Adv. Appl. Prob. **49,** 1–23 (2017)
*doi:*10.1017/apr.2016.76
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SHEARER'S POINT PROCESS, THE HARD-SPHERE MODEL, AND A CONTINUUM LOVÁSZ LOCAL LEMMA

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Abstract

A point process is R-dependent if it behaves independently beyond the minimum distance R. In this paper we investigate uniform positive lower bounds on the avoidance functions of R-dependent simple point processes with a common intensity. Intensities with such bounds are characterised by the existence of Shearer's point process, the unique R-dependent and R-hard-core point process with a given intensity. We also present several extensions of the Lovász local lemma, a sufficient condition on the intensity and R to guarantee the existence of Shearer's point process and exponential lower bounds. Shearer's point process shares a combinatorial structure with the hard-sphere model with radius R, the unique R-hard-core Markov point process. Bounds from the Lovász local lemma convert into lower bounds on the radius of convergence of a high-temperature cluster expansion of the hard-sphere model. This recovers a classic result of Ruelle (1969) on the uniqueness of the Gibbs measure of the hard-sphere model via an inductive approach of Dobrushin (1996).

Keywords: Avoidance function; the Lovász local lemma; hard-sphere model; partition function; R-dependent point process; R-hard-core

2010 Mathematics Subject Classification: Primary 60G55
Secondary 60G60; 82B05; 05C69

1. Introduction

A *point process* (PP) ξ on a complete separable metric space is *R-dependent*, if events of ξ based on Borel sets having mutual distance greater than or equal to R are independent. In this paper we deal only with simple PPs. Natural examples of R-dependent PPs are as follows. A Poisson PP, which is even 0-*dependent*. Range $R/2$ dependent thinnings of Poisson PPs [26], [35]. Poisson cluster PPs [4], [10] with an offspring distribution supported on a sphere of radius $R/2$ around a cluster centre point. Local constructions based on a Poisson PP, such as taking the centres of circumscribed circles of radius less than $R/2$ of triangles formed by triples of points from the Poisson PP. Determinantal and permanental PPs [7], [15], [34] with a kernel of bounded range R are also R-dependent.

If the space is discrete then a simple PP is a *Bernoulli random field* (BRF), an at most countable collection of $\{0, 1\}$-valued random variables indexed by the space. The study of R-dependent BRFs has a long history in the theory of discrete stochastic processes [1], [2], [8], [9], [12], [13], [22], [25]. Further uses of R-dependent BRFs are within the scope of the probabilistic method in combinatorics building on the *Lovász local lemma* (LLL) [3], [16] and in graph colouring [21], [23].

Received 1 July 2014; revision received 9 February 2016.
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https://doi.org/10.1017/apr.2016.76 Published online by Cambridge University Press

Let $(\mathcal{X}, \delta)$ be the complete separable metric space. Without loss of generality, rescaling the metric reduces the discussion to 1-dependence. Let $\mathcal{B}^{\mathrm{b}}$ and $\mathcal{B}$ be the set of bounded and all Borel sets on $\mathcal{X}$, respectively. Let $\mathcal{M}^{\mathrm{b}}$ be the space of boundedly finite Borel measures. The intensity measure of a PP is the expected number of points in a given Borel set B. For $M \in \mathcal{M}^{\mathrm{b}}$, let $\mathcal{C}(M)$ be the class of *simple and* 1*-dependent PP laws with intensity measure* M. Where there is no danger of confusion, identify a PP and its law. For $\xi \in \mathcal{C}(M)$ and $B \in \mathcal{B}^{\mathrm{b}}$, the *avoidance probability* is the probability of ξ having no points in B. The *avoidance function* of ξ maps $\mathcal{B}^{\mathrm{b}}$ to the avoidance probabilities of ξ.

In this paper we study uniform lower bounds on the avoidance function of PPs in $\mathcal{C}(M)$. First, Section 2.1 extends a dichotomy by Shearer [33] from BRFs to PPs. For large intensity measure M, there exists a PP in $\mathcal{C}(M)$ with an avoidance function vanishing on some bounded Borel set of positive M-measure (*zero phase*). For small intensity measure M, there is a uniform positive lower bound on the avoidance function of PPs in $\mathcal{C}(M)$ (*positive phase*).

Second, for M in the positive phase, Section 2.3 generalises a construction by Shearer [33] of the unique PP in $\mathcal{C}(M)$ with minimal avoidance function. A set of points is r*-hard-core*, if its points have mutual distance at least r. A PP ξ is r-hard-core, if its realisations are almost-surely so. An r-hard-core and R-dependent PP must have $r \leq R$. Together R-dependence and R-hard-core imply uniqueness of the PP law and existence only for small intensity measures. If a PP with these properties exists, we call it *Shearer's PP*. Shearer's PP has the minimal avoidance function in $\mathcal{C}(M)$, because it avoids clusters of points and spreads its points all over space. The existence of Shearer's PP describes the positive phase.

In Section 2.4 we recall the *hard-sphere model*, the unique Markov PP with range R interaction and an R-hard-core. The partition function of the hard-sphere model and the avoidance function of Shearer's PP have the same algebraic structure. Shearer's PP exists for a given intensity measure M, if and only if the cluster expansion of the hard-sphere model converges uniformly and absolutely at negative fugacity $-M$, generalising the identification in the BRF case [**?**].

Finally, in Section 2.5 we generalise the LLL [16] to the case of PPs. Here, a LLL denotes a *sufficient condition* on $\mathcal{X}$ and M for a *uniform exponential lower bound* on the avoidance functions in $\mathcal{C}(M)$ and to be in the positive phase. The core idea is to derive global properties, i.e. being in the positive phase, from local properties. On $\mathbb{R}^d$, this yields an explicit and uniform upper bound on the *empty space functions* F [11, Section 15.1] and J functions [**?**] of isotropic and 1-dependent PPs. For the hard-sphere model, a LLL becomes a sufficient condition for the convergence of the cluster expansion and yields a lower bound on the radius of convergence. This improves a classic lower bound via cluster expansion techniques by Ruelle [31] by a short inductive argument of Dobrushin [14] of less than two pages.

In Section 2.6 we explain the relevance of exponential lower bounds on the avoidance function for uniform stochastic domination by a Poisson PP. In Section 3 we discuss variations of 1-dependence and a key inequality of the avoidance functions of 1-dependent PPs. The remaining sections contain proofs.

2. Results

2.1. Uniform bounds on the avoidance functions

The first question is the existence of 1-dependent PPs with a given intensity measure. Let $\mathcal{M}^{\varnothing} := \{M \in \mathcal{M}^{\mathrm{b}} \mid \mathcal{C}(M) = \varnothing\}$. In Proposition 3.4 we show that atoms of mass greater than 1 in the intensity measure are the only obstacle. Thus, $\mathcal{C}(M) \neq \varnothing$, if and only if $M \in \mathcal{M}^{\mathrm{b}} \setminus \mathcal{M}^{\varnothing} = \{M \in \mathcal{M}^{\mathrm{b}} \mid \text{for all } x \in \mathcal{X}\colon M(\{x\}) \leq 1\}$.

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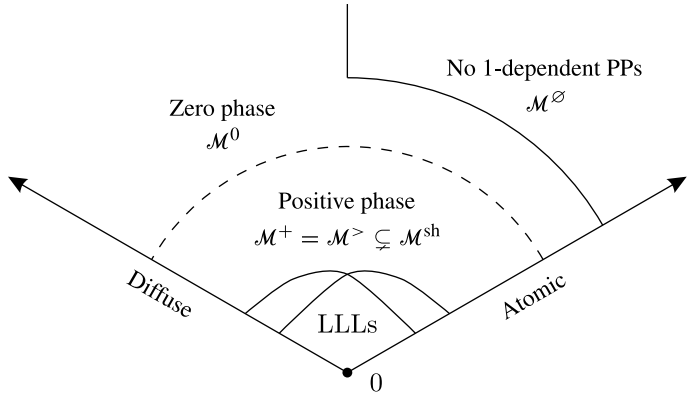


FIGURE 1: The cone of boundedly finite Borel measures $\mathcal{M}^{\mathrm{b}}$, with diffuse measures on the left axis, atomic measures on the right axis, and the zero measure in the apex. The positive and zero phases ($\mathcal{M}^+$ and $\mathcal{M}^0$) and $\mathcal{M}^{\varnothing}$ partition the cone. The sets $\mathcal{M}^+$, $\mathcal{M}^{\mathrm{sh}}$, and $\mathcal{M}^>$ are down-sets, whereas $\mathcal{M}^{\varnothing}$ and $\mathcal{M}^0 \sqcup \mathcal{M}^{\varnothing}$ are up-sets. The dashed line represents the atypical closure properties of $\mathcal{M}^>$ and contains $\mathcal{M}^{\mathrm{sh}} \setminus \mathcal{M}^>$. The LLLs carve out parts of the positive phase and guarantee exponential lower bounds.

In this paper we are concerned with lower bounds on the avoidance functions of 1-dependent PPs. The first result extends a dichotomy by Shearer [33] from BRFs to PPs.

Theorem 2.1. *If $M \in \mathcal{M}^{\mathrm{b}} \setminus \mathcal{M}^{\varnothing}$ then it falls into one of two phases. In the zero phase, there is a $\xi \in \mathcal{C}(M)$ with zero avoidance probability on some $B \in \mathcal{B}^{\mathrm{b}}$ of positive measure. That is, $M(B) > 0$ and $\mathbb{P}(\xi(B) = 0) = 0$. In the positive phase, there is a unique $\mu \in \mathcal{C}(M)$ minimizing the (conditional) avoidance probabilities uniformly in space and the class. That is, for every $\xi \in \mathcal{C}(M)$ and all $A, B \in \mathcal{B}^{\mathrm{b}}$, we have*

$$\mathbb{P}(\xi(B) = 0 \mid \xi(A) = 0) \geq \mathbb{P}(\mu(B) = 0 \mid \mu(A) = 0) > 0.$$

Theorem 2.1 follows from Corollary 2.1 and Theorem 2.4. In Section 2.3 we describe the distinguished PP μ in the positive phase. Let $\mathcal{M}^0$ and $\mathcal{M}^+$ be the subsets of $\mathcal{M}^{\mathrm{b}} \setminus \mathcal{M}^{\varnothing}$ being in the zero phase and positive phase, respectively. See also Figure 1.

2.2. The generating function

A *configuration* is a countable collection of points in $\mathcal{X}$. A configuration C is 1-*hard-core*, if its points have mutual distance at least 1, i.e. for all $\{x, y\} \subseteq C$: $\delta(x, y) \geq 1$. In the classic case of the metric space $(V, 2d)$ derived from a graph $G := (V, E)$ with geodesic metric d, the 1-hard-core configurations are the graph-theoretic *independent* sets of G. For $n \in \mathbb{N}_0$, let h_n be the indicator function of 1-hard-core tuples in $\mathcal{X}^n$.

Definition 2.1. The *generating function Z of weighted* 1-*hard-core configurations* is $\mathcal{B}^{\mathrm{b}} \times \mathcal{M}^{\mathrm{b}} \to \mathbb{R}$ and

$$(B, M) \mapsto \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} h_n(x_1, \ldots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i).$$

In Proposition 4.1 we show that Z is well-defined. If M is diffuse then $Z(B, M)$ is the expectation of a functional of a Poisson PP of intensity M. Such a representation fails for intensity measures containing atoms. The alternating sign in $Z(B, M)$ is a convenience to avoid using a negative measure as an argument to Z throughout most of this paper. For unambiguous

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choices of M, $Z(B)$ abbreviates to $Z(B, M)$. While $Z(\varnothing) = 1$ always holds, a central topic is which M admit $Z(B, M) \geq 0$ uniformly in B. For all $A, B \in \mathcal{B}^{\mathrm{b}}$ with $Z(B) > 0$, a key quantity is $z(A, B) := Z(A \cup B)/Z(B)$. In Section 4 we discuss the properties of Z and z in detail.

