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Understanding Portfolio Efficiency with Conditioning Information

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Abstract

I develop two new types of portfolio efficiency when returns are predictable. The first type maximizes the unconditional Sharpe ratio of excess returns and differs from unconditional efficiency unless the safe asset return is constant over time. The second type maximizes conditional mean-variance preferences and differs from unconditional efficiency unless, additionally, the maximum conditional Sharpe ratio is constant. Using stock data, I quantify and test their performance differences with respect to unconditionally and fixed-weight efficient returns. I also show the relevance of the two new portfolio strategies to test conditional asset pricing models.

I. Introduction

The seminal paper of Hansen and Richard (1987) analyzes the tension between the conditional implications of asset pricing theory and the use of unconditional moments in empirical work. Perhaps their best-known result is that *unconditionally efficient* (UE) returns are a subset of *conditionally efficient* (CE) returns. For instance, this result shows that the conditional capital asset pricing model (CAPM) implies that the market portfolio is CE but not necessarily UE.

The difference between CE and UE returns is well known in empirical asset pricing,[1] but the latter represent the only subset of CE returns that has been studied. Several papers have used UE returns to guide portfolio choice. Examples are Ferson and Siegel (2001) and, adding a benchmark, Chiang (2015).

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[1]Lettau and Ludvigson (2001) is an influential reference on the distinction between conditional and unconditional asset pricing models.

Other papers, such as those by Brandt and Santa-Clara (2006) and Bansal, Dahlquist, and Harvey (2004), approximate UE returns through managed portfolios. My main contribution is a comprehensive analysis of two new types of CE returns that are more relevant from empirical and theoretical perspectives.

Unconditional Sharpe (1966), (1994) ratios and Jensen's (1968) alphas of excess returns with respect to the safe asset return are commonly used in empirical work. For this reason, I characterize a new set of efficient returns that achieve the maximum unconditional Sharpe ratio or, equivalently, display zero unconditional Jensen's alphas as a pricing factor. I therefore refer to this new subset of CE returns as *performance-efficient* (PE) returns. I show that for a given target of expected return, PE returns minimize the variance of the excess return instead of the total return variance, which is minimized by UE returns. These two variances do not coincide in the presence of a safe asset such as the Treasury bill, whose return is conditionally riskless but unconditionally risky.[2]

Ferson and Siegel (2009) construct an efficiency test based on the maximum unconditional Sharpe ratio with conditioning information. My results indicate that such a test is actually testing whether a particular return is PE instead of UE. The authors assume a constant safe asset return when they develop their test and, in this case, both types of efficiency are equivalent. In fact, I show that this is the only case where UE and PE returns coincide.

I also develop a second type of efficient returns that can be rationalized by conditional mean-variance preferences that are commonly used in finance theory. If the unconditional variance of a return is decomposed as the average conditional variance plus the variance of the conditional mean, these optimal returns minimize only the first component. I use the term *residually efficient* (RE) returns because the average conditional variance is equal to the variance of the residual from a predictive regression. RE returns achieve the maximum Sharpe ratio and display zero Jensen's alphas as a pricing factor if the required variances and covariances are based on the residuals instead of the returns themselves. I define these new performance measures as the residual Sharpe ratio and alpha, respectively.

Abhyankar, Basu, and Stremme (2012) empirically evaluate several return predictors. For this purpose, they approximate the slope of the UE frontier asymptotes by the second moment of the slope of the CE frontier, that is, the maximum conditional Sharpe ratio. My results indicate that this second moment has an exact relation with the slope of the RE frontier instead, which is given by the maximum residual Sharpe ratio. I also show that the UE and RE return frontiers are equal if and only if both the safe asset return and the maximum conditional Sharpe ratio are constant over time.

Interest rates and Sharpe ratios change over time and hence the UE, PE, and RE returns represent different ways of exploiting conditioning information. However, determining how different they are is an empirical question and I use stock data to quantify and test their performance differences. I work with the 3

[2]As a by-product, I carefully describe UE returns in the presence of this safe asset. Hansen and Richard (1987) study a general setup that may or may not include a safe asset. However, they make the role of a safe asset explicit only to clarify certain ideas, such as the safe return is CE but not necessarily UE.

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Fama–French (1993) factors as excess returns and 3 prominent predictors: the dividend–price ratio, the default spread, and the term spread. To study the role of the investment set, I also work with the 6 and 25 size and book-to-market sorted Fama–French portfolios.

I study monthly returns during two periods, 1954–1983 and 1984–2012, because some features of return predictability may differ across data periods. The first period can be associated with stronger market predictability and the second with weaker market predictability. However, the size and value effects are relatively more predictable in the second period.

I estimate and test the differences in unconditional Sharpe ratios between PE and UE returns. My theoretical results show that UE returns do not display a unique unconditional Sharpe ratio. This ratio may change considerably for return targets around the safe asset, but as the target is increased, this ratio converges toward its value for PE returns. For this reason, in the empirical application I study low and high targets for UE returns, 6% and 10%, respectively, in annualized terms. The performance gaps are economically and statistically significant with the 25 portfolios and the low target 6%, with a stronger gap in the first period than in the second.

I extend the standard econometric setup of unconditional Sharpe ratios to residual Sharpe ratios. I estimate and test the differences in the residual Sharpe ratios of RE returns with respect to UE returns. Once again, the performance gap is especially strong for the low target and the 25 portfolios.

The return frequency is another relevant dimension to the differences across efficiency types. Annual returns show larger differences across the subsets of CE returns. In fact, UE returns are considerably different with respect to PE or RE returns for a small investment set, namely, the 3 Fama–French (1993) factors.

Finally, I connect my theoretical results and empirical methodology to the testing of conditional asset pricing models. I use the same data to revisit the evidence against the CAPM, which is a classic example of efficiency and asset pricing tests that still attracts attention. For instance, controversy remains about the relative performance of the unconditional and conditional CAPM, where the former model is the textbook one with *fixed-weight efficient* (FE) returns that do not exploit return predictability.

When using only the 3 factors as the investment set, in the first period the market is much more inefficient with FE returns than with PE or RE returns, but the second period provides the opposite situation. Then, the increases in Sharpe ratios when adding the size and value portfolios to the PE and RE strategies are much higher than those with FE strategies, which are not statistically significant. My findings also indicate that it is good empirical practice to compute the performance of both PE and RE returns when testing asset pricing models.

The remainder of the article is organized as follows: Section II describes the theoretical framework with conditioning information and defines CE returns. Section III provides a theoretical analysis of two new types of efficient returns as subsets of CE returns that generally differ from UE returns. Next, Section IV develops the empirical application. Finally, Section V summarizes the conclusions and discusses some avenues of further research. The auxiliary results are gathered in the Internet Appendix (available at www.jfqa.org).

II. Conditionally Efficient Returns

The investment set is defined by a safe asset with return R_0 and some risky assets with return vector $\mathbf{R}$. If I subtract R_0 from each return in $\mathbf{R}$, I obtain the vector of excess returns $\mathbf{r} = \mathbf{R} - R_0\mathbf{1}$, where $\mathbf{1}$ is a vector of ones. The case where no safe asset is available is studied in Internet Appendix C. The investors' information set is given by a vector $\mathbf{z}$ of predictors that are informative about future asset payoffs.

The safe asset return is conditionally riskless, $\text{var}\,(R_0|\mathbf{z}) = 0$, but it may be unconditionally risky with $\text{var}\,(R_0) > 0$. For ease of exposition, I assume the conditional variance $\text{var}\,(\mathbf{r}|\mathbf{z})$ is nonsingular with probability 1. This implies that none of the primitive risky assets is actually conditionally riskless or redundant. I also assume that the vector of risk premia $\text{E}\,(\mathbf{r}|\mathbf{z})$ has at least one nonzero entry to avoid trivial efficient returns.

If an investor is endowed with positive wealth, which can be normalized to 1 without loss of generality, the investor will be interested in portfolio strategies that cost 1 for every possible value of $\mathbf{z}$. Let $\mathbf{w}$ denote the vector of portfolio weights on the risky returns $\mathbf{R}$, where each weight is a decision variable of the investor conditional on her information set and hence a function of $\mathbf{z}$. The weight on the safe asset return R_0 must be $1 - \mathbf{w}'\mathbf{1}$ in a unit-cost portfolio. Therefore, the payoff of such a portfolio can be represented by the return

$$R \quad = \quad R_0\,(1 - \mathbf{w}'\mathbf{1}) + \mathbf{R}'\mathbf{w} \quad = \quad R_0 + (\mathbf{R} - R_0\mathbf{1})'\,\mathbf{w},$$

which can be interpreted as a unit weight on the safe return and weights $\mathbf{w}$ on the vector of excess returns

$$R \quad = \quad R_0 + \mathbf{r}'\mathbf{w}. \tag{1}$$

In this setting, the *conditionally efficient* (CE) returns of Hansen and Richard (1987) are the returns with minimum conditional variance for a given target of conditional expected returns, that is, the set of returns in equation (1) that solve the optimization problem

$$\min_{R} \text{var}\,(R|\mathbf{z}) \ \text{ for a given } \text{E}\,(R|\mathbf{z})\,, \tag{2}$$

which I denote R_C.

This problem is the conditional counterpart of the classic Markowitz (1952) portfolio problem, where the mean and variance are unconditional and the weights $\mathbf{w}$ in return (1) are real numbers instead of functions of $\mathbf{z}$. Similarly, the safe asset is interpreted as unconditionally riskless. In that well-known context, the efficient returns can be represented by

$$R_0 + \omega\mathbf{r}'\boldsymbol{\varphi}, \tag{3}$$

where[3]

$$\boldsymbol{\varphi} \quad = \quad [\text{var}\,(\mathbf{r})]^{-1}\,\text{E}\,(\mathbf{r}) \tag{4}$$

[3]Stevens (1998) interprets the vector $\boldsymbol{\varphi}$ in terms of hedging regressions. Each entry of $\boldsymbol{\varphi}$ is equal to the ratio of the intercept and the residual variance of the regression of that excess return onto a constant and the rest of excess returns in $\mathbf{r}$.

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and ω is a real number that depends on the mean target $\mathrm{E}(R)$. I refer to these returns as *fixed-weight efficient* (FE) because investors are constrained to fixed-weight strategies.

