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Upper and lower Birkhoff spectra for the Gauss map

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Abstract. In this paper, we study the irregular multifractal spectrum of the Gauss map, by investigating the upper and lower Birkhoff spectra. For potentials satisfying a weakly tempered distortion property, we establish that the Hausdorff dimensions of the classical, upper, and lower Birkhoff level sets are identical. Furthermore, we derive an exact formula for the dimension of the intersection of the upper and lower Birkhoff level sets, proving that this dimension equals the minimum of the two individual dimensions. These findings broaden the multifractal theory previously developed for conformal repellers and finite branch expanding systems to the infinite branch context of the Gauss map. As applications, we furnish complete multifractal descriptions of the upper and lower Lyapunov spectra of the Gauss map, as well as the upper and lower arithmetic-mean spectra of continued fractions.

Key words: multifractal analysis, upper and lower Birkhoff spectra, Gauss map, continued fractions
2020 Mathematics subject classification: 37E05 (Primary); 11K50, 28A80 (Secondary)

1. *Introduction*

In this paper, we investigate the irregular multifractal spectrum of Birkhoff averages arising from the Gauss map and continued fractions.

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The Gauss map $G\colon [0,1) \to [0,1)$ is defined by

$$G(0) := 0 \quad \text{and} \quad G(x) := \frac{1}{x} (\mathrm{mod}\ 1) \quad \text{for all } x \in (0,1).$$

This map is intimately connected with the continued fraction expansion of real numbers. Let $\lfloor \cdot \rfloor$ denote the floor function and G^n the nth iterate of G. For any $x \in [0,1)$, define $a_1(x) := \lfloor 1/x \rfloor$ and $a_{n+1}(x) := a_1(G^n(x))$ for $n \geq 1$ such that $G^n(x) > 0$. Then, x admits the continued fraction expansion

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} =: [a_1(x), a_2(x), a_3(x), \ldots],$$

where the positive integers $a_n(x)$ are referred to as the partial quotients of x. The Gauss map acts as the shift on partial quotients, namely $G([a_1, a_2, a_3, \ldots]) = [a_2, a_3, a_4, \ldots]$. For a detailed treatment of the Gauss map and continued fractions, we refer the reader to [6, 24].

Let $\psi\colon [0,1) \to \mathbb{R}$ be a potential. For any $x \in [0,1)$ and $n \geq 1$, we define the nth Birkhoff sum as

$$S_n\psi(x) := \sum_{k=0}^{n-1} \psi(G^k(x)). \tag{1}$$

Let

$$\alpha_{\inf} := \inf \left\{ \liminf_{n\to\infty} \frac{1}{n} S_n\psi(x) \colon x \in [0,1) \right\}$$

and

$$\alpha_{\sup} := \sup \left\{ \limsup_{n\to\infty} \frac{1}{n} S_n\psi(x) \colon x \in [0,1) \right\}.$$

Note that it is possible for $\alpha_{\inf}$ and $\alpha_{\sup}$ to be minus infinity and infinity, respectively. In this paper, the notation $[\alpha_{\inf}, \alpha_{\sup}]$ denotes a subset of the real line. If $\alpha_{\inf} = -\infty$, the interval is understood to be open on the left; similarly, if $\alpha_{\sup} = \infty$, the interval is open on the right. For any $\lambda \in [\alpha_{\inf}, \alpha_{\sup}]$, let

$$E_{\sup}(\lambda) := \left\{ x \in [0,1) \colon \limsup_{n\to\infty} \frac{1}{n} S_n\psi(x) = \lambda \right\}$$

and

$$E_{\inf}(\lambda) := \left\{ x \in [0,1) \colon \liminf_{n\to\infty} \frac{1}{n} S_n\psi(x) = \lambda \right\},$$

and define $E(\lambda) := E_{\sup}(\lambda) \cap E_{\inf}(\lambda)$. Denote by $\dim_{\mathrm{H}}$ the Hausdorff dimension (see [7] for the definition). We are interested in the upper and lower Birkhoff spectra

$$\lambda \mapsto \dim_{\mathrm{H}} E_{\sup}(\lambda) \quad \text{and} \quad \lambda \mapsto \dim_{\mathrm{H}} E_{\inf}(\lambda), \tag{2}$$

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and their connections to the classical Birkhoff spectrum

$$\lambda \mapsto \dim_{\mathrm{H}} E(\lambda). \tag{3}$$

As a canonical example of a uniformly expanding map with countably many branches, the Gauss map has attracted considerable attention in multifractal analysis since the 1990s. The methods developed in its study have been systematically extended to the broader class of countable Markov interval maps. In a seminal work from 1990, Mayer [29] investigated the thermodynamic formalism of the Gauss map and showed that its topological pressure exhibits a logarithmic singularity at $1/2$. This type of phase transition stands in sharp contrast to the situation for hyperbolic systems with finitely many branches, and plays a crucial role in determining the behavior of the Birkhoff spectra given in (2) and (3). In particular, it gives rise to a series of new and delicate phenomena in multifractal analysis (see, e.g., [10, 11, 19, 25, 28, 32, 35]), which distinguish the Gauss map from finite-branch dynamical systems, and reveal the rich geometric complexity of the Gauss map and continued fractions.

Specializing to the geometric potential $\psi(x) = \log |G'(x)|$, the set $E(\lambda)$ reduces to the level set of the Lyapunov exponent of the Gauss map:

$$J(\lambda) := \left\{ x \in [0, 1) : \lim_{n\to\infty} \frac{1}{n} \log |(G^n)'(x)| = \lambda \right\}.$$

The associated Birkhoff spectrum is referred to as the Lyapunov spectrum. The study of the Lyapunov spectrum for the Gauss map was initiated by Pollicott and Weiss [32, Proposition 4], and was later developed in detail by Kesseböhmer and Stratmann [25, Theorem 1.3], as well as independently by Fan *et al* [11, Theorem 1.3]. Let $\lambda_0 := 2\log((1+\sqrt{5})/2)$ and $\gamma_0 := \pi^2/6\log 2$. It was proved that the Lyapunov spectrum

$$L(\lambda) := \dim_{\mathrm{H}} J(\lambda)$$

has an unbounded domain $[\lambda_0, \infty)$, and is real-analytic on (λ_0, ∞), strictly increasing on $[\lambda_0, \gamma_0]$ and strictly decreasing on $[\gamma_0, \infty)$. In particular, L attains its unique maximum value 1 at γ_0 and has an inflection point at some point greater than γ_0.

Denote by $J_{\sup}(\lambda)$ and $J_{\inf}(\lambda)$ the sets obtained from $J(\lambda)$ by replacing the limit with the limsup and liminf, respectively. The resulting Birkhoff spectra are referred to as the upper and lower Lyapunov spectra. The general framework developed in the present paper, when applied to the Lyapunov exponents, leads to a comprehensive multifractal characterization of the upper and lower Lyapunov spectra.

THEOREM 1. (Upper and lower Lyapunov spectra) *The following statements hold.*

(i) *For any* $\lambda \in [\lambda_0, \infty)$,

$$\dim_{\mathrm{H}} J(\lambda) = \dim_{\mathrm{H}} J_{\inf}(\lambda) = \dim_{\mathrm{H}} J_{\sup}(\lambda) = L(\lambda).$$

(ii) *For any* $\lambda, \xi \in [\lambda_0, \infty)$ *with* $\lambda \le \xi$,

$$\dim_{\mathrm{H}}(J_{\inf}(\lambda) \cap J_{\sup}(\xi)) = \min\{L(\lambda),\ L(\xi)\}.$$

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The upper and lower Lyapunov spectra serve as key tools for understanding the multifractal nature of the Lochs index associated with comparisons between the continued fraction and decimal expansions of real numbers.

For any $x \in [0, 1)$ and $n \geq 1$, we define the Lochs index $k_n(x)$ as the maximum number of partial quotients in the continued fraction expansion of x that can be determined by the first n decimal digits of x. It was first proved by Lochs [27] that for Lebesgue-almost every $x \in [0, 1)$,

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log 10}{\gamma_0} = \frac{6 \log 2 \log 10}{\pi^2}.$$

Later, Wu [37, Theorem 1.3] established the asymptotic behavior of $k_n(x)$ in terms of the upper and lower Lyapunov exponents. Specifically, for each irrational $x \in [0, 1)$,

$$\liminf_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log 10}{\ell_{\sup}(x)} \quad \text{and} \quad \limsup_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log 10}{\ell_{\inf}(x)},$$

where $\ell_{\sup}(x)$ and $\ell_{\inf}(x)$ denote the upper and lower Lyapunov exponents of x, respectively. Combining this result with Theorem 1, we obtain a complete multifractal analysis of the Lochs index.

For any $\lambda \in [\lambda_0, \infty)$, define

$$K_{\sup}(\lambda) := \left\{ x \in [0, 1) : \limsup_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log 10}{\lambda} \right\}$$

and

$$K_{\inf}(\lambda) := \left\{ x \in [0, 1) : \liminf_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log 10}{\lambda} \right\},$$

and let $K(\lambda) := K_{\inf}(\lambda) \cap K_{\sup}(\lambda)$. The Hausdorff dimensions of these level sets and their intersections are determined as follows, which provides a refinement of [37, Corollary 2.5]. See [1, Theorem 11] for related results.