2.3. Shearer's PP

In this section we are concerned with 1-dependent PPs with a 1-hard-core. For a given intensity measure, there is at most one such PP. If it exists, we call it *Shearer's PP*, as it generalises the BRF construction of Shearer [33].

Theorem 2.2. *If a* 1*-hard-core* $\eta_M \in \mathcal{C}(M)$ *exists then its law is unique. Its avoidance function is* Z*, i.e. for each* $B \in \mathcal{B}^{\mathrm{b}}$, $\mathbb{P}(\eta_M(B) = 0) = Z(B)$*. Such a PP exists for all intensity measures in* $\mathcal{M}^{\mathrm{sh}} := \{M \in \mathcal{M}^{\mathrm{b}} \mid \text{for all } B \in \mathcal{B}^{\mathrm{b}} \colon Z(B, M) \geq 0\}$.

The proofs of this section's statements are in Section 5. If Shearer's PP exists then it minimises the (conditional) avoidance probabilities within the class of 1-dependent PPs with the same intensity measure.

Theorem 2.3. *If* $M \in \mathcal{M}^{\mathrm{sh}}$ *then, for all* $\xi \in \mathcal{C}(M)$ *and* $A, B \in \mathcal{B}^{\mathrm{b}}$ *with* $Z(B) > 0$*,*

$$\mathbb{P}(\xi(A) = 0 \mid \xi(B) = 0) \geq z(A, B) \geq 0.$$

If Shearer's PP has a positive avoidance function then it is the unique PP μ from the positive phase in Theorem 2.1. Let $\mathcal{M}^{>} := \{M \in \mathcal{M}^{\mathrm{b}} \mid \text{for all } B \in \mathcal{B}^{\mathrm{b}} \colon Z(B, M) > 0\}$.

Corollary 2.1. *If* $M \in \mathcal{M}^{>}$ *then* $M \in \mathcal{M}^{+}$ *and Shearer's PP* η_M *has the minimal (conditional) avoidance function in* $\mathcal{C}(M)$*. In short,* $\mathcal{M}^{>} \subseteq \mathcal{M}^{+}$.

Corollary 2.1 is a direct consequence of Theorem 2.3. On the other hand, if Shearer's PP does not have a positive avoidance function on $\mathcal{B}^{\mathrm{b}}$ or does not exist, then there is a PP with zero avoidance probability on some bounded Borel set. This puts the intensity measure M in the zero phase.

Theorem 2.4. *If* $M \notin \mathcal{M}^{>}$ *and* $M \notin \mathcal{M}^{\varnothing}$ *then* $M \in \mathcal{M}^{0}$.

Theorem 2.4 and Corollary 2.1 together imply that $\mathcal{M}^{>} = \mathcal{M}^{+}$.

Independent thinning of Shearer's PP decreases the intensity measure and preserves the two characterising properties of 1-dependence and 1-hard-core. A larger intensity measure decreases the avoidance function or inhibits the existence of 1-dependent PPs.

Theorem 2.5. *The sets* $\mathcal{M}^{\mathrm{sh}}$ *and* $\mathcal{M}^{>} = \mathcal{M}^{+}$ *are down-sets, i.e. closed under decreasing the measure. The set* $\mathcal{M}^{\varnothing}$ *is an up-set, i.e. closed under increasing the measure. The set* $\mathcal{M}^{0}$ *is an up-set, as long as atoms do not increase beyond mass* 1.

In general, Shearer's PP differs from all other R-independent models in the introduction. Probabilistic constructions of Shearer's PP are known only in special cases. Details are given in Section 5.4.

2.4. The hard-sphere model

Another simple 1-hard-core PP related to the function Z is the *hard-sphere model*. It is a Markov PP with the most repulsive range-1 interaction [31, Section 1.2.2]. A common visualisation of the hard-sphere model is as a collection of nonoverlapping open spheres with radius $\frac{1}{2}$ representing the hard cores of atoms.

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The hard-sphere model $h_{B,M}$ in a *finite volume* $B \in \mathcal{B}^{\mathrm{b}}$ with *fugacity* $M \in \mathcal{M}^{\mathrm{b}}$ and empty boundary conditions has Janossy intensity

$$\mathbb{P}(h_{B,M} = \mathrm{d}(x_1, \dots, x_n)) = \frac{h_n(x_1, \dots, x_n)}{Z(B, -M)} \prod_{i=1}^{n} M(\mathrm{d}x_i).$$

The normalising factor $Z(B, -M)$ is the *partition function* of the hard-sphere model. The argument $-M$ cancels the alternating sign in the definition of Z. For diffuse M, $h_{B,M}$ is equal to a Poisson(M) PP on B conditioned to be 1-hard-core.

The analysis of the hard-sphere model centres on the partition function and derived quantities, in particular ratios (*reduced correlations*) and its logarithm (*free energy*). Lower bounds on Z and z and their logarithms at negative fugacity play a key role in the low fugacity case (the *high temperature* case) and establish uniqueness of the Gibbs measure [17], [**?**]. A well-known tool is the *cluster expansion*, a series expansion of $\log Z$ [**?**]. It fails first at negative fugacities, that is along the boundary of $\mathcal{M}^{>}$. Details are given in Section 4.2.

2.5. Sufficient conditions for exponential bounds

This section contains several LLLs in Theorems 2.6 and 2.8 and Corollary 2.2. In Theorem 2.7, we discuss the relationship between $\mathcal{M}^{>}$ and $\mathcal{M}^{\mathrm{sh}}$. The proofs are given in Section 6.

Let $\mathcal{B}^1$ be the set of Borel sets of diameter less than 1. Let $\sqcup$ denote the disjoint union. The *unit partition number* of a Borel set $B \in \mathcal{B}$ is

$$\kappa(B) := \inf\left\{ k \in \mathbb{N} \;\middle|\; \text{there exist } A_1, \dots, A_k \in \mathcal{B}^1 \colon B = \bigsqcup_{i=1}^{k} A_i \right\}.$$

Let $U(x)$ be the *open unit sphere* around x. The most straightforward case are spaces where the unit-scale structure has a *uniform exponential growth bound* on spheres of radius r. Spaces such as $\mathbb{R}^d$, the hyperbolic plane or graphs with uniformly bounded degree fulfil this, but infinite-dimensional metric spaces such as l_2 do not. This is equivalent to the bound

$$K := \sup\{\kappa(U(x)) \mid x \in \mathcal{X}\} < \infty. \tag{2.1}$$

The first sufficient condition generalises the symmetric LLL [16] and Dobrushin's condition [14], respectively.

Theorem 2.6. *Let $M \in \mathcal{M}^{\mathrm{b}}$. If (2.1) holds and, for each $A \in \mathcal{B}^1$,*

$$M(A) \le \frac{(K+1)^{K+1}}{(K+2)^{K+2}}, \tag{2.2a}$$

then $M \in \mathcal{M}^{>}$ and, for all $A, B \in \mathcal{B}^{\mathrm{b}}$,

$$z(A, B) \ge \left(\frac{K+1}{K+2}\right)^{\kappa(A \setminus B)} > 0. \tag{2.2b}$$

Under condition (2.1), a slight loss of precision sharpens the positive lower bound in the positive phase to exponential lower bounds. In Proposition 2.7 we show that $\mathcal{M}^{\mathrm{sh}}$ is a subset of the closure of $\mathcal{M}^{>}$. In general, $\mathcal{M}^{\mathrm{sh}}$ is not the closure of $\mathcal{M}^{>}$ [32, Chapter 8]. It is so on a finite graph, but already fails on a connected infinite graph.

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Theorem 2.7. *Let $\alpha > 0$. If $(1+\alpha)M \in \mathcal{M}^{\mathrm{sh}}$ and (2.1) holds, then $M \in \mathcal{M}^{>}$ and, for all A, $B \in \mathcal{B}^{\mathrm{b}}$, $z(A, B) \geq (\alpha/(1+\alpha))^{\kappa(A\setminus B)} > 0$.*

The second sufficient condition generalises the asymmetric LLL [16]. It resembles a continuous version of the Kotecký–Preiss condition [24].

Theorem 2.8. *Let $M, N \in \mathcal{M}^{\mathrm{b}}$ with N being absolutely continuous with respect to M. If, for each $A \in \mathcal{B}^1$,*

$$\int_A \exp(N(U(x) \setminus A))M(\mathrm{d}x) \leq 1 - \exp(-N(A))\,, \tag{2.3a}$$

then $M \in \mathcal{M}^{>}$ and, for all $A, B \in \mathcal{B}^{\mathrm{b}}$,

$$z(A, B) \geq \exp(-N(A \setminus B)) > 0. \tag{2.3b}$$

A stronger and more practical version of condition (2.3a) is

$$M(A)\exp(N(U(x) \setminus A)) \leq 1 - \exp(-N(A)) \quad \text{for all } A \in \mathcal{B}^1,\ M\text{-a.e. } x \in A, \tag{2.4}$$

where M-a.e. stands for to M-almost everywhere. Specialising (2.4) to the space $\mathbb{R}^d$ with the Lebesgue measure yields an explicit bound.

Corollary 2.2. *Consider $\mathcal{X} = \mathbb{R}^d$ with the Lebesgue measure $\mathcal{L}$. Let V be the volume of the d-dimensional unit sphere. For $\lambda \leq 1/(\mathrm{e}V)$, let α be the unique solution of $\lambda = \alpha\exp(-\alpha V)$ in $[0, 1/(\mathrm{e}V)]$. This implies that $\lambda\mathcal{L} \in \mathcal{M}^{>}$ and, for all $A, B \in \mathcal{B}^{\mathrm{b}}$, $z(A, B) \geq \exp(-\alpha\mathcal{L}(A \setminus B)) > 0$.*

In the context of the hard-sphere model, Corollary 2.2 yields a uniform upper bound on finite-volume free energies. That is, $\sup\{-(\log Z(B, -\lambda\mathcal{L}))/\mathcal{L}(B) \mid B \in \mathcal{B}^{\mathrm{b}}\} \leq \alpha$. Together with taking the limit of $-(\log Z(B, -\lambda\mathcal{L}))/\mathcal{L}(B)$ in the van Hove sense [31, Definition 2.1.1], this implies the existence and complete analyticity of the infinite-volume free energy for fugacities less than $1/(\mathrm{e}V)$. Thus, from Corollary 2.2 we have an alternative proof, avoiding cluster expansion, of a classic result for uniqueness of the Gibbs measure of the hard-sphere model on $\mathbb{R}^d$ for small fugacities [31, Section 4.5, Equation (5.2)].

The LLLs and derived bounds here are not optimal. For example, the optimal bound on $\mathbb{R}$ in the context of Corollary 2.2 is $1/\mathrm{e}$ instead of $1/(2\mathrm{e})$ [19]. There are two reasons not to pursue further improvements of the LLLs here. First, improvements in the main cases of interest are already present in the literature on the hard-core and hard-sphere models [17], [18]. Second, in the context of Corollary 2.2, the best bounds differ by at most a factor of e [31, Section 4.5, Equation (5.17)].

2.6. Stochastic domination and order

A PP law φ *stochastically dominates* a PP law ξ, if there is a coupling of them such that φ contains almost surely all of ξ's points. Stochastic domination implies that φ's avoidance function is smaller than ξ's avoidance function. In the context of 1-dependent BRFs on locally finite graphs, the existence of Shearer's PP is equivalent to uniform stochastic domination by a Bernoulli product field [25], [**?**]. Uniform exponential lower bounds from the LLLs are a first step towards extending the stochastic domination result to 1-dependent PPs.

