The dice-and-numbers game

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Dedicated to my secondary school mathematics teacher: Károly Kovács

1. *Introduction*

In maths lessons in secondary school, on certain occasions my teacher entertained our class with the following game. Each player starts with an empty 2×5 grid, as shown in Figure 1(a). In each round of the game, a die is rolled, and each player has to choose one of the empty boxes in their grid and fill it with the number rolled. For example, after 3 rounds, a player's grid may look like the one in Figure 1(b). After 10 rounds everyone's grid is filled, forming two 5-digit numbers. A player's score is the sum of her/his two 5-digit numbers. Players with the largest sum win the game. We shall call this the 'dice-and-numbers game'. It is easy to think of other variants. Any child who is comfortable with column addition and understands how place value affects the size of numbers will develop a reasonable strategy for the game. But what is the *optimal* strategy?

6		3		
			2	

FIGURE 1: (a) Empty grid at the start; (b) A possible configuration after 3 rounds.

In Section 3 we work out the optimal strategy in a few smaller examples; based on this, readers will be able to extend it for any version of the game, in principle. The optimal strategy in complete generality was found in [1]. Smaller cases can be computed by hand, and readers familiar with basic computer programming can write code, without much difficulty, that solves larger examples.

In the second part of the article we turn to a slightly different question. Imagine that we play the game on a very long grid, and the first few numbers have already been played. How *sensitive* is the final outcome to the *initial* play? We illustrate this question with the following example, that we are going to comment on more in Section 4. In order to simplify the example, let us assume that the six sides of the dice have the numbers 1, 1, 2, 2, 3, 3 on them, so that only three numbers can be rolled, and they are equally likely. Let us play the game on a 1×25 grid, aiming to maximise the score; see Figure 2. It is clear that if we roll a 3, we should place it at the leftmost available place, and if we roll a 1, we should place it at the rightmost available place. It is also intuitive, that when we first roll a 2, we should aim for the middle. But how much risk is involved if we do not, and how does it depend on the amount by which the 2 is off centre? We give a brief discussion in Section 4.

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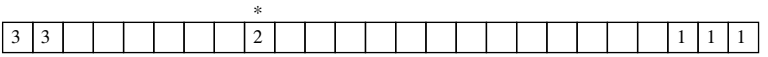


FIGURE 2: Example of initial play on a 1 × 25 grid. If the next number is 2, and we play the location marked with '*', is that risky for the goal of obtaining the maximal score?

Our presentation will only assume intuitive probability notions, and the main one we need, that of mathematical expectation, is reviewed in Section 2. Readers who wishes to flesh out our arguments into formal mathematical proofs, can do so based on introductory texts on probability. For example, the early chapters of the classic text [2] do not require specialist mathematical background.

2. *The notion of expected value*

Suppose we have a random quantity, such as the value rolled on a dice, or the final score we achieve in the dice-and-numbers game under a given strategy. The *expected value*, sometimes called the *mathematical expectation*, is an overall average of the random quantity over all possible realisations of the randomness involved. It can be computed by multiplying each possible value of the random quantity by the probability of the eventuality producing that value, and summing. If the random quantity is denoted X, its expected value is denoted $\mathrm{E}(X)$.

Example 1: Let X be the value rolled on a die with sides 1, 2, 3, 4, 5, 6. Each value will occur with probability $\frac{1}{6}$, so we have

$$\mathrm{E}(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$$

$$= \frac{1}{6} \times (1 + 2 + 3 + 4 + 5 + 6) = \frac{7}{2} = 3.5.$$

As the next example shows, we often want to compute expected values by breaking things down according to different cases, computing expected values under each case, and then averaging them according to the probabilities of each case. This process is called conditioning, and the expected values are, in this case, conditional expected values, given the cases considered.

Example 2: Suppose we play the dice-and-numbers game on a 1 × 2 grid, and we follow the following strategy: if the first roll is 4, 5 or 6, we place it in the left box, if it is 1, 2 or 3, we place it in the right box. (It is intuitive that this is the optimal play, and we will show this in Section 3.) Let Y be the score we achieve under this strategy. What is the expected score? We break things down according to the value of the first roll; see Figure 3.