In the more general setup where investors exploit return predictability, the returns in equation (1) that solve problem (2) have a similar structure

$$R_C = R_0 + \omega_C \mathbf{r}' \boldsymbol{\varphi}_{\mathbf{C}}, \tag{5}$$

where

$$\boldsymbol{\varphi}_{\mathbf{C}} = [\mathrm{var}(\mathbf{r}|\mathbf{z})]^{-1} \mathrm{E}(\mathbf{r}|\mathbf{z}), \tag{6}$$

and ω_C is a function of $\mathbf{z}$ that depends on the conditional mean target $\mathrm{E}(R|\mathbf{z})$.

We can also define Sharpe ratios and Jensen's alphas in this setting. We can translate a return R into an excess return $r = R - R_0$ and define the conditional Sharpe ratio of r as

$$S_C = \frac{\mathrm{E}(r|\mathbf{z})}{\sqrt{\mathrm{var}(r|\mathbf{z})}},$$

which is a function of $\mathbf{z}$. These Sharpe ratios and other types in the following sections are defined for r different from 0 or, equivalently, for R different from R_0. The conditional Jensen's alpha of r with respect to the pricing factor r_β is defined as

$$\alpha_C = \mathrm{E}(r|\mathbf{z}) - \beta_C \mathrm{E}(r_\beta|\mathbf{z}), \qquad \beta_C = \frac{\mathrm{cov}(r, r_\beta|\mathbf{z})}{\mathrm{var}(r_\beta|\mathbf{z})}.$$

Similarly to Sharpe ratios, Jensen's alphas are defined for r_β different from 0.

The CE returns have interesting properties in terms of the previous two measures. The excess returns of CE returns different from R_0 are characterized by achieving the maximum squared conditional Sharpe ratio, denoted $\mathcal{S}_C^2$ and given by

$$\mathcal{S}_C^2 = \mathrm{E}(\mathbf{r}|\mathbf{z})' [\mathrm{var}(\mathbf{r}|\mathbf{z})]^{-1} \mathrm{E}(\mathbf{r}|\mathbf{z}). \tag{7}$$

These excess returns are also characterized by $\alpha_C = 0$ when they are used as a factor to price any r.

These properties translate the textbook Markowitz properties into conditional moments. The CE returns lie along two straight lines in the $[\sqrt{\mathrm{var}(R|\mathbf{z})}, \mathrm{E}(R|\mathbf{z})]$ space for each possible value of $\mathbf{z}$. These two lines intersect on the vertical axis at R_0 and their slope is given by $\mathcal{S}_C$. I refer to this conditionally linear relation between the optimal $\sqrt{\mathrm{var}(R|\mathbf{z})}$ and $\mathrm{E}(R|\mathbf{z})$ as the CE frontier.

III. Two New Types of CE Returns

This section develops the theory of two subsets of CE returns, each one associated with a different construction of ω_C in return (5). I define these two types of efficient returns, show their properties, and study their differences with respect to a third subset of CE returns, the only one that has been studied in the literature.

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A. Performance-Efficient Returns

The analysis of unconditional moments is also relevant when returns are predictable. Unconditional moments, estimated by sample averages, are often used in empirical work. The performance evaluation of a portfolio manager is another example of the use of unconditional moments because the evaluator may not have access to the manager's information set.

In particular, unconditional Sharpe ratios and Jensen's alphas of excess returns are commonly used in empirical finance. For instance, see Ferson and Siegel (2009) and the classic references therein such as Jobson and Korkie (1982) and Gibbons, Ross, and Shanken (1989). The unconditional Sharpe ratio of an excess return r is defined as the real number

$$S_U = \frac{\mathrm{E}(r)}{\sqrt{\mathrm{var}(r)}},$$

and the unconditional Jensen's alpha of an excess return r with respect to a pricing factor r_β is defined as

$$\alpha_U = \mathrm{E}(r) - \beta_U \mathrm{E}(r_\beta), \qquad \beta_U = \frac{\mathrm{cov}(r, r_\beta)}{\mathrm{var}(r_\beta)}.$$

The Sharpe ratio above is computed from the variance of the excess return $\mathrm{var}(r) = \mathrm{var}(R - R_0)$, which is not equal to the variance of the return $\mathrm{var}(R)$ when the safe asset is unconditionally risky. Conversely, even though $\mathrm{var}(R_0) > 0$, the safe asset is riskless in terms of $\mathrm{var}(r)$ because $\mathrm{var}(R_0 - R_0) = 0$. Following the use of $\mathrm{var}(R - R_0)$ in empirical finance, I define the *performance-efficient* (PE) returns as the returns in equation (1) that solve the problem

$$\min_R \mathrm{var}(R - R_0) \text{ subject to } \mathrm{E}(R) = \nu, \tag{8}$$

and I denote them R_P. The following proposition analyzes the PE returns.

Proposition 1. The representation and properties of PE returns R_P defined by problem (8).

1) The PE returns are the CE returns in equation (5) with

$$\omega_C = \left[\frac{\nu - \mathrm{E}(R_0)}{\mathrm{E}\left(\frac{\mathcal{S}_C^2}{1+\mathcal{S}_C^2}\right)}\right] \frac{1}{1+\mathcal{S}_C^2}, \tag{9}$$

which is a function of $\mathbf{z}$ because it depends on $\mathcal{S}_C^2$.

2) The excess returns of PE returns different from R_0 are characterized by achieving the maximum S_U^2, denoted $\mathcal{S}_U^2$ and given by

$$\mathcal{S}_U^2 = \frac{\mathrm{E}\left(\frac{\mathcal{S}_C^2}{1+\mathcal{S}_C^2}\right)}{1 - \mathrm{E}\left(\frac{\mathcal{S}_C^2}{1+\mathcal{S}_C^2}\right)}. \tag{10}$$

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These excess returns are also characterized by $\alpha_U = 0$ when they are used as a factor to price any r.

Point 2 of Proposition 1 shows that the PE returns have similar properties to the classic Markowitz setup with a safe asset in terms of Sharpe ratios and alphas. However, this is not the case for *unconditionally efficient* (UE) returns, the only subset of CE returns that has been studied before. Hansen and Richard (1987) define them as the returns with minimum unconditional variance $\text{var}(R)$ for each target of unconditional expected return $\text{E}(R)$. Thus, the UE returns are the set of returns in equation (1) that solve the problem

$$\min_R \text{var}(R) \text{ subject to } \text{E}(R) = \nu, \tag{11}$$

and I denote them R_U. Adapting their results to the existence of a safe return R_0 that may be unconditionally risky, the UE returns R_U are the CE returns in equation (5) with

$$\omega_C = \left[\frac{\nu + \text{E}\left(R_0 \frac{\mathcal{S}_C^2}{1+\mathcal{S}_C^2}\right) - \text{E}(R_0)}{\text{E}\left(\frac{\mathcal{S}_C^2}{1+\mathcal{S}_C^2}\right)} - R_0 \right] \frac{1}{1+\mathcal{S}_C^2}, \tag{12}$$

which is a function of $\mathbf{z}$ because it depends on R_0 and $\mathcal{S}_C^2$.

From point 1 of Proposition 1, the difference between PE and UE returns with the same mean is

$$R_P - R_U = \left[R_0 - \frac{\text{E}\left(R_0 \frac{\mathcal{S}_C^2}{1+\mathcal{S}_C^2}\right)}{\text{E}\left(\frac{\mathcal{S}_C^2}{1+\mathcal{S}_C^2}\right)} \right] \frac{1}{1+\mathcal{S}_C^2} \mathbf{r}' \boldsymbol{\varphi}_\mathbf{C}. \tag{13}$$

When the safe asset return is unconditionally riskless, there is no conflict between the criteria $\text{var}(R)$ and $\text{var}(R - R_0)$, and $R_P - R_U = 0$ in equation (13). The following corollary to Proposition 1 states that this is the only case where the subsets of UE and PE returns are equal.

Corollary 1. Given the representations of PE and UE returns in equations (9) and (12), respectively, these two subsets of CE returns are equal if and only if

$$\text{var}(R_0) = 0.$$

In this special case the intercept of the CE frontier R_0 cannot change over time, but the slope $\mathcal{S}_C$ may change. In this regard, Proposition 1 provides a general relation (10) between $\mathcal{S}_U^2$ and $\mathcal{S}_C^2$. Jagannathan (1996) obtains a similar relation for the unconditional Sharpe ratio of UE returns with a constant safe asset return. My results show that PE returns are the only ones that achieve $\mathcal{S}_U^2$ independently of the behavior of R_0.

Ferson and Siegel (2001) also study UE returns under the assumption of a constant R_0. Their equation (12) provides the optimal vector of weights $\mathbf{w}$ in return (1) that solves the UE problem (11), which is equal to

$$\left[\frac{\nu - R_0}{\mathrm{E}\left[\mathrm{E}\left(\mathbf{r}|\mathbf{z}\right)' \left[\mathrm{E}\left(\mathbf{r}\mathbf{r}'|\mathbf{z}\right)\right]^{-1} \mathrm{E}\left(\mathbf{r}|\mathbf{z}\right)\right]}\right] \left[\mathrm{E}\left(\mathbf{r}\mathbf{r}'|\mathbf{z}\right)\right]^{-1} \mathrm{E}\left(\mathbf{r}|\mathbf{z}\right).$$

When the safe asset is such that $R_0 = \mathrm{E}\left(R_0\right)$, it is easy to show that these optimal weights are equal to $\omega_C \boldsymbol{\varphi}_{\mathbf{C}}$, with ω_C from equation (9), by exploiting the relation[4]

$$\mathrm{E}\left(\mathbf{r}\mathbf{r}'|\mathbf{z}\right) = \mathrm{var}\left(\mathbf{r}|\mathbf{z}\right) + \mathrm{E}\left(\mathbf{r}|\mathbf{z}\right)\mathrm{E}\left(\mathbf{r}|\mathbf{z}\right)'.$$

The weights must be equal because Corollary 1 states that UE and PE returns coincide in this special case.

Similarly, Ferson and Siegel (2009) construct an efficiency test that is based on $\mathcal{S}_U$. My theoretical results indicate that such a test is actually testing whether a particular return is PE, not UE. The empirical application in Section IV.C shows that the performance gap between UE and PE returns can be sizable.