Stochastic domination of Shearer's PP by a Poisson PP would permit a probabilistic construction and simulation by thinning the dominating Poisson PP. The intrinsic coupling of Shearer's PP by independent thinning implies that stochastic domination for the largest intensities in $\mathcal{M}^{\mathrm{sh}}$ suffices [30].

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Using the terminology from [5], η_M is *weakly sub-Poisson*. This means that its moments and avoidance function are both smaller than those of a Poisson(M) PP. The first follows from Proposition 5.1 and (4.1). The second follows by applying [5, Proposition 3.1] to the implication of Proposition 4.4, that, for disjoint $A, B \in \mathcal{B}^{\mathrm{b}}$,

$$\mathbb{P}(\eta_M(A \sqcup B) = 0) = Z(A \sqcup B, M) \leq \mathbb{P}(\eta_M(A) = 0)\mathbb{P}(\eta_M(B) = 0).$$

Being weakly sub-Poisson yields concentration inequalities [6, Section 3.3].

3. About 1-dependent PPs

In this section we discuss different notions of 1-dependence in (3.1). In Proposition 3.1 we characterise the avoidance functions of 1-dependent PPs. In Proposition 3.3 we present a key inequality of avoidance functions of 1-dependent PPs. In Proposition 3.4 we investigate the existence of simple 1-dependent PPs.

For $A \in \mathcal{B}$ and a PP ξ, let $A\xi$ be the restriction of ξ to A. Recall that the metric is δ. A PP ξ is *strong* 1-*dependent*, if, for all $A, B \in \mathcal{B}$,

$$\delta(A, B) := \inf\{\delta(x, y) \mid x \in A,\ y \in B\} \geq 1 \quad \Longrightarrow \quad A\xi \text{ is independent of } B\xi. \tag{3.1a}$$

All the examples in the introduction, the PP in the zero phase of Theorem 2.1, the PP counterexample in the proof of Theorem 2.4, Shearer's PP, and the PP in Proposition 3.4 are strong 1-dependent.

Proposition 3.1. *A PP is strong* 1-*dependent, if and only if its avoidance function Q is* 1-*multiplicative. That is, for all $A, B \in \mathcal{B}$, $\delta(A, B) \geq 1$ implies that $Q(A \cup B) = Q(A)Q(B)$.*

Proof. The necessity is evident, and the sufficiency follows from the fact that the avoidance function determines the law of a simple PP [11, Section 9.2.XIII]. □

A PP ξ is *weak* 1-*dependent*, if, for M-a.e. x, and every $B \in \mathcal{B}$,

$$\delta(x, B) := \delta(\{x\}, B) \geq 1 \quad \Longrightarrow \quad \mathbb{P}_x(\xi(B) = 0) = \mathbb{P}(\xi(B) = 0). \tag{3.1b}$$

In other words, weak 1-dependence reduces Palm probabilities for events at distance more than one away from the base point to normal probabilities.

Proposition 3.2. *A strong* 1-*dependent PP is weak* 1-*dependent.*

Examples of weak, but not strong, 1-dependent PPs are mixtures of random shifts of strong 1-dependent BRFs in [25, Section 5].

Proof of Proposition 3.2. Let $\xi \in \mathcal{C}(M)$. The *Campbell measure* C [11, Equation (13.1.1a)] on $(B, E) \in \mathcal{B} \times \sigma(\xi)$ is $C(B \times E) := \sum_{n=1}^{\infty} n\mathbb{P}(\xi(B) = n,\ \xi \in E)$. If $A, B \in \mathcal{B}^{\mathrm{b}}$ with $\delta(A, B) \geq 1$ and $E \in \sigma(A\xi)$ with embedding E' into $\sigma(\xi)$, then strong 1-dependence (3.1a) allows us to factorise $C(B \times E') = \sum_{n=1}^{\infty} n\mathbb{P}(\xi(B) = n,\ A\xi \in E) = \mathbb{E}(\xi(B))\mathbb{P}(A\xi \in E)$. Hence, M-a.e., the Palm density on $\sigma(A\xi)$ simplifies to $\mathbb{P}_x(\xi \in E) = \mathrm{d}C(\cdot \times E)/\mathrm{d}\mathbb{E}(\xi(\{\cdot\})) = \mathbb{P}(\xi \in E)$. As the σ-algebras of the form $\sigma(A\xi)$, for $A \in \mathcal{B}^{\mathrm{b}}$ with $A \cap U(x) = \varnothing$, generate $\sigma((X \setminus U(x))\xi)$, weak 1-dependence follows. □

The inequality in Proposition 3.3 lies at the core of the dichotomy in Theorem 2.1 and the minimality of the avoidance function of Shearer's PP in Theorem 2.3. In Proposition 4.2 we show that the 1-hard-core of Shearer's PP makes it the only 1-dependent PP to fulfil the

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inequality uniformly as an equality. The inequality motivates the relaxation from strong to weak 1-dependence. Thus, $\mathcal{C}(M)$ may be extended to be the class of weak 1-dependent simple PPs with intensity measure M.

Proposition 3.3. *For all $\xi \in \mathcal{C}(M)$, $A \in \mathcal{B}^1$, and $B \in \mathcal{B}^{\mathrm{b}}$ with $\mathbb{P}(\xi(B) = 0) > 0$,*

$$\frac{\mathbb{P}(\xi(A \cup B) = 0)}{\mathbb{P}(\xi(B) = 0)} \geq 1 - \int_{A \setminus B} \frac{\mathbb{P}(\xi(B \setminus U(x)) = 0)}{\mathbb{P}(\xi(B) = 0)} M(\mathrm{d}x).$$

Proof. With the Campbell measure C and the event $E := \{\omega \mid \xi(B) = 0\}$,

$$\begin{aligned}
\mathbb{P}(\xi(A \setminus B) \geq 1,\ \xi(B) = 0) &\leq \sum_{n=1}^{\infty} n \mathbb{P}(\xi(A \setminus B) = n,\ \xi \in E) \\
&= C((A \setminus B) \times E) \\
&= \int_{A \setminus B} \mathbb{P}_x(\xi(B) = 0) M(\mathrm{d}x) \\
&\leq \int_{A \setminus B} \mathbb{P}_x(\xi(B \setminus U(x)) = 0) M(\mathrm{d}x) \\
&\overset{(3.1\mathrm{b})}{=} \int_{A \setminus B} \mathbb{P}(\xi(B \setminus U(x)) = 0) M(\mathrm{d}x)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(\xi(A \cup B) = 0) &= \mathbb{P}(\xi(B) = 0) - \mathbb{P}(\xi(A \setminus B) \geq 1,\ \xi(B) = 0) \\
&\geq \mathbb{P}(\xi(B) = 0) - \int_{A \setminus B} \mathbb{P}(\xi(B \setminus U(x)) = 0) M(\mathrm{d}x).
\end{aligned}$$

□

Proposition 3.4. *Let $M \in \mathcal{M}^{\mathrm{b}}$. If M has no atom of mass greater than 1 then there exists a strong 0-dependent PP with intensity measure M. If M has an atom of mass greater than 1 then no simple PP with intensity measure M exists.*

Proof. For each $\xi \in \mathcal{C}(M)$ and atom x of M, $1 \geq \mathbb{E}(\xi(\{x\})) = M(\{x\}) =: m_x$. Hence, an atom of mass greater than 1 contradicts the simpleness of the PP.

For the converse, let $M \in \mathcal{M}^{\mathrm{b}}$ without atoms greater than 1. Let $\mathcal{A}$ and $\mathcal{D}$ be the atomic and diffuse support domains of M, respectively. Let $\mathcal{A}_=$ and $\mathcal{A}_<$ be the locations of atoms of mass equal to or less than 1, respectively. Construct a measure N with the same atomic and diffuse domains as follows. On $\mathcal{D}$, let $N \cap \mathcal{D} := M \cap \mathcal{D}$. On $\mathcal{A}$, let $n_x := N(\{x\}) := -\log(1 - m_x)$, if $x \in \mathcal{A}_<$, and 0 otherwise. For $B \in \mathcal{B}^{\mathrm{b}}$, let $m_B := \max\{m_x \mid x \in B \cap \mathcal{A}_<\}$. As $M(B) < \infty$, so is $m_B < 1$. Also, $N(B \setminus \mathcal{A}) = M(B \setminus \mathcal{A}) < \infty$. The inequality

$$-\log(1 - y) = \sum_{n=1}^{\infty} \frac{y^n}{n} \leq \sum_{n=1}^{\infty} y^n = \frac{y}{1 - y} \quad \text{for all } y \in [0, 1[, \tag{3.2}$$

entails the bounded finiteness of N on $\mathcal{A}$ via

$$N(B \cap \mathcal{A}) = -\sum_{x \in B \cap \mathcal{A}_<} \log(1 - m_x) \overset{(3.2)}{\leq} \sum_{x \in B \cap \mathcal{A}_<} \frac{m_x}{1 - m_x} \leq \frac{M(B \cap \mathcal{A}_<)}{1 - m_B} < \infty.$$

The aim is to construct a strong 0-dependent PP with intensity M. Let φ be the Poisson PP with intensity N. It may not be simple because of atoms in N. Let $\varphi^\bullet$ be its simple support PP,

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collapsing multiple points of φ. Let $\xi := \varphi^\bullet + \sum_{x\in\mathcal{A}_=}\delta_x$. The 0-dependence of ξ holds by construction and it remains to verify that ξ has intensity M. On the diffuse part of M, φ is simple, whence the intensity of ξ is M. On $\mathcal{A}_=$, φ is almost surely 0, but the atoms of mass 1 are present deterministically in ξ. For all $x \in \mathcal{A}_<$,

$$\mathbb{E}(\xi(\{x\})) = \mathbb{P}(\xi(\{x\}) = 1) = 1 - \mathbb{P}(\varphi(\{x\}) = 0) = 1 - \mathrm{e}^{-n_x} = m_x.$$

Hence, for each $B \in \mathcal{B}^{\mathrm{b}}$, the intensity measure is

$$\mathbb{E}(\xi(B \cap \mathcal{A}_<)) = \sum_{x\in B\cap\mathcal{A}_<} m_x = M(B \cap \mathcal{A}_<). \qquad \square$$

4. Properties of the generating function

Most properties of Z and z are trivial, if one knows that Z is the avoidance function of Shearer's PP. The properties are needed to establish the existence of Shearer's PP first, though. The equality in Proposition 4.2 (cf. the inequality in Proposition 3.3) and monotonicity in Proposition 4.4 are the most important ones.

Let $[n] := \{1, \dots, n\}$, with $[0] := \varnothing$. For a set S, let S^n be the Cartesian product of n copies of S, with $S^0 := \varnothing$. Empty products evaluate to 1 and empty sums to 0. For $B \in \mathcal{B}^{\mathrm{b}}$, $M \in \mathcal{M}^{\mathrm{b}}$, and $\lambda \in [0, \infty)$, let λM be the scaling of M by the factor λ and $M|_B$ the restriction of M to B.

4.1. Basic properties

For $B \in \mathcal{B}^{\mathrm{b}}$ and $M \in \mathcal{M}^{\mathrm{b}}$, a basic bound is

$$\int_{B^n} h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i) \le M(B)^n. \tag{4.1}$$

Also, for each $B \in \mathcal{B}^\kappa$, $n \in \mathbb{N}_0$, and $x_1, \dots, x_n \in B$,

$$h_n(x_1, \dots, x_n) = 1 \quad \Longleftrightarrow \quad n \le \kappa(B). \tag{4.2}$$

Proposition 4.1. *It holds that Z is well defined and 1-multiplicative as in Proposition 3.1.*

Proof. For $k \in \mathbb{N}_0$, the bound (4.1) implies that

$$\left|\sum_{n=k}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i)\right| \le \sum_{n=k}^{\infty} \frac{M(B)^n}{n!}.$$

For $k = 0$, this yields $|Z(B)| \le \exp(M(B))$. For $k \to \infty$, this shows the convergence of the series $Z(B)$.