THEOREM 2. *The following statements hold.*

(i) *For any* $\lambda \in [\lambda_0, \infty)$,

$$\dim_{\mathrm{H}} K(\lambda) = \dim_{\mathrm{H}} K_{\inf}(\lambda) = \dim_{\mathrm{H}} K_{\sup}(\lambda) = L(\lambda).$$

(ii) *For any* $\lambda, \xi \in [\lambda_0, \infty)$ *with* $\lambda \geq \xi$,

$$\dim_{\mathrm{H}}(K_{\inf}(\lambda) \cap K_{\sup}(\xi)) = \min\{L(\lambda),\ L(\xi)\}.$$

Iommi and Jordan [20] further developed this line of research by studying the Birkhoff spectrum for a wider class of potentials beyond the geometric potential. In the particular case of the potential $\psi(x) = \lfloor 1/x \rfloor$, the Birkhoff sum $S_n\psi(x)$ in (1) becomes the arithmetic sum of the partial quotients of x, and the set $E(\lambda)$ reduces to the level set of the arithmetic mean of partial quotients:

$$A(\lambda) := \left\{ x \in [0, 1) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k(x) = \lambda \right\}.$$

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Iommi and Jordan [20, Proposition 6.7] proved that the corresponding Birkhoff spectrum, called the arithmetic-mean spectrum,

$$\lambda \mapsto S(\lambda) := \dim_{\mathrm{H}} A(\lambda),$$

is defined on the unbounded interval $[1, \infty)$. Moreover, S is real-analytic, strictly increasing, and strictly concave on $(1, \infty)$, with $S(1) = 0$ and $\lim_{\lambda\to\infty} S(\lambda) = 1$.

Let $A_{\inf}(\lambda)$ and $A_{\sup}(\lambda)$ denote the sets obtained from $A(\lambda)$ by replacing the limit with the liminf and limsup, respectively. The general framework established in this paper provides a complete characterization of the Hausdorff dimension of these sets.

THEOREM 3. (Upper and lower arithmetic-mean spectra) *The following statements hold.*

(i) *For any* $\lambda \in [1, \infty)$,

$$\dim_{\mathrm{H}} A(\lambda) = \dim_{\mathrm{H}} A_{\inf}(\lambda) = \dim_{\mathrm{H}} A_{\sup}(\lambda) = S(\lambda).$$

(ii) *For any* $\lambda, \xi \in [1, \infty)$ *with* $\lambda \le \xi$,

$$\dim_{\mathrm{H}}(A_{\inf}(\lambda) \cap A_{\sup}(\xi)) = S(\lambda).$$

We remark that Theorem 3(i) is analogous to the classical results of Besicovitch [4] and Eggleston [5, Theorem 14] concerning the multifractal analysis of the arithmetic mean of digits in the dyadic expansion of real numbers, but the corresponding spectrum exhibits a different behavior.

Our investigation is motivated by the following two aspects. First, Feng *et al* [15] obtained the upper and lower Birkhoff spectra for conformal repellers on smooth Riemannian manifolds under continuous potentials. (See also earlier work of Fan *et al* [8] in the setting of subshifts of finite type.) More precisely, Feng *et al* [15, Theorem 1.1 and Propositions 6.3–6.4] proved that for every repeller J of an expanding $C^{1+\delta}$ conformal topologically mixing map and for every continuous potential $\psi : J \to \mathbb{R}$, the following properties hold.

(1) For any $\lambda \in [\lambda_{\min}, \lambda_{\max}]$,

$$\dim_{\mathrm{H}} E(\lambda) = \dim_{\mathrm{H}} E_{\inf}(\lambda) = \dim_{\mathrm{H}} E_{\sup}(\lambda). \tag{4}$$

(2) For any $\lambda, \xi \in [\lambda_{\min}, \lambda_{\max}]$ with $\lambda \le \xi$,

$$\dim_H(E_{\inf}(\lambda) \cap E_{\sup}(\xi)) = \min\{\dim_{\mathrm{H}} E(\lambda),\ \dim_{\mathrm{H}} E(\xi)\}. \tag{5}$$

Second, Gelfert *et al* studied the upper and lower Lyapunov spectra for several non-uniformly expanding systems beyond $C^{1+\delta}$ conformal expanding maps, including parabolic systems [18], rational maps [16], and multimodal maps [17]. Due to the presence of parabolic fixed points and critical points, the geometric potential is less regular, and they showed that only certain inequalities corresponding to (5) can be obtained for the upper and lower Lyapunov spectra, e.g., [16, Theorem 2] and [17, Theorems 1 and 2]. They further remark [17, p. 1450] that 'the question of whether these equalities for geometric potentials hold in these non-uniformly expanding settings remains open.' All these results naturally raise the question of whether (4) and (5) can be proved or disproved for the Gauss map under mildly regular potentials.

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2. *Statement of the main results*

For any $x \in [0, 1)$ and $n \geq 1$, let $I_n(x)$ denote the cylinder of order n that contains x. Let $\psi : [0, 1) \to \mathbb{R}$ be a potential. We define the *nth variation* of ψ by

$$\mathrm{Var}_n(\psi) := \sup\{|\psi(x) - \psi(y)| : x, y \in [0, 1), I_n(x) = I_n(y)\}.$$

Many conditions have been introduced in the literature via variations to describe different types of regularity or decay properties of potentials. For example, we have the following examples.

- Sarig [34, p. 1567] defined a potential ψ to be *locally Hölder continuous* if there exist constants $A > 0$ and $0 < r < 1$ such that $\mathrm{Var}_n(\psi) \leq Ar^n$ for all $n \geq 1$.
- Iommi and Jordan [20, p. 2562] defined a potential ψ to have *summable variation* if $\sum_{n=1}^{\infty} \mathrm{Var}_n(\psi) < \infty$.
- Seuret and Wang [36, Definition 2.3] introduced the notion of the *tempered distortion property* for potentials ψ satisfying $\mathrm{Var}_1(\psi) < \infty$ and $\mathrm{Var}_n(\psi) \to 0$ as $n \to \infty$. (Fan *et al* [9, Definition 2.1] refer to the tempered distortion property as the property that the potential has variations *uniformly converging to zero*.)

Throughout this paper, a potential $\psi : [0, 1) \to \mathbb{R}$ is said to satisfy the *weakly tempered distortion property* if

$$\mathrm{Var}_1(\psi) < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n}\mathrm{Var}_n(S_n\psi) = 0. \tag{6}$$

The weakly tempered distortion property plays a crucial role in our argument. It guarantees that the Birkhoff averages have uniformly small oscillations on sufficiently long cylinders, which allows us to pass from symbolic constructions to dimensional estimates. This condition is considerably weaker than summable variation or local Hölder continuity and, therefore, permits a broader class of potentials. In particular, both potentials $\psi(x) = \log |G'(x)|$ and $\psi(x) = \lfloor 1/x \rfloor$ considered in §1 are locally Hölder continuous, and thus satisfy the weakly tempered distortion property.

Our main theorems in this paper are stated as follows.

THEOREM A. *Let $\psi : [0, 1) \to \mathbb{R}$ be a potential satisfying the weakly tempered distortion property. Then, for any $\lambda \in [\alpha_{\inf}, \alpha_{\sup}]$,*

$$\dim_{\mathrm{H}} E(\lambda) = \dim_{\mathrm{H}} E_{\inf}(\lambda) = \dim_{\mathrm{H}} E_{\sup}(\lambda). \tag{7}$$

Theorem A qualitatively demonstrates that the classical, lower, and upper Birkhoff spectra coincide. Denote their common value in (7) by $b_\psi(\lambda)$. We remark that the value of the classical Birkhoff spectrum has been extensively studied; see, e.g., [9, 11, 20–23, 25, 32, 33] and references therein. As a result, the properties of the function $b_\psi(\lambda)$ are now relatively well understood. Specifically, Iommi and Jordan [20] introduced the following classes of potentials:

$$\mathcal{R} := \{\psi : [0, 1) \to \mathbb{R} : \psi < 0,\ \psi \text{ has summable variations},\ \lim_{x\to 0} \psi(x) = -\infty\}$$

and

$$\bar{\mathcal{R}} := \{\psi : [0, 1) \to \mathbb{R} : \psi \in \mathcal{R},\ \psi \text{ is locally Hölder continuous}\}.$$

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Let $\mathcal{T}$ denote the class of potentials satisfying the tempered distortion property and $\mathcal{W}$ the class of potentials satisfying the weakly tempered distortion property. Then,

$$\bar{\mathcal{R}} \subsetneq \mathcal{R} \subsetneq \mathcal{T} \subsetneq \mathcal{W}.$$

Let $\mathcal{M}_G$ denote the set of all G-invariant probability measures on $[0, 1)$. For each $\mu \in \mathcal{M}_G$, we write $h(\mu)$ for the measure-theoretic entropy of μ and $\lambda(\mu) := \int \log |G'| \, d\mu$ for the Lyapunov exponent of μ. Combining Theorem A with [20, Theorems 1.3 and 1.4] and [9, Theorem 1.1], we have the following hierarchical structure of $b_\psi(\lambda)$. To be more precise, we have the following.

