \left(1+e^{\hat{R}+\theta_{11}W}\right)^2}, \tag{4.12}$$

$$\lambda_1'(u) - \lambda_1'(\hat{u}) = \frac{4(1-\sigma^2)\sigma e^{\hat{R}+\theta_{12}W} W}{\left[1 - \left(\frac{e^{\hat{R}+\theta_{12}W}-1}{e^{\hat{R}+\theta_{12}W}+1}\right)\sigma\right]^3 \left(1+e^{\hat{R}+\theta_{12}W}\right)^2}, \tag{4.13}$$

$$\lambda_2'(u) - \lambda_2'(\hat{u}) = \frac{4(1-\sigma^2)\sigma e^{\hat{R}+\theta_{13}W} W}{\left[1 + \left(\frac{e^{\hat{R}+\theta_{13}W}-1}{e^{\hat{R}+\theta_{13}W}+1}\right)\sigma\right]^3 \left(1+e^{\hat{R}+\theta_{13}W}\right)^2}, \tag{4.14}$$

where $\widetilde{W} = \widetilde{W}_1 + \widetilde{W}_2$.

4.2 Boundary conditions of $\widetilde{W}_1$ and $\widetilde{W}_2$ on Γ_p

Let $\partial_\alpha = \partial_r + \alpha'(r)\partial_t$. Then by (3.26), we have

$$\partial_\alpha W_2 = -\partial_\alpha W_1 + \frac{2\alpha''(r)}{1-[\alpha'(r)]^2} - \partial_\alpha(\hat{R}_1 + \hat{R}_2) \quad \text{on } t = \alpha(r).$$

Inserting (4.2) into this, we have

$$\begin{aligned}\frac{1-\alpha'(r)\lambda_2}{1-\frac{\lambda_2}{u_p}}\widetilde{W}_2 + \frac{\alpha'(r)-\frac{1}{u_p}}{1-\frac{\lambda_2}{u_p}}\partial_2 W_2 = &-\frac{1-\alpha'(r)\lambda_1}{1-\frac{\lambda_1}{u_p}}\widetilde{W}_1 - \frac{\alpha'(r)-\frac{1}{u_p}}{1-\frac{\lambda_1}{u_p}}\partial_1 W_1 \\ &+ \frac{2\alpha''(r)}{1-[\alpha'(r)]^2} - \frac{1}{r}\left(\alpha'(t) - \frac{t}{r}\right)(\hat{R}_1 + \hat{R}_2)'\left(\frac{t}{r}\right) \quad \text{on } \Gamma_p.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\widetilde{W}_2 = &-\widetilde{W}_1 + \left(\alpha'(r) - \frac{1}{u_p}\right)\left(\frac{\lambda_2\widetilde{W}_2}{1-\frac{\lambda_2}{u_p}} + \frac{\lambda_1\widetilde{W}_1}{1-\frac{\lambda_1}{u_p}}\right) - \frac{\alpha'(r)-\frac{1}{u_p}}{1-\frac{\lambda_2}{u_p}}\partial_2 W_2 - \frac{\alpha'(r)-\frac{1}{u_p}}{1-\frac{\lambda_1}{u_p}}\partial_1 W_1 \\ &+ \frac{2\alpha''(r)}{1-[\alpha'(r)]^2} - \frac{1}{r}\left(\alpha'(t) - \frac{t}{r}\right)(\hat{R}_1 + \hat{R}_2)'\left(\frac{t}{r}\right) \quad \text{on } \Gamma_p.\end{aligned} \tag{4.15}$$

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By (A2) and (3.29), we also have

$$\left|\alpha'(r)-\frac{t}{r}\right|\le\left|\alpha'(r)-\frac{1}{u_p}\right|+\left|\frac{1}{u_p}-\frac{t}{r}\right|\le\frac{3\varepsilon}{r^{\delta}}\quad\text{on }\Gamma_p. \tag{4.16}$$

