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Distance Hereditary Graphs and the Interlace Polynomial

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The vertex-nullity interlace polynomial of a graph, described by Arratia, Bollobás and Sorkin in [3] as evolving from questions of DNA sequencing, and extended to a two-variable interlace polynomial by the same authors in [5], evokes many open questions. These include relations between the interlace polynomial and the Tutte polynomial and the computational complexity of the vertex-nullity interlace polynomial. Here, using the medial graph of a planar graph, we relate the one-variable vertex-nullity interlace polynomial to the classical Tutte polynomial when $x = y$, and conclude that, like the Tutte polynomial, it is in general #P-hard to compute. We also show a relation between the two-variable interlace polynomial and the topological Tutte polynomial of Bollobás and Riordan in [13].

We define the γ invariant as the coefficient of x^1 in the vertex-nullity interlace polynomial, analogously to the β invariant, which is the coefficient of x^1 in the Tutte polynomial. We then turn to distance hereditary graphs, characterized by Bandelt and Mulder in [9] as being constructed by a sequence of adding pendant and twin vertices, and show that graphs in this class have γ invariant of 2^{n+1} when n true twins are added in their construction. We furthermore show that bipartite distance hereditary graphs are exactly the class of graphs with γ invariant 2, just as the series-parallel graphs are exactly the class of graphs with β invariant 1. In addition, we show that a bipartite distance hereditary graph arises precisely as the circle graph of an Euler circuit in the oriented medial graph of a series-parallel graph. From this we conclude that the vertex-nullity interlace polynomial is polynomial time to compute for bipartite distance hereditary graphs, just as the Tutte polynomial is polynomial time to compute for series-parallel graphs.

1. Introduction

In [3], Arratia, Bollobás and Sorkin defined a one-variable graph polynomial $q_N(G)$ (denoted q there, but we follow their later work [5], reserving q for the two-variable generalization)

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motivated by questions arising from DNA sequencing by hybridization addressed by Arratia, Bollobás, Coppersmith and Sorkin in [2]. This polynomial models the interlaced repeated subsequences of DNA that can interfere with the unique reconstruction of the original DNA strand. This work promptly generated further interest and other applications by Arratia, Bollobás and Sorkin [4, 5], Aigner and van der Holst [1], Balister, Bollobás, Cutler and Pebody [7], and Balister, Bollobás, Riordan and Scott [8]. In [5], Arratia, Bollobás and Sorkin define a two-variable interlace polynomial, and show that the original polynomial of [3] is a specialization of it, renaming the original interlace polynomial the vertex-nullity interlace polynomial due to its relationship with the two-variable generalization. In [3], and again in [4], Arratia, Bollobás and Sorkin provide some tantalizing properties of the vertex-nullity interlace polynomial, particularly for circle graphs, but describe it as ' ... territory newly invented, but not yet explored.' They present several open questions, some reiterated in [5] in the context of the two-variable interlace polynomial, including the interlace polynomial's relation to known graph polynomials and the computational complexity of the vertex-nullity interlace polynomial.

In Section 3 we note that, for planar graphs, the vertex-nullity interlace polynomial is related to the Tutte polynomial via medial graphs, and from this it follows that the vertex-nullity interlace polynomial, like the Tutte polynomial, is computationally intractable in the general case. We similarly provide a relationship between the two-variable interlace polynomial and the topological Tutte polynomial of Bollobás and Riordan from [13], again via a medial graph construction, thus showing that the interlace polynomial in some sense is encoding topological information.

After establishing that a polynomial is #P-hard to compute in general, a natural question is whether there are interesting classes of graphs for which it is tractable. We identify such a class of graphs, bipartite distance hereditary graphs, for which the vertex-nullity interlace polynomial is polynomial time to compute. In Section 4 we show that these graphs are characterized by an invariant γ (the coefficient of x^1 in q_N) analogously to the way series-parallel graphs are characterized by the β invariant (the common coefficient of x^1 and y^1 in the Tutte polynomial). This completes the characterization of graphs for which the coefficient of x^1 in q_N is 2, initiated by Aigner and van der Holst in [1].

We also show that the γ invariant of a (not necessarily bipartite) distance hereditary graph G is 2^{n+1} if n true twins are added in some construction sequence of G, but that distance hereditary graphs do not comprise the entire class for which γ is a power of 2.

We show in Section 5 that bipartite distance hereditary graphs arise precisely as circle graphs derived from Eulerian circuits in the oriented medial graphs of series-parallel graphs. From this characterization, and that distance hereditary graphs may be recognized in polynomial time, we conclude that the vertex-nullity interlace polynomial is polynomial time to compute, on the class of bipartite distance hereditary graphs.

2. The interlace and circuit partition polynomials

The vertex-nullity interlace polynomial of a graph was defined recursively by Arratia, Bollobás and Sorkin in [3] via a pivoting operation and was seen by them in [5] to be a specialization of a much richer two-variable interlace polynomial, $q(G; x, y)$, with a similar pivot recursion. Let vw be an edge of a graph G, and let A_v, A_w and A_{vw} be the sets of vertices in $V(G)\backslash\{v, w\}$ adjacent to v only, w only, and to both v and w, respectively. The pivot operation 'toggles' the edges between

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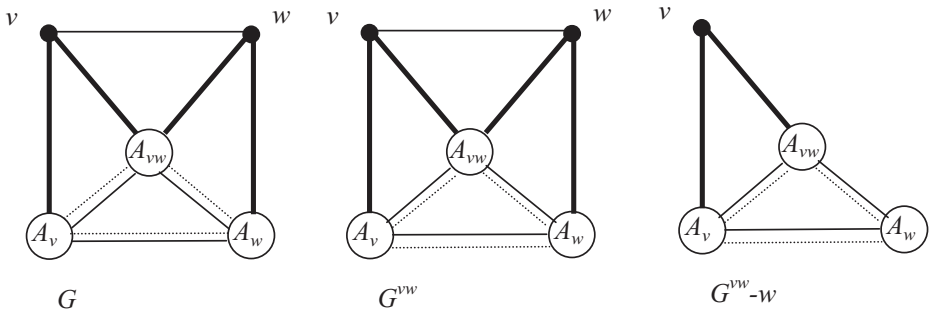


Figure 1. Pivoting on the edge vw. A_v, A_w and A_{vw} are the sets of vertices of G adjacent to v only, w only, and to both v and w, respectively. These sets are constant in all the diagrams. Vertices of G adjacent to neither v nor w are omitted. Heavy lines indicate that all edges are present. and dotted lines represent non-edges. (Note the interchange of edges and non-edges among A_v, A_w and A_{vw}.)

A_v, A_w and A_{vw}, by deleting existing edges and inserting edges between previously non-adjacent vertices. The result of this operation is denoted G^{vw}. More formally, G^{vw} has the same vertex set as G, and edge set equal to the symmetric difference $E(G)\Delta S$, where S is the complete tripartite graph with vertex classes A_v, A_w and A_{vw}. See Figure 1.