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FIGURE 3: Cases for computing the expected score on a 1×2 grid. The first term in each sum represents the contribution of the first roll to the score. The second term is the average contribution of the second roll, given the location it will go into.

Conditional on the first roll being 4, 5 or 6, this contributes 40, 50 or 60 to the score, since we place it in the left box. The average value of the second roll, that will be placed in the right box, contributes 3.5, due to Example 1. Conditional on the first roll being 1, 2 or 3, this contributes 1, 2 or 3 to the score, since we place it in the right box. The average value of the second roll contributes $10 \times 3.5 = 35$, since it is placed in the left box. The contributions in each of the six cases of the first roll are shown in Figure 3. Since each case occurs with probability $\frac{1}{6}$, we multiply by these probabilities and add:

$$\begin{aligned} \mathrm{E}(Y) &= \frac{1}{6}(60 + 3.5) + \frac{1}{6}(60 + 3.5) + \frac{1}{6}(60 + 3.5) \\ &\qquad + \frac{1}{6}(3 + 35) + \frac{1}{6}(2 + 35) + \frac{1}{6}(1 + 35) \\ &= \frac{1}{6}(63.5 + 53.5 + 43.5 + 38 + 37 + 36) \\ &= 45.25. \end{aligned}$$

The conditional expected values of Y, given the first roll being 6, 5, 4, 3, 2, 1, respectively, are 63.5, 53.5, 43.5, 38, 37, 36, respectively. The expected value of Y is the average of these, taken according to the probabilities of the first roll being 6, 5, 4, 3, 2, 1, respectively.

For later use, it will be convenient to rewrite the result in the following form:

$$\begin{aligned} \mathrm{E}(Y) &= \frac{1}{6}(60 + 50 + 40 + 35 + 35 + 35) + \frac{1}{6}(3.5 + 3.5 + 3.5 + 3 + 2 + 1) \\ &= 10 \times \frac{1}{6}(6 + 5 + 4 + 3.5 + 3.5 + 3.5) + \frac{1}{6}(3.5 + 3.5 + 3.5 + 3 + 2 + 1) \\ &= 10 \times 4.25 + 2.75. \end{aligned} \tag{1}$$

The way we want to think about this formula is as follows: playing two rolls, the average reward we get from the left box is 4.25 times the place value 10, and the average reward we get from the right box is 2.75 times its place value.

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3. *Computing the optimal strategy*

Let us take Example 2 a step further, and show that the strategy considered there is optimal. First, as an example, consider the possibility that the first roll is 4, and we place it into the right box, instead of the left box. The conditional expected score then is: $4 + 35 = 39$. The term 4 represents the contribution to the score from the digit 4 in the right box, and the term 35 is the average contribution of the second roll to be placed in the left box: 10×3.5. Note that this is smaller than the conditional expected score we get if we placed the 4 in the left box: that was 43.5. This shows that there is a disadvantage in placing the 4 into the right box.

We can similarly determine the conditional expected scores in the remaining five cases of the first roll if we make a choice different from the one in Example 2. The following table summarises these, and shows that in each case the alternative is worse.

First roll	6	5	4	3	2	1
Example 2	63.5	53.5	43·5	38	37	36
Alternative	41	40	39	33.5	23.5	13.5

A valuable observation of a more general nature can be gleaned from this table. A careful examination of the computations shows that the deciding factor in where a digit should be placed is whether it is greater than or less than 3.5, the average score we get in the game on the 1×1 grid. In other words, digits greater than 3.5 should be placed on the left, and those smaller than 3.5 should be placed on the right. Indeed: for a digit $d > 3.5$, the immediate reward we get from placing d to the left is $d \times 10$, as opposed to the average reward 3.5×10 we would get from the second roll under the alternative. The difference between the two choices is expressed by the inequality:

$$d \times 10 + 3.5 \times 1 > 3.5 \times 10 + d \times 1, \tag{2}$$

which holds if, and only if, $d > 3.5$.