I refer to the relations between the standard deviations of UE and PE returns and the mean target as the UE and PE frontiers, respectively. Figure 1 displays the UE and PE frontiers in two different spaces, using the data and empirical methods of Section IV. In Graph A, the PE frontier is a straight line with slope $\mathcal{S}_U$ for positive risk premia in the $\left[\sqrt{\mathrm{var}\left(r\right)}, \mathrm{E}\left(r\right)\right]$ space and provides the best performance in that space.

I can relate the excess returns of UE and PE returns with the same mean ν by

$$R_U - R_0 = \left(R_P - R_0\right) - \left(R_P - R_U\right)$$

with $R_P - R_U$ displayed in equation (13). This term is uncorrelated with $R_P - R_0$ and hence I have the following variance decomposition

$$\mathrm{var}\left(R_U - R_0\right) = \mathrm{var}\left(R_P - R_0\right) + \mathrm{var}\left(R_P - R_U\right).$$

The difference between $\mathrm{var}\left(R_U - R_0\right)$ and $\mathrm{var}\left(R_P - R_0\right)$ is the same across different values of ν because $R_P - R_U$ does not depend on the chosen ν. Conversely, as is common practice, the plots in Figure 1 are based on standard deviations instead of variances. In Graph A the unconditional Sharpe ratio of $R_U - R_0$ converges to the ratio for $R_P - R_0$ as ν increases.

Graph B of Figure 1 compares the PE and UE frontiers on the $\left[\sqrt{\mathrm{var}\left(R\right)}, \mathrm{E}\left(R\right)\right]$ space, where the latter is the most efficient. I can relate the PE and UE returns with the same mean ν by

$$R_P = R_U + \left(R_P - R_U\right)$$

[4]This relation implies that

$$\left[\mathrm{E}\left(\mathbf{r}\mathbf{r}'|\mathbf{z}\right)\right]^{-1} \mathrm{E}\left(\mathbf{r}|\mathbf{z}\right) = \frac{\boldsymbol{\varphi}_{\mathbf{C}}}{1 + \mathcal{S}_C^2}$$

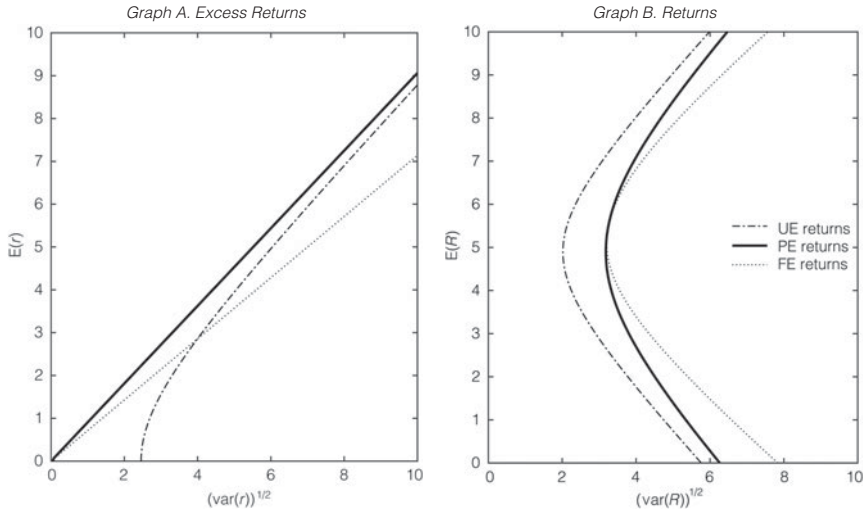
and

$$\mathrm{E}\left[\mathrm{E}\left(\mathbf{r}|\mathbf{z}\right)' \left[\mathrm{E}\left(\mathbf{r}\mathbf{r}'|\mathbf{z}\right)\right]^{-1} \mathrm{E}\left(\mathbf{r}|\mathbf{z}\right)\right] = \mathrm{E}\left[\frac{\mathcal{S}_C^2}{1 + \mathcal{S}_C^2}\right].$$

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FIGURE 1
UE, PE, and FE Returns

In Figure 1, the means and standard deviations are measured in annual percentage. UE stands for unconditionally efficient, PE stands for performance-efficient, and FE stands for fixed-weight efficient.



with $R_P - R_U$ displayed in equation (13). This term is uncorrelated with R_U and hence the difference between both frontiers follows from

$$\text{var}\,(R_P) \quad = \quad \text{var}\,(R_U) + \text{var}\,(R_P - R_U)\,.$$

The PE returns are located on a parabola that is parallel to the parabola of UE returns in the $[\text{var}(R), \text{E}\,(R)]$ space. The difference between the two parabolas is equal to $\text{var}\,(R_P - R_U)$. However, Graph B is based on standard deviations instead of variances. In this plot, the UE and PE frontiers share the location of the minimum and the asymptotes, being more different for lower values of ν.

For comparison, the location of the Markowitz frontier is also displayed in Figure 1. This frontier is defined by the FE returns in expression (3). In Graph A, the FE frontier is represented by a straight line and there is a region where FE returns perform better than UE returns.

Finally, the unconditional alphas of PE and UE returns can be compared. The former yield zero α_U as a factor when pricing any excess return, but α_U is not generally 0 when UE returns are used as pricing factors. In fact, the value of α_U depends on the particular mean target of the UE return.

Nevertheless, Hansen and Richard (1987) show that UE returns satisfy a different beta pricing equation. In a general setup that may not include a safe asset, a return R_β different from the minimum unconditional variance return is UE if and only if, for every R,

$$\text{E}\,(R) - E_U \quad = \quad \frac{\text{cov}\,(R, R_\beta)}{\text{var}\,(R_\beta)}\,[\text{E}\,(R_\beta) - E_U]\,, \tag{14}$$

where E_U is a real number. This number is interpreted as the unconditional mean of the corresponding zero-beta return and depends on the chosen UE pricing factor R_β.

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B. Residually Efficient Returns

Internet Appendix A analyzes the link between mean-variance preferences and frontiers with conditioning information. Ferson and Siegel (2001) show that UE returns can be rationalized by preferences $\mathrm{E}(R|\mathbf{z}) - (b/2)\,\mathrm{E}\left(R^2|\mathbf{z}\right)$ for some positive real number b. However, the most common conditional mean-variance preferences in finance theory are probably

$$\mathrm{E}(R|\mathbf{z}) - (\theta/2)\,\mathrm{var}(R|\mathbf{z})$$

for some positive real number θ. Areas such as market microstructure and rational expectations equilibria often rely on these preferences. See, for example, Easley and O'Hara (2004) or Brunnermeier (2001) for a survey of asset pricing theory under asymmetric information.[5]

If I average these preferences over $\mathbf{z}$, I obtain $\mathrm{E}(R) - (\theta/2)\,\mathrm{E}[\mathrm{var}(R|\mathbf{z})]$ and hence these preferences penalize only a component of the unconditional variance. Specifically, I can decompose the unconditional variance of a return as

$$\mathrm{var}(R) \quad = \quad \mathrm{E}[\mathrm{var}(R|\mathbf{z})] + \mathrm{var}[\mathrm{E}(R|\mathbf{z})]\,.$$

I refer to the first component $\mathrm{E}[\mathrm{var}(R|\mathbf{z})]$ as the residual variance because it is the variance of the residual $R - \mathrm{E}(R|\mathbf{z})$ from a predictive regression,

$$\mathrm{var}(R - \mathrm{E}(R|\mathbf{z})) \quad = \quad \mathrm{E}\left[(R - \mathrm{E}(R|\mathbf{z}))^2\right] \quad = \quad \mathrm{E}[\mathrm{var}(R|\mathbf{z})]\,.$$

Note that the safe asset return may be risky from the perspective of $\mathrm{var}(R_0) > 0$, but not from the perspective of $\mathrm{E}[\mathrm{var}(R_0|\mathbf{z})] = 0$.

Following the link between conditional mean-variance preferences and $\mathrm{E}[\mathrm{var}(R|\mathbf{z})]$, I define the *residually efficient* (RE) returns as the set of returns in equation (1) that solve the problem

$$\min_R \mathrm{E}[\mathrm{var}(R|\mathbf{z})] \text{ subject to } \mathrm{E}(R) \quad = \quad \nu, \tag{15}$$

which I denote R_R. This link also motivates the use of residual Sharpe ratios and alphas, where the required variances and covariances are based on the residuals of predictive regressions instead of the returns themselves. I define the residual Sharpe ratio of an excess return r as the real number

$$S_R \quad = \quad \frac{\mathrm{E}(r)}{\sqrt{\mathrm{E}[\mathrm{var}(r|\mathbf{z})]}}$$

and the residual Jensen's alpha of an excess return r with respect to a pricing factor r_β as

$$\alpha_R \quad = \quad \mathrm{E}(r) - \beta_R \mathrm{E}(r_\beta)\,, \qquad \beta_R \quad = \quad \frac{\mathrm{E}[\mathrm{cov}(r, r_\beta|\mathbf{z})]}{\mathrm{E}\left[\mathrm{var}\left(r_\beta|\mathbf{z}\right)\right]}.$$

[5]They are also used in continuous time asset allocation (Basak and Chabakauri (2010) and the references therein) and to study the complexity of the performance evaluation of an informed manager by an uninformed agent (Dybvig and Ross (1985)).

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The following proposition analyzes RE returns.

Proposition 2. The representation and properties of RE returns R_R defined by problem (15).

1) The RE returns are the CE returns in equation (5) with

$$\omega_C = \frac{\nu - \mathrm{E}(R_0)}{\mathrm{E}(\mathcal{S}_C^2)}, \tag{16}$$

which is a real number.

2) The excess returns of RE returns different from R_0 are characterized by achieving the maximum S_R^2, denoted $\mathcal{S}_R^2$ and given by

$$\mathcal{S}_R^2 = \mathrm{E}(\mathcal{S}_C^2). \tag{17}$$

These excess returns are also characterized by $\alpha_R = 0$ when they are used as a factor to price any r.