4.3 Boundary conditions of $\widetilde{W}_1$ and $\widetilde{W}_2$ on Γ_s

Let $\partial_\beta=\partial_r+\beta'(r)\partial_t$. Then by (3.21), we have

$$\partial_\beta W_1=A_1\partial_\beta W_2+A_2\eta'(r)+\partial_{W_2}I\partial_\beta W_2+\partial_\eta I\eta'(r)\quad\text{on }\Gamma_s.$$

Inserting (4.2) into this, we have

$$\begin{aligned}&\frac{1-\beta'(r)\lambda_1}{1-\frac{\lambda_1}{u_p}}\widetilde{W}_1+\frac{\beta'(r)-\frac{1}{u_p}}{1-\frac{\lambda_1}{u_p}}\partial_1W_1\\&=(A_1+\partial_{W_2}I)\left(\frac{1-\beta'(r)\lambda_2}{1-\frac{\lambda_2}{u_p}}\widetilde{W}_2+\frac{\beta'(r)-\frac{1}{u_p}}{1-\frac{\lambda_2}{u_p}}\partial_2W_2\right)+\big(A_2+\partial_\eta I\big)\eta'(r)\quad\text{on }\Gamma_s.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\widetilde{W}_1\;=\;&(A_1+\partial_{W_2}I)\widetilde{W}_2+\Big(\beta'(r)-\frac{1}{u_p}\Big)\left(\frac{\lambda_1\widetilde{W}_1}{1-\frac{\lambda_1}{u_p}}-(A_1+\partial_{W_2}I)\frac{\lambda_2\widetilde{W}_2}{1-\frac{\lambda_2}{u_p}}\right)\\&+(A_1+\partial_{W_2}I)\left(\frac{\beta'(r)-\frac{1}{u_p}}{1-\frac{\lambda_2}{u_p}}\right)\partial_2W_2-\frac{\beta'(r)-\frac{1}{u_p}}{1-\frac{\lambda_1}{u_p}}\partial_1W_1+\big(A_2+\partial_\eta I\big)\eta'(r)\\=\;&(A_1+\partial_{W_2}I)\widetilde{W}_2+\Big(\beta'(r)-s_0\Big)\left(\frac{\lambda_1\widetilde{W}_1}{1-\frac{\lambda_1}{u_p}}-(A_1+\partial_{W_2}I)\frac{\lambda_2\widetilde{W}_2}{1-\frac{\lambda_2}{u_p}}\right)\\&+\frac{\Big(s_0-\frac{1}{u_p}\Big)\lambda_1\widetilde{W}_1}{1-\frac{\lambda_1}{u_p}}-(A_1+\partial_{W_2}I)\frac{(s_0-\frac{1}{u_p})\lambda_2\widetilde{W}_2}{1-\frac{\lambda_2}{u_p}}\\&+(A_1+\partial_{W_2}I)\left(\frac{\beta'(r)-\frac{1}{u_p}}{1-\frac{\lambda_2}{u_p}}\right)\partial_2W_2-\frac{\beta'(r)-\frac{1}{u_p}}{1-\frac{\lambda_1}{u_p}}\partial_1W_1+\big(A_2+\partial_\eta I\big)\eta'(r)\end{aligned} \tag{4.17}$$

on Γ_s. By computation, we also have

$$\Big|\beta'(r)-s_0\Big|<|\eta|+|r\eta'|. \tag{4.18}$$

5 Global solution of the free boundary problem (1.6), (3.1)–(3.2)

5.1 Local solution of the free boundary problem

Let $r=\bar{r}(t)$ be defined by

$$\begin{cases}\dfrac{\mathrm{d}\bar{r}(t)}{\mathrm{d}t}=\dfrac{\hat{u}(\frac{t}{\bar{r}(t)})+\sigma}{1+\sigma\hat{u}(\frac{t}{\bar{r}(t)})},\ t>\frac{r_0}{u_p};\\ \bar{r}(\frac{r_0}{u_p})=r_0.\end{cases}$$

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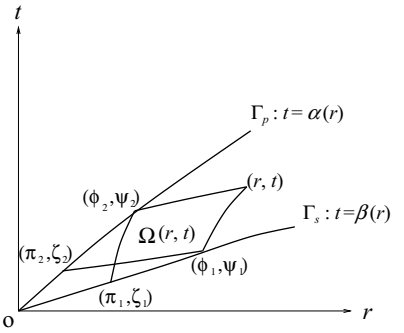


FIGURE 4. Reflected characteristic curves.