A second observation we can make is that the inequality (2) is *unaffected* if the factor 10 is replaced by 100, say (and in fact by any other value that is greater than 1). Indeed, suppose that we have a 1×3 grid, and the middle box is pre-filled with a digit a:

	a	

Then our score will be $a \times 10$, plus the result of playing the game with reward 100 for the left box and reward 1 for the right box. Given this position, the optimal strategy is still given by placing digits 6, 5, 4 into the left box, and digits 3, 2, 1 into the right box. The table of conditional expected scores now reads (in excess of the term $a \times 10$ that will not change):

First roll	6	5	4	3	2	1
Optimal	603.5	503.5	403·5	353	352	351
Alternative	356	355	354	303.5	203.5	103.5

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This time the choice of box boils down to the inequality

$$d \times 100 + 3.5 \times 1 > 3.5 \times 100 + d \times 1, \tag{3}$$

i.e. $d > 3.5$. The expected score (in excess of the term $a \times 10$), can be written, by analogy with (1), as follows:

$$\frac{1}{6}(600 + 500 + 400 + 350 + 350 + 350) + \frac{1}{6}(3.5 + 3.5 + 3.5 + 3 + 2 + 1)$$

$$= 100 \times \frac{1}{6}(6 + 5 + 4 + 3.5 + 3.5 + 3.5) + \frac{1}{6}(3.5 + 3.5 + 3.5 + 3 + 2 + 1)$$

$$= 100 \times 4.25 + 2.75 = 427.75. \tag{4}$$

Our expected score for two rolls is again of the same form as in (1): we have an expected reward from the left box equal to 4.25 times its place value 100, and an expected reward from the right box equal to 2.75 times its place value. By similar considerations, in any set-up, where we are left with two boxes to play, the expected reward we get from the left one will be 4.25 times its place value, and the expected reward we get from the right one will be 2.75 times its place value.

The above considerations allow us now to find the optimal strategy for a 1×3 grid as follows. Suppose that the first digit rolled is a. We first consider the alternative of placing a in the left versus the middle box. In either case, we arrive at a problem with two boxes, which we have already solved. We can write the inequality analogous to (2) and (3) as

$$a \times 100 + 4.25 \times 10 + 2.75 \times 1 > 4.25 \times 100 + a \times 10 + 2.75 \times 1, \tag{5}$$

i.e. $a > 4.25$. This shows that the left box is preferable to the middle box if the number rolled exceeds 4.25, that is, if it is 6 or 5. We can similarly compare the alternative of placing a in the middle box, versus in the right box:

$$4.25 \times 100 + a \times 10 + 2.75 \times 1 > 4.25 \times 100 + 2.75 \times 10 + a \times 1, \tag{6}$$

i.e. $a > 2.75$. Thus the middle box is preferable to the right box if the number rolled is 3 or 4, otherwise, we should choose the right box.

We can also find the expected value if we use this strategy, analogously to (1):

$$\frac{1}{6}(600 + 500 + 425 + 425 + 425 + 425)$$

$$+ \frac{1}{6}(42.5 + 42.5 + 40 + 30 + 27.5 + 27.5)$$

$$+ \frac{1}{6}(2.75 + 2.75 + 2.75 + 2.75 + 2 + 1)$$

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$$= 100 \times \frac{1}{6}(6 + 5 + 4.25 + 4.25 + 4.25 + 4.25)$$
$$+ 10 \times \frac{1}{6}(4.25 + 4.25 + 4 + 3 + 2.75 + 2.75)$$
$$+ 1 \times \frac{1}{6}(2.75 + 2.75 + 2.75 + 2.75 + 2 + 1)$$
$$= 100 \times 4\tfrac{2}{3} + 10 \times 3\tfrac{1}{2} + 1 \times 2\tfrac{1}{3} = 504. \quad (7)$$

This formula also shows more generally, that whenever we have three boxes left to play, the expected rewards we get from the left, middle and right one, respectively, are $4\frac{2}{3}$, $3\frac{1}{2}$ and $2\frac{1}{3}$ times the respective place values of the boxes. We will refer to these values as 'breakpoints'.