Point 2 of Proposition 2 shows that RE returns have standard properties in terms of residual Sharpe ratios and alphas. Once again, this is not the case for UE returns. I can decompose the difference between RE and UE returns with the same mean as

$$R_R - R_U = (R_R - R_P) + (R_P - R_U),$$

where R_P is the PE return with the same mean. From point 1 of Propositions 1 and 2, the second component $R_P - R_U$ is computed in equation (13) and the first component is

$$R_R - R_P = [\nu - \mathrm{E}(R_0)] \left[\frac{1}{\mathrm{E}(\mathcal{S}_C^2)} - \frac{1}{\mathrm{E}\left(\frac{\mathcal{S}_C^2}{1+\mathcal{S}_C^2}\right)(1+\mathcal{S}_C^2)} \right] \mathbf{r}'\boldsymbol{\varphi}_\mathbf{C}. \tag{18}$$

The next corollary characterizes the special cases where the subset of RE returns is equal to the subsets of PE or UE returns.[6]

Corollary 2. Given the representations of PE, UE, and RE returns in equations (9), (12), and (16), respectively,

1) The RE and PE returns are equal if and only if

$$\mathrm{var}(\mathcal{S}_C) = 0.$$

[6]There are other theoretically possible cases. The RE and UE returns share one element if and only if there are two real numbers (a, b) such that

$$R_0 = a + b\mathcal{S}_C^2,$$

in which case the shared return has a constant conditional mean a. A simple example is a constant R_0, as this return is also UE.

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2) The RE and UE returns are equal if and only if

$$\text{var}\,(R_0) \quad = \quad \text{var}\,(\mathcal{S}_C) \quad = \quad 0.$$

In this case, these two subsets of CE returns are also equal to the PE returns.

Corollary 2 indicates the relevance of the Sharpe ratio $\mathcal{S}_C$, that is, the slope of the CE frontier, in the relation between RE returns and the other two efficiency types. For instance, the difference $R_R - R_P$ in equation (18) is driven by the fact that the RE returns are constructed by scaling $\mathbf{r}'\boldsymbol{\varphi}_\mathbf{C}$ with real numbers, and the PE returns are constructed by scaling $\mathbf{r}'\boldsymbol{\varphi}_\mathbf{C}/(1+\mathcal{S}_C^2)$ instead.

Abhyankar et al. (2012) approximate the slope of the UE frontier asymptotes by the second moment of the slope of the CE frontier in their analysis of ex ante gains from return predictability.[7] That is, they approximate $\mathcal{S}_U$ in equation (10) by the second moment of $\mathcal{S}_C$. However, equation (17) in Proposition 2 shows that the second moment of $\mathcal{S}_C$ has an exact relation with the slope of the RE frontier $\mathcal{S}_R$. The relation between $\mathcal{S}_R$ and $\mathcal{S}_C$ is more direct than the relation between $\mathcal{S}_U$ and $\mathcal{S}_C$. Moreover, point 2 in Corollary 2 states that the UE and RE returns are different subsets of CE returns unless both R_0 and $\mathcal{S}_C$ are constant over time. The empirical application in Section IV.C shows that the performance gap between UE and RE returns can be sizable.

The RE frontier is given by two straight lines in the $[\sqrt{\text{E}\,[\text{var}\,(R|\mathbf{z})]}, \text{E}\,(R)]$ space with zero residual variance at $\text{E}\,(R) = \text{E}\,(R_0)$, as displayed in Graph A of Figure 2. The PE frontier is also linear in that space and shares the safe asset return with the RE frontier.

The RE returns provide the best risk–return trade-off in that space, whereas UE returns provide the best frontier in Graph B of Figure 2 for the $[\sqrt{\text{var}\,(R)}, \text{E}\,(R)]$ space. I can relate the RE and UE returns with the same mean ν by

$$R_R \quad = \quad R_U + (R_R - R_U)\,,$$

with $R_R - R_U$ equal to the sum of equations (13) and (18) and thus dependent on the chosen ν. The relative location of the RE and UE frontiers in Graph A of Figure 2 follows from the residual variance decomposition

$$\text{E}\,[\text{var}\,(R_U|\mathbf{z})] \quad = \quad \text{E}\,[\text{var}\,(R_R|\mathbf{z})] + \text{E}\,[\text{var}\,(R_R - R_U|\mathbf{z})]\,,$$

which follows itself from the zero residual covariance between the term $R_R - R_U$ and R_R. Conversely, the lack of correlation between the term $R_R - R_U$ and R_U yields the variance decomposition

$$\text{var}\,(R_R) \quad = \quad \text{var}\,(R_U) + \text{var}\,(R_R - R_U)$$

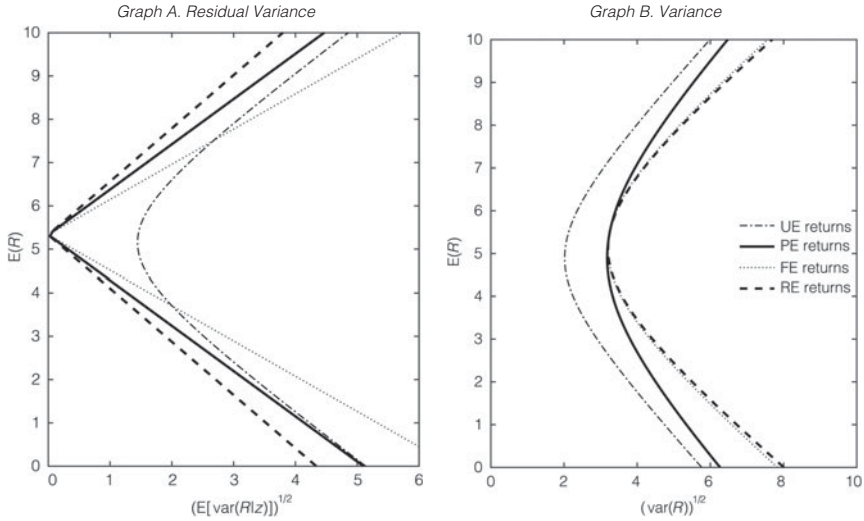
and thus the location of RE returns to the right of UE returns in Graph B.

[7] Abhyankar et al. (2012) also evaluate the ex post unconditional performance of UE returns and claim that these returns improve the performance with respect to a constant ω_C in return (5). My results show that such CE returns are actually RE returns and hence they are efficient in terms of S_R, but not S_U.

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FIGURE 2
UE, PE, RE, and FE Returns

In Figure 2, the means and standard deviations are measured in annual percentage. UE stands for unconditionally efficient, PE stands for performance-efficient, RE stands for residually efficient, and FE stands for fixed-weight efficient.



The location of the FE frontier is also displayed in Figure 2. In this example, in some regions the performance of FE returns is better than that of some CE returns. In fact, RE returns look very similar to FE returns in Graph B, even though they are the optimal returns in Graph A.

Finally, Internet Appendix B derives the implications of the properties of PE and RE returns for testing mean-variance efficiency and asset pricing models. In particular, conditional asset pricing models can be tested by comparing the maximum unconditional or residual Sharpe ratios from two investment sets. The validity of models with both traded and nontraded factors can be tested. The conditional CAPM is an example of a traded factor because in this model the market portfolio must be CE in equilibrium.[8] In the case of nontraded factors, the associated mimicking portfolios can be used.

IV. Empirical Application

This section develops an econometric framework for the estimation of unconditional and residual Sharpe ratios. This framework is applied to testing the performance gaps between the different efficiency types that are studied in Section III and to testing asset pricing models.

A. Unconditional and Residual Sharpe Ratios

I extend the standard setup in empirical finance for testing the difference between the unconditional Sharpe ratios of two investment strategies (see Ledoit

[8]The CAPM was originally developed by Sharpe (1964), Lintner (1965), and Mossin (1966) in a context without conditioning information.

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and Wolf (2008) and the references therein) to residual Sharpe ratios. For some excess return r_i and a vector of predictors $\mathbf{z}$, I define the influence functions

$$\mathbf{h}\left(r_i, \mathbf{z}\,;\, \mu_i, \mathbf{b}_i', \sigma_i^2\right) \quad = \quad \begin{pmatrix} \begin{pmatrix} 1 \\ \mathbf{z} \end{pmatrix} \left(r_i - \mu_i - \mathbf{b}_i'\mathbf{z}\right) \\ \left(r_i - \mu_i - \mathbf{b}_i'\mathbf{z}\right)^2 - \sigma_i^2 \end{pmatrix}$$

with parameters $\left(\mu_i, \mathbf{b}_i', \sigma_i^2\right)$. The first block of influence functions pins down the conditional mean of r_i with $\mu_i + \mathbf{b}_i'\mathbf{z}$, and the last influence function pins down the residual variance of r_i. The linearity of the conditional mean is not really a constraint because I could add powers of $\mathbf{z}$ or other functions of the predictors. Given time-series data on $(r_i, \mathbf{z})$, I can estimate the residual Sharpe ratio of r_i by means of the moment conditions

$$\mathrm{E}\left[\mathbf{h}\left(r_i, \mathbf{z}\,;\, \mu_i, \mathbf{b}_i', \sigma_i^2\right)\right] \quad = \quad \mathbf{0},$$

which identify the ratio as $\mu_i / \left(\sigma_i^2\right)^{1/2}$ if the mean is subtracted from the predictors $\mathbf{z}$ and thus μ_i identifies the expectation of r_i.

I can jointly estimate the residual Sharpe ratio of two excess returns r_1 and r_2 by means of the moment conditions

$$\mathrm{E}\begin{bmatrix} \mathbf{h}\left(r_1, \mathbf{z}\,;\, \mu_1, \mathbf{b}_1', \sigma_1^2\right) \\ \mathbf{h}\left(r_2, \mathbf{z}\,;\, \mu_2, \mathbf{b}_2', \sigma_2^2\right) \end{bmatrix} \quad = \quad \mathbf{0}, \tag{19}$$

with parameters to estimate $\left(\mu_1, \mathbf{b}_1', \sigma_1^2, \mu_2, \mathbf{b}_2', \sigma_2^2\right)$. If the mean is subtracted from the predictors $\mathbf{z}$, then the difference in residual Sharpe ratios can be identified by

$$\frac{\mu_2}{\left(\sigma_2^2\right)^{1/2}} - \frac{\mu_1}{\left(\sigma_1^2\right)^{1/2}},$$

and I can test if its true value is 0 with a Wald test.