(1) (Theorem A) If $\psi \in \mathcal{W}$, then $b_\psi(\lambda)$ is well defined on $[\alpha_{\inf}, \alpha_{\sup}]$.

(2) [9, Theorem 1.1] If $\psi \in \mathcal{T}$, then for any $\lambda \in [\alpha_{\inf}, \alpha_{\sup}]$,

$$b_\psi(\lambda) = \lim_{\varepsilon \to 0} \sup \left\{ \frac{h(\mu)}{\lambda(\mu)} : \mu \in \mathcal{M}_G, \ \left| \int \psi \, d\mu - \lambda \right| < \varepsilon, \ \lambda(\mu) < \infty \right\}. \tag{8}$$

(3) [20, Theorem 1.3] If $\psi \in \mathcal{R}$, then for any $\lambda \in [\alpha_{\inf}, \alpha_{\sup}]$,

$$b_\psi(\lambda) = \sup \left\{ \frac{h(\mu)}{\lambda(\mu)} : \mu \in \mathcal{M}_G, \ \int \psi \, d\mu = \lambda, \ \lambda(\mu) < \infty \right\}. \tag{9}$$

(4) [20, Theorem 1.4] If $\psi \in \bar{\mathcal{R}}$, then the conditional variational principle (9) remains valid. Moreover, depending on the limiting behavior of $\psi(x)/(-2 \log x)$ as $x \to 0$, one further obtains additional regularity properties of $b_\psi(\lambda)$.

As shown in [10], the right-hand sides of (8) and (9) do not necessarily coincide. We also point out that, when considering the fast Birkhoff spectra, the analogous identities among the classical, lower, and upper spectra no longer hold. This discrepancy was first observed by Fan *et al* [11, 12], and Liao and Rams [26] in their study of the Khintchine spectrum for the Gauss map. In a recent work, the present authors and Wang [13] further extended these findings to the setting of countable Markov interval maps.

THEOREM B. *Let $\psi : [0, 1) \to \mathbb{R}$ be a potential satisfying the weakly tempered distortion property. Then, for any $\lambda, \xi \in [\alpha_{\inf}, \alpha_{\sup}]$ with $\lambda \le \xi$,*

$$\dim_{\mathrm{H}}(E_{\inf}(\lambda) \cap E_{\sup}(\xi)) = \min\{\dim_{\mathrm{H}} E(\lambda), \ \dim_{\mathrm{H}} E(\xi)\}. \tag{10}$$

Theorem B establishes a precise relationship between the upper and lower Birkhoff level sets for the Gauss map by showing that the Hausdorff dimension of their intersection equals the smaller of the two individual dimensions. This result extends the identity established by Feng *et al* [15, Proposition 6.4] and Gelfert and Rams [18, Theorem 1] from finite-branch systems to the infinite-branch context of the Gauss map. Moreover, the intersections of the upper and lower Birkhoff level sets considered here yield a finer multifractal decomposition of $[0, 1)$. In the classical setting, the multifractal decomposition of $[0, 1)$ induced by the Birkhoff level sets takes the form

$$[0, 1) = E' \cup \bigcup_{\lambda \in [\alpha_{\inf}, \alpha_{\sup}]} E(\lambda), \tag{11}$$

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where E' is the *irregular set* defined by

$$E' := \left\{ x \in [0,1) \colon \liminf_{n\to\infty} \frac{1}{n} S_n \psi(x) < \limsup_{n\to\infty} \frac{1}{n} S_n \psi(x) \right\}.$$

However, this decomposition provides little information about the internal structure of E', for which essentially no multifractal information is captured. Although in many dynamical systems the irregular set is known to have full topological entropy and full Hausdorff dimension (see, e.g., [2, 3, 8, 15, 20, 31]), its finer geometric and dimensional properties remain largely mysterious. Considering the intersections $E_{\inf}(\lambda) \cap E_{\sup}(\xi)$, we obtain the following refined multifractal decomposition of (11):

$$[0,1) = \bigcup_{\substack{\lambda,\, \xi \in [\alpha_{\inf},\, \alpha_{\sup}] \\ \lambda \le \xi}} (E_{\inf}(\lambda) \cap E_{\sup}(\xi)).$$

In this new decomposition, when $\lambda = \xi$, the intersections reduce to the classical Birkhoff level set $E(\lambda)$, while for $\lambda < \xi$, they provide a finer stratification of the irregular set E', capturing how points with different upper and lower Birkhoff averages contribute to its structure. Theorem B quantifies the Hausdorff dimensions of these intersections, thereby giving a refined multifractal description of the irregular set.

The rest of this paper is organized as follows. In §3, we establish a key lemma and use it to prove Theorems A and B. Section 4 is devoted to a detailed proof of the key lemma.

3. *Key Lemma and proofs of Theorems A and B*

Let $\lambda, \xi \in [\alpha_{\inf}, \alpha_{\sup}]$ with $\lambda \le \xi$ be fixed. For convenience, denote

$$E(\lambda, \xi) := E_{\inf}(\lambda) \cap E_{\sup}(\xi).$$

For any $\alpha \in \mathbb{N}^n$, we denote by $I_n(\alpha)$ the cylinder of order n associated with α. That is, if $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n$, then

$$I_n(\alpha) = \{x \in [0,1) \colon a_1(x) = a_1, \ldots, a_n(x) = a_n\}.$$

For any $n, r \in \mathbb{N}$ and $a, b \in \mathbb{Q}$ with $a < b$, let

$$Z_n(a,b) := \left\{ \alpha \in \mathbb{N}^n \colon a < \frac{1}{n} S_n \psi(x) < b \text{ for all } x \in I_n(\alpha) \right\} \tag{12}$$

and

$$Z_{n,r}(a,b) := \left\{ \alpha \in Z_n(a,b) \colon \frac{1}{2^r} \le |I_n(\alpha)| < \frac{1}{2^{r-1}} \right\}.$$

Since $|I_n(\alpha)| \le 2^{-n}$ for all $n \ge 1$, we have $r \ge n$ whenever $Z_{n,r}(a,b) \ne \emptyset$.

Key Lemma. Let $\lambda, \xi \in [\alpha_{\inf}, \alpha_{\sup}]$ with $\lambda \le \xi$. Suppose that there exists $s > 0$ such that for any $a, b \in \mathbb{Q}$ with $a < \lambda < b$, for any $c, d \in \mathbb{Q}$ with $c < \xi < d$, and for any $n_0 \ge 1$, there exist integers $n \ge n_0$, $\tilde{n} \ge n_0$, $r \ge n$, and $\tilde{r} \ge \tilde{n}$ such that

$$\sum_{\alpha \in Z_{n,r}(a,b)} |I_n(\alpha)|^s \ge 1 \quad \text{and} \quad \sum_{\alpha \in Z_{\tilde{n},\tilde{r}}(c,d)} |I_{\tilde{n}}(\alpha)|^s \ge 1.$$

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Then,

$$\dim_{\mathrm{H}} E(\lambda, \xi) \geq s.$$

In particular, when $\lambda = \xi$, the Key Lemma reduces to the case of a single level set $E(\lambda)$. In other words, if there exists $s > 0$ such that for any $a, b \in \mathbb{Q}$ with $a < \lambda < b$ and for any $n_0 \geq 1$, there exist integers $n \geq n_0$ and $r \geq n$ such that

$$\sum_{\alpha \in Z_{n,r}(a,b)} |I_n(\alpha)|^s \geq 1,$$

then $\dim_{\mathrm{H}} E(\lambda) \geq s$.

The proof of the Key Lemma follows a Cantor-type construction combined with Falconer's dimension estimate and occupies the whole of §4. In the rest of this section, we first proceed with the proofs of Theorems A and B, assuming the Key Lemma.

3.1. *Proof of Theorem A assuming the Key Lemma.*

Proof. Fix $\lambda \in [\alpha_{\inf}, \alpha_{\sup}]$. For simplicity, we denote by τ, $\tau_{\inf}$ and $\tau_{\sup}$ the Hausdorff dimension of the sets $E(\lambda)$, $E_{\inf}(\lambda)$, and $E_{\sup}(\lambda)$, respectively. Then,

$$\tau \leq \tau_{\inf} \quad \text{and} \quad \tau \leq \tau_{\sup}.$$

Hence, it suffices to prove that $\tau_{\inf} \leq \tau$ and $\tau_{\sup} \leq \tau$. We prove only the first inequality, as the second follows analogously.