Let Ω_{local} be a domain closed by $r = u_p t$, $t = s_0 r$ and $r = \bar{r}(t)$. It is obvious that the solution of free boundary value problem (1.6), (3.1)–(3.2) in the region Ω_{local} is $(\hat{u}, \hat{\rho})(\frac{t}{r})$. In what follows, we are going to extend the local solution to a global solution.

5.2 Uniform estimates for W, η and their derivatives

In this section, we will derive the uniform decay estimates for W, η and their derivatives. Using these estimates, we can obtain the global classical solution of the free boundary value problem (1.6), (3.1) and (3.2).

Lemma 5.1 (Uniform decay estimates) *Let $D_T = \{(t, r) \mid r_0 \le r \le T, \alpha(r) \le t \le \beta(r)\}$ for any large $T > r_0$. When σ and ε are sufficiently small, there exist two positive constants $\mathcal{C}_0$ and $\mathcal{C}_1$ independent of ε and T, such that the solution of the problem (1.6), (3.1)–(3.2) satisfies*

$$|W_1| < \frac{\mathcal{C}_0\varepsilon}{r^\delta}, \ |W_2| < \frac{\mathcal{C}_0\varepsilon}{r^\delta}, \ |\widetilde{W}_1| < \frac{\mathcal{C}_1\varepsilon}{r^{1+\delta}}, \quad \text{and} \quad |\widetilde{W}_2| < \frac{\mathcal{C}_1\varepsilon}{r^{1+\delta}} \quad \text{in } D_T,$$

and

$$|\eta| < \frac{17(1-u_p^2)\mathcal{C}_0\varepsilon}{48u_p^2 r^\delta} \quad \text{and} \quad |\eta'| < \frac{3(1-u_p^2)\mathcal{C}_0\varepsilon}{u_p^2 r^{1+\delta}} \quad \text{in } [r_0, T].$$

Proof In what follows, we are going to use the reflected characteristics method to prove these estimates.

For any $(r_1, t_1) \in D_T$, we shall denote by $\gamma_j(\tau; r_1, t_1)$ $(j = 1, 2)$ the backward j-th characteristic curve passing through the point (r_1, t_1). Thus, we have

$$\begin{cases} \dfrac{\mathrm{d}\gamma_j(\tau; r_1, t_1)}{\mathrm{d}\tau} = \lambda_j\big(u\big(\gamma_j(\tau; r_1, t_1), \tau\big)\big), \ \tau \le t_1; \\ \gamma_j(t_1; r_1, t_1) = r_1. \end{cases}$$

Referring to Figure 4. The characteristic curve $r = \gamma_1(\tau; r_1, t_1)$ intersects with Γ_s at a point $\big(\phi_1(r_1, t_1), \psi_1(r_1, t_1)\big)$. The characteristic curve $r = \gamma_2(\tau; r_1, t_1)$ intersects with Γ_p at a point $\big(\phi_2(r_1, t_1), \psi_2(r_1, t_1)\big)$. The backward C_2 characteristic curve issuing from $\big(\phi_1(r_1, t_1), \psi_1(r_1, t_1)\big)$ intersects with Γ_p at a point $\big(\pi_2(r_1, t_1), \zeta_2(r_1, t_1)\big)$. The backward C_1 characteristic curve issuing from $\big(\phi_2(r_1, t_1), \psi_2(r_1, t_1)\big)$ intersects with Γ_s at a point $\big(\pi_1(r_1, t_1), \zeta_1(r_1, t_1)\big)$.