Let $A, B \in \mathcal{B}^{\mathrm{b}}$ with $\delta(A, B) \ge 1$. The 1-hard-core condition trivially holds for pairs in $A \times B$. For $n, m \in \mathbb{N}_0$, $x_1, \dots, x_n \in A$ and $y_1, \dots, y_m \in B$, $h_{n+m}(x_1, \dots, x_n, y_1, \dots, y_m) = h_n(x_1, \dots, x_n)h_m(y_1, \dots, y_m)$. Hence, 1-multiplicativity follows from

$$\begin{aligned} Z(A \cup B) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(A\cup B)^n} (-1)^n h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} \left(\int_{A^j} (-1)^j h_j(x_1, \dots, x_j) \prod_{i=1}^{j} M(\mathrm{d}x_i)\right) \\ &\qquad \times \left(\int_{B^{n-j}} (-1)^{n-j} h_{n-j}(x_1, \dots, x_{n-j}) \prod_{i=1}^{n-j} M(\mathrm{d}x_i)\right) \end{aligned}$$

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$$= \left(\sum_{n=0}^{\infty} \frac{1}{n!} \int_{A^n} (-1)^n h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i)\right)$$
$$\times \left(\sum_{n=0}^{\infty} \frac{1}{n!} \int_{B^n} (-1)^n h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i)\right)$$
$$= Z(A)Z(B). \qquad \square$$

Proposition 4.2. *For all $A \in \mathcal{B}^1$ and $B \in \mathcal{B}^{\mathrm{b}}$ with $Z(B) > 0$,*

$$z(A, B) = 1 - \int_{A \setminus B} z(B \cap U(x),\ B \setminus U(x))^{-1} M(\mathrm{d}x). \tag{4.3}$$

Proof. By (4.2), at most one point of a 1-hard-core configuration C lies in A. If $y \in A \cap C$ then the 1-hard-core implies that $C \cap (B \setminus U(y)) = \varnothing$. This leads to

$$Z(B) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Bigg(\int_{(B \setminus A)^n} h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i)$$
$$+ n \int_{A \times B^{n-1}} h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i) \Bigg).$$

A point in A excludes the possibility of other points in A. Thus,

$$Z(B) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{(B \setminus A)^n} h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i)$$
$$- \int_A \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} \int_{B^{n-1}} h_n(x_1, \dots, x_{n-1}, y) \prod_{i=1}^{n-1} M(\mathrm{d}x_i) M(\mathrm{d}y)$$
$$= Z(B \setminus A) - \int_A \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{(B \setminus U(y))^n} h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i) M(\mathrm{d}y)$$
$$= Z(B \setminus A) - \int_A Z(B \setminus U(y)) M(\mathrm{d}y). \qquad \square$$

4.2. Cluster expansion and monotonicity

Cluster expansion is a series expansion of the logarithm of a generating series. It is a classic technique from statistical mechanics [31, Section 4.4].

Proposition 4.3. *Let $A, B \in \mathcal{B}^{\kappa}$, and $M \in \mathcal{M}^{\mathrm{b}}$ with $M|_{\mathcal{X} \setminus (A \cup B)} = 0$, i.e. it is concentrated on $A \cup B$. The statement that for all $N \leq M$: $N \in \mathcal{M}^{>}$ holds, if and only if*

$$\log z(A, B, M) = -\sum_{n=1}^{\infty} \frac{1}{n!} \int_{(A \cup B)^n \setminus B^n} P(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i)$$

is a convergent series, with $P(x_1, \dots, x_n) \in \mathbb{N}_0$ well defined.

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Proof. A cluster expansion of the partition function of a hard-sphere gas with radius 1 at negative fugacity [**?**] with an application of Penrose's identity [29] implies that the coefficients $P(x_1, \dots, x_n)$ have the desired properties. □

Proposition 4.4. *As long as they are positive, the functions Z and z are monotone decreasing in both space and measure.*

Proof. As $-\log z(A, B, M)$ is a sum over integrals over nonnegative integrands, it is monotone increasing in both the integration domains and the measure. Since $Z(\varnothing) = 1$, the same holds for the cluster expansion of Z. □

Monotonicity implies the following telescoping identity. For all $A, B \in \mathcal{B}^{\mathrm{b}}$ with $Z(B) > 0$ and every partition $\{A_i\}_{i=1}^n$ of $A \setminus B$ by elements of $\mathcal{B}^{\mathrm{b}}$,

$$z(A, B) = \prod_{i=1}^{n} z\Bigg(A_i, B \sqcup \bigsqcup_{j=1}^{i-1} A_j\Bigg). \tag{4.4}$$

4.3. Continuity properties

In this section we investigate the continuity properties of Z in both space and measure.

Proposition 4.5. *For $B \in \mathcal{B}^{\mathrm{b}}$ and $M \in \mathcal{M}^{\mathrm{b}}$, the function $f_B \colon [0, \infty] \to \mathbb{R}$, $\lambda \mapsto Z(B, \lambda M)$ is continuous.*

Proof. The scaling of M by λ implies that continuity of f_B at 1 suffices. This follows from the bound, for each ε with $|\varepsilon| < 1$,

$$\begin{aligned}
&|Z(B, (1+\varepsilon)M) - Z(B, M)| \\
&\quad = \left| \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} h_n(x_1, \dots, x_n)(1 - (1-\varepsilon)^n) \prod_{i=1}^{n} M(\mathrm{d}x_i) \right| \\
&\quad \le \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j=1}^{n} \binom{n}{j} |\varepsilon|^j \int_{B^n} \prod_{i=1}^{n} M(\mathrm{d}x_i) \\
&\quad \overset{(4.1)}{\le} \sum_{n=0}^{\infty} \frac{1}{n!} 2^n |\varepsilon| M(B)^n \\
&\quad = |\varepsilon| \exp(2M(B)).
\end{aligned}$$

□

Proposition 4.6. *For $B \in \mathcal{B}^{\mathrm{b}}$, let $\lambda_B := \inf\{\lambda \mid f_B(\lambda) < 0\}$. If $B \supseteq A \in \mathcal{B}^{\mathrm{b}}$ then $\lambda_B \le \lambda_A$. If $M(B) > 0$ then $\lambda_B = \min\{\lambda \mid f_B(\lambda) = 0\} \in (0, \infty)$ and $\lambda_B M|_B \in \mathcal{M}^{\mathrm{sh}}$.*

Proof. First prove the contra-variance of λ_B. Let $\Lambda < \lambda_B$. Proposition 4.3 implies that $f_B(\lambda) \le f_A(\lambda)$ for all $\lambda \le \Lambda$. Hence, $\lambda_A \ge \lambda_B$.

If $M(B) > 0$ then there is $B \supseteq A \in \mathcal{B}^1$ with $M(A) > 0$. Since $f_A(\lambda) = 1 - \lambda M(A)$, $\lambda_A = 1/M(A)$. Contra-variance yields $\lambda_B \le \lambda_A = 1/M(A) < \infty$. The continuity of f_B from Proposition 4.5 implies, together with $f_B(0) = Z(\varnothing) = 1$, that $\lambda_B > 0$.

The continuity of f_B renders the infimum a minimum. Contra-variance implies that, for all $\lambda \le \lambda_B$ and $B \supseteq A \in \mathcal{B}^{\mathrm{b}}$, $f_A(\lambda) \ge 0$. Proposition 4.3 implies that $\lambda_B M|_B \in \mathcal{M}^{\mathrm{sh}}$. □

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Proposition 4.7. *Let $B \in \mathcal{B}^{\mathrm{b}}$ be a continuity set of M, i.e. $M(\overline{B} \setminus B) = 0$, where $\overline{B}$ is the closure of B. For each sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{B}^{\mathrm{b}}$ decreasing to B with $M(B_1) > 0$, the limit $Z(B_n) \to Z(B)$ as $n \to \infty$ holds.*

Proof. For $A, B \in \mathcal{B}^{\mathrm{b}}$ with $A \subseteq B$ and $M(B) > 0$, bound the difference as

$$\begin{aligned}
|Z(B) - Z(A)| &= \left| \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n \setminus A^n} h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i) \right| \\
&\overset{(4.1)}{\leq} \sum_{n=1}^{\infty} \frac{1}{n!} (M(B)^n - M(A)^n) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j=1}^{n} \binom{n}{j} M(B \setminus A)^j M(A)^{n-j} \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n!} M(B \setminus A) 2^n \max\{1, M(B)\}^{n-1} \\
&\leq 2M(B \setminus A) \exp(\max\{1, M(B)\}).
\end{aligned}$$

Thus, $|Z(B_n) - Z(B)| \leq 2M(B_n \setminus B) \exp(\max\{1, M(B_1)\}) \to 0$ by the continuity of M at B as $n \to \infty$. □

4.4. Behaviour under the set difference operator

The *set difference operator* Δ at $A \in \mathcal{B}$ transforms a set function $\phi\colon \mathcal{B} \to \mathbb{R}$ into $\Delta(A)\phi\colon \mathcal{B} \to \mathbb{R},\ B \mapsto \phi(B) - \phi(B \cup A)$.

Lemma 4.1. *For all $n \in \mathbb{N}$ and $A_1, \dots, A_n, B \in \mathcal{B}$, and $\phi\colon \mathcal{B} \to \mathbb{R}$, iterated application of difference operators commutes and has the canonical form*

$$\Delta(\{A_1, \dots, A_n\}) := \Delta(A_1)(\cdots(\Delta(A_n)\phi)\cdots)(B) = \sum_{I\subseteq[n]} (-1)^{|I|} \phi\left(B \cup \bigcup_{i\in I} A_i\right). \tag{4.5}$$

In particular, $\Delta(\{A\}) = \Delta(A)$ and $\Delta(\varnothing)$ is the identity.