Definition 2.1. The vertex-nullity interlace polynomial is defined recursively as:

$$q_N(G;x) = \begin{cases} x^n \text{ if } G = E_n, \text{ the edgeless graph on } n \text{ vertices,} \\ q_N(G-v;x) + q_N(G^{vw}-w;x) \text{ if } vw \in E(G). \end{cases}$$

This polynomial was shown to be well-defined on all simple graphs in [3].

Definition 2.2. The two-variable interlace polynomial is defined, for a graph G of order n, which may have loops, but no multiple loops or multiple edges, by

$$q(G;x,y) = \sum_{S \subseteq V(G)} (x-1)^{r(G[S])}(y-1)^{n(G[S])}, \tag{2.1}$$

where $r(G[S])$ and $n(G[S]) = |S| - r(G[S])$ are, respectively, the $\mathbb{F}_2$-rank and nullity of the adjacency matrix of $G[S]$, the subgraph of G induced by S.

Equivalently, the two-variable interlace polynomial can be defined by the following reduction formulas from Arratia, Bollobás and Sorkin [5]. For a graph G, for any edge ab where neither a nor b has a loop,

$$q(G) = q(G-a) + q(G^{ab}-b) + ((x-1)^2-1)q(G^{ab}-a-b), \tag{2.2}$$

for any looped vertex a,

$$q(G) = q(G-a) + (x-1)q(G^a-a), \tag{2.3}$$

and, for the edgeless graph E_n on $n \geqslant 0$ vertices, $q(E_n) = y^n$. Here G^a is the *local complementation* of G, and is defined as follows. Let $N(a)$ be the neighbours of a, that is, the set $\{w \in V :$

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a and w are joined by an edge$\}$. The graph G^a is equal to G except that $G^a[N(a)] = \overline{G[N(a)]}$, *i.e.*, we 'toggle' the edges among the neighbours of a, switching edges to non-edges and *vice versa*. We refer the reader to [5] for details of the looped case, as we will only apply the interlace polynomial to simple graphs in the following sections.

Arratia, Bollobás and Sorkin show in [5] that the vertex-nullity interlace polynomial is a specialization of the two-variable interlace polynomial as follows:

$$q_N(G;y) = q(G;2,y) = \sum_{W \subseteq V(G)} (y-1)^{n(G[W])}. \tag{2.4}$$

An equivalent formulation for $q_N(G;x)$ is given by Aigner and van der Holst in [1].

Proposition 2.3. *If G is a simple graph, then $q(G;x,y)$ has no constant term unless G is the empty graph (no vertices or edges), in which case $q(G;x,y) = 1$.*

Proof. This observation follows from noting that this is true for a simple graph with one vertex, and applying induction on the number of vertices with equation (2.2). □

Proposition 2.4. *If G is a simple graph, then $q_N(G;x,y)$ has no constant term unless G is the empty graph (no vertices or edges), in which case $q_N(G;x,y) = 1$.*

Proof. This follows by induction on the number of vertices and Definition 2.1 □

Proposition 2.5. *Let G be a simple graph. Then the coefficients of $q_N(G;x)$ are non-negative integers.*

Proof. The proof is by induction on the number of vertices of G. If G has one vertex and no edges, then $q_N(G;x) = x$. Now assume the hypothesis holds for all simple graphs on $n-1$ vertices, and let G be a simple graph with n vertices. If G has no edges, $q_N(G) = x^n$. Otherwise, let uv be an edge of G. Thus, $q_N(G;x) = q_N(G-u;x) + q_n(G^{uv}-v;x)$. By induction, the coefficients of $q_N(G-u;x)$ and $q_N(G^{uv}-v;x)$ are non-negative integers, and thus the coefficients of $q_N(G;x)$ are sums of non-negative integers. □

Proposition 2.6. *For any graph G with more than one vertex, the coefficients of x^1 and y^1 in the two-variable interlace polynomial are negatives of one another.*

Proof. We collect terms and write $q(G;x,y) = \sum_{i,j \geqslant 0} a_{ij} x^i y^j$ for the two-variable interlace polynomial. Thus, we want to prove that $a_{10} = -a_{01}$. That is, that

$$\frac{\partial q}{\partial x}\Big|_{x=y=0} = a_{10} = -a_{01} = -\frac{\partial q}{\partial y}\Big|_{x=y=0}.$$

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Note that

$$\left.\frac{\partial q}{\partial x}\right|_{x=y=0} = \sum_{S\subseteq V(G)} (-1)^{n(G[S])+r(G[S])-1} r(G[S]), \quad \text{and}$$

$$\left.\frac{\partial}{\partial y}\right|_{x=y=0} = \sum_{S\subseteq V(G)} (-1)^{n(G[S])+r(G[S])-1} n(G[S]).$$

Thus

$$\begin{aligned} a_{10} + a_{01} &= \sum_{S\subseteq V(G)} (-1)^{n(G[S])+r(G[S])-1}(r(G[S]) + n(G[S])) \\ &= -\sum_{S\subseteq V(G)} (-1)^{|S|}|S| \\ &= -\sum_{l=0}^{|V(G)|} (-1)^l \binom{|V(G)|}{l} l \\ &= (-1)^{|V(G)|+1} \frac{d}{dt}(t-1)^{|V(G)|}\big|_{t=1} = 0. \end{aligned}$$

Therefore $a_{10} + a_{01} = 0$, so $a_{10} = -a_{01}$. □

Proposition 2.7. *Let G be a simple graph with more than one vertex. Writing $q_N(G;x) = \sum a_i x^i$ and $q(G;x,y) = \sum a_{ij}x^i y^j$, then $a_1 = a_{01} = -a_{10}$.*

Proof. Direct computation shows that the proposition holds when G has 2 vertices, so we now assume G has at least 3 vertices and proceed by induction on the number of edges of G. The proposition clearly holds if G has no edges, so now suppose it holds when G has $n-1$ edges, and now let G be a graph with n edges. We write $a_{01}(G)$ for the coefficient of y^1 in $q(G;x,y)$, and $a_1(G)$ for the coefficient of x^1 in $q_n(G;x)$.

Observe that since G has at least 3 vertices, then for any edge ab, there is no constant term in $q(G^{ab} - a - b)$. We now apply equation (2.2) and Definition 2.1 to an edge ab of G, together with the induction to see that:

$$a_{01}(G) = a_{01}(G-a) + a_{01}(G^{ab}-b) = a_1(G-a) + a_1(G^{ab}-b) = a_1(G). \qquad \square$$

This common value of $a_1 = a_{01} = a_{10}$ we will call γ and discuss further in Section 4, showing that it characterizes a well-known class of graphs. In the case of the Tutte polynomial, the coefficients of x^1 and y^1 are equal for graphs with 2 or more edges, rather than negatives of one another, and their common value is the β-invariant also discussed further in Section 4. Although we do not pursue it here, we would not be surprised by further relations among the coefficients of q analogous to those for the Tutte polynomial found by Brylawski in [29]. For example, since by equation (2.4), and writing $q(G;x,y) = \sum a_{ij}x^i y^j$ so that $a_1 = \sum_{i=0} a_{i1}2^i$, it follows from Proposition 2.7 that if G is a simple graph, then $\sum_{i=1} a_{i1}2^i = 0$.