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Let $\Omega(r_1,t_1)$ be a closed domain encircled by Γ_s, Γ_p, $r=\gamma_1(\tau;r_1,t_1)$ and $r=\gamma_2(\tau;r_1,t_1)$. By the continuity induction, we only need to prove that if

- $|W_1|<\frac{\mathcal{C}_0\varepsilon}{r^\delta}$, $|W_2|<\frac{\mathcal{C}_0\varepsilon}{r^\delta}$, $|\widetilde{W}_1|<\frac{\mathcal{C}_1\varepsilon}{r^{1+\delta}}$, and $|\widetilde{W}_2|<\frac{\mathcal{C}_1\varepsilon}{r^{1+\delta}}$ in $\Omega(r_1,t_1)\setminus\{(r_1,t_1)\}$;
- $|\eta(r)|<\frac{17(1-u_p^2)\mathcal{C}_0\varepsilon}{48u_p^2r^\delta}$ and $|\eta'(r)|<\frac{3(1-u_p^2)\mathcal{C}_0\varepsilon}{u_p^2r^{1+\delta}}$ as $r\in[r_0,\phi_1(r_1,t_1))$,

then

$$|W_1(r_1,t_1)|<\frac{\mathcal{C}_0\varepsilon}{r_1^\delta},\quad |W_2(r_1,t_1)|<\frac{\mathcal{C}_0\varepsilon}{r_1^\delta},\quad |\widetilde{W}_1(r_1,t_1)|<\frac{\mathcal{C}_1\varepsilon}{r_1^{1+\delta}},\quad |\widetilde{W}_2(r_1,t_1)|<\frac{\mathcal{C}_1\varepsilon}{r_1^{1+\delta}},$$

$$|\eta(\phi_1(r_1,t_1))|<\frac{17(1-u_p^2)\mathcal{C}_0\varepsilon}{48u_p^2\phi_1^\delta(r_1,t_1)},\quad\text{and}\quad |\eta'(\phi_1(r_1,t_1))|<\frac{3(1-u_p^2)\mathcal{C}_0\varepsilon}{u_p^2\phi_1^{1+\delta}(r_1,t_1)}.$$

In what follows, we shall let ϵ be an arbitrary small positive constant.

Step 1. Estimations of $|\eta|$ and $|\eta'|$.
Using (2.8),(3.4), (3.15) and (3.24), we know that when σ and ε are sufficiently small,

$$-\Big(\frac{1-u_p^2}{2u_p^2}+\epsilon\Big)\frac{2\mathcal{C}_0\varepsilon}{r^\delta}+(-2+\epsilon)\eta(r)<r\eta'(r)+\eta(r)<\Big(\frac{1-u_p^2}{2u_p^2}+\epsilon\Big)\frac{2\mathcal{C}_0\varepsilon}{r^\delta}+(-2+\epsilon)\eta(r). \tag{5.1}$$

Thus by integration, we know that there exists a small $\epsilon>0$ which is independent of σ and ε, such that when σ and ε are sufficiently small,

$$\begin{aligned}|\eta(\phi_1)| &< e^{-\int_{r_0}^{\phi_1}\frac{3-\epsilon_1}{r}\mathrm{d}r}\left(\int_{r_0}^{\phi_1}\Big(\frac{1-u_p^2}{2u_p^2}+\epsilon\Big)\frac{2\mathcal{C}_0\varepsilon}{r^{1+\delta}}e^{\int_{r_0}^{r}\frac{3-\epsilon}{r}\mathrm{d}r}\mathrm{d}r\right)\\ &= \frac{1}{3-\epsilon-\delta}\Big(\frac{1-u_p^2}{2u_p^2}+\epsilon\Big)\frac{2\mathcal{C}_0\varepsilon}{\phi_1^\delta}\Big(1-\big(\frac{r_0}{\phi_1}\big)^{3-\epsilon-\delta}\Big)\\ &< \frac{17(1-u_p^2)}{48u_p^2}\cdot\frac{\mathcal{C}_0\varepsilon}{\phi_1^\delta}.\end{aligned}$$