If I am interested in the difference in unconditional Sharpe ratios instead, I can simply change the influence functions in equation (19) to

$$\mathbf{h}\left(r_i, \mathbf{z}\,;\, \mu_i, \mathbf{b}_i', \sigma_i^2\right) \quad = \quad \begin{pmatrix} \begin{pmatrix} 1 \\ \mathbf{z} \end{pmatrix} \left(r_i - \mu_i - \mathbf{b}_i'\mathbf{z}\right) \\ \left(r_i - \mu_i\right)^2 - \sigma_i^2 \end{pmatrix}.$$

I could also simplify the upper block to $r_i - \mu_i$ and hence I would not use $\mathbf{z}$ in the influence functions. This is the standard setup in empirical finance.[9]

I use the generalized method of moments (GMM) of Hansen (1982) with the system of moment conditions (19) to develop the corresponding inference. I can compute standard errors of estimators and p-values of tests without assuming Gaussian or independent and identically distributed returns. In particular,

[9] I need a good approximation to the true conditional mean of r_1 and r_2 to identify the residual Sharpe ratios, which is not required for the computation of unconditional ratios. However, neither of the two ratios requires a good approximation of the true conditional variances of r_1 and r_2.

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I use Newey–West (1987) standard errors and p-values, which are robust to heteroskedasticity and autocorrelation.

In case the GMM asymptotic p-values do not provide a good approximation in these tests, I complement them with block bootstrap p-values. For that purpose, I sample Wald statistics divided by their standard errors as follows. I run a first-order vector autoregression (VAR(1)) of $(r_1, r_2, \mathbf{z})$ and store the residuals. This VAR is my data-generating process (DGP) after changing the value of the estimated μ_2 to satisfy the null hypothesis of equal Sharpe ratios. I impose $\mu_2 = (\sigma_2^2)^{1/2}\mu_1/(\sigma_1^2)^{1/2}$ in the DGP, which changes only the level of r_2 without altering its variance or dynamics. I sample blocks of residuals with replacement to consider additional nonlinear dynamics in the data, such as generalized autoregressive conditional heteroskedasticity (GARCH) effects. For each sample of $(r_1, r_2, \mathbf{z})$ generated in this way, I store the associated Wald statistic divided by its standard error. I repeat this process several times, which provides a sample of t-statistics. Then, I compute the corresponding p-value of the t-statistic obtained from the actual data.

This block bootstrap on the residuals of a VAR(1) is similar to that of Ledoit and Wolf (2008). However, these authors consider only unconditional Sharpe ratios and therefore resample (r_1, r_2) instead of $(r_1, r_2, \mathbf{z})$. They do not impose the null hypothesis in the DGP either. Another difference is that they consider automatic approaches to pin down the number of lags in the Newey–West (1987) standard errors and the number of blocks in the bootstrap. I consider several combinations of values instead to check the robustness of the computations.

B. Data Description

I use monthly returns from 1954 to 2012 available from Kenneth French's data library. In particular, I use the Treasury bill return as R_0 and different combinations of the Fama–French (1993) size and book-to-market sorted portfolios as the vector $\mathbf{r}$. To make the analysis more transparent, I often focus on the 3 Fama–French factors, whose first factor is the excess return on the market, and the two additional excess returns are associated with the size and value effects. These are called SMB (long in small capitalization stocks and short in big stocks) and HML (long in high book-to-market stocks and short in low book-to-market stocks), respectively. Given their widespread use in empirical work, I also consider the 6 and 25 Fama–French portfolios to study the role of the investment set in the differences across types of efficient returns. I subtract the safe asset return from their returns to obtain the corresponding excess returns. See Kenneth French's Web page (http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html) and Fama and French (1993) for further details.

I use 3 predictors to define the vector $\mathbf{z}$: the U.S. dividend–price ratio, the default spread, and the term spread. The first predictor is taken from Robert Shiller's Web page (http://www.econ.yale.edu/∼shiller/data.htm), and the other two predictors are constructed from the Federal Reserve Economic Data. The default spread uses yields on AAA- and BAA-rated bonds and the term spread uses 10- and 1-year constant maturity Treasury bond yields. These predictors are widely used in empirical finance (e.g., see Ferson and Siegel (2009)).

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As a first description of the data, Table 1 analyzes the mean predictability in each one of the 3 Fama–French (1993) factors. I divide the full sample into two halves (1954–1983 and 1984–2012) because some features of return predictability may differ across periods. A formal empirical analysis of predictability changes across time is beyond the scope of this article.

Panel A of Table 1 reports the coefficient of determination R^2 from each predictive regression as a statistical measure of predictability. In the case of the market factor, we can associate the first period with stronger predictability and the second period with weaker predictability. See Ferson and Siegel (2009) and the references therein for similar evidence up to the early 2000s, with weaker predictability in the 1990s. In particular, these authors study the periods 1963–1994 and 1995–2002 and find lower predictability in the second period. However, the weaker market predictability in the second period does not necessarily mean that predictability is disappearing, as SMB and HML do not follow the same pattern. The predictive regressions of SMB and HML show a stable or even increasing R^2.

TABLE 1

Predictability in the Fama–French (1993) Factors

Table 1 shows the predictability in the monthly excess returns on the Fama–French (1993) factors, which are the market portfolio (MMR) and the portfolios that capture the size and value effects (SMB and HML, respectively). Panel A reports the R^2 of the predictive regression of each factor on the predictors, which are the dividend–price ratio, the default spread, and the term spread. Panel B displays the unconditional Sharpe ratios of performance-efficient (PE) and fixed-weight efficient (FE) returns on each factor, and the residual Sharpe ratios of residually efficient (RE) and FE returns. The PE and RE returns are constructed with the conditional mean of excess returns given by the predictive regressions and a constant conditional variance. The differences in Sharpe ratios and their statistical significance with Newey–West (1987) standard errors are also displayed. *, **, and *** indicate significance with asymptotic *p*-values at the 10%, 5%, and 1% levels, respectively; $^{+}$, $^{++}$, and $^{+++}$ indicate significance with block bootstrap *p*-values at the 10%, 5%, and 1% levels, respectively.

	1954–1983			1984–2012		
	MMR	SMB	HML	MMR	SMB	HML
Panel A. Predictive Regressions						
R^2	0.071	0.024	0.001	0.008	0.023	0.011
Panel B. Unconditional and Residual Sharpe Ratios (S_U and S_R)						
S_U						
PE	1.023	0.644	0.693	0.617	0.651	0.464
FE	0.433	0.419	0.678	0.444	0.075	0.331
PE − FE	$0.590^{***,++}$	0.225	0.015	0.174	$0.576^{***,+++}$	0.134
S_R						
RE	1.073	0.701	0.695	0.620	0.647	0.471
FE	0.450	0.424	0.678	0.445	0.076	0.333
RE − FE	$0.623^{***,++}$	0.276	0.017	0.175	$0.571^{***,+++}$	0.138

Panel B of Table 1 displays an economic measure of return predictability for each Fama–French (1993) factor. I test the gains from conditioning information by using FE returns as r_1 and PE or RE returns as r_2 in the econometric framework of Section IV.A. If I apply the representation of FE returns in expression (3) to a single factor r, the excess returns of FE returns are spanned by scaling $r\varphi$ with real numbers, or equivalently by scaling r because φ is also a real number. The unconditional and residual Sharpe ratios do not depend on such scaling as long as it does not change the sign of the ratio. Therefore, I can simplify the computation

of Sharpe ratios of FE returns by using $r_1 = r$ directly (i.e., associate r_1 with holding the factor). Regarding r_2, if I apply the representation of PE returns in equations (5) and (9) to a single factor r, the excess returns of PE returns are spanned by scaling $r\varphi_C/(1 + \mathcal{S}_C^2)$ with real numbers. Therefore, I can use $r_2 = r\varphi_C/(1 + \mathcal{S}_C^2)$ to compute the unconditional and residual Sharpe ratios of PE returns. Following the representation of RE returns in equations (5) and (16), I can choose $r_2 = r\varphi_C$ to compute the Sharpe ratios of RE returns.

The computation of r_2 depends on objects like φ_C and $\mathcal{S}_C^2$ (defined in equations (6) and (7), respectively) and hence it requires the conditional mean and variance of the Fama–French (1993) factors. If I use the true conditional mean and variance, I identify the true maximum Sharpe ratios. If this is not the case, r_2 still exploits conditioning information, albeit not optimally. Following the spirit of predictive regressions, I model the conditional mean vector as linear in the predictors and the conditional variance matrix as constant. I can identify the former vector by running least squares of excess returns on a constant and the predictors and the latter matrix by the unconditional variance of the residuals. Here I focus on mean predictability, but I consider GARCH effects in the next sections.

Statistical significance with asymptotic p-values is denoted by an asterisk (*), and a cross ($^+$) is used to denote bootstrap p-values based on 1,000 bootstrap samples.[10] I compute the p-values for 6 and 12 lags in the Newey–West (1987) standard errors (the cubic root of the sample size is around 7) and, similarly, blocks of size 6 and 12 in the bootstrap. To conserve space, in Table 1 I report the statistical significance only for 6 lags and blocks of size 6 for bootstrap p-values. I do the same in the following tables. The significance is similar for other combinations of lags and block sizes, which are available from the author.

The findings in Panel B of Table 1 follow the patterns of R^2 in Panel A. In terms of both unconditional and residual Sharpe ratios, I find high and statistically significant gains from conditioning information for the market in the first period and high and statistically significant gains for SMB in the second period. The increase in annualized Sharpe ratios is around 0.6 in both cases. The low unconditional and residual Sharpe ratios of holding SMB in the second period are due to a sharp decrease in its unconditional risk premium. The gains for HML are also higher in the second period, but I find no statistical significance.

C. Sharpe Ratios of Efficient Returns

This section studies the Sharpe ratios of efficient returns constructed from the 3 Fama–French (1993) factors and the associated 6 and 25 sorted portfolios. I keep the conditional mean of the corresponding vector $\mathbf{r}$ linear in the predictors, but I study both a constant and a time-varying conditional variance of $\mathbf{r}$. In the second case, that random matrix should be positive semidefinite at every realization. For this purpose, I follow the spirit of Bollerslev's (1990) constant

[10]The econometric framework is the standard one in which the econometrician compares the Sharpe ratios of two time series of excess returns, r_1 and r_2. The econometrician does not have information on or simply does not consider how these returns were constructed. To consider the estimation of their portfolio weights in my inference, I should add the corresponding moments to system (19). This would complicate the computations, especially the bootstrap.