We note that analogous forms of Propositions 2.4 and 2.7 do not hold for the *rank polynomial*, defined as $q_R(G;x) = q(G;x,2)$ in [5], as can be seen from computing

$$q(K_3;x,y) = 2x^2 + yx^2 - 4x - 2yx + 4y \quad \text{and} \quad q_R(K_3;x,y) = 4x^2 - 8x + 8.$$

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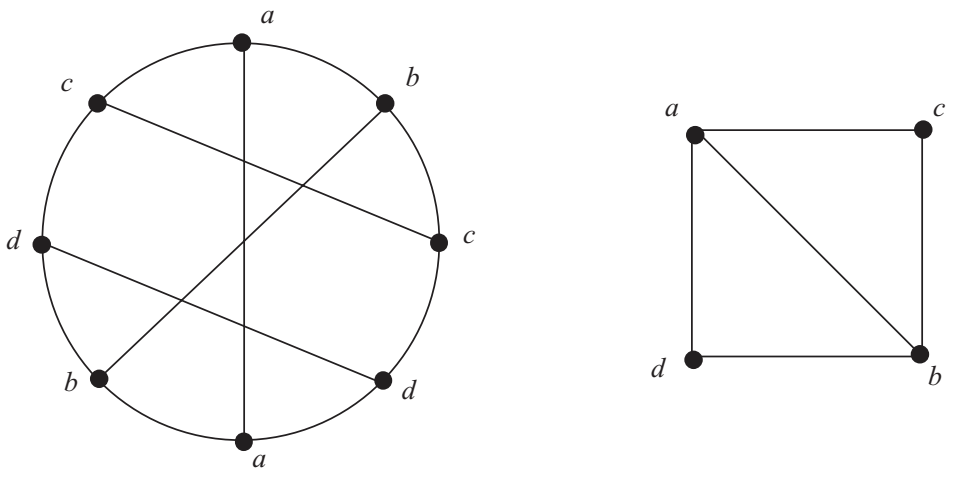


Figure 2. A chord diagram (left) and its associated circle graph (right).

In [3] and [5], Arratia, Bollobás and Sorkin give an interpretation of the vertex-nullity interlace polynomial of a circle graph in terms of the circuit partition, or Martin, polynomial of a related 4-regular Eulerian digraph. We briefly review these and related concepts in order to state Theorem 2.10, which is an essential tool for relating the interlace and Tutte polynomials in Section 3.

Recall that a *circle graph* on n vertices is a graph G derived from a chord diagram, where two copies of each of the symbols 1 to n are arranged on the perimeter of a circle, and a chord is drawn between like symbols. Two vertices v and w in G share an edge if and only if their corresponding chords intersect in the chord diagram. Note that G is necessarily simple. See Figure 2. Circle graphs have also been called alternance graphs by Bouchet [19] and interlace graphs by Arratia, Bollobás and Sorkin [3]. Research on circle graphs includes a complete characterization and a polynomial time algorithm for identifying them. For example, see Bouchet [15, 17, 18, 23], Czemerinski, Durán and Gravano [35], Durán [37], Fraysseix [42], Gasse [43], Read and Rosenstiehl [60, 61], and Wessel and Pöschel [70].

A 4-*regular Eulerian* digraph is a 4-regular directed graph such that, at each vertex, two edges are oriented inward, and two are oriented outward. It is not necessarily connected, and may be a multigraph, that is, have loops or multiple edges. A 4-regular Eulerian digraph is called a 2-in, 2-out graph in [3]. Note that if C is an Eulerian circuit of a 4-regular Eulerian digraph, and we write the vertices along the perimeter of a circle in the order that they are visited by C (each is visited exactly 2 times), and then draw a chord between like vertices, the result is a chord diagram.

Definition 2.8. A *graph state* of a 4-regular Eulerian digraph $\vec{G}$ is the result of replacing each 4-valent vertex v of $\vec{G}$ with two 2-valent vertices each joining an incoming and an outgoing edge originally adjacent to v. Thus a graph state is a disjoint union of consistently oriented cycles. See Figure 3.

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A 4-regular Eulerian digraph $\vec{G}$ A graph state of $\vec{G}$ with 2 components

Figure 3. A graph state.

Note that graph states (see [38]) are equivalent to the circuit partitions of Arratia, Bollobás and Sorkin [3] and Bollobás [12], the Eulerian decompositions of Bouchet [19], and the Eulerian k-partitions of Martin [52] and Las Vergnas [49].

Definition 2.9. The *circuit partition polynomial* of a 4-regular Eulerian digraph $\bar{G}$ is $f(\vec{G};x) = \sum_{k\geqslant 0} f_k(\vec{G})x^k$, where $f_k(\vec{G})$ is the number of graph states of $\bar{G}$ with k components, defining $f_0(\vec{G})$ to be 1 if $\vec{G}$ has no edges, and 0 otherwise.

The circuit partition polynomial is a simple translation of the Martin polynomial $m(\vec{G};x)$, defined recursively for 4-regular digraphs by Martin in his 1977 thesis [52], with $f(\vec{G};x) = xm(\vec{G};x+1)$.

Las Vergnas found closed forms for the Martin polynomials (for both graphs and digraphs). He also extended their properties to general Eulerian digraphs and further developed their theory (see [47, 50, 49]). The transforms of the Martin polynomials, for arbitrary Eulerian graphs and digraphs, were given in [38], and then aptly named circuit partition polynomials by Bollobás in [12], with splitting identities provided in [12] and [40]. The circuit partition polynomial is also a specialization of a much broader multivariable polynomial, the generalized transition polynomial of [41], which assimilates such graph invariants as the Penrose polynomial that are not evaluations of the Tutte polynomial.

For circle graphs, the vertex-nullity interlace polynomial and the circuit partition polynomial are related by the following theorem, noting that although $\vec{G}$ may be a multigraph, H is necessarily simple.