Combining this with (5.1), we have

$$|\eta'(\phi_1)|<\frac{3(1-u_p^2)\mathcal{C}_0\varepsilon}{u_p^2\phi_1^{1+\delta}}. \tag{5.2}$$

Step 2. Estimations of $|W_1|$ and $|W_2|$.
It follows from (3.11)–(3.17) that if σ and ε are sufficiently small, then

$$\Big|\frac{F_j(u)-F_j(\hat{u})}{r\lambda_j(u)}\Big|+\Big|\frac{(\lambda_j(\hat{u})-\lambda_j(u))\partial_r\hat{R}_j}{\lambda_j(u)}\Big|\le\frac{1}{r}(1+\epsilon)\big(|W_1|+|W_2|\big)\quad\text{in }\Omega(r_1,t_1). \tag{5.3}$$

Using Lemmas 2.2 and 2.3, it can be easily seen that if σ and ε are sufficiently small, then

$$\frac{\phi_j(r_1,t_1)}{r_1}>1-\frac{\sigma}{(3-3\epsilon)u_p},\quad \frac{\pi_j(r_1,t_1)}{r_1}>1-\frac{2\sigma}{(3-3\epsilon)u_p},\quad j=1,2. \tag{5.4}$$

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Using the characteristic method, the boundary conditions (3.21) and (3.26), and (5.3), we get

$$\begin{aligned}
&|W_1(r_1,t_1)|\\
\le\ & \big|W_1(\phi_1,\psi_1)\big|+\int_{\phi_1}^{r_1}\frac{2(1+\epsilon)\mathcal{C}_0\varepsilon}{r^{1+\delta}}\mathrm{d}r\\
\le\ & |A_1W_2(\phi_1,\psi_1)|+|A_2\eta(\phi_1)|+|I(W_2(\phi_1,\psi_1),\eta(\phi_1))|+\int_{\phi_1}^{r_1}\frac{2(1+\epsilon)\mathcal{C}_0\varepsilon}{r^{1+\delta}}\mathrm{d}r\\
\le\ & |A_1W_2(\pi_2,\zeta_2)|+|A_2\eta(\phi_1)|+|I(W_2(\phi_1,\psi_1),\eta(\phi_1))|+\int_{\pi_2}^{r_1}\frac{2(1+\epsilon)\mathcal{C}_0\varepsilon}{r^{1+\delta}}\mathrm{d}r\\
\le\ & |A_1W_1(\pi_2,\zeta_2)|+|G_{11}(\pi_2)|+|G_{12}(\pi_2,\zeta_2)|+|A_2\eta(\phi_1)|\\
&+|I(W_2(\phi_1,\psi_1),\eta(\phi_1))|+\int_{\pi_2}^{r_1}\frac{2(1+\epsilon)\mathcal{C}_0\varepsilon}{r^{1+\delta}}\mathrm{d}r.
\end{aligned}$$

Thus, by (3.23), (3.25), (3.27) and (5.4), we have that there exists a small $\epsilon>0$ which is independent of σ and ε, such that when σ and ε are sufficiently small, there holds