Lemma 4.2. *Let $\{A_i\}_{i=1}^n$ be disjoint Borel sets. For $I \subseteq [n]$, let $A_I := \bigsqcup_{i\in I} A_i$. For each $\phi\colon \mathcal{B} \to \mathbb{R}$, we have $\sum_{I\subseteq[n]} \Delta(\{A_i\}_{i\in I})\phi(A_{[n]\setminus I}) = \phi(\varnothing)$.*

Proposition 4.8. *If $M \in \mathcal{M}^{\mathrm{sh}}$ then Z is* completely monotone*, i.e. for each $n \in \mathbb{N}_0$ and all $A_1, \dots, A_n, B \in \mathcal{B}^{\kappa}$, the iterated difference $\Delta(\{A_i\}_{i=1}^n)Z(B)$ is nonnegative.*

Proof. Proceed by induction over n. For $n = 0$, $M \in \mathcal{M}^{\mathrm{sh}}$ implies that $\Delta(\varnothing)Z(B) = Z(B) \geq 0$. For $n = 1$, the monotonicity of Z from Proposition 4.4 implies that $\Delta(\{A_1\})Z(B) = \Delta(A_1)Z(B) = Z(B) - Z(B \cup A) \geq 0$. The induction step from n to $n + 1$ needs more preparation. For $I \subseteq [n]$, let $A_I := \bigcup_{i\in I} A_i$. For $x \in A_{n+1}$, let $B^x := B \setminus U(x)$ and, for $i \in [n + 1]$, let $A_i^x := A_i \setminus U(x)$. In particular, $A_{n+1}^x = A_{n+1} \setminus U(x) = \varnothing$. As $\Delta(\varnothing)\phi = \phi$, the degree of the iterated difference decreases in

$$\begin{aligned}
&\Delta(\{A_i\}_{i=1}^{n+1})Z(B) \\
&\quad = \Delta(A_{n+1})(\Delta(\{A_i\}_{i=1}^{n})Z(B)) \\
&\quad = \Delta(\{A_i\}_{i=1}^{n})Z(B) - \Delta(\{A_i\}_{i=1}^{n})Z(B \cup A_{n+1})
\end{aligned}$$

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$$\overset{(4.5)}{=} \sum_{I\subseteq[n]} (-1)^{|I|} Z(B \cup A_I) - \sum_{I\subseteq[n]} (-1)^{|I|} Z(B \cup A_I \cup A_{n+1})$$

$$\overset{(4.3)}{=} \sum_{I\subseteq[n]} (-1)^{|I|} \Big(Z(B \cup A_I) - \underbrace{Z(B \cup A_I \cup A_{n+1})}_{} \Big)$$

$$= \sum_{I\subseteq[n]} (-1)^{|I|} \Big(Z(B \cup A_I) - \overbrace{Z(B \cup A_I) + \int_{A_{n+1}} Z((B \cup A_I) \setminus U(x)) M(\mathrm{d}x)}^{} \Big)$$

$$= \sum_{I\subseteq[n]} (-1)^{|I|} \int_{A_{n+1}} Z(B^x \cup A_I^x) M(\mathrm{d}x)$$

$$= \int_{A_{n+1}} \underbrace{\Delta(\{A_i^x\}_{i=1}^n) Z(B^x)}_{\geq 0 \text{ by the induction hypothesis}} \quad M(\mathrm{d}x) \geq 0. \qquad \square$$

Proposition 4.9. *Let $A_1, \dots, A_n, B \in \mathcal{B}^\kappa$ be disjoint. If $\kappa(B \sqcup \bigsqcup_{i=1}^n A_i) < n$ then $\Delta(\{A_i\}_{i=1}^n) Z(B) = 0$.*

Proof. For $I \subseteq [n]$, let $A_I := \bigsqcup_{i\in I} A_i$. Let $A := A_{[n]}$. Then

$$\Delta(\{A_i\}_{i=1}^n) Z(B) = \sum_{I\subseteq[n]} (-1)^{|I|} Z(B \sqcup A_I)$$

$$= \sum_{I\subseteq[n]} (-1)^{|I|} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{(B\sqcup A_I)^m} h_m(x_1, \dots, x_m) \prod_{i=1}^m M(\mathrm{d}x_i).$$

For $m \in \mathbb{N}_0$ and $x_1, \dots, x_m \in A \sqcup B$, regard the indices $I(x_1, \dots, x_m) := \{i \in [n] \mid \{x_1, \dots, x_m\} \cap A_i \neq \varnothing\}$ of the partition elements containing the points

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{(B\sqcup A_I)^m} h_m(x_1, \dots, x_m) \sum_{I(x_1,\dots,x_m)\subseteq I\subseteq[n]} (-1)^{|I|} \prod_{i=1}^m M(\mathrm{d}x_i)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{(B\sqcup A_I)^m} h_m(x_1, \dots, x_m) \sum_{I(x_1,\dots,x_m)=[n]} (-1)^n \prod_{i=1}^m M(\mathrm{d}x_i).$$

Using (4.2) yields $n \leq m \leq \kappa(A \sqcup B) < n$ and a zero integrand. $\square$

Proposition 4.10. *Let $\{A_i\}_{i=1}^n$ be disjoint elements of $\mathcal{B}^1$. For $I \in [n]$, let $A_I := \bigsqcup_{i\in I} A_i$. For all $r \geq \kappa(A_{[n]})$, $\sum_{I\subseteq[n],|I|\leq r} \Delta(\{A_i\}_{i\in I}) Z(A_{[n]\setminus I}) = 1$ holds.*

Proof. We use Proposition 4.9 and evaluate the sum with Lemma 4.2. $\square$

5. Proofs around Shearer's PP

Section 5.1 contains the uniqueness and characterisation in Proposition 5.1 and the existence in Proposition 5.2. Together they imply Theorem 2.2. In Section 5.2 we prove the minimality of Shearer's PP for the avoidance function in Theorem 2.3. In Section 5.3 we discuss intrinsic couplings between Shearer's PP at different intensities and the monotonicity properties of the sets of measures. In Section 5.4 we show that Shearer's PP law differs from well known hard-core or 1-dependent PP laws and reference probabilistic constructions. The notation of Section 4 applies.

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Besides the σ-algebra of all Borel sets $\mathcal{B}$, there are also the algebras of bounded, κ-finite, and less than unit-diameter Borel sets $\mathcal{B}^{\mathrm{b}}$, $\mathcal{B}^{\kappa}$, and $\mathcal{B}^{1}$, respectively. Note that $\mathcal{B}^{\kappa}$ is an algebra of $\mathcal{B}^{1}$. Both algebras $\mathcal{B}^{\mathrm{b}}$ and $\mathcal{B}^{\kappa}$ generate the σ-algebra $\mathcal{B}$. The distinction between $\mathcal{B}^{\kappa}$ and $\mathcal{B}^{\mathrm{b}}$ plays a key role, as key proofs employ induction over κ. The strategy is to first establish results on $\mathcal{B}^{\kappa}$, extend them by σ-finiteness to $\mathcal{B}$ and project them onto $\mathcal{B}^{\mathrm{b}}$. Sufficient conditions for $\mathcal{B}^{\kappa}$ and $\mathcal{B}^{\mathrm{b}}$ to coincide are that $(\mathcal{X}, \delta)$ is either σ-compact or total (bounded sets are pre-compact). The structure of the space below the distance 1 plays no role.

5.1. Characterisation, uniqueness, and existence

For $a, n \in \mathbb{N}_0$, let the *falling factorial* be $a^{[n]} := \prod_{i\in[n]}(a - i + 1)$. The *factorial moment measure* of ξ of order n on $B \in \mathcal{B}$ is $\mathbb{E}(\xi(B)^{[n]})$ [11, Section 9.5].

Proposition 5.1. *If there exists a strong* 1*-dependent and* 1*-hard-core PP* η_M *with intensity measure* $M \in \mathcal{M}^{\mathrm{b}}$*, then its factorial moment measure of order* n *at* $B \in \mathcal{B}^{\mathrm{b}}$ *fulfils*

$$\mathbb{E}(\eta_M(B)^{[n]}) = \int_{B^n} h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i)$$

and its avoidance function is Z*. There is at most one PP with these three properties in* $\mathcal{C}(M)$*.*

Proof. If η_M has finite factorial moment measures of all orders then its avoidance function is $\mathbb{P}(\eta_M(B) = 0) = \sum_{n=0}^{\infty}((-1)^n/n!)\mathbb{E}(\xi(B)^{[n]}) = Z(B)$ [10, Equation (5.4.10)]. Because the avoidance function determines the PP's law [11, Section 9.2.XIII], uniqueness follows.

For all $r, n_1, \dots, n_r \in \mathbb{N}_0$ with $n := \sum_{i=1}^{r} n_i$ and disjoint $A_1, \dots, A_r \in \mathcal{B}^{1}$, we need to show that

$$\mathbb{E}\left(\prod_{i=1}^{r} \eta_M(A_i)^{[n_i]}\right) = \int_{\prod_{i=1}^{r} A_i^{n_i}} h_n(x_1, \dots, x_n) \prod_{l=1}^{n} M(\mathrm{d}x_l). \tag{5.1}$$

The 1-hard-core of η_M and (4.2) imply that $\eta_M(A_i)^{[n_i]} = 1$, if both $\eta_M(A_i) = 1$ and $n_i = 1$, and 0 otherwise. Suppose that there is an $j \in [r]$ with $n_j \geq 2$. On the one hand, the jth factor of the left-hand side of (5.1) is equal to 0, whence the left-hand side of (5.1) is equal to zero. On the other hand, this gives an upper bound on the right-hand side of (5.1) of

$$\left(\prod_{j\neq i\in[r]} M(A_i)^{n_i}\right) \int_{A_j^{n_j}} \underbrace{h_{n_j}(x_1, \dots, x_{n_j})}_{=0} \prod_{l=1}^{n_j} M(\mathrm{d}x_l) = 0.$$

The remaining case has $r = n$ and all $n_i = 1$. We proceed by induction over n. If $n = 1$ and $A \in \mathcal{B}^{1}$ then $\mathbb{E}(\eta_M(A)) = \mathbb{P}(\eta_M(A) = 1) = M(A) = \int_A h_1(x)M(\mathrm{d}x)$. The induction step from $(n - 1)$ to n is

$$\begin{aligned}
\mathbb{E}\left(\prod_{i=1}^{n} \eta_M(A_i)\right) &\overset{(4.2)}{=} \mathbb{P}(\text{for all } i \in [n]\colon \eta_M(A_i) = 1) \\
&= \int_{A_n} \mathbb{P}_{x_n}(\text{for all } i \in [n-1]\colon \eta_M(A_i) = 1) M(\mathrm{d}x_n) \\
&\overset{(3.1\mathrm{b})}{=} \int_{A_n} \mathbb{P}(\text{for all } i \in [n-1]\colon \eta_M(A_i \setminus U(x_n)) = 1) M(\mathrm{d}x_n)
\end{aligned}$$

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$$= \int_{A_n} \left(\int_{\prod_{i=1}^{n-1}(A_i \setminus U(x_n))} h_{n-1}(x_1, \dots, x_{n-1}) \prod_{i=1}^{n-1} M(\mathrm{d}x_i) \right) M(\mathrm{d}x_n)$$

$$= \int_{A_n} \left(\int_{\prod_{i=1}^{n-1} A_i} h_n(x_1, \dots, x_n) \prod_{i=1}^{n-1} M(\mathrm{d}x_i) \right) M(\mathrm{d}x_n)$$

$$= \int_{\prod_{i=1}^{n} A_i} h_n(x_1, \dots, x_n) \prod_{i=1}^{n} M(\mathrm{d}x_i).$$

Let $k := \kappa(B)$ and $\{A_i\}_{i=1}^k$ be a, possibly countable, partition of B into elements of $\mathcal{B}^1$. If $k = \infty$ then $[k] = \mathbb{N}$. Let $\mathcal{N}_{k,n} := \{(n_1, \dots, n_k) \in \mathbb{N}_0^k \mid \sum_{i=1}^k n_i = n\}$. For $a \in \mathbb{R}^k$, the multinomial Chu–Vandermonde identity is

$$\left(\sum_{i \in [k]} a_i \right)^{[n]} = \sum_{(n_1,\dots,n_k) \in \mathcal{N}_{k,n}} \binom{n}{n_1, \dots, n_k} \prod_{i=1}^{k} a_i^{[n_i]}.$$

The multinomial Chu–Vandermonde identity together with (5.1) yields

$$\mathbb{E}(\eta_M(B)^{[n]}) = \mathbb{E}\left(\sum_{(n_1,\dots,n_k) \in \mathcal{N}_{k,n}} \binom{n}{n_1, \dots, n_k} \prod_{i=1}^{k} \eta_M(A_i)^{[n_i]} \right)$$

$$= \sum_{(n_1,\dots,n_k) \in \mathcal{N}_{k,n}} \binom{n}{n_1, \dots, n_k} \mathbb{E}\left(\prod_{i=1}^{k} \eta_M(A_i)^{[n_i]} \right)$$

$$= \sum_{(n_1,\dots,n_k) \in \mathcal{N}_{k,n}} \binom{n}{n_1, \dots, n_k} \int_{\prod_{i=1}^{k} A_i^{n_i}} h_n(x_1, \dots, x_n) \prod_{l=1}^{n} M(\mathrm{d}x_l)$$

$$= \int_{B^n} h_n(x_1, \dots, x_n) \prod_{l=1}^{n} M(\mathrm{d}x_l). \qquad \square$$

Proposition 5.2. *If $M \in \mathcal{M}^{\mathrm{sh}}$ then a strong* 1*-dependent and* 1*-hard-core PP with intensity M exists.*

Proof. Four sufficient conditions [11, Section 9.2.XV] jointly guarantee the existence of a simple PP with avoidance function Z.