We first claim that for any $\varepsilon > 0$, there exist $\widetilde{a}, \widetilde{b} \in \mathbb{Q}$ with $\widetilde{a} < \lambda < \widetilde{b}$ and $N \in \mathbb{N}$ such that for all $n \geq N$ and all $r \geq n$,

$$\sum_{\alpha \in Z_{n,r}(\widetilde{a},\widetilde{b})} |I_n(\alpha)|^{\tau+\varepsilon} < 1. \tag{13}$$

In fact, if the claim is false, then there exists $\varepsilon_0 > 0$ such that for any $a, b \in \mathbb{Q}$ with $a < \lambda < b$ and any integer $n_0 \geq 1$, there exist integers $n \geq n_0$ and $r \geq n$ satisfying

$$\sum_{\alpha \in Z_{n,r}(a,b)} |I_n(\alpha)|^{\tau+\varepsilon_0} \geq 1.$$

Applying the Key Lemma with $\lambda = \xi$, we obtain $\dim_{\mathrm{H}} E(\lambda) \geq \tau + \varepsilon_0$, which contradicts the definition of τ.

We now show that

$$E_{\inf}(\lambda) \subset \bigcap_{K=1}^{\infty} \bigcup_{n=K}^{\infty} \bigcup_{\alpha \in Z_n(\widetilde{a},\widetilde{b})} I_n(\alpha) = \bigcap_{K=1}^{\infty} \bigcup_{n=K}^{\infty} \bigcup_{r=n}^{\infty} \bigcup_{\alpha \in Z_{n,r}(\widetilde{a},\widetilde{b})} I_n(\alpha). \tag{14}$$

The equality in (14) follows directly from the definitions of $Z_n(\widetilde{a}, \widetilde{b})$ and $Z_{n,r}(\widetilde{a}, \widetilde{b})$. To prove the inclusion, take $a', b' \in \mathbb{R}$ such that $\widetilde{a} < a' < \lambda < b' < \widetilde{b}$, and define

$$\widetilde{\varepsilon} := \min\{a' - \widetilde{a},\ \widetilde{b} - b'\}.$$

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Since the potential ψ satisfies the weakly tempered distortion property given in (6), there exists $N_0 \in \mathbb{N}$ with $N_0 \geq N$ such that for all $n \geq N_0$,

$$\frac{1}{n}\mathrm{Var}_n(S_n\psi) < \widetilde{\varepsilon}. \tag{15}$$

For any $x \in E_{\inf}(\lambda)$, we have

$$\liminf_{n\to\infty} \frac{1}{n} S_n\psi(x) = \lambda.$$

Hence, for any $j \in \mathbb{N}$, there exists $N_j \in \mathbb{N}$ with $N_j \geq N_0$ such that

$$a' < \frac{1}{N_j} S_{N_j}\psi(x) < b'. \tag{16}$$

For any $j \in \mathbb{N}$, let $\alpha_j := a_1(x)a_2(x)\cdots a_{N_j}(x)$. Then, $x \in I_{N_j}(\alpha_j)$. We next prove that $\alpha_j \in Z_{N_j}(\widetilde{a}, \widetilde{b})$ for all $j \in \mathbb{N}$. Fix $j \in \mathbb{N}$. For any $y \in I_{N_j}(\alpha_j)$, it follows from (15) that

$$\left|\frac{1}{N_j} S_{N_j}\psi(x) - \frac{1}{N_j} S_{N_j}\psi(y)\right| \leq \frac{1}{N_j}\mathrm{Var}_{N_j}(S_{N_j}\psi) < \widetilde{\varepsilon}.$$

Combining this with (16), we derive that

$$\frac{1}{N_j} S_{N_j}\psi(y) > \frac{1}{N_j} S_{N_j}\psi(x) - \widetilde{\varepsilon} > a' - \widetilde{\varepsilon} \geq \widetilde{a}$$

and

$$\frac{1}{N_j} S_{N_j}\psi(y) < \frac{1}{N_j} S_{N_j}\psi(x) + \widetilde{\varepsilon} < b' + \widetilde{\varepsilon} \leq \widetilde{b}.$$

Thus,

$$\widetilde{a} < \frac{1}{N_j} S_{N_j}\psi(y) < \widetilde{b},$$

which implies that $\alpha_j \in Z_{N_j}(\widetilde{a}, \widetilde{b})$. Therefore, the inclusion in (14) holds.

Since $|I_n(\alpha)| \leq 2^{-(r-1)}$ for all $\alpha \in Z_{n,r}(\widetilde{a}, \widetilde{b})$, for any $K \geq N$, we deduce from (13) that

$$\begin{aligned}\sum_{n=K}^{\infty}\sum_{r=n}^{\infty}\sum_{\alpha\in Z_{n,r}(\widetilde{a},\widetilde{b})} |I_n(\alpha)|^{\tau+2\varepsilon} &\leq \sum_{n=K}^{\infty}\sum_{r=n}^{\infty}\Bigg(\sum_{\alpha\in Z_{n,r}(\widetilde{a},\widetilde{b})} |I_n(\alpha)|^{\tau+\varepsilon}\Bigg)2^{-(r-1)\varepsilon} \\ &\leq \sum_{n=K}^{\infty}\sum_{r=n}^{\infty} 2^{-(r-1)\varepsilon}.\end{aligned}$$

By (14), we conclude that

$$\begin{aligned}\mathcal{H}^{\tau+2\varepsilon}(E_{\inf}(\lambda)) &\leq \liminf_{K\to\infty}\sum_{n=K}^{\infty}\sum_{r=n}^{\infty} 2^{-(r-1)\varepsilon} \\ &= (1-2^{-\varepsilon})^{-1}\liminf_{K\to\infty}\sum_{n=K}^{\infty} 2^{-(n-1)\varepsilon} = 0.\end{aligned}$$

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Hence, $\dim_{\mathrm{H}} E_{\inf}(\lambda) \le \tau + 2\varepsilon$; that is, $\tau_{\inf} \le \tau + 2\varepsilon$. Letting $\varepsilon \to 0$, we obtain $\tau_{\inf} \le \tau$. This completes the proof of Theorem A. □

3.2. *Proof of Theorem B assuming the Key Lemma.* The proof of Theorem B proceeds similarly to that of Theorem A.

Proof. Let $\lambda, \xi \in [\alpha_{\inf}, \alpha_{\sup}]$ with $\lambda \le \xi$ be fixed. Define

$$\rho := \dim_{\mathrm{H}} E(\lambda, \xi).$$

Since $E(\lambda, \xi) \subset E_{\inf}(\lambda)$ and $E(\lambda, \xi) \subset E_{\sup}(\xi)$, it follows from Theorem A that

$$\dim_{\mathrm{H}} E(\lambda) = \dim_{\mathrm{H}} E_{\inf}(\lambda) \ge \rho \quad \text{and} \quad \dim_{\mathrm{H}} E(\xi) = \dim_{\mathrm{H}} E_{\sup}(\xi) \ge \rho,$$

which implies that

$$\min\{\dim_{\mathrm{H}} E(\lambda),\ \dim_{\mathrm{H}} E(\xi)\} \ge \rho.$$

Hence, it remains to prove that

$$\min\{\dim_{\mathrm{H}} E(\lambda),\ \dim_{\mathrm{H}} E(\xi)\} \le \rho. \tag{17}$$

By the Key Lemma, we derive that for any $\varepsilon > 0$, there exist $a_0, b_0 \in \mathbb{Q}$ with $a_0 < \lambda < b_0$, $c_0, d_0 \in \mathbb{Q}$ with $c_0 < \xi < d_0$, and $\widetilde{N} \in \mathbb{N}$ such that, for all $n \ge \widetilde{N}$ and all $r \ge n$,

$$\sum_{\alpha \in Z_{n,r}(a_0,b_0)} |I_n(\alpha)|^{\rho+\varepsilon} < 1, \tag{18}$$

or for all $\tilde{n} \ge \widetilde{N}$ and all $\tilde{r} \ge \tilde{n}$,

$$\sum_{\alpha \in Z_{\tilde{n},\tilde{r}}(c_0,d_0)} |I_{\tilde{n}}(\alpha)|^{\rho+\varepsilon} < 1. \tag{19}$$

Moreover, we claim that

$$E_{\inf}(\lambda) \subset \bigcap_{K=1}^{\infty} \bigcup_{n=K}^{\infty} \bigcup_{r=n}^{\infty} \bigcup_{\alpha \in Z_{n,r}(a_0,b_0)} I_n(\alpha) \tag{20}$$

and

$$E_{\inf}(\xi) \subset \bigcap_{K=1}^{\infty} \bigcup_{\tilde{n}=K}^{\infty} \bigcup_{\tilde{r}=\tilde{n}}^{\infty} \bigcup_{\alpha \in Z_{\tilde{n},\tilde{r}}(c_0,d_0)} I_{\tilde{n}}(\alpha). \tag{21}$$

The justification of the inclusions in (20) and (21) is analogous to that of (14), and is therefore omitted.