$$\begin{aligned}
|W_1(r_1,t_1)|\ \le\ & \frac{\left(1-\frac{(4-\epsilon)}{u_p}\sigma\right)}{(1-\frac{2\sigma}{(3-3\epsilon)u_p})^{\delta}}\cdot\frac{\mathcal{C}_0\varepsilon}{r_1^{\delta}}+\left(\frac{4u_p^2}{1-u_p^2}\right)\frac{\varepsilon}{(1-\frac{2\sigma}{(3-3\epsilon)u_p})^{\delta}r_1^{\delta}}+\frac{8u_p^2}{1-u_p^2}\cdot\frac{\varepsilon}{\left(1-\frac{2\sigma}{(3-3\epsilon)u_p}\right)^{\delta}r^{\delta}}\\
&+\frac{(4+\epsilon)17\sigma}{48(1-\frac{\sigma}{(3-3\epsilon)u_p})^{\delta}u_p}\cdot\frac{\mathcal{C}_0\varepsilon}{r_1^{\delta}}+\mathcal{C}\left[\left(\frac{\mathcal{C}_0\varepsilon}{r_1^{\delta}}\right)^2+\left(\frac{17(1-u_p^2)}{48u_p^2}\cdot\frac{\mathcal{C}_0\varepsilon}{(1-\frac{\sigma}{(3-3\epsilon)u_p})^{\delta}r_1^{\delta}}\right)^2\right]\\
&+\frac{2(1+\epsilon)}{\delta}\left[\frac{1}{(1-\frac{2\sigma}{(3-3\epsilon)u_p})^{\delta}}-1\right]\frac{\mathcal{C}_0\varepsilon}{r_1^{\delta}}\\
<\ & \left(1-\frac{\sigma}{2u_p}\right)\frac{\mathcal{C}_0\varepsilon}{r_1^{\delta}}+\frac{16u_p^2}{1-u_p^2}\frac{\varepsilon}{r_1^{\delta}}.
\end{aligned}$$

So, if we choose $\mathcal{C}_0=\frac{32u_p^3}{(1-u_p^2)\sigma}$, then we have $|W_1(r_1,t_1)|<\frac{\mathcal{C}_0\varepsilon}{r_1^{\delta}}$.

Similarly, we can get $|W_2(r_1,t_1)|<\frac{\mathcal{C}_0\varepsilon}{r_1^{\delta}}$.

Step 3. Estimations of $|\widetilde{W}_1|$ and $|\widetilde{W}_2|$.

Using (4.4)–(4.14) and Lemmas 2.1 and 3.1, we know that for any given small $\epsilon>0$, when σ and ε are sufficiently small, we have

$$|\partial_z u-\partial_z\hat{u}|<\left(\frac{1-u_p^2}{2}+\epsilon\right)|\widetilde{W}|+\frac{\epsilon}{r}|W|, \tag{5.5}$$

$$\left|\partial_z\big[F_j(u)-F_j(\hat{u})\big]\right|<\epsilon\left(|\widetilde{W}|+\frac{|W|}{r}\right), \tag{5.6}$$

$$\left|\partial_z\big[\lambda_j(u)-\lambda_j(\hat{u})\big]\partial_r\hat{R}_j\right|<(u_p+\epsilon)\frac{|\widetilde{W}|}{r}+\frac{\epsilon|W|}{r^2}, \tag{5.7}$$

$$\left|\partial_z\big[\lambda_j(u)-\lambda_j(\hat{u})\big]\partial_r W_j\right|<\epsilon\left(\frac{|\widetilde{W}|}{r}+\frac{|W|}{r^2}\right)+\frac{2u_p}{\sigma}|\widetilde{W}|^2, \tag{5.8}$$

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$$\left|\left[\lambda_j(u)-\lambda_j(\hat{u})\right]\partial_z\partial_r\hat{R}_j\right| < (u_p+\epsilon)\frac{|W|}{r^2} \tag{5.9}$$

and

$$\left|\partial_z\lambda_j(\hat{u})\partial_r W_j\right| < \epsilon\left(\frac{|\widetilde{W}|}{r}+\frac{|W|}{r^2}\right) \tag{5.10}$$

in $\Omega(r_1,t_1)$. Thus, we have

$$\left|\frac{H_i}{\lambda_i(u)}\right| < (1+\epsilon)\left(\frac{|\widetilde{W}|}{r}+\frac{|W|}{r^2}+\frac{|\widetilde{W}|^2}{\sigma}\right) \quad \text{in } \Omega(r,t), \tag{5.11}$$

where $i=1,2$.

In what follows, we denote by $\mathcal{D}$ various constants that depend only on u_p and σ.