- Proposition 4.8 implies the complete monotonicity of Z.
- Unit at $\varnothing$, i.e. $Z(\varnothing) = 1$, holds trivially.
- Continuity in space at $\varnothing$ follows from Proposition 4.7 combined with the fact that $\varnothing$ is a continuity set of M, as $M(\overline{\varnothing} \setminus \varnothing) = M(\varnothing) = 0$.
- Almost sure bounded finiteness of the PP. Let $\{A_{i,n}\}_{i \in [k_n],\, n \in \mathbb{N}}$ be a *dissecting system* [10, Proposition A2.1.V] of $B \in \mathcal{B}^\kappa$. By intersecting every partition of the dissecting system with a fixed $\mathcal{B}^1$-partition of B, we may assume that the dissecting system contains only finite $\mathcal{B}^1$-partitions. Let

$$F(n, r) := \sum_{I \subseteq [k_n],\, |I| \le r} \Delta(\{A_{i,n}\}_{i \in I}) Z\left(B \setminus \bigsqcup_{i \in I} A_{i,n} \right).$$

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By Proposition 4.10, $F(n,r) = 1$ for $n \geq r \geq \kappa(B)$. Hence,

$$\lim_{r\to\infty} \lim_{n\to\infty} F(n,r) = 1.$$

Recall that the algebra $\mathcal{B}^\kappa$ generates the σ-algebra $\mathcal{B}$. Thus, there exists a simple PP η_M on $\mathcal{X}$ with avoidance function Z on $\mathcal{B}^\kappa$. It remains to show that the PP η_M is simple, strong 1-dependent, 1-hard-core, and has intensity measure M. The characterisation in Proposition 5.1 shows that there is a unique extension of its law to all of $\mathcal{B}^{\mathrm{b}}$.

Simpleness follows from the 1-hard-core. By Proposition 4.1, the function Z is 1-multiplicative. Proposition 3.1 asserts strong 1-dependence.

Let $A \in \mathcal{B}^1$. The PP $\varphi_{\overline{A}}$ on the closure $\overline{A}$ of A chooses no point with probability $1 - M(\overline{A})$ and one point with probability $M(\overline{A})$ distributed with the density $M(\mathrm{d}x)/M(\overline{A})$. The avoidance functions of $\varphi_{\overline{A}}$ and $\overline{A}\eta_M$ coincide, since, for each $A \supseteq B \in \mathcal{B}^{\mathrm{b}}$,

$$\begin{aligned}\mathbb{P}(\varphi_{\overline{A}}(B) = 0) &= 1 - M(\overline{A}) + M(\overline{A}) \int_{\overline{A}\setminus B} \frac{M(\mathrm{d}x)}{M(\overline{A})} \\ &= 1 - M(\overline{A}) + M(\overline{A} \setminus B) \\ &= 1 - M(B) \\ &= Z(B).\end{aligned}$$

For a countable dense subset S of $\mathcal{X}$, consider the following countable subset of $\mathcal{B}^1$.

$$\mathcal{H} := \{\{x \in \mathcal{X} \mid \delta(x,s) \leq \alpha \text{ or } \delta(x,t) \leq \alpha\} \mid s,t \in S \text{ with } 1 - 3\alpha := \delta(s,t) < 1\}.$$

If $\delta(x,y) < 1$ then there is a closed $A \in \mathcal{H}$ containing both x and y. Therefore,

$$\begin{aligned}\mathbb{P}(\eta_M \text{ is not 1-hard-core}) &= \mathbb{P}(\inf\{\delta(x,y) \mid x,y \in \eta_M\} < 1) \\ &= \mathbb{P}(\text{there exists } A \in \mathcal{H}\colon \eta_M(A) \geq 2) \leq \sum_{A\in\mathcal{H}} \mathbb{P}(\varphi_A \geq 2) = 0.\end{aligned}$$

For closed $A \in \mathcal{B}^1$ and all $A \supseteq B \in \mathcal{B}^{\mathrm{b}}$,

$$\mathbb{E}(\eta_M(B)) = \mathbb{E}(A\eta_M(B)) = \mathbb{E}(\varphi_A(B)) = \int_B \frac{1}{M(A)} M(\mathrm{d}x) M(A) = M(B).$$

Linearity of expectations extends this to the intensity measure of η_M. □

5.2. Proofs of Theorems 2.3 and 2.4

Proof of Theorem 2.3. First, we prove the statement only over $\mathcal{B}^\kappa$. The general statement over $\mathcal{B}^{\mathrm{b}}$ follows by taking limits along sequences in $\mathcal{B}^\kappa$ to a limit in $\mathcal{B}^{\mathrm{b}}$. Let $A, B \in \mathcal{B}^\kappa$. Assume that $Z(B) > 0$. Use induction over $k := \kappa(A \cup B)$. Let $\xi \in \mathcal{C}(M)$ with avoidance function Q. If $Q(B) > 0$, let $q(A,B) := Q(A \cup B)/Q(B)$. If $k = 0$ then $A = B = \varnothing$ and $q(\varnothing,\varnothing) = 1 = z(\varnothing,\varnothing)$. If $k > 0$ then telescoping (4.4) restricts to the case $A \in \mathcal{B}^1$ and $A \cap B = \varnothing$. Let $\{A_i\}_{i=1}^k$ be a $\mathcal{B}^1$-partition of $A \sqcup B$. For $x \in A$, let $A(x)$ be the unique partition element containing x. Apply Proposition 3.3 twice to obtain

$$\begin{aligned}q(A,B) &= 1 - \int_A q(B, B \setminus U(x))^{-1} M(\mathrm{d}x) \\ &= 1 - \int_A q(B, B \setminus A(x))^{-1} q(B \setminus A(x), B \setminus U(x))^{-1} M(\mathrm{d}x) \qquad (5.2a)\end{aligned}$$

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and, for $x \in A$,

$$q(B, B \setminus A(x)) = 1 - \int_{B \cap A(x)} q(B \setminus A(x), B \setminus A(x) \setminus U(y))^{-1} M(\mathrm{d}y). \tag{5.2b}$$

In the second half of this proof we show that the expansions (5.2) are well defined. For $x \in A$ and $y \in A(x)$, we have $A(x) \subseteq U(y)$ and $\kappa(B \setminus A(x)) \leq k - 1$. Hence, the inductive hypothesis applies to the integrand in (5.2b) and the second factor of the integrand in (5.2a). For $x \in A$, the inductive hypothesis implies that $Q(B \setminus A(x)) > 0$. Applying (4.3) to (5.2b), we see that

$$q(B, B \setminus A(x)) \geq 1 - \int_{B \cap A(x)} z(B \setminus A(x), B \setminus A(x) \setminus U(y))^{-1} M(\mathrm{d}y) = z(B, B \setminus A(x)).$$

Substitute this into (5.2a), multiply and we see that this implies that $Q(B) > 0$. Applying (4.3) to (5.2a), we obtain

$$\begin{aligned} q(A, B) &\geq 1 - \int_A z(B, B \setminus A(x))^{-1} z(B \setminus A(x), B \setminus U(x))^{-1} M(\mathrm{d}x) \\ &= 1 - \int_A z(B, B \setminus U(x))^{-1} M(\mathrm{d}x) = z(A, B). \end{aligned}$$

□

Proof of Theorem 2.4. If M has an atom of mass one at x then the strong 0-dependent PP ξ from Proposition 3.4 has $\mathbb{P}(\xi(\{x\}) = 0) = 0$. Thus, it suffices to consider only measures with atoms smaller than one. For every $M \in \mathcal{M}^{\mathrm{sh}} \setminus \mathcal{M}^{>}$, the avoidance function of Shearer's PP η_M vanishes on some bounded Borel set and Proposition 2.4 follows trivially.

For every $M \notin \mathcal{M}^{\mathrm{sh}}$, there exists $B \in \mathcal{B}^{\mathrm{b}}$ with $Z(B, M) < 0$ and $M(B) > 0$. Let $\overline{B}$ be the closure of B. Proposition 4.6 implies that $\lambda_B = \min\{\lambda \mid Z(B, \lambda M) \leq 0\}$, $0 < \lambda_{\overline{B}} \leq \lambda_B < 1$, and $\lambda_{\overline{B}} M|_{\overline{B}} \in \mathcal{M}^{\mathrm{sh}}$. Proposition 4.4 asserts that, for each $\overline{B} \supseteq A \in \mathcal{B}^{\mathrm{b}}$ and $N \leq \lambda_B M$, we have $Z(A, N) \geq 0$. From here on, assume that B is closed with $Z(B, M) < 0$. Let $\Lambda := \lambda_B$.

Consider three independent PPs.

- Proposition 3.4 guarantees the existence of a strong 1-dependent $\varphi \in \mathcal{C}(M)$.
- For a yet undetermined $N \in \mathcal{M}^{\mathrm{b}}$, a (maybe nonsimple) Poisson(N) PP ϑ on B.
- As $\Lambda M|_B \in \mathcal{M}^{\mathrm{sh}}$, Proposition 5.2 guarantees the existence Shearer's PP $\eta_{\Lambda M}$.

Recall that $\psi^\bullet$ is the simple support PP of a general PP ψ. The target PP is $\xi := (\mathcal{X} \setminus B)\varphi + (\eta_{\Lambda M} + \vartheta)^\bullet$. As all three component PPs are strong 1-dependent, so is ξ. Since $\mathbb{P}(\xi(B) = 0) = \mathbb{P}(\eta_{\Lambda M}(B) = 0,\ \vartheta(B) = 0) \leq \mathbb{P}(\eta_{\Lambda M}(B) = 0) = 0$, the avoidance probability of ξ vanishes on B.