Theorem 2.10 (Arratia, Bollobás and Sorkin [3], Theorem 6.1). *If $\vec{G}$ is a 4-regular Eulerian digraph, C is any Eulerian circuit of $\vec{G}$, and H is the circle graph of the chord diagram determined by C, then $f(\vec{G};x) = xq_N(H;x+1)$.*

3. Relation to the classical and topological Tutte polynomials and computational complexity

Theorem 2.10, combined with a relationship between the Martin and Tutte polynomials for planar graphs, relates the vertex-nullity interlace polynomial to the Tutte polynomial and consequently resolves the computational complexity question raised by Arratia, Bollobás and Sorkin in [3]

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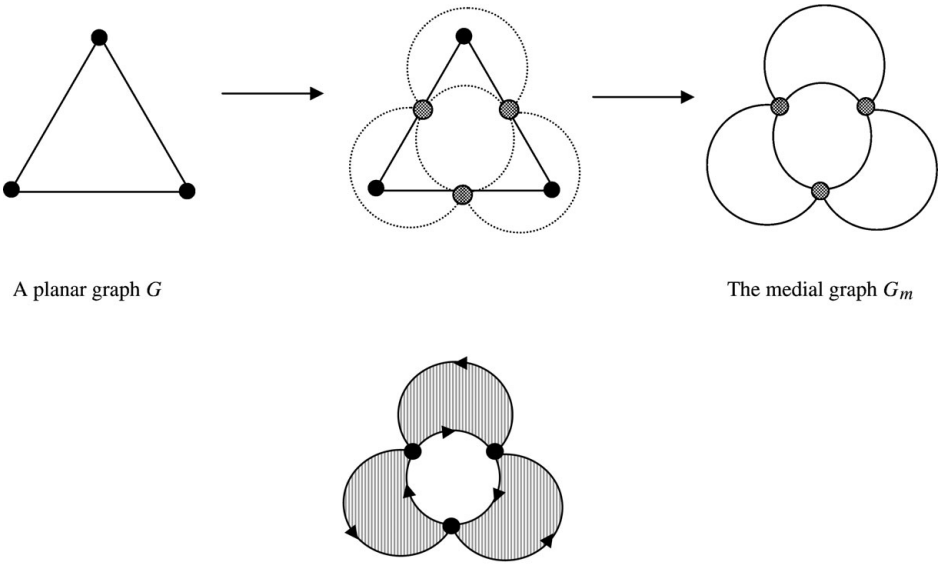


Figure 4. The oriented medial graph.

and [5]. Arratia, Bollobás and Sorkin prove in [5] that the two-variable interlace and the one-variable vertex-rank polynomials are $\#P$-hard to compute; here we show that the vertex-nullity polynomial is also $\#P$-hard (Corollary 3.2).

Let G be a graph embedded in the plane. Its medial graph, G_m, has vertices corresponding to the edges of G. Two vertices of G_m are joined by an edge if the corresponding edges of G are incident to the same vertex v of G and are sequential in the cyclic order around v.

Let G_m be the medial graph of a plane graph G. Colour the faces of the medial graph black or white, depending on whether they contain or do not contain, respectively, a vertex of the original graph G. This face-2-colours the medial graph. The edges of the medial graph are then directed so that the black face is on the left of an incident edge. See Figure 4. Denote this oriented medial graph by $\vec{G}_m$. Both G and $\vec{G}_m$ may be multigraphs.

The *Tutte polynomial* of a multigraph, $t(G; x, y)$, may be defined by the linear recursion relation $t(G; x, y) = t(G - e; x, y) + t(G/e; x, y)$ if G has an edge e that is neither an isthmus (*i.e.*, cut-edge or bridge) nor a loop of G, and by $t(G; x, y) = x^i y^j$ if G consists of i isthmuses and j loops. Here G/e is the contraction of e in G. See Brylawski [29] or Brylawski and Oxley [30], for example, for an in-depth treatment of the Tutte polynomial, including generalizations to matroids.

Martin [52, 53] found the relationship $m(\vec{G}_m; x) = t(G; x, x)$, which was further explored by Las Vergnas in [47, 50]. This now allows us to relate the vertex-nullity interlace polynomial to the Tutte polynomial, a relation also observed by Arratia, Bollobás and Sorkin at the end of Section 7 in [4].

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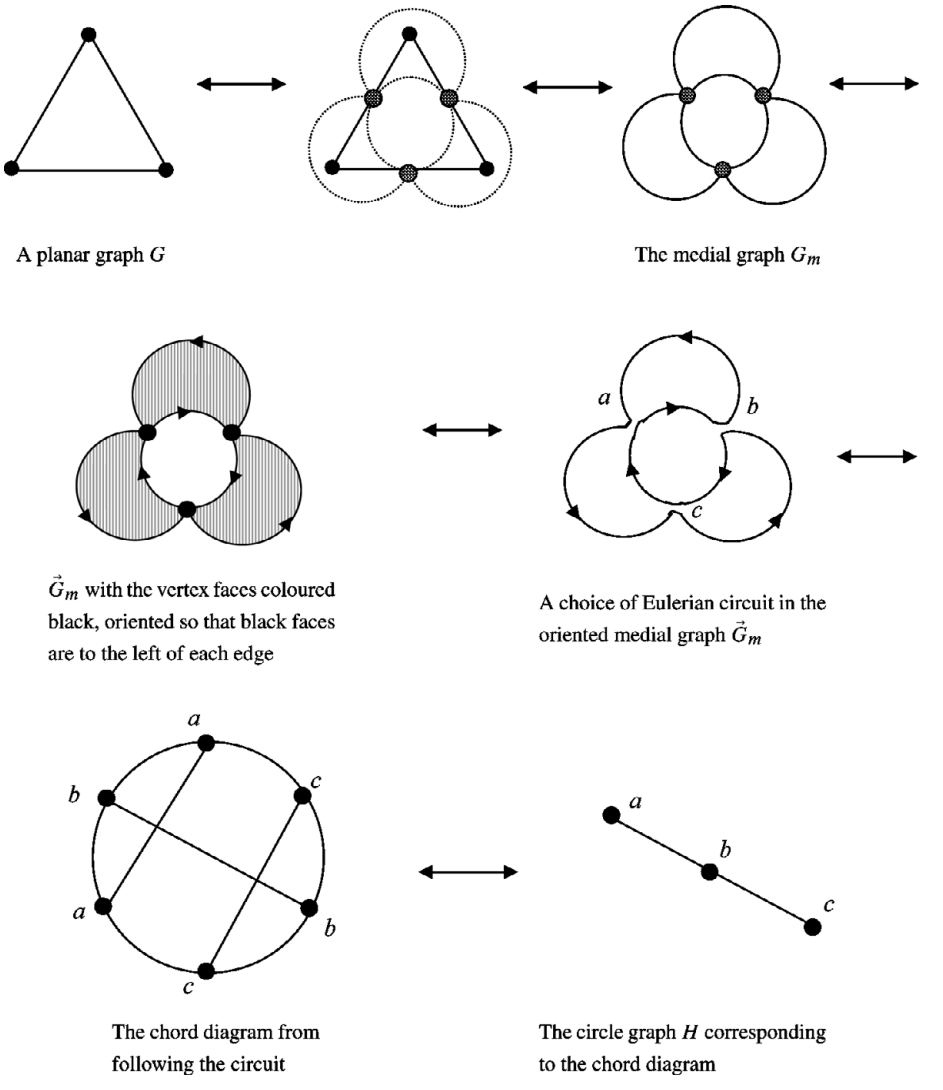


Figure 5. Transformations between a planar graph and a circle graph.