We are now in a position to prove inequality (17). If (18) holds, then, by the inclusion in (20), we obtain

$$\mathcal{H}^{\rho+2\varepsilon}(E_{\inf}(\lambda)) \le \liminf_{K\to\infty} \sum_{n=K}^{\infty} \sum_{r=n}^{\infty} \sum_{\alpha \in Z_{n,r}(a_0,b_0)} |I_n(\alpha)|^{\rho+2\varepsilon}$$

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$$\leq \liminf_{K\to\infty} \sum_{n=K}^{\infty} \sum_{r=n}^{\infty} \Bigg(\sum_{\alpha\in Z_{n,r}(a_0,b_0)} |I_n(\alpha)|^{\rho+\varepsilon} \Bigg) 2^{-(r-1)\varepsilon}$$

$$\leq \liminf_{K\to\infty} \sum_{n=K}^{\infty} \sum_{r=n}^{\infty} 2^{-(r-1)\varepsilon} = 0.$$

This implies that $\dim_{\mathrm{H}} E_{\inf}(\lambda) \leq \rho + 2\varepsilon$. By Theorem A, it follows that

$$\dim_{\mathrm{H}} E(\lambda) \leq \rho + 2\varepsilon.$$

Letting $\varepsilon \to 0$, we have $\dim_{\mathrm{H}} E(\lambda) \leq \rho$. Similarly, if (19) holds, then, by the inclusion in (21), we conclude that $\dim_{\mathrm{H}} E(\xi) \leq \rho$. Therefore, inequality (17) follows. We complete the proof. □

4. *Proof of the Key Lemma*

In this section, we present the proof of the Key Lemma, following an approach previously developed by the present authors in [14]. Since our goal is to establish a lower bound for $\dim_{\mathrm{H}} E(\lambda, \xi)$, we will do this by applying Falconer's formula (see [7, Example 4.6]).

LEMMA 4. (Falconer's formula) *Let* $[0, 1] =: \mathrm{E}_0 \supset \mathrm{E}_1 \supset \mathrm{E}_2 \supset \cdots$ *be a decreasing sequence of sets, where each* E_n *is a union of finitely many disjoint closed intervals (called nth level basic intervals), and the maximum length of the nth level basic intervals tends to zero as* $n \to \infty$. *Define*

$$\mathrm{E} := \bigcap_{n=0}^{\infty} \mathrm{E}_n. \tag{22}$$

Assume that each $(n-1)$*th level basic interval contains at least* m_n *nth level basic intervals, which are separated by gaps of length at least* ε_n. *If* $m_n \geq 2$ *and* $0 < \varepsilon_{n+1} < \varepsilon_n$ *for all sufficiently large n, then*

$$\dim_{\mathrm{H}} \mathrm{E} \geq \liminf_{n\to\infty} \frac{\log(m_1 m_2 \cdots m_{n-1})}{-\log(m_n \varepsilon_n)}. \tag{23}$$

To apply Lemma 4, we proceed in the following four steps. Let $\lambda, \xi \in [\alpha_{\inf}, \alpha_{\sup}]$ with $\lambda \leq \xi$ be fixed.

4.1. *Step I: Construction of the sequence* $\{\mathrm{E}_n\}_{n\geq 1}$. In the rest of this paper, when the order of a cylinder set is clear from the context or does not need to be emphasized, we omit the corresponding subscript. This convention is adopted mainly for typographical clarity, as the expressions appearing in these subscripts can be rather long.

Define $a_1 = \lceil \lambda \rceil - 1$, $b_1 = \lfloor \lambda \rfloor + 1$, $c_2 = (\lceil 2\xi \rceil - 1)/2$, $d_2 = (\lfloor 2\xi \rfloor + 1)/2$ and, for any $k > 1$,

$$a_{2k-1} := \max \left\{ a_{2k-3}, \frac{\lceil (2k-1)\lambda \rceil - 1}{2k-1} \right\}, \quad b_{2k-1} := \min \left\{ b_{2k-3}, \frac{\lfloor (2k-1)\lambda \rfloor + 1}{2k-1} \right\}$$

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and

$$c_{2k} := \max\left\{c_{2k-2}, \frac{\lceil 2k\xi\rceil - 1}{2k}\right\}, \quad d_{2k} := \min\left\{d_{2k-2}, \frac{\lfloor 2k\xi\rfloor + 1}{2k}\right\}.$$

Then, $a_{2k-1}, b_{2k-1}, c_{2k}, d_{2k} \in \mathbb{Q}$ and

$$\lambda - \frac{1}{2k-1} \le a_{2k-1} \le a_{2k+1} < \lambda < b_{2k+1} \le b_{2k-1} \le \lambda + \frac{1}{2k-1}$$

and

$$\xi - \frac{1}{2k} \le c_{2k} \le c_{2k+2} < \xi < d_{2k+2} \le d_{2k} \le \xi + \frac{1}{2k}.$$

By the condition of the Key Lemma, there exist strictly increasing sequences $\{n_k\}_{k\ge1}$ and $\{r_k\}_{k\ge1}$ of positive integers with $r_k \ge n_k$ such that, for all $k \in \mathbb{N}$,

$$\sum_{\alpha\in Z_{n_{2k-1},r_{2k-1}}(a_{2k-1},b_{2k-1})} |I_{n_{2k-1}}(\alpha)|^s \ge 1 \quad \text{and} \quad \sum_{\alpha\in Z_{n_{2k},r_{2k}}(c_{2k},d_{2k})} |I_{n_{2k}}(\alpha)|^s \ge 1. \tag{24}$$

For notational convenience, let

$$\widetilde{a}_k := \begin{cases} a_{2m-1} & \text{if } k = 2m-1, \\ c_{2m} & \text{if } k = 2m, \end{cases} \quad \text{and} \quad \widetilde{b}_k := \begin{cases} b_{2m-1} & \text{if } k = 2m-1, \\ d_{2m} & \text{if } k = 2m. \end{cases}$$

Under this notation, the inequalities in (24) can be rewritten as

$$\sum_{\alpha\in Z_{n_k,r_k}(\widetilde{a}_k,\widetilde{b}_k)} |I_{n_k}(\alpha)|^s \ge 1 \quad \text{for all } k \ge 1. \tag{25}$$

The cardinality of $Z_{n_k,r_k}(\widetilde{a}_k, \widetilde{b}_k)$ is estimated as follows: since $|I_{n_k}(\alpha)| < (1/2^{r_k-1})$ for all $\alpha \in Z_{n_k,r_k}(\widetilde{a}_k, \widetilde{b}_k)$, by (25), we have

$$\sharp Z_{n_k,r_k}(\widetilde{a}_k, \widetilde{b}_k) \cdot \frac{1}{2^{s(r_k-1)}} > 1,$$

and therefore,

$$\sharp Z_{n_k,r_k}(\widetilde{a}_k, \widetilde{b}_k) > 2^{s(r_k-1)}.$$

Let $0 < \varepsilon < s$ be fixed. It follows that, for any sufficiently large k,

$$\sharp Z_{n_k,r_k}(\widetilde{a}_k, \widetilde{b}_k) > 2^{(s-\varepsilon)(r_k+1)}.$$

Hence, there exists $0 < C \le 1$ such that

$$\sharp Z_{n_k,r_k}(\widetilde{a}_k, \widetilde{b}_k) > C \cdot 2^{(s-\varepsilon)(r_k+1)} \quad \text{for all } k \ge 1. \tag{26}$$

For simplicity, let $X_k := Z_{n_k,r_k}(\widetilde{a}_k, \widetilde{b}_k)$.

Let

$$M_k := \max\left\{\left|\sum_{k=0}^{j-1} \psi(G^k(x))\right| : x \in I_{n_k}(\alpha), \alpha \in X_k, 1 \le j \le n_k\right\}. \tag{27}$$

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Notice that, since $\mathrm{Var}_1(\psi) < \infty$, we derive that ψ is bounded on the interval $I(w)$ for any finite sequence w of positive integers.

Choose a sequence $\{u_k\}_{k\geq 0}$ of integers such that $u_0 = 0$ and

$$u_k > (k+1)\left(r_{k+1} + M_{k+1} + \sum_{j=0}^{k-1} u_j n_j\right) \quad \text{for all } k \geq 1. \tag{28}$$

We proceed to construct the sequence $\{\mathrm{E}_n\}_{n\geq 1}$ of subsets of $[0, 1]$. For this purpose, we first introduce some notation.

