The area of $ABCD$ is given by

$$g(x, y) = \tfrac{1}{2}(ad \sin x + bc \sin y).$$

Thus we are to maximize $g(x, y)$ given the constraint $f(x, y) = 0$. According to the method of Lagrange, we are to solve the system $[\nabla g = \lambda \nabla f, f(x, y) = 0]$, that is, the system

$$\text{(i)}\ \frac{ad\cos x}{2} = \lambda(2ad\sin x), \qquad \text{(ii)}\ \frac{bc\cos y}{2} = \lambda(-2bc\sin y), \qquad \text{(iii)}\ f = 0.$$

If $\lambda = 0$, then $\cos x = \cos y = 0$, and hence $x = y = \frac{1}{2}\pi$ and the quadrilateral is cyclic. This actually happens, in view of (iii), when $a^2 + d^2 = b^2 + c^2$. So we assume that $\lambda \neq 0$. In this case, $\cos x \neq 0$; otherwise, $\sin x = \pm 1$ and (i) implies that $\lambda = 0$. Therefore $\cos x \neq 0$, and similarly $\cos y \neq 0$. It now follows from (i) and (ii) that $\tan x = -\tan y$, each being equal to $\dfrac{1}{4\lambda}$. Since $x, y > 0$ and $x + y < 2\pi$, it follows that $x + y = \pi$, and hence the optimal quadrilateral is cyclic, as claimed.

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100.23 A quartet of isogonal conjugates

Given a triangle ABC and a point P in the plane, we can reflect the lines AP, BP and CP in the internal angle bisectors of $\angle CAB$, $\angle ABC$ and $\angle BCA$ respectively to produce three more lines. These lines are concurrent at a point known as the *isogonal conjugate* of P with respect to $\triangle ABC$. This configuration has many well-known properties which are summarised in [1].

However, the following elegant result appears to be new.

Theorem: Let $ABDC$ be a quadrilateral which is not cyclic. Let A^* be the isogonal conjugate of A with respect to $\triangle BDC$ and similarly let B^*, C^* and D^* be the isogonal conjugates of B, C and D with respect to $\triangle ADC$, $\triangle ABD$ and $\triangle ABC$ respectively. Then A, B, C and D are the circumcentres of $\triangle B^*C^*D^*$, $\triangle A^*C^*D^*$, $\triangle A^*D^*B^*$ and $\triangle A^*C^*B^*$.

Figure 1 shows the circumcircle of $\triangle B^*C^*D^*$ which has centre A.

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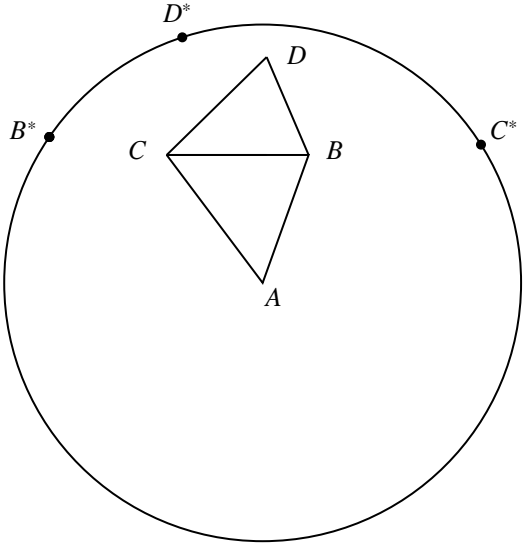


FIGURE 1

The proof begins with a lemma, which is a simple angle calculation. We will focus on the points A^* and D^* but the argument applies equally to the other pairs.

Lemma: The line BC is the perpendicular bisector of A^*D^*.

Proof: The dotted lines in Figure 2 represent the internal angle bisectors of $\angle ABC$ and of $\angle CBD$. The line BD is reflected in the first of these to give the line BF which contains D^*, and similarly the line BA is reflected in the second to give the line BE which contains A^*.

Letting $\angle ABC = 2\theta$ and $\angle CBD = 2\phi$, we see that $\angle CBE = \angle CBF = 2(\theta + \phi)$. For the analogous calculation at C, let $\angle ACB = 2\alpha$ and $\angle DCB = 2\beta$, and we obtain a common angle of $2(\alpha + \beta)$. The lines at B and C defining the isogonal conjugates will be parallel if, and only if, $\alpha + \beta + \theta + \phi = 180°$, which does not happen since the quadrilateral is not cyclic. It follows that $\triangle BCD^* \cong \triangle BCA^*$, and this implies that BC is the perpendicular bisector of A^*D^*.

Note that it is possible for D^* to be above BC and A^* to be below.

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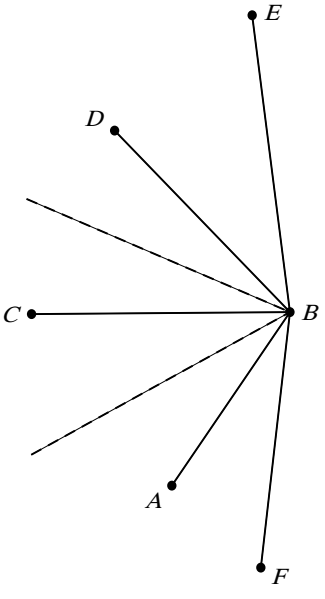


FIGURE 2

The proof of the theorem relies on the fact that isogonal conjugates share the same pedal circle.

The centre of the common pedal circle of A and A^* with respect to ΔBDC is the midpoint of AA^*, which is labelled O in Figure 3. Note that this circle goes through the point X which is the intersection of BC and A^*D^*, since, by the lemma, these segments are perpendicular.