Theorem 3.1. *If G is a planar graph, and H is the circle graph of some Eulerian circuit of $\vec{G}_m$, then $q_N(H;x) = t(G;x,x)$.*

Proof. By Theorem 2.10, $f(\vec{G}_m;x) = xq_N(H;x+1)$, but recalling that the circuit partition and Martin polynomials are simple translations of each other, we have that $f(\vec{G}_m;x) = xm(\vec{G}_m;x+1)$, and hence $q_N(H;x) = m(\vec{G}_m;x) = t(G;x,x)$. See Figure 5. □

Since the Tutte polynomial is known to be #P-hard for planar graphs except at few isolated points and along the curves $(x-1)(y-1) = 1$ and $(x-1)(y-1) = 2$ (see Jaeger, Vertigan and Welsh [46] and Welsh [69]), we have the following immediate corollary.

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Corollary 3.2. *The vertex-nullity interlace polynomial is #P-hard in general.*

Of course, this leads immediately to the question of whether there might be classes of graphs for which the vertex-nullity interlace polynomial might be more tractable. We provide one such class in Sections 4 and 5.

The remainder of this section addresses the natural question of extending Theorem 3.1. The extension may indeed be applied elsewhere, and interestingly, to the topological Tutte polynomial of Bollobás and Riordan [13], where the classical Tutte polynomial is generalized to encode topological information about graphs embedded on orientable surfaces.

The topological Tutte polynomial of Bollobás and Riordan for cyclic graphs was defined in [13], and generalized for non-orientable ribbon graphs by Bollobás and Riordan in [14]. A *cyclic graph* $\mathbf{G}$ is a graph with a *rotation system*, that is, a family of local rotations around each vertex of an abstract graph G. A *local rotation* around a vertex v is a cyclic order of the edges incident with v, and a rotation system determines a topological embedding of a graph G.

Contraction and deletion for cyclic graphs are described by Bollobás and Riordan in [13]. If e is an edge of a cyclic graph $\mathbf{G}$, then $\mathbf{G} - e$ is the cyclic graph obtained by deleting the edge e from the underlying graph and from whichever local rotations in which it occurs. The contraction $\mathbf{G}/e$ of a non-loop edge $e = uv$ has G/e as its underlying graph. Let w be the vertex of G/e obtained by identifying u and v. The local rotation at w is obtained by concatenating the rotation systems of u and v at e. That is, we order the edges after e using the local rotation at u until we get to e again. From there, we add the edges after e using the order from the local rotation around v.

A cyclic graph $\mathbf{G}$ with a single vertex v is given by the cyclic order of the half-edges around v. Therefore we can identify $\mathbf{G}$ with the chord diagram D that has labels corresponding to the edges of $\mathbf{G}$ around the boundary in exactly the same order given by the cyclic permutation of the edges around v. Cyclic graphs correspond to graphs embedded in oriented surfaces (see Bollobás and Riordan [13] and the references therein).

The $\mathbb{F}_2$-rank of a chord diagram D was defined by Bollobás and Riordan in [13] as $r(D) = \frac{1}{2}r(M(H))$, where $M(H)$ is the adjacency matrix of the circle graph H of D, and $n(D)$ is the number of chords.

Definition 3.3 ([13]). Let $\mathbf{G}$ be a cyclic graph. The polynomial $C(\mathbf{G}; X, Y, Z)$ is defined by

$$C(\mathbf{G}; X, Y, Z) = C(\mathbf{G}/e; X, Y, Z) + C(\mathbf{G} - e; X, Y, Z)$$

if e is neither a bridge nor a loop of $\mathbf{G}$, and $C(\mathbf{G}; X, Y, Z) = XC(\mathbf{G}/e; X, Y, Z)$ if e is a bridge in $\mathbf{G}$.

Note that if $\mathbf{G}$ is a cyclic graph with one vertex and D is the chord diagram corresponding to the cyclic order of the loop half-edges about the single vertex of $\mathbf{G}$, then $\mathbf{G}$ and D determine each other up to isomorphism. With this we write $C(D; X, Y, Z)$ for $C(\mathbf{G}; X, Y, Z)$ when $\mathbf{G}$ is a one-vertex cyclic graph.

For a one-vertex cyclic graph $\mathbf{G}$ we have $C(\mathbf{G}; X, Y, Z) = \sum_{D' \subseteq D} Y^{n(D')} Z^{r(D')}$, where D is the chord diagram corresponding to $\mathbf{G}$.

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Theorem 3.4. *Let H be the circle graph of a chord diagram D. Then*

$$q(H; YZ^{\frac{1}{2}} + 1, Y + 1) = C(D; X, Y, Z),$$

and

$$q(H; x, y) = C\left(D; x, y-1, \left(\frac{x-1}{y-1}\right)^2\right).$$

Proof. The vertices of H correspond to the chords of D. A set $S \subseteq V(H)$ corresponds to a subdiagram D_S of D. The induced subgraph $H[S]$ is the circle graph of D_S.

It follows that $r(D_S) = \frac{1}{2}r(H[S])$. Thus

$$\begin{aligned} C(D; X, Y, Z) &= \sum_{D' \subseteq D} Y^{n(D')} Z^{r(D')} \\ &= \sum_{S \subseteq V(H)} Y^{|S|} Z^{\frac{1}{2}r(H[S])} \\ &= \sum_{S \subseteq V(H)} (YZ^{\frac{1}{2}})^{r(H[S])} Y^{n(H[S])} \\ &= q(H; YZ^{\frac{1}{2}} + 1, Y + 1). \end{aligned}$$

Therefore,

$$q(H; YZ^{\frac{1}{2}} + 1, Y + 1) = C(D; X, Y, Z), \quad \text{and}$$
$$q(H; x, y) = C\left(D; x, y-1, \left(\frac{x-1}{y-1}\right)^2\right). \qquad \square$$

By Theorem 3.4 we obtain a reduction formula for the C-polynomial of a chord diagram.

Definition 3.5. Let D be a chord diagram and let a, b be two intersecting chords. Let H be the circle graph of D. Then we write D^{ab} for the chord diagram whose circle graph is H^{ab}.