Using (4.15), (4.16) and Lemma 2.4, we know that for any given small $\epsilon>0$, if σ and ε are sufficiently small then,

$$|\widetilde{W}_2| < |\widetilde{W}_1| + \frac{\varepsilon}{r^\delta(1-u_p^2-\epsilon)\sigma}|\widetilde{W}| + \frac{\mathcal{D}\varepsilon}{r^{1+\delta}} \quad \text{on } \Gamma_p.$$

Hence, when ε is sufficiently small, we have

$$|\widetilde{W}_2| < (1+\epsilon\sigma)|\widetilde{W}_1| + \frac{\mathcal{D}\varepsilon}{r^{1+\delta}} \quad \text{on } \Gamma_p. \tag{5.12}$$

From (4.17) and (4.18), we know that for any given small $\epsilon>0$, when σ and ε are sufficiently small, there hold

$$\begin{aligned}
|\widetilde{W}_1| &< \Big(1-\frac{4-\epsilon}{u_p}\sigma+\frac{\mathcal{C}'\varepsilon}{r^\delta}\Big)|\widetilde{W}_2| + \frac{\mathcal{C}'\varepsilon}{r^\delta\sigma}|\widetilde{W}| + \frac{\sigma}{(3-3\epsilon)u_p}|\widetilde{W}| \\
&\quad + \Big(\frac{\mathcal{C}'\varepsilon}{r^\delta}+\frac{\sigma}{(3-3\epsilon)u_p}\Big)\frac{|W|}{r} + \Big(\frac{\mathcal{C}'\varepsilon}{r^\delta}+\frac{(4u_p+\epsilon)\sigma}{1-u_p^2}\Big)|\eta'| \\
&< \Big(1-\frac{4-\epsilon}{u_p}\sigma+\frac{\mathcal{C}'\varepsilon}{r_0^\delta}\Big)|\widetilde{W}_2| + \frac{\mathcal{C}'\varepsilon}{r_0^\delta\sigma}|\widetilde{W}| + \frac{\sigma}{(3-3\epsilon)u_p}|\widetilde{W}| \\
&\quad + \Big(\frac{\mathcal{C}'\varepsilon}{r_0^\delta}+\frac{\sigma}{(3-3\epsilon)u_p}\Big)\frac{|W|}{r} + \Big(\frac{\mathcal{C}'\varepsilon}{r_0^\delta}+\frac{(4u_p+\epsilon)\sigma}{1-u_p^2}\Big)|\eta'| \quad \text{on } \Gamma_s,
\end{aligned}$$

where $\mathcal{C}'$ depends only on $\mathcal{C}$ and u_p. Hence, when ε is sufficiently small, we have

$$|\widetilde{W}_1| < \Big(1-\frac{10(1-\epsilon)}{3u_p}\sigma\Big)|\widetilde{W}_2| + \mathcal{D}\frac{\varepsilon}{r^{1+\delta}} \quad \text{as } (r,t)\in\big\{t=\beta(r),\ r\le\phi_1(r_1,t_1)\big\}. \tag{5.13}$$

Using the characteristic method and the boundary conditions (5.11), (5.12) and (5.13), we get

$$\begin{aligned}
&|\widetilde{W}_1(r_1,t_1)| \\
&\le \big|\widetilde{W}_1(\phi_1,\psi_1)\big| + \int_{\phi_1}^{r_1}\frac{2(1+\epsilon)(\mathcal{C}_0+\mathcal{C}_1)\varepsilon}{r^{2+\delta}} + \frac{5\mathcal{C}_1^2\varepsilon^2}{r^{2+2\delta}}\mathrm{d}r \\
&< \Big(1-\frac{10(1-\epsilon)}{3u_p}\sigma\Big)|W_2(\phi_1,\psi_1)| + \frac{\mathcal{D}\varepsilon}{\phi_1^{1+\delta}} + \int_{\phi_1}^{r_1}\frac{2(1+\epsilon)(\mathcal{C}_0+\mathcal{C}_1)\varepsilon}{r^{2+\delta}} + \frac{5\mathcal{C}_1^2\varepsilon^2}{r^{2+2\delta}}\mathrm{d}r
\end{aligned}$$