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conditional correlation model. I proceed in two steps: First, I fit a GARCH$(1,1)$ to each residual from the predictive regressions to obtain the conditional variance of each excess return in $\mathbf{r}$.[11] Second, I compute the conditional correlations of $\mathbf{r}$ as the unconditional correlations of the residuals scaled by their GARCH volatilities.

Table 2 compares the unconditional performance of PE returns against FE and UE returns. The statistical significance of the differences in Sharpe ratios follows the econometric framework of Section IV.A. For instance, when I compare the Sharpe ratios of FE and PE returns, I can use $r_1 = \mathbf{r}'\boldsymbol{\varphi}$ from the representation of FE returns in expression (3) and $r_2 = \mathbf{r}'\boldsymbol{\varphi}_C/(1+\mathcal{S}_C^2)$ from the representation of PE returns in equations (5) and (9). Panel A displays the results when PE returns do not exploit GARCH effects, and Panel B displays the results when they do.

TABLE 2

Unconditional Sharpe Ratios

Table 2 shows unconditional Sharpe ratios from the monthly excess returns on 3 investment sets: the 3 Fama–French (1993) factors (FF3) and the 6 and 25 Fama–French portfolios (FF6 and FF25, respectively). In particular, the table reports the unconditional Sharpe ratios of performance-efficient (PE), fixed-weight efficient (FE), and unconditionally efficient (UE) returns. The UE returns are reported for annualized mean targets of 6% and 10%. Panel A displays the results when PE and UE returns are constructed with the conditional mean of excess returns given by the predictive regressions and a constant conditional variance, and Panel B displays the results for time-varying variance. The differences in Sharpe ratios and their statistical significance with Newey–West (1987) standard errors are also displayed. *, **, and *** indicate significance with asymptotic p-values at the 10%, 5%, and 1% levels, respectively; $^{+}$, $^{++}$, and $^{+++}$ indicate significance with block bootstrap p-values at the 10%, 5%, and 1% levels, respectively.

	1954–1983			1984–2012		
	FF3	FF6	FF25	FF3	FF6	FF25
Panel A. Constant Conditional Variance of Excess Returns						
PE	1.288	1.436	2.256	1.047	1.508	2.517
FE	0.955	1.092	1.484	0.670	1.390	2.211
PE − FE	0.334$^{**,++}$	0.344$^{**,++}$	0.772$^{***,+++}$	0.376$^{*,+}$	0.118	0.306$^{**,++}$
UE6	1.203	1.325	1.281	1.005	1.366	1.909
PE − UE6	0.085	0.111	0.975$^{***,+++}$	0.042	0.142**	0.608$^{***,++}$
UE10	1.300	1.460	2.220	1.035	1.472	2.375
PE − UE10	−0.011	−0.025	0.036	0.012	0.036	0.142**
Panel B. Time-Varying Conditional Variance of Excess Returns						
PE	1.264	1.450	2.308	1.032	1.659	2.614
FE	0.955	1.092	1.484	0.670	1.390	2.211
PE − FE	0.309$^{**,+}$	0.358$^{**,+}$	0.824$^{***,+++}$	0.362	0.270$^{**,+}$	0.403$^{**,++}$
UE6	1.148	1.304	1.162	1.004	1.525	2.026
PE − UE6	0.116	0.145	1.147$^{***,+++}$	0.028$^{***,+++}$	0.134*	0.588$^{***,++}$
UE10	1.264	1.479	2.205	1.025	1.628	2.498
PE − UE10	0.000	−0.029	0.104	0.007	0.031	0.116*

In Panel A of Table 2 the performance gap of PE returns with respect to FE returns is high and statistically significant in the first period and increases with the number of assets. This pattern does not hold in the second period, where the gap becomes lower than in the first period for the 6 and 25 portfolios. In fact,

[11] I compute the unconditional variance of the residuals and use it as the steady state of the conditional variance. I estimate the remaining two parameters of the GARCH$(1,1)$, usually denoted α and β, by Gaussian quasi-maximum likelihood. The results are similar with the joint estimation of the conditional mean and variance, and the estimation of the steady state variance without variance targeting, but the computations are much more time consuming.

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the performance gap is not statistically significant for the 6 portfolios. Similarly, Ferson and Siegel (2009) reject the lack of gains from conditioning information in 1963–1994 but not in 1995–2002. Considering GARCH effects in Panel B increases the performance of PE returns with the 6 portfolios and the 25 portfolios, both economically and statistically.

Corollary 1 states that UE and PE returns are equivalent if and only if the intercept of the CE frontier is constant. However, the unconditional variance of R_0 is not 0. The rate of return of the safe asset has a mean of 0.430 and a standard deviation of 0.264 in the first period. Both statistics decrease in the second period to 0.340 and 0.216, respectively. Table 2 quantifies the difference between PE and UE returns in terms of Sharpe ratios. It reports the performance gap for two mean targets $\mathrm{E}(R) = \nu$, a low target of 6% and a high target of 10% in annualized terms, because the Sharpe ratio of UE returns depends on the chosen target. Following the representation of UE returns in equations (5) and (12), I can use $r_1 = \omega_C \mathbf{r}' \boldsymbol{\varphi}_{\mathbf{C}}$ in the econometric framework of Section IV.A. The scaling ω_C depends on the target ν and on some expectations that are estimated by their sample counterparts.

In Graph A of Figure 1 the unconditional Sharpe ratios of UE returns converge to the ratio for PE returns as ν increases. Therefore, the performance gap should be more relevant for targets closer to the safe asset. Panel A of Table 1 shows that the gaps are economically and statistically significant with the 25 portfolios and the low target of 6%, with a stronger gap in the first period than in the second. For that target, the gap is higher than 0.1 for the 6 portfolios, but statistically significant only in the second period, with asymptotic p-values. In that period, I also find a statistically significant difference for the 10% target and the 25 portfolios, but only with the asymptotic p-value. In the first period, some differences for that target become slightly negative, but this simply reflects how similar these UE returns are to PE returns.

The main patterns from Panel A of Table 2 continue to hold when considering GARCH effects in Panel B. PE and UE returns are especially different for the low mean target and the 25 portfolios, with a larger gap in the first period. This investment set is widely used in empirical work (e.g., Ferson and Siegel (2009)).

Table 3 studies the residual performance of RE returns against FE and UE returns. In the econometric framework of Section IV.A, I can use $r_2 = \mathbf{r}' \boldsymbol{\varphi}_{\mathbf{C}}$ to compute the Sharpe ratios of RE returns following their representation in equations (5) and (16). Similarly to PE returns in Table 2, the performance gap with respect to FE strategies is especially large in the first period. Interestingly, Panel B of Table 3 shows that considering GARCH effects clearly improves RE returns with the 6 and 25 portfolios, especially in the second period. This translates into large and significant performance gaps with respect to FE returns in the second period, which are not found without considering the GARCH effects in Panel A.

Corollary 2 states that UE and RE returns are equivalent if and only if both the intercept and slope of the CE frontier are constant, but none of these variables has zero unconditional variance. In particular, when I work with the 3 Fama–French (1993) factors and do not consider GARCH effects, the slope of the CE frontier (whose square is given by equation (7)) has a mean of 0.373 and a standard deviation of 0.172 in the first period. Both statistics decrease in the second period to 0.285 and 0.117, respectively.

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TABLE 3
Residual Sharpe Ratios

Table 3 shows residual Sharpe ratios from the monthly excess returns on 3 investment sets: the 3 Fama–French (1993) factors (FF3) and the 6 and 25 Fama–French portfolios (FF6 and FF25, respectively). In particular, the table reports the residual Sharpe ratios of residually efficient (RE), fixed-weight efficient (FE), and unconditionally efficient (UE) returns. The UE returns are reported for annualized mean targets of 6% and 10%. Panel A displays the results when RE and UE returns are constructed with the conditional mean of excess returns given by the predictive regressions and a constant conditional variance, and Panel B displays the results for time-varying variance. The differences in Sharpe ratios and their statistical significance with Newey–West (1987) standard errors are also displayed. *, **, and *** indicate significance with asymptotic *p*-values at the 10%, 5%, and 1% levels, respectively; $^{+}$, $^{++}$, and $^{+++}$ indicate significance with block bootstrap *p*-values at the 10%, 5%, and 1% levels, respectively.

	1954–1983			1984–2012		
	FF3	FF6	FF25	FF3	FF6	FF25
Panel A. Constant Conditional Variance of Excess Returns						
RE	1.382	1.515	2.298	0.675	1.391	2.226
FE	0.969	1.112	1.494	1.046	1.491	2.316
RE − FE	0.413**,++	0.403**,++	0.805***,+++	0.371*,+	0.100	0.090
UE6	1.281	1.416	1.443	1.019	1.408	2.102
RE − UE6	0.100	0.099	0.855***,+++	0.027	0.083	0.214
UE10	1.348	1.513	2.259	1.046	1.496	2.462
RE − UE10	0.034	0.002	0.039	0.000	−0.005	−0.146
Panel B. Time-Varying Conditional Variance of Excess Returns						
RE	1.353	1.542	2.354	1.011	1.656	2.652
FE	0.969	1.112	1.494	0.675	1.391	2.226
RE − FE	0.384**,++	0.430**,++	0.860***,+++	0.336	0.264*,+	0.426***,++
UE6	1.219	1.382	1.327	1.010	1.543	2.164
RE − UE6	0.133	0.160	1.026***,++	0.001	0.113	0.488***,++
UE10	1.306	1.534	2.285	1.030	1.634	2.535
RE − UE10	0.047	0.008	0.068	−0.020	0.022	0.118

Because UE and RE returns are not equivalent for these data, Table 3 shows the estimates and tests of their performance differences. The performance gap is particularly large for the low target and the 25 portfolios, as in Table 2. In Panel A of Table 3 the gap is not statistically significant in the second period, but this changes when considering GARCH effects in Panel B. The performance gaps with respect to UE returns increase considerably, becoming strongly significant with the 25 portfolios and the low target in both periods.

Tables 2 and 3 focus on monthly returns, but the return frequency is another relevant dimension for the differences across efficiency types. Internet Appendix D reports the counterpart of these tables with annual returns of the 3 factors. Table D1 shows that the differences across the subsets of CE returns are larger for annual returns. In fact, UE returns are clearly different with respect to PE or RE returns for this small investment set. In this regard, Figures 1 and 2, which illustrate Section III, correspond to these annual returns in the first period, so that the different efficiency types are easier to distinguish.