Corollary 3.6. *Let D be a chord diagram and let a, b be two intersecting chords. Then*

$$\begin{aligned} C(D; X, Y, Z) = {} & C(D - a; X, Y, Z) + C(D^{ab} - a; X, Y, Z) \\ & + (Y^2 Z - 1) C(D^{ab} - a - b; X, Y, Z). \end{aligned}$$

An oriented medial graph $\vec{G}_m$ may be defined for a cyclic graph G as in the plane case. Loops homotopic to zero or bridges of $\mathbf{G}$ are cut vertices of $\vec{\mathbf{G}}_m$. Thus they correspond to isolated chords in the chord diagram D of any Eulerian circuit of $\vec{\mathbf{G}}_m$ and to isolated chords in H, the circle graph of D. However, by Theorem 3.4, $q(H; x, y) = C(D; x, y-1, (\frac{x-1}{y-1})^2)$. But if e is a loop homotopic to zero or a bridge, $D = D_1 + D_2$, where D_1 is the isolated chord corresponding to e. By Corollary 2 in Bollobás and Riordan [13], we have that $C(D) = C(D_1)C(D_2)$. Thus $q(H; x, y) = C(D_1; x, y-1, (\frac{x-1}{y-1})^2)C(D_2; x, y-1, (\frac{x-1}{y-1})^2) = yq(H[V - e]; x, y)$, and this is independent of whether e is a loop homotopic to zero or a bridge. Therefore, if e is a loop homotopic to zero or a bridge, $q(H; x, y) = yq(H[V - e]; x, y)$. It follows that the $q(H; x, y)$ does not distinguish loops

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homotopic to zero and bridges of **G**. Thus, in the general case, $q(H; x, y)$ is not an evaluation of Bollobás and Riordan's topological Tutte polynomial of **G**.

4. Distance hereditary graphs and the γ invariant

We turn our attention now to the question raised in Section 3 about classes of graphs for which the vertex-nullity interlace polynomial is polynomial time computable. Series-parallel graphs and their characterization by the β invariant lead to the considerations of this section.

Definition 4.1. A *series-parallel* graph is a graph constructed from a digon (two vertices joined by two edges in parallel) by repeatedly adding an edge in parallel to an existing edge, or adding an edge in series with an existing edge by subdividing the edge. Series-parallel graphs are in general loopless multigraphs, and are planar.

For a graph G with 2 or more edges, the coefficients of x^1 and y^1 in the Tutte polynomial are equal, and this common value, $\beta(G)$, introduced by Crapo in [34], is called the β invariant of G. Brylawski [28] characterized series-parallel graphs (in the more general context of matroids) by the property that G is a series-parallel graph if and only if $\beta(G) = 1$. The β invariant has been explored further, for example by Oxley in [58] and by Benashki, Martin, Moore and Traldi in [10]. In analogy to the β invariant, we define the γ *invariant* as the coefficient of x^1 in $q_N(G; x)$ which by Proposition 2.7 is the common value of a_{01} and $|a_{10}|$ when G is a simple graph with more than one vertex. This section examines distance hereditary graphs (*DH graphs*) and characterizes them in terms of the γ invariant in that $\gamma(G) = 2^n$ if G is a DH graph, and $\gamma = 2$ if and only if G is a bipartite DH graph (*BDH graph*). *En route*, we present several general reduction formulas for $\gamma(G)$, $q_N(G; x)$ and $q(G; x, y)$.

We show in Proposition 5.1 that H is a BDH graph if and only if it is a circle graph corresponding to an Eulerian cycle in the medial graph of some series-parallel graph, and this suffices to show $\gamma(G) = 2$ if G is a BDH graph, but not *vice versa*. In Section 5 we show that q_N is polynomial time to compute for BDH graphs.

Definition 4.2. For G a simple graph and writing $q_N(G; x) = \sum a_i x^i$, we call a_1 the γ *invariant* of G and denote it $\gamma(G)$.

We recall from Proposition 5.3 in Arratia, Bollobás and Sorkin [3] that q_N is multiplicative on disjoint unions, and then note some very simple properties of γ that will be used repeatedly throughout this section.

Property 4.3. *If vw is an edge of G, then $\gamma(G) = \gamma(G - v) + \gamma(G^{vw} - w)$.*

Property 4.4. *If vw is an edge of G, then $\gamma(G) = \gamma(G^{vw})$.*

Proof. The property is immediate from $q_N(G; x) = q_N(G^{uv}; x)$, Proposition 8.1 in Arratia, Bollobás and Sorkin [3]. □

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Property 4.5. *$\gamma(G) = 0$ if and only if G has more than one component.*

Proof. The property is immediate from Remark 20 of Arratia, Bollobás and Sorkin [4], which states that the degree of the lowest-degree term of $q_N(G)$ is the number of components of G. □

Property 4.6. *If G is a connected graph, then $\gamma(G - v) = 0$ if and only if v is a cut vertex of G or G consists of a single vertex.*

Property 4.7. *If G is a connected graph, then $\gamma(G) = 1$ if and only if G is an isolated vertex.*

Proof. That $\gamma(G) \neq 1$ if $|V(G)| \geqslant 2$ follows from Property 4.3 and induction on $|E(G)|$, since both $\gamma(G - v)$ and $\gamma(G^{uv} - u)$ are either 0 or greater than 1, unless $G = K_2$, in which case both are 1. If G is an isolated vertex, then $\gamma(G) = 1$. □

We also note that we immediately derive a new interpretation for the β invariant of a planar graph.

Proposition 4.8. *Let G be a planar graph and let H be the circle graph of some Eulerian circuit of $\vec{G}_m$. Then*

$$\beta(G) = -\frac{1}{2} \sum_{S \subseteq V(G)} n(G[S])(-1)^{n(G[S])} = \frac{1}{2} a_1 = \sum_i a_{i,1} 2^{i-1}.$$

Proof. By Theorem 3.1, $q_N(H; x) = t(G; x, x)$, so $\gamma(H) = 2\beta(G)$. But $\gamma(H) = q'_N(H; 0)$, so the result follows from the expression for q_N in equation (2.4) and the observation after Proposition 2.6. □

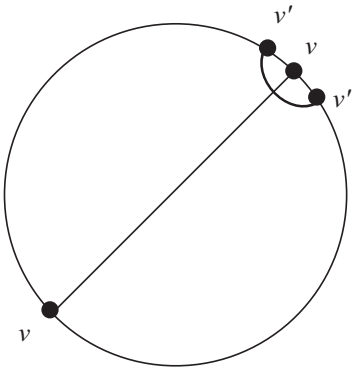
We now review distance hereditary graphs. This important class of graphs, introduced by E. Howorka, has a number of characterizations (see [44, 45]), the one responsible for the name being that any connected induced subgraph of a distance hereditary graph G inherits its distance function from G. See Brandstädt, Le and Spinrad [27] and McKee and McMorris [54] for detailed overviews of this and related classes of graphs.

However, for our purposes, we will use the formulation of Definition 4.9, due to Bandelt and Mulder in [9], although we do not allow infinite graphs. Twin vertices (see Burlet and Uhry's usage in [31]) are 'split pairs' in [9]; we use the terminology *twin* because of the emphasis on the differing roles of true versus false twins with respect to the interlace polynomial. Also, restricting twins to non-isolated vertices and starting with K_1, as we have in Definition 4.9, is equivalent to starting with K_2 as in [9], both formulations serving to ensure that the resulting graph is connected. In fact, our restriction forces K_2 to always be the next step after K_1.