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$$
\begin{aligned}
&< \Big(1-\frac{10(1-\epsilon)}{3u_p}\sigma\Big)|\widetilde{W}_2(\pi_2,\zeta_2)| + \frac{\mathcal{D}\varepsilon}{\phi_1^{1+\delta}} + \int_{\pi_2}^{r_1}\frac{2(1+\epsilon)(\mathcal{C}_0+\mathcal{C}_1)\varepsilon}{r^{2+\delta}} + \frac{5\mathcal{C}_1^2\varepsilon^2}{r^{2+2\delta}}\mathrm{d}r\\
&< \Big(1-\frac{10(1-\epsilon)}{3u_p}\sigma\Big)(1+\epsilon\sigma)|\widetilde{W}_1(\pi_2,\zeta_2)| + \frac{\mathcal{D}\varepsilon}{\pi_2^{1+\delta}} + \frac{\mathcal{D}\varepsilon}{\phi_1^{1+\delta}}\\
&\quad + \int_{\pi_2}^{r_1}\frac{2(1+\epsilon)(\mathcal{C}_0+\mathcal{C}_1)\varepsilon}{r^{2+\delta}} + \frac{5\mathcal{C}_1^2\varepsilon^2}{r^{2+2\delta}}\mathrm{d}r.
\end{aligned}
$$

Thus, using (1.9) we known that there exists a small $\epsilon > 0$ which is independent of σ and ε, such that when ε is sufficiently small there holds

$$
\begin{aligned}
|\widetilde{W}_1(r_1,t_1)| &< \Big(1-\frac{10(1-\epsilon)}{3u_p}\sigma\Big)\frac{(1+\epsilon\sigma)\mathcal{C}_1\varepsilon}{(1-\frac{2\sigma}{(3-3\epsilon)u_p})^{1+\delta}r_1^{1+\delta}} + \frac{2(1+\epsilon)\mathcal{C}_1\varepsilon}{r_1^{1+\delta}}\left(\frac{1}{(1-\frac{2\sigma}{(3-3\epsilon)u_p})^{1+\delta}}-1\right)\\
&\quad + \frac{\mathcal{D}\varepsilon}{r_1^{1+\delta}} + \frac{5\mathcal{C}_1\varepsilon}{r_1^{\delta}}\left(\frac{1}{(1-\frac{2\sigma}{(3-3\epsilon)u_p})^{1+2\delta}}-1\right)\frac{\mathcal{C}_1\varepsilon}{r_1^{1+\delta}}\\
&< \Big(1-\frac{\sigma}{2u_p}\Big)\frac{\mathcal{C}_1\varepsilon}{r_1^{1+\delta}} + \frac{\mathcal{D}\varepsilon}{r_1^{1+\delta}}.
\end{aligned}
$$

So, there exists a $\mathcal{C}_1$ depending only on σ and u_p, such that

$$
|\widetilde{W}_1(r_1,t_1)| < \frac{\mathcal{C}_1\varepsilon}{r_1^{1+\delta}}.
$$

Similarly, we can prove $|\widetilde{W}_2(r_1,t_1)| < \frac{\mathcal{C}_1\varepsilon}{r_1^{1+\delta}}$.

Therefore, by an argument of continuity, we can get Lemma 5.1. □

5.3 Global solution of the free boundary problem

Using (2.26), (4.2) and Lemma 5.1, we can get uniform a priori C^1 norm estimates of W_j $(j=1,2)$ and η. Therefore, by the local existence result and the standard continuity extension method, one can obtain the global existence of a classical solution; see for example [9, 10, 19]. We have the following theorem.

Theorem 5.1 *For sufficiently small σ and ε, the free boundary problem (1.6), (3.1) and (3.2) admits a global classical solution.*

From Theorem 5.1, we immediately have Theorem 1.1.

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Conflicts of interest

The author has no conflicts of interest.

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