The process can be iterated to give a strategy for any $1 \times N$ grid. The optimal strategy for the first move for $N = 4$ is as follows: if the first number rolled is 5 or 6, place it in the first box, if it is 4, place it in the second box, if it is 3, place it in the third box, and if it is 1 or 2, place it in the fourth box. These choices are determined by the size of the number rolled relative to the breakpoints $4\frac{2}{3} > 3\frac{1}{2} > 2\frac{1}{3}$. After each roll, we need to count the number of available spaces, and place the next number according to the table below.

	Dice throw					
Available spaces	1	2	3	4	5	6
1	1	1	1	1	1	1
2	2	2	2	1	1	1
3	3	3	2	2	1	1
4	4	4	3	2	1	1

Figure 4 shows the behaviour of the breakpoints for large N. When there are $N = 120$ empty boxes, the optimal strategy places the numbers 6, 5, 4, 3, 2, 1 in positions 1, 29, 50, 71, 92, 120, respectively. These correspond to imagining a 'perfect board' with 20 numbers of each kind, and placing the number rolled (when different from 1 and 6) roughly in the middle position of its stretch.

The iterative computation we outlined above can be applied to the 2×5 grid as well, or any other grid; the method is completely general. Readers are invited to work out the 2×2 case as an exercise, for which some of the necessary computations are already contained in the above. A complete mathematical proof, for any set of rewards instead of place values, can be found in [1], [3, Section VI.7]. The behaviour of the general model for large N was found in [4].

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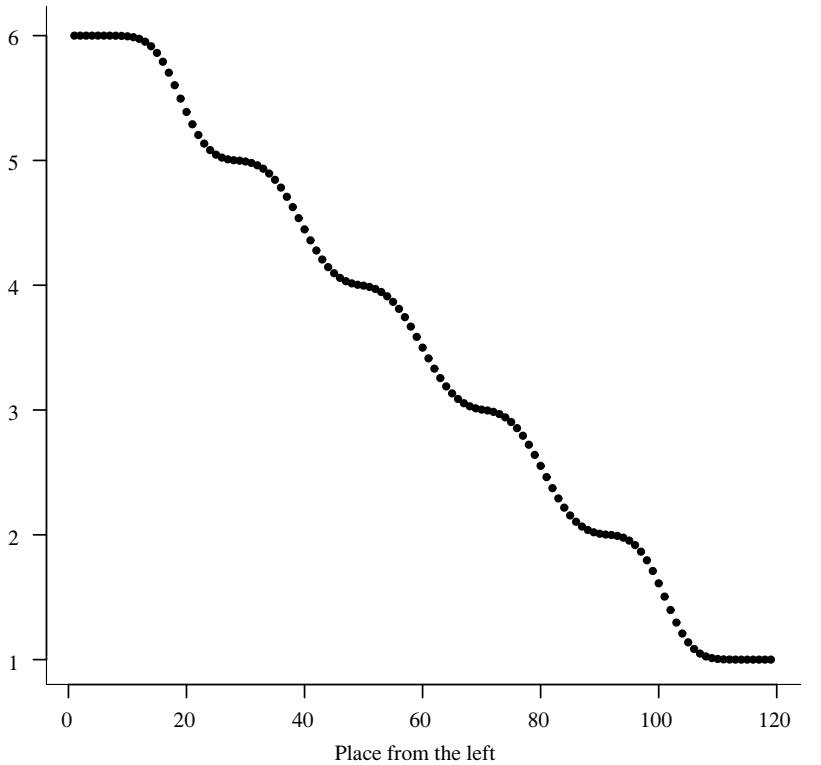


FIGURE 4: Breakpoints for $N = 120$

4. *Sensitivity of the outcome to initial play*

Let us now return to the simplified game where the dice had sides 1, 1, 2, 2, 3, 3, so only three values could be rolled (with equal probabilities). Suppose that our goal is to achieve the *maximum possible score*, that is, when the numbers at the end form a non-increasing sequence. The optimal strategy for such a game is as follows. Place any 3 in the first available box and place any 1 in the last available box. Place the first 2 as close as possible to the midpoint of the last 3 and the first 1. Place any subsequent 2's adjacent to a previous 2, in a way that balances the empty space on the left and right; see Figure 5.