An enlargement with centre A^* and scale factor 2 takes O to A, and also maps the pedal circle to a circle with centre A.

By the lemma, X is the midpoint of A^*D^*, so the image of the pedal circle passes through D^*. But there is nothing special about D^*, and it follows that B^* and C^* also lie on this circle.

Note again the significance of the condition that $ABCD$ is not cyclic. If it were, then the pedal triangles would degenerate to Simson lines and this proof would no longer be possible.

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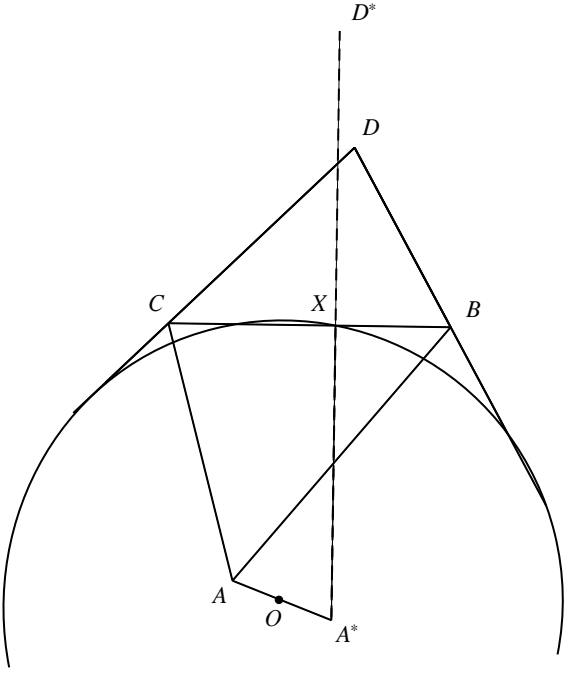


FIGURE 3

Corollary: Suppose that this process of creating isogonal conjugates A^*, B^*, C^* and D^* is repeated to produce another four points A^{**}, B^{**}, C^{**} and D^{**}. Then the quadrilaterals $ABCD$ and $A^{**}B^{**}C^{**}D^{**}$ are inversely homothetic.

Proof: Consider the circles with centres A and B, which share the points C^* and D^* (see Figure 4). Their line of centres AB is the perpendicular bisector of their common chord.

It follows that the sides of the quadrilateral $A^*B^*C^*D^*$ are perpendicular to those of $ABCD$, but the corresponding angles are the complements of those of $ABCD$.

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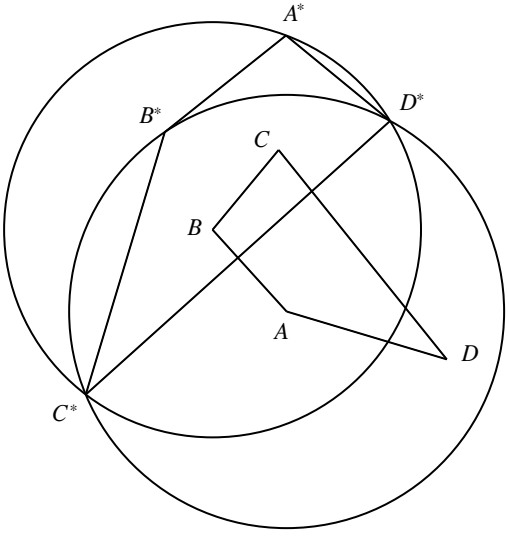


FIGURE 4

It follows that

$$\angle DCA = \angle A^*B^*D^* = \angle D^{**}C^{**}A^{**}$$

and

$$\angle CAD = \angle B^*D^*A^* = \angle C^{**}A^{**}D^{**}$$

and so ΔCDA and $\Delta C^{**}D^{**}A^{**}$ are similar (see Figure 5). As the same is true of ΔCBA and $\Delta C^{**}B^{**}A^{**}$ it follows that the quadrilaterals $ABCD$ and $A^{**}B^{**}C^{**}D^{**}$ are similar and similarly situated, so there is a centre of homothecy. Since directions are reversed this homothecy is inverse.

Note that it is not true in general that quadrilaterals $ABCD$ and $A^*B^*C^*D^*$ are similar.

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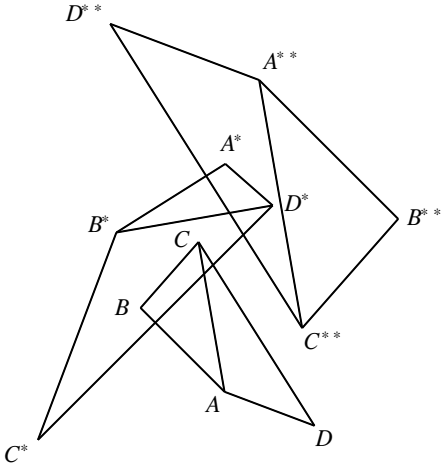


FIGURE 5

Acknowledgement: I am grateful to Geoff Smith for suggesting that this configuration might be worth investigating, and also to the referee who pointed out that the result in this Note is related (as a sort of converse process) to the construction in [2]. However, [2] is a lot longer and more technical than this Note since it explores a much more general case.

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100.24 A synthetic proof of Dao's generalisation of the Simson line theorem

Introduction

The Simson line theorem [1, p. 41] states that: The pedal triangle of a point P with respect to triangle ABC degenerates to a straight line if, and only if, P lies on the circumcircle of triangle ABC.

Several authors have written about generalisations of the Simson line [2, 3, 4].

In September 2014, Dao Thanh Oai (see [5]) conjectured the following remarkable generalisation of the Simson line theorem.

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