For any $k \in \mathbb{N}$ and $n \in \mathbb{N}$ with $\sum_{i=0}^{k-1} u_i < n \leq \sum_{i=0}^{k} u_i$, define

$$\mathcal{A}_k^{*(n)} := \left\{ w_k^{(1)} w_k^{(2)} \cdots w_k^{(n-\sum_{i=0}^{k-1} u_i)} \colon w_k^{(j)} \in X_k \text{ for all } 1 \leq j \leq n - \sum_{i=0}^{k-1} u_i \right\}.$$

The length of a word in $\mathcal{A}_k^{*(n)}$ is $(n - \sum_{i=0}^{k-1} u_i)n_k$. In particular, when $n = \sum_{i=0}^{k} u_i$, we write $\mathcal{A}_k^{*(n)}$ as $\mathcal{A}_k$:

$$\mathcal{A}_k := \{w_k^{(1)} w_k^{(2)} \cdots w_k^{(u_k)} \colon w_k^{(j)} \in X_k \text{ for all } 1 \leq j \leq u_k\}.$$

Define

$$\mathcal{W}_k^{*(n)} := \{w_1 \cdots w_{k-1} w_k^{*(n)} \colon w_1 \in \mathcal{A}_1, \ldots, w_{k-1} \in \mathcal{A}_{k-1}, w_k^{*(n)} \in \mathcal{A}_k^{*(n)}\}.$$

The length of a word in $\mathcal{W}_k^{*(n)}$ is $\sum_{j=1}^{k-1} u_j n_j + (n - \sum_{i=0}^{k-1} u_i)n_k$. When $n = \sum_{i=0}^{k} u_i$, we write $\mathcal{W}_k^{*(n)}$ as $\mathcal{W}_k$:

$$\mathcal{W}_k := \{w_1 \cdots w_{k-1} w_k \colon w_1 \in \mathcal{A}_1, \ldots, w_{k-1} \in \mathcal{A}_{k-1}, w_k \in \mathcal{A}_k\}.$$

Let v_k denote the maximum component among all vectors in X_k for each $k \in \mathbb{N}$. The sequence $\{\mathrm{E}_n\}_{n\geq 1}$ of sets is defined as follows.

(I-i) When $\sum_{i=0}^{k-1} u_i < n < \sum_{i=0}^{k} u_i$ for some $k \in \mathbb{N}$, for any $w_1 \cdots w_{k-1} w_k^{*(n)} \in \mathcal{W}_k^{*(n)}$, we define the nth level basic interval as

$$J(w_1 \cdots w_{k-1} w_k^{*(n)}) := \bigcup_{j=1}^{v_k} \mathrm{cl}(I(w_1 \cdots w_{k-1} w_k^{*(n)} j)),$$

where $\mathrm{cl}(A)$ denotes the closure of the set A. We then set

$$\mathrm{E}_n := \bigcup_{w_1 \cdots w_{k-1} w_k^{*(n)} \in \mathcal{W}_k^{*(n)}} J(w_1 \cdots w_{k-1} w_k^{*(n)}).$$

(I-ii) When $n = \sum_{i=0}^{k} u_i$ for some $k \in \mathbb{N}$, for any $w_1 \cdots w_{k-1} w_k \in \mathcal{W}_k$, we define the nth level basic interval as

$$J(w_1 \cdots w_{k-1} w_k) := \bigcup_{j=1}^{v_{k+1}} \mathrm{cl}(I(w_1 \cdots w_{k-1} w_k j)).$$

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We then set

$$\mathrm{E}_n := \bigcup_{w_1\cdots w_{k-1}w_k\in\mathcal{W}_k} J(w_1\cdots w_{k-1}w_k).$$

With this construction, the sequence $\{\mathrm{E}_n\}_{n\geq 1}$ is now fully defined.

4.2. *Step II: Inclusion of the limit set of* $\{\mathrm{E}_n\}_{n\geq 1}$ *in* $E(\lambda,\xi)$. Since $\mathrm{E}_{n+1}\subseteq \mathrm{E}_n$ for all $n\in\mathbb{N}$, we define $\mathrm{E}_0 := [0,1]$ and the limit set

$$\mathrm{E} := \bigcap_{n=0}^{\infty} \mathrm{E}_n.$$

We will prove that

$$\mathrm{E}\subset E(\lambda,\xi). \tag{29}$$

For any $x\in\mathrm{E}$, the word consisting of the partial quotients of the continued fraction expansion of x is

$$\underbrace{w_1^{(1)}\cdots w_1^{(u_1)}}_{=:w_1\in\mathcal{W}_1}\underbrace{w_2^{(1)}\cdots w_2^{(u_2)}}_{=:w_2\in\mathcal{W}_2}\cdots\underbrace{w_{k-1}^{(1)}\cdots w_{k-1}^{(u_{k-1})}}_{=:w_{k-1}\in\mathcal{W}_{k-1}}\cdots.$$

For any $k\in\mathbb{N}$ and $0\leq \ell\leq u_k-1$, we have

$$G^{\sum_{i=1}^{k-1}u_in_i+\ell n_k}(x)\in I_{n_k}(w_k^{(\ell+1)}),$$

which, together with (12), implies that

$$n_k\widetilde{a}_k < \sum_{j=0}^{n_k-1}\psi(G^{\sum_{i=1}^{k-1}u_in_i+\ell n_k+j}(x)) < n_k\widetilde{b}_k. \tag{30}$$

Thus,

$$u_kn_k\widetilde{a}_k < \sum_{j=0}^{u_kn_k-1}\psi(G^{\sum_{i=1}^{k-1}u_in_i+j}(x)) < u_kn_k\widetilde{b}_k. \tag{31}$$

For any $N\in\mathbb{N}$, there exist $k\in\mathbb{N}$, $0\leq \ell_k<u_k$, and $0\leq t_k<n_k$ such that

$$N = u_1n_1+\cdots+u_{k-1}n_{k-1}+\ell_kn_k+t_k.$$

Let us write

$$D(N) := \sum_{j=1}^{k-1}u_jn_j+\ell_kn_k.$$

By (30), (31), and the definition of M_k in (27), we derive that

$$S_N\psi(x) > \sum_{j=1}^{k-1}u_jn_j\widetilde{a}_j+\ell_kn_k\widetilde{a}_k-M_k$$

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and

$$S_N\psi(x) < \sum_{j=1}^{k-1} u_j n_j \widetilde{b}_j + \ell_k n_k \widetilde{b}_k + M_k.$$

By (28), we have

$$\sum_{j=1}^{k-1} u_j n_j + \ell_k n_k > u_{k-1} > k r_k \geq k n_k,$$

which implies that

$$D(N) \leq N < \sum_{j=1}^{k-1} u_j n_j + \ell_k n_k + n_k < \left(1+\frac{1}{k}\right)\left(\sum_{j=1}^{k-1} u_j n_j + \ell_k n_k\right) = \left(\frac{k+1}{k}\right) D(N).$$

Hence,

$$\begin{aligned}
\frac{1}{N} S_N\psi(x) &< \frac{\sum_{j=1}^{k-1} u_j n_j \widetilde{b}_j + \ell_k n_k \widetilde{b}_k + M_k}{N} \\
&\leq \max\left\{ \frac{\sum_{j=1}^{k-1} u_j n_j \widetilde{b}_j + \ell_k n_k \widetilde{b}_k + M_k}{D(N)}, \frac{k}{k+1} \cdot \frac{\sum_{j=1}^{k-1} u_j n_j \widetilde{b}_j + \ell_k n_k \widetilde{b}_k + M_k}{D(N)} \right\} \\
&< \max\left\{ \frac{\sum_{j=1}^{k-1} u_j n_j \widetilde{b}_j + \ell_k n_k \widetilde{b}_k}{D(N)}, \frac{k}{k+1} \cdot \frac{\sum_{j=1}^{k-1} u_j n_j \widetilde{b}_j + \ell_k n_k \widetilde{b}_k}{D(N)} \right\} + \frac{M_k}{u_{k-1}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{N} S_N\psi(x) &> \frac{\sum_{j=1}^{k-1} u_j n_j \widetilde{a}_j + \ell_k n_k \widetilde{a}_k - M_k}{N} \\
&> \min\left\{ \frac{\sum_{j=1}^{k-1} u_j n_j \widetilde{a}_j + \ell_k n_k \widetilde{a}_k}{D(N)}, \frac{k}{k+1} \cdot \frac{\sum_{j=1}^{k-1} u_j n_j \widetilde{a}_j + \ell_k n_k \widetilde{a}_k}{D(N)} \right\} - \frac{M_k}{u_{k-1}}.
\end{aligned}$$

Since $\liminf_{k\to\infty} \widetilde{a}_k = \lambda$ and $\limsup_{k\to\infty} \widetilde{b}_k = \xi$, it follows that

$$\liminf_{k\to\infty} \min_{0\leq \ell_k < u_k} \left\{ \frac{\sum_{j=1}^{k-1} u_j n_j \widetilde{a}_j + \ell_k n_k \widetilde{a}_k}{D(N)} \right\} \geq \lambda$$

and

$$\limsup_{k\to\infty} \max_{0\leq \ell_k < u_k} \left\{ \frac{\sum_{j=1}^{k-1} u_j n_j \widetilde{b}_j + \ell_k n_k \widetilde{b}_k}{D(N)} \right\} \leq \xi.$$

By the definition of u_{k-1} defined in (28), we have $M_k/u_{k-1} < (1/k)$ for all $k \in \mathbb{N}$. Therefore, we conclude that

$$\lambda \leq \liminf_{N\to\infty} \frac{1}{N} S_N\psi(x) \leq \limsup_{N\to\infty} \frac{1}{N} S_N\psi(x) \leq \xi.$$

Next, we show that

$$\liminf_{N\to\infty} \frac{1}{N} S_N\psi(x) = \lambda \quad \text{and} \quad \limsup_{N\to\infty} \frac{1}{N} S_N\psi(x) = \xi. \tag{32}$$