3	3							2	2	2	2	2	*								1	1	1	1

FIGURE 5: If the next roll is 2, it should be placed in the box marked *.

We ask the question: if the first 2 is *not* placed optimally, how sensitive will the chance of achieving the maximal score be to this initial choice, when we play on a large $1 \times N$ grid? For example, suppose that we place the first 2 as shown in Figure 6. Assuming optimal play from now on, will

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the chance of achieving the maximal score be significantly different compared to optimal initial play? In this section, we briefly analyse this question, and illustrate the answer numerically.

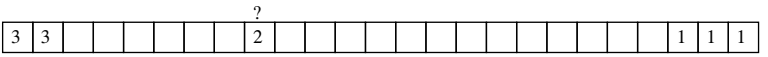


FIGURE 6: Example of initial play on a 1×25 grid.

In the grid shown in Figure 6 it is clear that if we placed the 2 in the leftmost available box, next to the 3's, we would only be able to achieve the maximal possible score if no further 3's are rolled. This would be extremely unlikely: as the probability that we roll a 1 or 2 on a given roll is $\frac{2}{3}$, the probability that this happens on each of the remaining 19 rolls after the 2 is played is $\left(\frac{2}{3}\right)^{19} \approx 0.00045$. That is less than 5 chances in 10 000. The same holds for the rightmost place, as then there is no space for any further 1's. Is there a clear boundary between the region where the probability is very small, and where the probability is nearly the optimal one? Figure 7 illustrates that the answer is 'yes', when N is large.

Let us first argue that for large N, when the first 2 is placed in the middle, the probability of achieving the maximal score is close to 0.5. Let $D_n \geqslant 0$ denote the difference between the number of free spaces on the left and the right after n rolls, with $n = 1$ corresponding to the time when the first 2 has been placed. Thus under the optimal initial move we have $D_1 = 0$ or $D_1 = 1$, depending on parity. Following the 'balancing strategy', whenever $D_n \geqslant 1$ we have $D_{n+1} = D_n - 1$ with probability $\frac{2}{3}$, and $D_{n+1} = D_n + 1$ with probability $\frac{1}{3}$. When $D_n = 0$, we always have $D_{n+1} = 1$. That is, writing 'P' for 'probability', the random quantity D_n evolves according to:

$$\mathrm{P}\big[D_{n+1} = k\big] = \begin{cases} \frac{2}{3}\,\mathrm{P}[D_n = k+1] + \frac{1}{3}\,\mathrm{P}[D_n = k-1] & \text{when } k \geqslant 1; \\ \frac{2}{3}\,\mathrm{P}[D_n = 1] & \text{when } k = 0. \end{cases} \tag{8}$$

Using (8) recursively, it can be checked numerically (and also proved mathematically using the more advanced theory of Markov chains [2]), that if $D_1 = 0$ or $D_1 = 1$, then for large n, we have $\mathrm{P}\big[D_n = k\big] \approx p(k)$, $k = 0, 1, 2, \ldots$, where the numbers $p(k)$ satisfy the equations

$$p(k) = \left\{ \begin{array}{ll} \frac{2}{3}p(k+1) + \frac{1}{3}p(k-1) & \text{when } k \geqslant 1; \\ \frac{2}{3}p(1) & \text{when } k = 0. \end{array} \right\}$$

The unique solution is given by $p(k) = \left(\frac{1}{2}\right)^{k+1}$, $k = 0, 1, 2, \ldots$. In particular, the probability that the balancing strategy achieves perfect balance at the time when we reach the last box is $\mathrm{P}\big[D_N = 0\big] \approx p(0) = 0.5$.