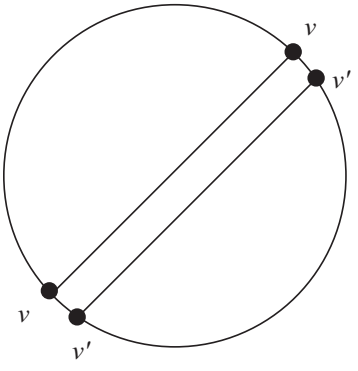
Definition 4.9. A *distance hereditary* graph (DH graph) is a graph that can be constructed from a single vertex by a finite number of applications of the following operations.

1 Adding a pendant vertex to a vertex v, *i.e.*, adding a new vertex v' and an edge vv'.
2 Adding a twin vertex of a vertex v.

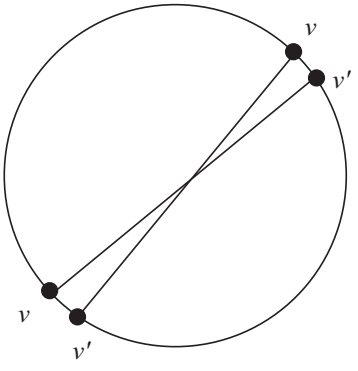
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Adding a pendant vertex to v



Adding a false twin to v



Adding a true twin to v

Figure 6. Effect of adding a pendant or twin vertex.

(a) True twin: adding a new vertex v' adjacent to v and edge uv' if and only if uv is an edge.
(b) False twin: adding a new vertex v', and edge uv' if and only if uv is an edge, but not the edge vv'.

Note that a DH graph is a circle graph with operation 1 corresponding to adding a small chord perpendicular to v in the chord diagram, operation 2(a) corresponding to adding a very close parallel chord, and 2(b) to adding a very close crossing chord as in Figure 6. DH graphs are simple.

Proposition 4.10 (Bandelt and Mulder [9], Corollary 3). *A DH graph is bipartite if and only if no true twins are added in its construction.*

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We abbreviate bipartite distance hereditary graphs as BDH graphs.

The following proposition highlights a duality among pendant and twin vertices with respect to the pivot operation.

Proposition 4.11. *If uv is an edge of G, then w is pendant on u in G if and only if w is a false twin of v in G^{uv}, and u, v are true twins in G if and only if u, v are true twins in G^{uv}.*

Proof. This follows because, if w is pendant on u, then $w \in A_u$ but is adjacent to no vertices in A_{uv} or A_v (A_u, A_w, A_{uw} as in the first paragraph of Section 2). Thus, in G^{uv}, the vertex w has edges joining it to all of the neighbours of v, and no others. Similarly, if w is a false twin of v in G^{uv}, it loses all its edges except the one joining it to u in $(G^{uv})^{uv} = G$. For true twins u, v, the sets A_u and A_v are empty, so no toggling occurs and in fact $G = G^{uv}$. □

Proposition 4.12. *If G' is the graph that results from adding a pendant vertex w to a vertex u of a loopless graph G, then $q(G'; x, y) = q(G; x, y) + (x^2 - 2x + y)q(G - u; x, y)$.*

Proof. We pivot on edge uw of G', noting that $G' - w = G$, and $G'^{uw} = G'$ since w pendant on u implies that A_w and A_{uw} are empty, and $G' - u = G - u \cup w$, where w is an isolated vertex. By (2.2), since G, and hence G', is loopless,

$$\begin{aligned} q(G') &= q(G' - w) + q(G'^{uw} - u) + (x^2 - 2x)q(G'^{uw} - u - w) \\ &= q(G) + q(G' - u) + (x^2 - 2x)q(G' - u - w) \\ &= q(G) + q(G - u \cup w) + (x^2 - 2x)q(G - u) \\ &= q(G) + (x^2 - 2x + y)q(G - u). \end{aligned}$$

□

Pivoting on an edge uv does not change the character of pendant and twin vertices not involving u or v.

Proposition 4.13. *A vertex w is pendant on $z \notin \{u, v\}$ in G if and only if w is pendant on z in G^{uv}. Similarly, w is a true or false twin of $z \notin \{u, v\}$ in G if and only if w is a true or false twin of z in G^{uv}.*

Proof. If w is pendant on $z \notin \{u, v\}$, then $w \notin A_v \cup A_u \cup A_{vw}$, so the adjacencies of w are not affected by the pivot. If w is a true or false twin of $z \notin \{u, v\}$, then w and z are in the same part of the partition A_u, A_v, A_{uv}, $V(G) - (A_u \cup A_v \cup A_{uv})$ of the vertices of G. Furthermore, since the neighbours of w and z are the same, the neighbours of w are distributed within this partition precisely as are the neighbours of z. Thus, the pivot operation does not change the adjacency between w and z. The pivot operation toggles an adjacency between w and a vertex x if and only if it toggles the adjacency between z and x, so w is a twin of z in G if and only if it is a twin of z in G^{uv}. □

Proposition 4.14. *If G'' is the graph that results from adding a false twin w to a vertex v of a loopless graph G, when v is not isolated in G, then*

$$q(G''; x, y) = q(G; x, y) + y(q(G; x, y) - q(G - v; x, y)).$$

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Proof. We pivot on edge uv of G'', noting that since v and w are false twins then G''^{uv} is G^{uv} with vertex w pendant on u. This follows because, except for w, A_u, A_v and A_{uv} are exactly the same in both G'' and G, but w, adjacent precisely to everything in $A_v \cup A_{uv}$, loses all edges except wu in the pivot. Also note that since v and w are duplicates $G'' - v = G'' - w = G$. Now by (2.2), we have

$$\begin{aligned} q(G'') &= q(G'' - v) + q(G''^{uv} - u) + (x^2 - 2x)q(G''^{uv} - u - v) \\ &= q(G) + q(G^{uv} - u \cup w) + (x^2 - 2x)q(G^{uv} - u - v \cup w) \\ &= q(G) + yq(G^{uv} - u) + y(x^2 - 2x)q(G^{uv} - u - v) \\ &= q(G) + y(q(G) - q(G - v)). \end{aligned}$$

□

Proposition 4.15. *If G''' is the graph that results from adding a true twin w to v of a loopless graph G, when v is not isolated in G, then*

$$q(G'''; x, y) = 2q(G; x, y) + ((x-1)^2 - 1)q(G - v; x, y).$$

Proof. Recall that if v and w are true twins, then $G'''^{vw} = G'''$ and hence $G'''^{vw} - w = G''' - w = G$, so the result follows from equation (2.2). □

The duality among pendant and twin vertices is particularly apparent in the context of the vertex-nullity interlace polynomial.