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For the lower limit, let $N_k := \sum_{j=1}^{2k-1} u_j n_j$ for all $k \in \mathbb{N}$. Since $u_{2k-1} > \sum_{j=1}^{2k-2} u_j n_j$, we have

$$1 \leq \frac{N_k}{u_{2k-1}n_{2k-1}} < 1 + \frac{1}{n_{2k-1}},$$

which implies that $\lim_{k\to\infty}(N_k/u_{2k-1}n_{2k-1}) = 1$. Hence,

$$\liminf_{k\to\infty} \frac{1}{N_k} S_{N_k}\psi(x) = \liminf_{k\to\infty} \frac{1}{u_{2k-1}n_{2k-1}} S_{N_k}\psi(x).$$

By (31), we obtain

$$S_{N_k}\psi(x) < \sum_{j=1}^{2k-1} u_j n_j \widetilde{b}_j,$$

which implies that

$$\frac{1}{u_{2k-1}n_{2k-1}} S_{N_k}\psi(x) < \widetilde{b}_{2k-1} + \frac{\sum_{j=1}^{2k-2} u_j n_j \widetilde{b}_j}{u_{2k-1}n_{2k-1}}.$$

Note that $u_{2k-1}n_{2k-1} > n_{2k-1}\sum_{j=1}^{2k-2} u_j n_j$ for all $k \in \mathbb{N}$, while $\widetilde{b}_{2k-1} \to \lambda$ and $\widetilde{b}_{2k} \to \xi$ as $k \to \infty$. It follows that $\{\widetilde{b}_{2k-1}\}_{k\geq 1}$ and $\{\widetilde{b}_{2k}\}_{k\geq 1}$ are bounded. Therefore,

$$\lim_{k\to\infty} \frac{\sum_{j=1}^{2k-2} u_j n_j \widetilde{b}_j}{u_{2k-1}n_{2k-1}} = 0,$$

and consequently,

$$\liminf_{k\to\infty} \frac{1}{N_k} S_{N_k}\psi(x) \leq \lambda.$$

Hence,

$$\liminf_{N\to\infty} \frac{1}{N} S_N\psi(x) = \lambda.$$

For the upper limit, let $\widetilde{N}_k := \sum_{j=1}^{2k} u_j n_j$ for all $k \in \mathbb{N}$. Since $\sum_{j=1}^{2k-1} u_j n_j < u_{2k}$, it follows that $\lim_{k\to\infty}(\widetilde{N}_k/u_{2k}n_{2k}) = 1$. Therefore,

$$\limsup_{k\to\infty} \frac{1}{\widetilde{N}_k} S_{\widetilde{N}_k}\psi(x) = \limsup_{k\to\infty} \frac{1}{u_{2k}n_{2k}} S_{\widetilde{N}_k}\psi(x).$$

From (31), we deduce that

$$S_{\widetilde{N}_k}\psi(x) > \sum_{j=1}^{2k} u_j n_j \widetilde{a}_j,$$

and thus,

$$\frac{1}{u_{2k}n_{2k}} S_{\widetilde{N}_k}\psi(x) > \widetilde{a}_{2k} + \frac{\sum_{j=1}^{2k-1} u_j n_j \widetilde{a}_j}{u_{2k}n_{2k}}.$$

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Since, as before, $\lim_{k\to\infty}(\sum_{j=1}^{2k-1} u_j n_j \widetilde{a}_j/u_{2k}n_{2k}) = 0$, we deduce that

$$\limsup_{N\to\infty} \frac{1}{N} S_N\psi(x) \geq \limsup_{k\to\infty} \frac{1}{\widetilde{N}_k} S_{\widetilde{N}_k}\psi(x) \geq \xi,$$

where the last inequality follows from $\widetilde{a}_{2k} \to \xi$ as $k \to \infty$. Hence,

$$\limsup_{N\to\infty} \frac{1}{N} S_N\psi(x) = \xi.$$

Therefore, (32) holds. That is, E is a subset of $E(\lambda, \xi)$.

By (29), we obtain

$$\dim_{\mathrm{H}} E(\lambda, \xi) \geq \dim_{\mathrm{H}} \mathrm{E}.$$

To calculate the lower bound of $\dim_{\mathrm{H}}$ E, by Lemma 4, it remains to estimate the values of m_n and ε_n.

4.3. *Step III: Estimation of m_n and ε_n in Lemma 4.* We first recall the meanings of m_n and ε_n in Lemma 4: m_n are chosen such that every $(n-1)$th level basic interval contains at least m_n nth level basic intervals, and ε_n is chosen such that the gaps between these nth level basic intervals have lengths at least ε_n.

The estimates of the values of m_n and ε_n are as follows.

(II-i) When $\sum_{i=0}^{k-1} u_i < n < \sum_{i=0}^{k} u_i$ for some $k \in \mathbb{N}$, equivalently,

$$\sum_{i=0}^{k-1} u_i \leq n-1 < \sum_{i=0}^{k} u_i - 1,$$

the $(n-1)$th level basic interval is of the form $J(w)$ for some $w \in \mathcal{W}_k^{*(n-1)}$, and it contains the nth level basic intervals:

$$J(ww') := \bigcup_{j=1}^{v_k} \mathrm{cl}(I(ww'j))$$

for all $w' \in X_k$. Hence, we may take $m_n = \sharp X_k$. In this case, the minimum length of the gaps between these nth level basic intervals is at least

$$\widetilde{\varepsilon}_n := \min\{|I(w(v_k+1))|\colon w \in \mathcal{W}_k^{*(n)}\}. \tag{33}$$

(II-ii) When $n = \sum_{i=0}^{k} u_i$ for some $k \in \mathbb{N}$, equivalently,

$$n-1 = \sum_{i=0}^{k} u_i - 1,$$

the $(n-1)$th level basic interval is of the form $J(w)$ for some $w \in \mathcal{W}_k^{*(n-1)}$, and it contains the nth level basic intervals:

$$J(ww') := \bigcup_{j=1}^{v_{k+1}} \mathrm{cl}(I(ww'j))$$

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for all $w' \in X_k$. Hence, we may take $m_n = \sharp X_k$. In this case, the minimum length of the gaps between these nth level basic intervals is at least

$$\widetilde{\varepsilon}_n := \min\{|I(w(v_{k+1}+1))|\colon w \in \mathcal{W}_k\}. \tag{34}$$

Therefore, for any $k \in \mathbb{N}$ and $n \in \mathbb{N}$ with $\sum_{i=0}^{k-1} u_i < n \le \sum_{i=0}^{k} u_i$, by (26), we have

$$m_n = \sharp X_k > C \cdot 2^{(s-\varepsilon)(r_k+1)}. \tag{35}$$

For ε_n, we may take

$$\varepsilon_n := \frac{1}{8 \cdot 2^{r_k}} \cdot \min\{|I(w)|\colon w \in \mathcal{W}_k^{*(n)}\} \tag{36}$$

for all $n \in \mathbb{N}$ with $\sum_{i=0}^{k-1} u_i < n < \sum_{i=0}^{k} u_i$, and take

$$\varepsilon_n := \frac{1}{8 \cdot 2^{r_{k+1}}} \cdot \min\{|I(w)|\colon w \in \mathcal{W}_k\} \tag{37}$$

for all $n = \sum_{i=0}^{k} u_i$. Since $r_{k+1} > r_k$, it follows that the sequence $\{\varepsilon_n\}_{n\ge1}$ is strictly decreasing. In fact, when $\sum_{i=0}^{k-1} u_i < n < \sum_{i=0}^{k} u_i - 1$, by (36), we obtain

$$\varepsilon_n = \frac{1}{8 \cdot 2^{r_k}} \cdot \min\{|I(u)|\colon u \in \mathcal{W}_k^{*(n)}\}, \quad \varepsilon_{n+1} = \frac{1}{8 \cdot 2^{r_k}} \cdot \min\{|I(w)|\colon w \in \mathcal{W}_k^{*(n+1)}\}.$$

Since $|I(u)| \ge 2|I(w)|$ for any $u \in \mathcal{W}_k^{*(n)}$ and $w \in \mathcal{W}_k^{*(n+1)}$, it follows that $\varepsilon_n > \varepsilon_{n+1}$. When $n = \sum_{i=0}^{k} u_i - 1$, we deduce from (36) and (37) that

$$\varepsilon_n = \frac{1}{8 \cdot 2^{r_k}} \cdot \min\{|I(u)|\colon u \in \mathcal{W}_k^{*(n)}\}, \quad \varepsilon_{n+1} = \frac{1}{8 \cdot 2^{r_{k+1}}} \cdot \min\{|I(w)|\colon w \in \mathcal{W}_k\}.$$

Since $r_{k+1} > r_k$, we have $\varepsilon_n > \varepsilon_{n+1}$. When $n = \sum_{i=0}^{k} u_i$, from (36) and (37), we derive that

$$\varepsilon_n = \frac{1}{8 \cdot 2^{r_{k+1}}} \cdot \min\{|I(u)|\colon u \in \mathcal{W}_k\}, \quad \varepsilon_{n+1} = \frac{1}{8 \cdot 2^{r_{k+1}}} \cdot \min\{|I(w)|\colon w \in \mathcal{W}_{k+1}^{*(n+1)}\},$$

and hence, $\varepsilon_n > \varepsilon_{n+1}$.