To determine N, verify that it has finite mass on B and that M is the intensity of ξ. Let $\mathcal{A}$ be the atoms of M in B. The diffuse domain is $\mathcal{D} := B \setminus \mathcal{A}$. Set $N|_{\mathcal{D}} := (1 - \Lambda)M|_{\mathcal{D}}$, which is finite. Because $\mathcal{D}\eta_{\Lambda M}$ and $\mathcal{D}\vartheta$ are simple and independent, for each $B \supseteq A \in \mathcal{B}^{\mathrm{b}}$,

$$\mathbb{E}(\xi(\mathcal{D} \cap A)) = \mathbb{E}((\eta_{\Lambda M} + \vartheta)^\bullet(\mathcal{D} \cap A)) = \mathbb{E}(\eta_{\Lambda M}(\mathcal{D} \cap A)) + \mathbb{E}(\vartheta(\mathcal{D} \cap A)) = M(\mathcal{D} \cap A).$$

For an atom $x \in \mathcal{A}$, let $n_x := \mathbb{E}(\vartheta(\{x\}))$. The construction demands that

$$\begin{aligned} m_x := \mathbb{E}(\xi(\{x\})) &= \mathbb{E}((\eta_{\Lambda M} + \vartheta)^\bullet(\{x\})) \\ &= \mathbb{P}(\eta_{\Lambda M}(\{x\}) = 1) + \mathbb{P}(\eta_{\Lambda M}(\{x\}) = 0,\ \vartheta(\{x\}) \geq 1) \end{aligned}$$

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$$= \mathbb{E}(\eta_{\Lambda M}(\{x\})) + Z(\{x\},\, \Lambda M)(1 - \mathbb{P}(\vartheta(\{x\}) = 0))$$
$$= \Lambda m_x + (1 - m_x)(1 - \exp(-n_x)).$$

Since $m_x < 1$ then so is $(1-\Lambda)m_x/(1-\Lambda m_x) < 1$ and

$$n_x = -\log\left(1 - \frac{(1-\Lambda)m_x}{1-\Lambda m_x}\right) \overset{(3.2)}{\leq} \frac{(1-\Lambda)m_x/(1-\Lambda m_x)}{1-(1-\Lambda)m_x/(1-\Lambda m_x)} = \frac{(1-\Lambda)m_x}{1-m_x}.$$

As $M(\mathcal{A}) \leq M(B) < \infty$ and all atoms of M are less than 1, let $m_B := \max\{m_x \mid x \in \mathcal{A}\} < 1$. The finiteness of the atomic part of N follows from

$$N(\mathcal{A}) = \sum_{x\in\mathcal{A}} n_x \leq \sum_{x\in\mathcal{A}} \frac{(1-\Lambda)m_x}{1-m_x} \leq \frac{(1-\Lambda)}{1-m_B} \sum_{x\in\mathcal{A}} m_x = \frac{(1-\Lambda)}{1-m_B} M(\mathcal{A}) < \infty. \qquad \square$$

5.3. Intrinsic coupling and monotonicity

Proposition 5.3. *Let η_M be Shearer's PP with intensity measure M. Let $p\colon \mathcal{X} \to [0,1]$ be measurable. Define $N \in \mathcal{M}^{\mathrm{b}}$ by $N(B) := \int_B p(x)M(\mathrm{d}x)$. The independent p-thinning [11, Section 11.3] of η_M has the same law as η_N.*

Proof. Independent thinning preserves strong 1-dependence and the 1-hard-core. It also implies the intensity measure N. We conclude by the uniqueness from Proposition 5.1. $\square$

Proof of Theorem 2.5. Choose the thinning in Proposition 5.3 with $p = \mathrm{d}N/\mathrm{d}M$. This proves that $\mathcal{M}^{\mathrm{sh}}$ and $\mathcal{M}^{>} = \mathcal{M}^{+}$ are *down-sets*. The proofs of Proposition 3.4 and Theorem 2.4 imply that $\mathcal{M}^{\varnothing}$ and $\mathcal{M}^{0} \sqcup \mathcal{M}^{\varnothing}$ are *up-sets*, respectively. $\square$

5.4. Shearer's PP is different

Except in trivial cases (zero intensity measure, space with isolated small components), Shearer's PP differs from other well known hard-core PPs. Because a hard-core radius lower bounds a dependence radius, the difficulty is combining 1-dependence and 1-hard-core. Shearer's PP is not a Poisson PP, nor one of Matérn's constructions [26], [35], [36], nor a hard-sphere model as in Section 2.4, and is neither a determinantal nor permanental PP [7], [15]. In special cases explicit constructions are possible, though. These are: on the graph $\mathbb{Z}$ for homogeneous intensity and all radii R [27], on $\mathbb{R}$ for homogeneous intensity and all radii R [19], and on chordal graphs for $R = 2$ and all admissible intensities [20].

The space $(\mathcal{X}, \delta)$ is *r-connected*, if each pair of points is part of a finite point sequence with consecutive pairwise distance less than r, and $(\mathcal{X}, \delta)$ is *r-disconnected*, if $\inf\{\delta(x,y) \mid \{x,y\} \subseteq \mathcal{X}\} \geq r$. If $(\mathcal{X}, \delta)$ is 1-disconnected then Shearer's PP is a product BRF. For the remainder of this section, we assume that $(\mathcal{X}, \delta)$ is 1-connected with diameter greater than 1 and that, for all $x \in \mathcal{X}$ and each neighbourhood B of x, $M(B) > 0$.

Proposition 5.4. *Shearer's PP is not a Matérn-style hard-core PP.*

Proof. Let $N \in \mathcal{M}^{\mathrm{b}}$. For the extreme cases of diffuse and atomic N, let φ be a Poisson(N) PP and a product BRF of intensity N, respectively. Additionally, attach independent and identically distributed Uniform($[0,1]$) marks to points of φ. There are disjoint $A_1, A_2, A_3 \in \mathcal{B}^1$ with positive M-measure with the diameters of $A_1 \sqcup A_2$ and $A_2 \sqcup A_3$ smaller than R and $\delta(A_1, A_3) \geq R$. This reduces the setting to BRFs on the graph $G := ([3], \{\{1,2\},\{2,3\}\})$ with the geodesic metric d, inducing the metric space $([3], 2d)$. The aim is to show that a 1-hard-core

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clashes with independence of the marginals at 1 and 3. Let (n_1, n_2, n_3) and (m_1, m_2, m_3) be the underlying and positive target atomic intensities, respectively.

For the Matérn I hard-core PP [26], [**?**], delete all points of φ having at least another point at distance less than 1. The target intensities are $m_1 = n_1(1-n_2)$, $m_2 = (1-n_1)n_2(1-n_3)$, and $m_3 = (1-n_2)n_3$. 1-dependence demands that $n_1(1-n_2)^2 n_3 = m_1 m_3 = n_1(1-n_2)n_3$. This implies that $n_2 = 0$ and contradicts $m_2 > 0$.

To obtain the Matérn II hard-core PP [26], [35], delete every point x of φ whose mark l fulfils $l = \max\{h \mid (y,h) \in \varphi,\ \delta(x,y) < 1\}$. By symmetry, the comparison between the labels on neighbouring sites yields a probability of $\frac{1}{2}$ in a site's favour. The target intensities are $m_1 = n_1(1-n_2/2)$, $m_2 = n_2(1-n_1/4)(1-n_3/4)$, and $m_3 = n_3(1-n_2/2)$. 1-dependence demands that $n_1(1-n_2/2)^2 n_3 = m_1 m_3 = n_1(1-3n_2/4)n_3$. This implies that $n_2 = 0$, a contradiction to $m_2 > 0$.

A marked point (x,l) inhibits a marked point (y,k), if $\delta(x,y) < 1$ and $l \le k$. A marked point (x,l) is uninhibited, if it fulfils $l = \min\{h \mid (y,h) \in \varphi,\ \delta(x,y) < 1\}$. Iteratively, delete all inhibited points. Uninhibited points only contribute once to the deletion and every 1-connected cluster of points contains at least one uninhibited point. Hence, the deletion procedure stabilises locally almost surely and the resulting PP is the Matérn III hard-core PP [36] with radius 1. The target intensities are $m_1 = n_1(1-n_2/2+n_2n_3/4)$ and $m_3 = n_3(1-n_2/2+n_2n_1/4)$. This implies that $n_1, n_3 \in]0,1[$. 1-dependence demands that $n_1n_3(1-n_2/2+n_2n_3/4)(1-n_2/2+n_2n_1/4) = m_1m_3 = n_1(1-n_2/4)n_3$. This is impossible, as the left-hand side is always larger than the right-hand side. □

Proposition 5.5. *Shearer's PP is not the hard-sphere model.*

Proof. Let $A, B \in \mathcal{B}^1$ with $A \subsetneq B$ and $0 < M(A) < M(B)$. Let h_B be the hard-sphere model with fugacity N on B. It has avoidance function $N/(1+N(B))$. Demanding equal laws for Shearer's PP of intensity N and the hard-sphere model with fugacity N on A, B and $B \setminus A$ leads to only trivial solutions of

$$\frac{N(B)}{1+N(B)} = M(B) = M(A) + M(B \setminus A) = \frac{N(A)}{1+N(A)} + \frac{N(B)-N(A)}{1+M(B)-M(A)}. \qquad \square$$

Proposition 5.6. *Shearer's PP is neither determinantal nor permanental.*

Proof. The higher moment densities of a determinantal PP are the determinants of a matrix with entries from a bivariate, symmetric and measurable kernel $K\colon \mathcal{X}^2 \to \mathbb{R}$. Consider the correlation function of n points, i.e. the Radon–Nikodyn derivative of the nth factorial moment measure of η_M with respect to the n-fold product of M. It depends only on the 1-connected graph structure of the n points and takes values 1 and 0, for 1-disconnected graphs and graphs containing at least one 1-edge, respectively.

For $x \in \mathcal{X}$, this yields $\det K(x,x) = 1$. For $\{x,y\} \subseteq \mathcal{X}$, this yields $K(x,y) = \pm 1$ and $K(x,y) = 0$ for $\delta(x,y) < 1$, and $\delta(x,y) \ge 1$, respectively. For $n = 3$, the graph $(\{x,y,z\}, \{\{x,y\},\{y,z\}\})$ yields the contradiction $0 = 1 - K(x,y)^2 - K(y,z)^2 = -1$.

The attraction of permanental PPs contradicts the 1-hard-core of Shearer's PP. □

6. Proofs of the LLLs

Lemma 6.1. *If (2.1) holds then* $\mathcal{B}^{\mathrm{b}} = \mathcal{B}^{\kappa}$.

Proof. Induction on $\operatorname{diam}(B) := \sup\{\delta(x,y) \mid x,y \in B\}$ shows that, for each $B \in \mathcal{B}^{\mathrm{b}}$, $\kappa(B) \le K^{\lfloor \operatorname{diam}(B) \rfloor} < \infty$. □

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Proof of Theorem 2.6. By (2.1) and Lemma 6.1, $\mathcal{B}^{\mathrm{b}} = \mathcal{B}^{\kappa}$. Let $A, B \in \mathcal{B}^{\kappa}$. This proof uses induction over $k := \kappa(A \cup B)$. If $k = 0$ then $A = B = \varnothing$ and $z(\varnothing, \varnothing) = 1$. If $k > 0$ then we telescope (4.4) to restrict to $A \setminus B \in \mathcal{B}^1$. Let $\{A_i\}_{i=1}^k$ be a $\mathcal{B}^1$-partition of $A \cup B$. For $x \in A \cup B$, let $A(x)$ be the partition element containing x. Applying (4.3) twice, we obtain

$$\begin{aligned} z(A, B) &= 1 - \int_{A\setminus B} z(B, B \setminus U(x))^{-1} M(\mathrm{d}x) \\ &= 1 - \int_{A\setminus B} z(B, B \setminus A(x))^{-1} z(B \setminus A(x), B \setminus U(x))^{-1} M(\mathrm{d}x) \end{aligned} \tag{6.1a}$$

and, for $x \in A \setminus B$,

$$z(B, B \setminus A(x)) = 1 - \int_{B\cap A(x)} z(B \setminus A(x), B \setminus U(y))^{-1} M(\mathrm{d}y). \tag{6.1b}$$

For $x \in A \setminus B$ and $y \in A(x)$, $A(x) \subseteq U(y)$, whence $\kappa(B \setminus U(y)) \leq \kappa(B \setminus A(x)) \leq k - 1$. Thus, the inductive hypothesis applies to the integrand of (6.1b) and the second factor in (6.1a). Bounding the integrand of (6.1b) by the inductive hypothesis (2.2b) leads to

$$\begin{aligned} z(B, B \setminus A(x)) &\geq 1 - \int_{B\cap A(x)} \left(\frac{K+2}{K+1}\right)^{\kappa(U(y))} M(\mathrm{d}y) \\ &\overset{(2.1)}{\geq} 1 - \left(\frac{K+2}{K+1}\right)^{K} M(B \cap A(x)) \\ &\overset{(2.2a)}{\geq} 1 - \left(\frac{K+2}{K+1}\right)^{K} \frac{(K+1)^{K+1}}{(K+2)^{K+2}} \\ &\geq 1 - \frac{1}{K+2} \\ &= \frac{K+1}{K+2}. \end{aligned}$$

Substituting this into the right-hand side of (6.1a) and bounding the right factor of the integrand by the inductive hypothesis (2.2b) leads to

$$\begin{aligned} z(A, B) &\geq 1 - \int_{A\setminus B} \frac{K+2}{K+1} \left(\frac{K+2}{K+1}\right)^{\kappa(U(x))} M(\mathrm{d}x) \\ &\overset{(2.1)}{\geq} 1 - \left(\frac{K+2}{K+1}\right)^{K+1} M(A \setminus B) \\ &\overset{(2.2a)}{\geq} 1 - \left(\frac{K+2}{K+1}\right)^{K+1} \frac{(K+1)^{K+1}}{(K+2)^{K+2}} \\ &= 1 - \frac{1}{K+2} \\ &= \frac{K+1}{K+2}. \end{aligned}$$

□

Proof of Theorem 2.7. By (2.1) and Lemma 6.1, we have $\mathcal{B}^{\mathrm{b}} = \mathcal{B}^{\kappa}$. Theorem 2.7 follows from telescoping the statement, for all $A \in \mathcal{B}^1, B \in \mathcal{B}^{\mathrm{b}}$: $z(A, B) \geq \alpha/(1 + \alpha)$.