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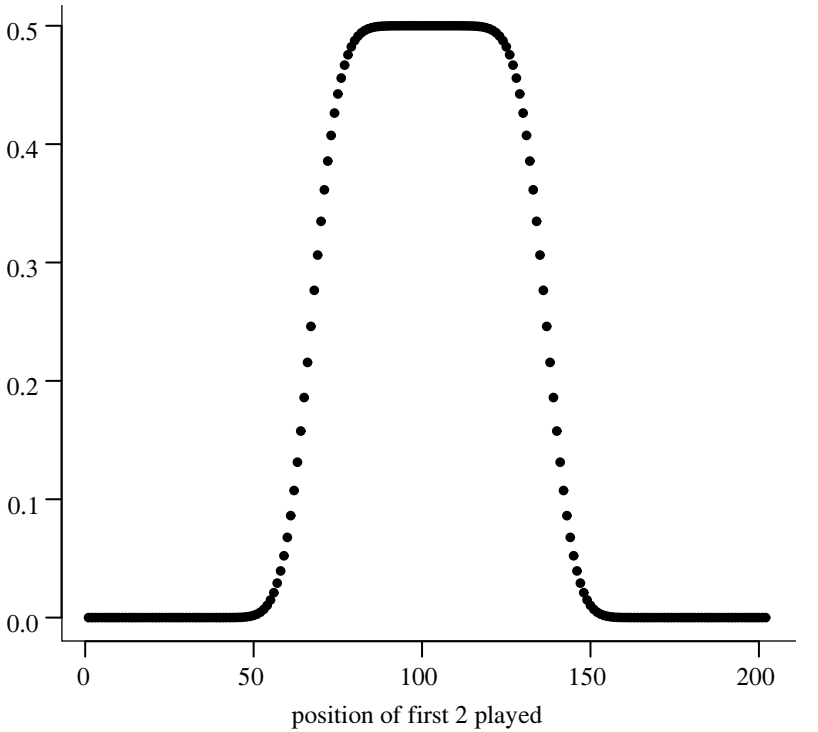


FIGURE 7: The probability of achieving the maximal score on a 1×200 grid, as a function of the position where the first 2 is played. Note the region in the middle third, where the probability is indistinguishable from a constant (almost exactly 0.5), and the sharp transition to the two regions at the sides, where the probability is indistinguishable from 0. (To be precise: the graph assumes that the first roll was 2. The number of initial 1's and 3's should be subtracted from 200, but this leaves the shape of the graph very similar.)

The equations (8) also allow the recursive computation of the probabilities shown in Figure 7. Suppose $N = 200$, and assume that the first roll is 2. If we place this 2 in box k, we have

$$D_1 = |(k - 1) - (200 - k)| = |201 - 2k|,$$

and Figure 7 shows a plot of $\mathrm{P}[D_{200} = 0]$ against k.

It is apparent from Figure 7 that if the first 2 is placed roughly in the middle third of the grid, the probability that the maximal score is achieved is almost the same as the optimum (very nearly 0.5). However, if the first 2 is placed in the left or right third of the grid, the probability that the maximal score can still be achieved is tiny. It is clear from the plot that there is a sharp transition between these two behaviours at around the points $\frac{1}{3}N$ and $\frac{2}{3}N$.

The location of the transition can be understood as follows. For large N, the number of 3's rolled is approximately $\frac{1}{3}N$ with high probability. (The

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mathematically precise version of this statement is called the Law of Large Numbers [2].) If the first 2 is placed in the left third, this does not leave 'enough space' for the approximately $\frac{1}{3}N$ digits 3 that will be rolled with quite high certainty, and the probability of achieving the maximal score is tiny. Similarly for the right third. It can also be shown, that in the middle third, the probability is very close to the optimal value 0.5, as in this case balancing is successful before space runs out, with high probability.

A closing reminiscence

In addition to the game analysed above, several maths lessons of Károly Kovács stay vividly in my mind. His teaching of mathematical proofs, whether it concerned geometry, or a careful construction of rational arithmetic, shared its beauty with Euclid's Elements. I thank him for being a forever patient and inspiring teacher.

Acknowledgement

I am grateful to an anonymous referee for useful advice.

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10.1017/mag.2016.105

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https://doi.org/10.1017/mag.2016.105 Published online by Cambridge University Press