We next show that

$$\widetilde{\varepsilon}_n > \varepsilon_n \quad \text{for all } n \ge 1.$$

To this end, we will make use of the following fact: for any two finite words u and v,

$$\tfrac{1}{2}|I(u)||I(v)| < |I(uv)| \le 2|I(u)||I(v)|, \tag{38}$$

see [30, Lemma A.2] for details. We also note that $v_k^2 < 2^{r_k}$ for all $k \ge 1$. In fact, since v_k is the maximum component among all vectors in X_k, there exists $\omega' \in X_k$ having v_k as one of its components. In this case, by the definition of X_k, we have

$$\frac{1}{2^{r_k}} \le |I(\omega')| < \frac{1}{v_k^2},$$

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which implies that $v_k^2 < 2^{r_k}$. When $\sum_{i=0}^{k-1} u_i < n < \sum_{i=0}^{k} u_i$, for any $w \in \mathcal{W}_k^{*(n)}$, by (38), we derive that

$$|I(w(v_k+1))| \geq \frac{1}{2(v_k+1)(v_k+2)}|I(w)| > \frac{1}{8v_k^2}|I(w)| \geq \frac{1}{8\cdot 2^{r_k}}|I(w)|,$$

which, together with (33), implies that

$$\widetilde{\varepsilon}_n > \frac{1}{8\cdot 2^{r_k}} \cdot \min\{|I(w)|\colon w \in \mathcal{W}_k^{*(n)}\} = \varepsilon_n.$$

When $n = \sum_{i=0}^{k} u_i$, for any $w \in \mathcal{W}_k$, it follows from (38) that

$$|I(w(v_{k+1}+1))| > \frac{1}{8\cdot 2^{r_{k+1}}}|I(w)|,$$

which, together with (34), implies that

$$\widetilde{\varepsilon}_n > \frac{1}{8\cdot 2^{r_{k+1}}} \cdot \min\{|I(w)|\colon w \in \mathcal{W}_k\} = \varepsilon_n.$$

4.4. *Step IV: Computation of the lower limit in Lemma 4.* We are ready to calculate the lower limit

$$\liminf_{n\to\infty} \frac{\log(m_1 m_2 \cdots m_{n-1})}{-\log(m_n \varepsilon_n)}. \tag{39}$$

Concerning the numerator in (39), for any $\sum_{i=0}^{k-1} u_i < n \leq \sum_{i=0}^{k} u_i$, it follows from (35) that

$$\begin{aligned} m_1 m_2 \cdots m_{n-1} &\geq (\sharp X_1)^{u_1} \cdots (\sharp X_{k-1})^{u_{k-1}} (\sharp X_k)^{n-1-\sum_{i=0}^{k-1} u_i} \\ &> C^{n-1} \cdot \left(\prod_{j=1}^{k-1} 2^{(s-\varepsilon)(r_j+1)u_j} \right) 2^{(s-\varepsilon)(r_k+1)(n-1-\sum_{i=0}^{k-1} u_i)}. \end{aligned}$$

Define

$$U_n := \sum_{j=1}^{k-1} (r_j+1)u_j + (r_k+1)\left(n-1-\sum_{i=0}^{k-1} u_i\right). \tag{40}$$

Then,

$$\log(m_1 m_2 \cdots m_{n-1}) > (s-\varepsilon)U_n \log 2 + (n-1)\log C. \tag{41}$$

Turning to the denominator in (39), for any $w_j = w_j^{(1)} w_j^{(2)} \cdots w_j^{(u_j)} \in \mathcal{A}_j$, by (38), we have

$$\begin{aligned} |I(w_j)| &\geq \frac{1}{2}|I(w_j^{(1)})| \cdot \frac{1}{2}|I(w_j^{(2)})| \cdots \frac{1}{2}|I(w_j^{(u_j-1)})| \cdot |I(w_j^{(u_j)})| \\ &> \frac{1}{2^{u_j-1}} \left(\frac{1}{2^{r_j}}\right)^{u_j} \\ &> \frac{1}{2^{(r_j+1)u_j}}. \end{aligned} \tag{42}$$

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When $\sum_{i=0}^{k-1} u_i < n < \sum_{i=0}^{k} u_i$, for any $w = w_1 w_2 \cdots w_{k-1} w_k^{*(n)} \in \mathcal{W}_k^{*(n)}$, from (38), we derive that

$$\begin{aligned} |I(w)| &\geq \frac{1}{2}|I(w_1)| \cdot \frac{1}{2}|I(w_2)| \cdots \frac{1}{2}|I(w_{k-1})| \cdot |I(w_k^{*(n)})| \\ &= \frac{1}{2^{k-1}}|I(w_1)| \cdot |I(w_2)| \cdots |I(w_{k-1})| \cdot |I(w_k^{*(n)})|. \end{aligned} \tag{43}$$

Note that

$$|I(w_k^{*(n)})| > \frac{1}{2^{(r_k+1)(n-\sum_{i=0}^{k-1} u_i)}}.$$

Combining this with (42) and (43), we conclude that

$$|I(w)| > \frac{1}{2^{U_n+r_k+k}},$$

which, together with (36), implies that

$$m_n \varepsilon_n > \varepsilon_n > \frac{1}{2^{U_n+2r_k+k+3}}. \tag{44}$$

When $n = \sum_{i=0}^{k} u_i$, for any $w = w_1 w_2 \cdots w_{k-1} w_k \in \mathcal{W}_k$, we deduce from (38) and (42) that

$$\begin{aligned} |I(w)| &\geq \frac{1}{2}|I(w_1)| \cdot \frac{1}{2}|I(w_2)| \cdots \frac{1}{2}|I(w_{k-1})| \cdot |I(w_k)| \\ &> \frac{1}{2^{k-1}} \cdot \prod_{j=1}^{k} \frac{1}{2^{(r_j+1)u_j}} \\ &= \frac{1}{2^{U_n+r_k+k}}, \end{aligned}$$

which, together with (37), implies that

$$m_n \varepsilon_n > \varepsilon_n > \frac{1}{2^{U_n+r_{k+1}+r_k+k+3}}. \tag{45}$$

We are now in a position to estimate the lower limit in (39). When $\sum_{i=0}^{k-1} u_i < n < \sum_{i=0}^{k} u_i$, by (28) and the definition of U_n in (40), we see that

$$U_n > u_{k-1} > k r_k.$$

Combining this with (41) and (44), we derive that

$$\begin{aligned} \frac{\log(m_1 m_2 \cdots m_{n-1})}{-\log(m_n \varepsilon_n)} &> \frac{(s-\varepsilon)U_n + (n-1)\log_2 C}{U_n + 2r_k + k + 3} \\ &> \frac{s-\varepsilon}{1 + (2r_k + k + 3)/k r_k} + \frac{n}{U_n}\log_2 C, \end{aligned} \tag{46}$$

where the last inequality comes from $0 < C \leq 1$. When $n = \sum_{i=0}^{k} u_i$, it follows from (28) and (40) that

$$U_n > u_k - 1 > (k+1)r_{k+1} - 1 > k r_{k+1}.$$

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Combining this with (41) and (45), we conclude that

$$\begin{aligned}\frac{\log(m_1m_2\cdots m_{n-1})}{-\log(m_n\varepsilon_n)} &> \frac{(s-\varepsilon)U_n+(n-1)\log_2 C}{U_n+r_{k+1}+r_k+k+3}\\ &> \frac{s-\varepsilon}{1+(r_{k+1}+r_k+k+3)/kr_{k+1}}+\frac{n}{U_n}\log_2 C. \qquad (47)\end{aligned}$$

Since $\{r_k\}_{k\geq 1}$ is strictly increasing and $r_k\to\infty$ as $k\to\infty$, we obtain

$$\lim_{k\to\infty}\frac{2r_k+k+3}{kr_k}=\lim_{k\to\infty}\frac{r_{k+1}+r_k+k+3}{kr_{k+1}}=0.$$

It follows from the definition of U_n in (40) that

$$\lim_{n\to\infty}\frac{U_n}{n}=\infty.$$

By (46) and (47), we derive that

$$\liminf_{n\to\infty}\frac{\log(m_1m_2\cdots m_{n-1})}{-\log(m_n\varepsilon_n)}\geq s-\varepsilon,$$

which, together with (29) and Lemma 4, implies that

$$\dim_{\mathrm{H}} E(\lambda,\xi)\geq \dim_{\mathrm{H}} \mathrm{E}\geq s-\varepsilon.$$

Letting $\varepsilon\to 0$, we have

$$\dim_{\mathrm{H}} E(\lambda,\xi)\geq s.$$

Combining the above four steps, we complete the proof of the Key Lemma.

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