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Let $N := (1+\alpha)M$. The equality (4.3) and the monotonicity of z in M from Proposition 4.4 imply that

$$\begin{aligned}
0 &\le z(A, B, N) \\
&= 1 - \int_{A\setminus B} z(B \setminus A, B \setminus U(x), N)^{-1} N(\mathrm{d}x) \\
&\le 1 - \int_{A\setminus B} z(B \setminus A, B \setminus U(x), M)^{-1} \frac{\mathrm{d}N}{\mathrm{d}M}(x) M(\mathrm{d}x) \\
&\le 1 - (1+\alpha) \int_{A\setminus B} z(B \setminus A, B \setminus U(x), M)^{-1} M(\mathrm{d}x) \\
&= 1 - (1+\alpha)(1 - z(A, B, M)).
\end{aligned}$$

□

Proof of Theorem 2.8. Assuming that (2.3b) holds for $A, B \in \mathcal{B}^\kappa$, the general case follows from a limiting argument. Let $A, B \in \mathcal{B}^{\mathrm{b}}$ with $A \cap B = \varnothing$. Take sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{B}^\kappa$ exhausting A and B, respectively. The monotonicity of z in space from Proposition 4.4 implies that

$$z(A, B) = \lim_{n\to\infty} z(A_n, B_n) \ge \lim_{n\to\infty} \exp(-N(A_n \setminus B_n)) = \exp(-N(A \setminus B)).$$

It remains to prove (2.3b) for $A, B \in \mathcal{B}^\kappa$. The first part of the proof is verbatim the same as the one of Theorem 2.6, leading to (6.1). For $x \in A$, bounding the integrand in (6.1b) by the inductive hypothesis (2.3b) leads to

$$\begin{aligned}
z(B, B \setminus A(x)) &\ge 1 - \int_{B\cap A(x)} \exp(N((B \setminus A(x)) \cap U(y))) M(\mathrm{d}y) \\
&\ge 1 - \int_{B\cap A(x)} \exp(N(U(y) \setminus (B \cap A(x)))) M(\mathrm{d}y) \\
&\overset{(2.3a)}{\ge} \exp(-N(B \cap A(x))).
\end{aligned}$$

Substituting this into the right-hand side of (6.1a) and using the inductive hypothesis on the right factor of the integrand leads to

$$\begin{aligned}
z(A, B) &\ge 1 - \int_{A\setminus B} \exp(N(B \cap A(x))) \exp(N((B \cup U(x)) \setminus A(x))) M(\mathrm{d}x) \\
&= 1 - \int_{A\setminus B} \exp(N(B \cap U(x))) M(\mathrm{d}x) \\
&\ge 1 - \int_{A\setminus B} \exp(N(U(x) \setminus (A \setminus B))) M(\mathrm{d}x) \\
&\overset{(2.3a)}{\ge} \exp(-N(A \setminus B)).
\end{aligned}$$

□

Proof of Corollary 2.2. Here $X = \mathbb{R}^d$, $M = \lambda\mathcal{L}$, and $N = \alpha\mathcal{L}$. For each $A \in \mathcal{B}^1$ with $L := \mathcal{L}(A)$, condition (2.4) can be written as $\lambda L\, \mathrm{e}^{\alpha(V-L)} \le 1 - \mathrm{e}^{-\alpha L}$. The identity $\lambda\, \mathrm{e}^{\alpha V} = \alpha$ simplifies this to $\alpha L\, \mathrm{e}^{-\alpha L} + \mathrm{e}^{-\alpha L} \le 1$. The last inequality is just $1 + z \le \mathrm{e}^z$ with $z := \alpha L \ge 0$. Thus, condition (2.3a) holds. We conclude via Theorem 2.8. □

https://doi.org/10.1017/apr.2016.76 Published online by Cambridge University Press

Acknowledgements

The author acknowledges the support of the VIDI project 'Phase transitions, Euclidean fields and random fractals' (grant number NWO 639.032.916) while working at the VU Amsterdam. The author thanks Marie-Colette van Lieshout for discussions about general PP theory and the anonymous reviewers and editor for their helpful comments.

References

[1] Aaronson, J., Gilat, D. and Keane, M. (1992). On the structure of 1-dependent Markov chains. *J. Theoret. Prob.* **5,** 545–561.
[2] Aaronson, J., Gilat, D., Keane, M. and de Valk, V. (1989). An algebraic construction of a class of one-dependent processes. *Ann. Prob.* **17,** 128–143.
[3] Alon, N. and Spencer, J. H. (2008). *The Probabilistic Method*, 3rd edn. John Wiley, Hoboken, NJ.
[4] Baddeley, A. J., van Lieshout, M. N. M. and Møller, J. (1996). Markov properties of cluster processes. *Adv. Appl. Prob.* **28,** 346–355.
[5] Błaszczyszyn, B. and Yogeshwaran, D. (2014). On comparison of clustering properties of point processes. *Adv. Appl. Prob.* **46,** 1–20.
[6] Błaszczyszyn, B. and Yogeshwaran, D. (2015). Clustering comparison of point processes, with applications to random geometric models. In *Stochastic Geometry, Spatial Statistics and Random Fields* (Lecture Notes Math. **2120**), Springer, Cham, pp. 31–71.
[7] Borodin, A. (2011). Determinantal point processes. In *The Oxford Handbook of Random Matrix Theory*, Oxford University Press, pp. 231–249.
[8] Broman, E. I. (2005). One-dependent trigonometric determinantal processes are two-block-factors. *Ann. Prob.* **33,** 601–609.
[9] Burton, R. M., Goulet, M. and Meester, R. (1993). On 1-dependent processes and k-block factors. *Ann. Prob.* **21,** 2157–2168.
[10] Daley, D. J. and Vere-Jones, D. (2003). *An introduction to the Theory of Point Processes: Elementary Theory and Methods*, Vol. I, 2nd edn. Springer, New York.
[11] Daley, D. J. and Vere-Jones, D. (2008). *An Introduction to the Theory of Point Processes: General Theory and Structure*, Vol. II, 2nd edn. Springer, New York.
[12] De Valk, V. (1988). The maximal and minimal 2-correlation of a class of 1-dependent 0-1 valued processes. *Israel J. Math.* **62,** 181–205.
[13] De Valk, V. (1993). Hilbert space representations of m-dependent processes. *Ann. Prob.* **21,** 1550–1570.
[14] Dobrushin, R. L. (1996). Estimates of semi-invariants for the Ising model at low temperatures. In *Topics in Statistical and Theoretical Physics* (Amer. Math. Soc. Transl. Ser. 2 **177**), American Mathematical Society, Providence, RI, pp. 59–81.
[15] Eisenbaum, N. (2012). Stochastic order for alpha-permanental point processes. *Stoch. Process. Appl.* **122,** 952–967.
[16] Erdős, P. and Lovász, L. (1975). Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and Finite Sets: To Paul Erdős on His 60th Birthday* (Keszthely, 1973; Colloq. Math. Soc. János Bolyai **10**), Vol. II, North-Holland, Amsterdam, pp. 609–627.
[17] Fernández, R. and Procacci, A. (2007). Cluster expansion for abstract polymer models. New bounds from an old approach. *Commun. Math. Phys.* **274,** 123–140.
[18] Fernández, R., Procacci, A. and Scoppola, B. (2007). The analyticity region of the hard sphere gas. Improved bounds. *J. Statist. Phys.* **128,** 1139–1143.
[19] Hofer-Temmel, C. (2015). Shearer's point process and the hard-sphere model in one dimension. Preprint. Available at http://arxiv.org/abs/1504.02672v1.
[20] Hofer-Temmel, C. and Lehner, F. (2015). Clique trees of infinite locally finite chordal graphs. Preprint. Available at http://arxiv.org/abs/1311.7001v3.
[21] Holroyd, A. E. (2014). One-dependent coloring by finitary factors. Preprint. Available at http://arxiv.org/abs/1411.1463v1.
[22] Janson, S. (1984). Runs in m-dependent sequences. *Ann. Prob.* **12,** 805–818.
[23] Jensen, T. R. and Toft, B. (1995). *Graph Coloring Problems*. John Wiley, New York.
[24] Kotecký, R. and Preiss, D. (1986). Cluster expansion for abstract polymer models. *Commun. Math. Phys.* **103,** 491–498.
[25] Liggett, T. M., Schonmann, R. H. and Stacey, A. M. (1997). Domination by product measures. *Ann. Prob.* **25,** 71–95.

https://doi.org/10.1017/apr.2016.76 Published online by Cambridge University Press

[26] Matérn, B. (1960). *Spatial Variation: Stochastic Models and Their Application to Some Problems in Forest Surveys and Other Sampling Investigations*. Meddelanden Från Statens Skogsforskningsinstitut 49, Stockholm.
[27] Mathieu, P. and Temmel, C. (2012). k-independent percolation on trees. *Stoch. Process. Appl.* **122,** 1129–1153.
[28] Miracle-Solé, S. (2010). On the theory of cluster expansions. *Markov Process. Relat. Fields* **16,** 287–294.
[29] Penrose, O. (1967). Convergence of fugacity expansions for classical systems. In *Statistical Mechanics: Foundations and Applications*, ed. T. Bak, Benjamin, New York, pp. 101–109.
[30] Rolski, T. and Szekli, R. (1991). Stochastic ordering and thinning of point processes. *Stoch. Process. Appl.* **37,** 299–312.
[31] Ruelle, D. (1969). *Statistical Mechanics: Rigorous Results*. Benjamin, New York.
[32] Scott, A. D. and Sokal, A. D. (2005). The repulsive lattice gas, the independent-set polynomial, and the Lovász local lemma. *J. Statist. Phys.* **118,** 1151–1261.
[33] Shearer, J. B. (1985). On a problem of Spencer. *Combinatorica* **5,** 241–245.
[34] Soshnikov, A. (2000). Determinantal random point fields. *Uspekhi Mat. Nauk* **55,** 107–160.
[35] Stoyan, D. and Stoyan, H. (1985). On one of Matérn's hard-core point process models. *Math. Nachr.* **122,** 205–214.
[36] Teichmann, J., Ballani, F. and van den Boogaart, K. (2013). Generalizations of Matérn's hard-core point processes. *Spatial Statist.* **3,** 33–53.
[37] Temmel, C. (2014). Shearer's measure and stochastic domination of product measures. *J. Theoret. Prob.* **27,** 22–40.
[38] van Lieshout, M. N. M. and Baddeley, A. J. (1996). A nonparametric measure of spatial interaction in point patterns. *Statist. Neerlandica* **50,** 344–361.

https://doi.org/10.1017/apr.2016.76 Published online by Cambridge University Press