Internet Appendix D also reports an out-of-sample exercise on the gains from conditioning information during the second period. Table D2 confirms the weaker market predictability and stronger SMB predictability in that period. I also find gains of dynamic strategies with respect to the equal-weighted portfolios of the 3 factors and the 6 and 25 portfolios.

Finally, the previous tables and discussion focus on Sharpe ratios, but I can run an equivalent analysis in terms of Jensen's alphas. These computations are

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available from the author. For instance, when I use UE returns as pricing factors, the value of α_U depends on the particular mean target and can be very different from 0.[12] Similarly, the zero-beta returns of UE returns, E_U in the beta-pricing equation (14), can be very different from the average safe asset return.

D. Testing Asset Pricing Models

I also apply the econometric methodology of Section IV.A to test conditional asset pricing models. In particular, I use the previous data to test whether the conditional CAPM holds for a given vector $\mathbf{r}$. The main implication of this model is that the market return is CE, but PE and RE returns can also be used. To test the conditional CAPM with RE returns, I apply the representation of RE returns in equations (5) and (16) to the market excess return r and the vector $\mathbf{r}$ to obtain $r_1 = r\varphi_C$ and $r_2 = \mathbf{r}'\boldsymbol{\varphi}_\mathbf{C}$, respectively. To test the conditional CAPM with PE returns, which are represented in equations (5) and (9), the computations are similar but I use excess returns like $\mathbf{r}'\boldsymbol{\varphi}_\mathbf{C}/(1+\mathcal{S}_C^2)$ instead. I do not use UE returns because they do not have a unique unconditional or residual Sharpe ratio. Alternatively, I can test the unconditional CAPM by means of the FE returns in expression (3), using $r_1 = r$ and $r_2 = \mathbf{r}'\boldsymbol{\varphi}$.

If I use the true conditional mean and variance of r and $\mathbf{r}$ in the previous computations, I identify the true efficient returns and test the conditional CAPM. If this is not the case, I am not strictly testing the CAPM but am still testing the efficiency of the market portfolio for some (not necessarily optimal) portfolio strategy. Even with misspecified conditional moments, finding significant improvements in performance should cast some doubts on the validity of the CAPM.

Table 4 reports the tests of the CAPM,[13] comparing the performance of the market portfolio against the 3 investment sets of Section IV.C. Panel A displays the tests with unconditional Sharpe ratios, and Panel B displays the tests with residual ratios. In each panel, PE and RE returns are constructed for both a constant and a time-varying conditional variance (denoted C and V, respectively).

Start with the first period and the 3 Fama–French (1993) factors in both panels of Table 4. The market is more inefficient with FE returns in that period. Both the unconditional and residual Sharpe ratios of the FE returns with the 3 factors are more than double those with the market only and this increase is statistically significant. But the increase for PE and RE returns is not that strong. Table 1 shows that the market return is relatively more predictable than the other factors in this period and hence the latter may not add value in a dynamic strategy. I find some statistical significance with RE returns in Panel B, however, which suggests it would be good empirical practice to report tests for both unconditional and residual Sharpe ratios.

[12] Lewellen and Nagel (2006) analyze the unconditional alphas of CE returns. These alphas are generally not 0, even though the conditional alphas of CE returns must be 0. However, Lewellen and Nagel do not study UE returns in particular.

[13] To consider the estimation of portfolio weights in the test, I should add the corresponding moments to system (19). Similarly, to test an asset pricing model with nontraded factors, I should add the moments that estimate the mimicking portfolios.

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TABLE 4

Efficiency of the Market Portfolio against the Fama–French Portfolios

Table 4 shows efficiency tests from the monthly excess returns on the market portfolio (MMR) and 3 investment sets: the 3 Fama–French (1993) factors (FF3) and the 6 and 25 Fama–French portfolios (FF6 and FF25, respectively). Panel A displays the unconditional Sharpe ratios of performance-efficient (PE) and fixed-weight efficient (FE) returns and Panel B displays the residual Sharpe ratios of residually efficient (RE) and FE returns. Here PE C and RE C are constructed with the conditional mean of excess returns given by the predictive regressions and a constant conditional variance, and PE V and RE V are constructed with time-varying variance. The differences in Sharpe ratios and their statistical significance with Newey–West (1987) standard errors are also displayed. *, **, and *** indicate significance with asymptotic *p*-values at the 10%, 5%, and 1% levels, respectively; $^{+}$, $^{++}$, and $^{+++}$ indicate significance with block bootstrap *p*-values at the 10%, 5%, and 1% levels, respectively.

	1954–1983			1984–2012		
	FE	PE C	PE V	FE	PE C	PE V
Panel A. Unconditional Sharpe Ratios						
MMR	0.433	1.023	1.000	0.444	0.617	0.574
FF3	0.955	1.288	1.264	0.670	1.047	1.032
FF3 − MMR	0.521**,++	0.265	0.264	0.227	0.429**,++	0.458**,++
FF6	1.092	1.436	1.450	1.390	1.508	1.659
FF6 − MMR	0.659***,+++	0.412**,+	0.450**,+	0.946***,+++	0.891***,+++	1.086***,+++
FF25	1.484	2.256	2.308	2.211	2.517	2.614
FF25 − MMR	1.051***,+++	1.232***,+++	1.309***,+++	1.768***,+++	1.900***,+++	2.040***,+++
	1954–1983			1984–2012		
	FE	RE C	RE V	FE	RE C	RE V
Panel B. Residual Sharpe Ratios						
MMR	0.450	1.073	1.029	0.445	0.620	0.577
FF3	0.969	1.382	1.353	0.675	1.046	1.011
FF3 − MMR	0.519**,++	0.309*	0.323*,+	0.230	0.426**,+	0.434**,++
FF6	1.112	1.515	1.542	1.391	1.491	1.656
FF6 − MMR	0.662***,+++	0.442**,+	0.513**,++	0.946***,+++	0.871***,+++	1.079***,+++
FF25	1.494	2.298	2.354	2.226	2.316	2.652
FF25 − MMR	1.044***,+++	1.225***,+++	1.324***,+++	1.781***,+++	1.696***,+++	2.075***,+++

Importantly, the second period represents the opposite situation. Then the increases in Sharpe ratios when the SMB and HML returns are added to the PE and RE strategies are much higher than those with FE strategies, the latter not being statistically significant. Table 1 shows that the size effect is more predictable than the market in this period and hence it may add value in a dynamic strategy.

If I extend the investment set to the 6 or 25 sorted portfolios, the CAPM is rejected with any of the strategies in both periods. Nevertheless, I can still find a similar pattern between the two periods with the 6 portfolios. The market's performance gap is greater with FE strategies than with PE or RE strategies in the first period, but the gap is higher with the latter strategies in the second period if GARCH effects are considered. If I use the 25 portfolios, the PE and RE strategies also provide stronger evidence against the efficiency of the market portfolio than FE strategies in the first period.

Ferson and Siegel (2009) also reject the conditional CAPM in 1963–1994 and 1995–2002 using the 25 Fama–French (1993) portfolios. Lewellen and Nagel (2006) use average alphas from short-window regressions to conclude that the conditional CAPM and the classic unconditional CAPM yield similar bad performance in 1964–2001. My empirical evidence shows that such a conclusion depends on the investment set and the period.

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V. Conclusions and Further Research

My main contribution is a thorough theoretical and empirical analysis of two new types of efficient returns when returns are predictable. The first type maximizes the unconditional Sharpe ratio that is commonly used in empirical work. Equivalently, these efficient returns display zero unconditional Jensen's alphas when they are used as a pricing factor. The second type maximizes the conditional mean-variance preferences that are commonly used in theoretical work. These efficient returns motivate the definition of residual Sharpe ratios and alphas, where the required variances and covariances are based on the residuals of predictive regressions instead of the returns themselves.

I show that the two efficiency types differ from unconditional efficiency, which is the only type that has been studied in the literature, when the safe asset return and the maximum conditional Sharpe ratio change over time. Because interest rates and Sharpe ratios change over time, there are three ways of exploiting conditioning information, but the magnitude of their differences is an empirical question. I therefore develop an econometric framework to quantify and test differences in both unconditional and residual Sharpe ratios.

In particular, I study monthly stock returns during two periods (1954–1983 and 1984–2012). These data show that the differences between UE returns and the two new types of efficient returns are more relevant the more assets that are considered in the investment set and the lower the mean target, with a larger gap in the first period than in the second. These differences are stronger for annual returns, which suggests that the return frequency is another relevant dimension.

My theoretical results and empirical methodology can also be used to test conditional asset pricing models. These data show that if the CAPM is tested with the 3 Fama–French (1993) factors, there is more evidence against the unconditional CAPM in the first period and against the conditional CAPM in the second period. This change in the relative evidence against the two variants of the model can be associated with the change in the relative predictability across the Fama–French factors. The first period can be associated with stronger market predictability and the second with weaker market predictability. However, the size and value effects are relatively more predictable in the second period.

Finally, there are some interesting avenues for further research. I derive the efficiency properties for an investment set that includes a safe asset, which is common in empirical work. If such an asset is not available to investors, Sharpe ratios cannot be used as in my empirical application. The relevant tests are studied by Peñaranda and Sentana (2011), (2012) in a framework that does not explicitly take into account information, and I plan to develop the corresponding extensions. In this regard, Internet Appendix C studies the relation between mean-variance frontiers with and without a safe asset for the different types of efficient returns.

My analysis has provided new efficiency measures that can be useful in performance evaluation and model selection. Similarly, the new efficient returns could be used to compute mimicking portfolios following Ferson, Siegel, and Xu (2006). The analysis of more general preferences that consider higher order moments and intertemporal efficiency are additional topics of further research. Internet Appendix D provides an out-of-sample analysis, but not a formal real-time

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Bayesian framework such as the one of Avramov and Chordia (2006) or Johannes, Korteweg, and Polson (2014). It would be interesting to study the new types of efficient returns as a practical portfolio strategy in such settings.

Appendix. Proofs

The weight vector $\boldsymbol{\varphi}_{\mathbf{C}}$ and the squared Sharpe ratio $\mathcal{S}_C^2$ are defined in equations (6) and (7), respectively. I can use them to construct two excess returns

$$\text{(A-1)} \qquad \begin{aligned} r_v &= \mathbf{r}' \left[\operatorname{var}(\mathbf{r}|\mathbf{z})\right]^{-1} \mathrm{E}(\mathbf{r}|\mathbf{z}) = \mathbf{r}'\boldsymbol{\varphi}_{\mathbf{C}}, \\ r_e &= \mathbf{r}' \left[\mathrm{E}\left(\mathbf{r}\mathbf{r}'|\mathbf{z}\right)\right]^{-1} \mathrm{E}(\mathbf{r}|\mathbf{z}) = \frac{1}{1+\mathcal{S}_C^2}\mathbf{r}'\boldsymbol{\varphi}_{\mathbf{C}} \end{aligned}$$

that will simplify the proofs. Importantly, r_e is the unique excess return that satisfies $\mathrm{E}(r_e r|\mathbf{z}) = \mathrm{E}(r|\mathbf{z})$ and r_v is the unique excess return that satisfies $\operatorname{cov}(r_v, r|\mathbf{z}) = \mathrm{E}(r|\mathbf{z})$ for every excess return r. The means of these excess returns are related to $\mathcal{S}_C^2$ as follows

$$\text{(A-2)} \qquad \mathrm{E}(r_v) = \mathrm{E}\left(\mathcal{S}_C^2\right), \qquad \mathrm{E}(r_e) = \mathrm{E}\left(\frac{\mathcal{S}_C^2}{1+\mathcal{S}_C^2}\right).$$

Proof of Proposition 1

1) For any return R, the excess return $r = R - R_0$ can be decomposed into two components

$$r = \eta r_e + \epsilon, \qquad \eta = \frac{\mathrm{E}(r_e r)}{\mathrm{E}(r_e^2)} = \frac{\mathrm{E}(r)}{\mathrm{E}(r_e)},$$

where the first component is the unconditional projection of r onto the unconditional span of r_e and ϵ is the projection error. The error satisfies $\mathrm{E}(\epsilon) = 0$ because $\mathrm{E}(r_e\epsilon) = 0$ and thus the error does not affect $\mathrm{E}(r)$ but increases $\mathrm{E}\left(r^2\right)$.

The returns that solve problem (8) cannot have an error term in the previous projection and hence the solution can be represented as

$$R_P - R_0 = \eta r_e, \qquad \eta = \frac{\nu - \mathrm{E}(R_0)}{\mathrm{E}(r_e)}.$$

Substituting from equations (A-1) and (A-2), I have the equivalent expression

$$R_P = R_0 + \left[\frac{\nu - \mathrm{E}(R_0)}{\mathrm{E}\left(\frac{\mathcal{S}_C^2}{1+\mathcal{S}_C^2}\right)}\right] \frac{1}{1+\mathcal{S}_C^2}\mathbf{r}'\boldsymbol{\varphi}_{\mathbf{C}}.$$

2) First, any PE return $R_P \neq R_0$ satisfies these properties. From the proof of point 1, the excess returns of a PE return $R_P \neq R_0$ can be expressed as $R_P - R_0 = \eta r_e$ with $\eta \neq 0$. The squared unconditional Sharpe ratio S_U^2 of these excess returns is

$$\frac{\mathrm{E}^2(R_P - R_0)}{\operatorname{var}(R_P - R_0)} = \frac{\eta^2 \mathrm{E}^2(r_e)}{\eta^2 \mathrm{E}(r_e)(1 - \mathrm{E}(r_e))} = \frac{\mathrm{E}(r_e)}{1 - \mathrm{E}(r_e)}$$

and, as a pricing factor, they provide $\alpha_U = 0$ for any excess return r because

$$\frac{\operatorname{cov}(r, R_P - R_0)}{\operatorname{var}(R_P - R_0)}\mathrm{E}(R_P - R_0) = \frac{\eta \mathrm{E}(r)(1 - \mathrm{E}(r_e))}{\eta^2 \mathrm{E}(r_e)(1 - \mathrm{E}(r_e))}\eta \mathrm{E}(r_e) = \mathrm{E}(r).$$

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Second, any return that satisfies these properties must be a PE return. From the decomposition of excess returns in the proof of point 1, any excess return $r_\beta \neq 0$ can be decomposed into an underlying $R_P - R_0 = \eta r_e$ plus an error ϵ. Thus, the Sharpe ratio of r_β satisfies

$$\frac{\mathrm{E}^2(r_\beta)}{\mathrm{var}(r_\beta)} = \frac{\mathrm{E}^2(R_P - R_0)}{\mathrm{var}(R_P - R_0) + \mathrm{var}(\epsilon)}$$

and the beta pricing of the underlying $R_P - R_0$ with r_β as the pricing factor yields

$$\frac{\mathrm{cov}(R_P - R_0, r_\beta)}{\mathrm{var}(r_\beta)}\mathrm{E}(r_\beta) = \frac{\mathrm{var}(R_P - R_0)}{\mathrm{var}(R_P - R_0) + \mathrm{var}(\epsilon)}\mathrm{E}(R_P - R_0).$$

Therefore, an excess return r_β such that

$$\frac{\mathrm{E}^2(r_\beta)}{\mathrm{var}(r_\beta)} = \frac{\mathrm{E}(r_e)}{1 - \mathrm{E}(r_e)}$$

or such that

$$\frac{\mathrm{cov}(r, r_\beta)}{\mathrm{var}(r_\beta)}\mathrm{E}(r_\beta) = \mathrm{E}(r)$$

for any excess return r must satisfy $\mathrm{var}(\epsilon) = 0$, which translates into $\epsilon = \mathrm{E}(\epsilon) = 0$. I conclude that r_β must be equal to a particular $R_P - R_0$. □

Proof of Proposition 2

1) This proof relies on the residual inner product $\mathrm{E}[\mathrm{cov}(x, y|\mathbf{z})]$ between random variables x and y and its corresponding residual norm $\sqrt{\mathrm{E}[\mathrm{var}(x|\mathbf{z})]}$. This may not be a proper norm in the sense that $\mathrm{var}(x|\mathbf{z}) = 0$ implies $x = \mathrm{E}(x|\mathbf{z})$ but not necessarily $x = 0$. However, this is not a concern in my setting because in the space of excess returns I must have $x = 0$ under the usual assumption of no arbitrage opportunities. As commented in Section III.B, I use the term *residual* because $\mathrm{E}[\mathrm{var}(x|\mathbf{z})]$ is equal to the variance of the residual $x - \mathrm{E}(x|\mathbf{z})$ from a predictive regression.

For any return R, I can decompose the excess return $r = R - R_0$ into two components:

$$r = \lambda r_v + \varepsilon, \qquad \lambda = \frac{\mathrm{E}[\mathrm{cov}(r, r_v|\mathbf{z})]}{\mathrm{E}[\mathrm{var}(r_v|\mathbf{z})]} = \frac{\mathrm{E}(r)}{\mathrm{E}(r_v)},$$

where the first component is the residual projection of r onto the unconditional span of r_v and ε is the projection error. The error satisfies $\mathrm{E}(\varepsilon) = 0$ because, by construction, $\mathrm{E}[\mathrm{cov}(r_v, \varepsilon|\mathbf{z})] = 0$. The error does not affect $\mathrm{E}(r)$ but increases $\mathrm{E}[\mathrm{var}(r|\mathbf{z})] = \mathrm{E}[\mathrm{var}(R|\mathbf{z})]$.

Therefore, the optimal returns that solve problem (15) cannot have an error term in the previous projection and they can be represented as

$$R_R - R_0 = \lambda r_v, \qquad \lambda = \frac{\nu - \mathrm{E}(R_0)}{\mathrm{E}(r_v)}.$$

Substituting from equations (A-1) and (A-2), I have the equivalent expression

$$R_R = R_0 + \frac{\nu - \mathrm{E}(R_0)}{\mathrm{E}(\mathcal{S}_C^2)}\mathbf{r}'\boldsymbol{\varphi}_\mathbf{C}.$$

2) First, any RE return $R_R \neq R_0$ satisfies these properties. From the proof of point 1, the excess return of an RE return $R_R \neq R_0$ can be expressed as $R_R - R_0 = \lambda r_v$ with $\lambda \neq 0$. The squared residual Sharpe ratio $\mathcal{S}_R^2$ of these excess returns is

$$\frac{\mathrm{E}^2(R_R - R_0)}{\mathrm{E}[\mathrm{var}(R_R - R_0|\mathbf{z})]} = \frac{\lambda^2\mathrm{E}^2(r_v)}{\lambda^2\mathrm{E}(r_v)} = \mathrm{E}(r_v)$$

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and, as a pricing factor, they yield $\alpha_R = 0$ for any excess return r because

$$\frac{\mathrm{E}\left[\mathrm{cov}\left(r, R_R - R_0|\mathbf{z}\right)\right]}{\mathrm{E}\left[\mathrm{var}\left(R_R - R_0|\mathbf{z}\right)\right]}\mathrm{E}\left(R_R - R_0\right) \quad = \quad \frac{\lambda \mathrm{E}\left(r\right)}{\lambda^2 \mathrm{E}\left(r_v\right)}\lambda \mathrm{E}\left(r_v\right) \quad = \quad \mathrm{E}\left(r\right).$$

Second, any return that satisfies these properties must be an RE return. From the decomposition of excess returns in the proof of point 1, I can decompose any excess return $r_\beta \neq 0$ into an underlying $R_R - R_0 = \lambda r_v$ plus an error term. Following a similar argument to the proof of point 2 of Proposition 1, I can show that if r_β satisfies

$$\mathrm{E}\left(r_v\right) \quad = \quad \frac{\mathrm{E}^2\left(r_\beta\right)}{\mathrm{E}\left[\mathrm{var}\left(r_\beta|\mathbf{z}\right)\right]}$$

or

$$\mathrm{E}\left(r\right) \quad = \quad \frac{\mathrm{E}\left[\mathrm{cov}\left(r, r_\beta|\mathbf{z}\right)\right]}{\mathrm{E}\left[\mathrm{var}\left(r_\beta|\mathbf{z}\right)\right]}\mathrm{E}\left(r_\beta\right)$$

for any excess return r, the error term must be 0 and thus r_β must be equal to a particular $R_R - R_0$. □

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