Corollary 4.16. *The vertex-nullity interlace polynomial has the following duality identities for a simple graph G.*

(1) *If G' is the graph that results from adding a pendant vertex w to a vertex u of G, then*

$$q_N(G'; x) = q_N(G; x) + xq_N(G - u; x).$$

(2) *If G'' is the graph that results from adding a false twin w to a non-isolated vertex v with u adjacent to v, then*

$$q_N(G''; x) = q_N(G; x) + xq_N(G^{uv} - u; x).$$

(3) *If G''' is the graph that results from adding a true twin w to a non-isolated vertex v, then*

$$q_N(G'''; x) = 2q_N(G; x).$$

Proof. Parts (1), (2), (3) follow from equation (2.4) and Propositions 4.12, 4.14 and 4.15, respectively. Part (2) is also a restating of Arratia, Bollobás and Sorkin [4, Proposition 40], using an expression just before the final form given there. □

Corollary 4.17. *γ is invariant under pendant or false twins, and doubles for true twins, as follows.*

(1) *If G' is the graph that results from adding a pendant vertex to G with $|V(G)| \geqslant 2$, then $\gamma(G') = \gamma(G)$, i.e., γ is invariant under the addition of pendant vertices.*

(2) *If G'' is the graph that results from adding a false twin to a non-isolated vertex of G, then $\gamma(G'') = \gamma(G)$, so γ is invariant under the addition of false twin vertices.*

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(3) *If G''' is the graph that results from adding a true twin to a non-isolated vertex of G, then $\gamma(G''') = 2\gamma(G)$, so γ is doubled by the addition of true twin vertices.*

Definition 4.18. Let G and F be graphs with $u \in V(G)$, $v \in V(F)$. Then the one-point join of G and F, denoted by $G_{u\cdot v} F$, is formed by identifying u and v, resulting in a cut vertex of $G_{u\cdot v} F$.

If H is a graph with a cut vertex v, then H is the one-point joint of G and F where $G = v \cup G'$, for G' a component of $H - v$ containing at least one neighbour of v in H, and $F = v \cup F'$, for F' the complement of G' in $H - v$.

Note that adding a pendant edge to G at v is equivalent to taking the one-point join of K_2 to G at v. Also note that if v is an isolated vertex of F, then $G_u \cdot_v F$ is just the disjoint union of G and $F - v$.

Proposition 4.19. *If H is the one-point join $G_u \cdot_v F$ where neither u nor v are isolated vertices, then $2\gamma(H) = \gamma(G)\gamma(F)$.*

Proof. H has more than one component if and only if at least one of G or F has, in which case both sides of the equation are zero. Thus, we may assume H is connected, and we proceed by induction on the number of vertices of G. Since u is not isolated and G is connected, the base case is $G = K_2$ (*i.e.*, $G_u \cdot_v F$ adds a pendant vertex to F), and the result follows from Corollary 4.17 and that $\gamma(K_2) = 2$.

Now suppose G has n vertices. If every edge of H is incident with $u = v$, then $H = K_{1,r}$, with $G = K_{1,n}$, and $F = K_{1,s}$ where $r = n + s$. Since, by Arratia, Bollobás and Sorkin [3, Proposition 7.1], $\gamma(K_{1,m}) = 2$ for $m \geqslant 1$, the result follows.

Otherwise, without loss of generality, there is an edge ab in G with $u \notin \{a, b\}$. Note that toggling in H with respect to ab occurs only among the edges of G not F, so $H - a = (G - a)_u \cdot_v F$ and $H^{ab} - b = (G^{ab} - b)_u \cdot_v F$. Thus, $\gamma(H) = \gamma((G - a)_u \cdot_v F) + \gamma((G^{ab} - b)_u \cdot_v F)$. By induction, this is

$$\frac{1}{2}\gamma(G - a)\gamma(F) + \frac{1}{2}\gamma(G^{ab} - b)\gamma(F) = \frac{1}{2}(\gamma(G - a) + \gamma(G^{ab} - b))\gamma(F) = \frac{1}{2}\gamma(G)\gamma(F).$$

Therefore, $2\gamma(H) = \gamma(G)\gamma(F)$. □

Definition 4.20. Let G and F be graphs with $u \in V(G)$ and $v \in V(F)$. Then the two-point join of G and F, denoted $G_u :_v F$, is formed by adding an edge au whenever $av \in E(F)$ and an edge bv whenever $bu \in E(G)$.

Note that the vertices u and v are false twins in $G_u :_v F$. Also adding a false twin v of a vertex u of G is equivalent to taking the two-point join $G_u :_v F$ where F consists of just the single vertex v.

Proposition 4.21. *If H is the two-point join $G_u :_v F$, then $2\gamma(H) = \gamma(G)\gamma(F)$.*

Proof. Since u and v are duplicate vertices in H, by Corollary 4.17, $\gamma(H) = \gamma(H - u)$ but $H - u = G_u \cdot_v F$, so by Proposition 4.19, $\gamma(H - u) = \frac{1}{2}\gamma(G)\gamma(F)$, so $2\gamma(H) = \gamma(G)\gamma(F)$. □

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For the following, we need to recall alternative characterizations of BDH graphs, with parts (i)–(iii) due to Bandelt and Mulder [9], and for the equivalence of (iii) and (iv). See also Ausiello, D'Atri and Moscarini [6]. A $(6,2)$-chordal graph is a graph such that every cycle of length at least 6 has at least 2 chords. We write $\bar{C}_6$ for a 6-cycle with a single antipodal chord.

Proposition 4.22 (Bandelt and Mulder [9], Corollaries 3 and 4). *The following are equivalent.*

(i) *G is a BDH graph.*
(ii) *G is constructed from a single vertex by a sequence of adding pendant vertices and false twins, but no true twins.*
(iii) *G is connected, triangle-free, and contains as an induced subgraph no C_n with $n \geqslant 5$ nor $\bar{C}_6$.*
(iv) *G is a connected bipartite $(6,2)$-chordal graph.*

Proof. The equivalence of definitions (i)–(iii) appears in Bandelt and Mulder [9]. That (iv) implies (iii) comes from noting that bipartite implies triangle-free, and if $H \subseteq G$ is a cycle of length greater than 4, then by bipartness, it has length at least 6, and hence contains at least 2 chords, ruling out both induced cycles and $\bar{C}_6$. On the other hand, if G satisfies (iii), then it is bipartite since (iii) is equivalent to (i). If C_n is a cycle in G with $n > 6$, then it has at least one chord, which creates a C_m with $m > 4$, so it has a chord, and thus C_n has at least two chords. Any subgraph C_6 in G must have a chord, which (since there are no triangles) must be an antipodal chord, and since $\bar{C}_6$ is forbidden, must have 2 subchords. Thus G is bipartite $(6,2)$-chordal. □

Corollary 4.23. *If $H = G_{u \cdot v} F$, then H is a BDH graph if and only if both G and F are BDH graphs.*

Proof. This follows immediately from the characterization of a BDH graph as a bipartite $(6,2)$-chordal graph, since, by definition of one-point join, any cycle of H must be entirely contained in either G or F. □

We are now ready to prove the main theorem of this section, which completes the classification of graphs for which $\gamma = 2$ begun by Aigner and van der Holst [1].

Theorem 4.24. *H is a simple, connected graph with $\gamma(H) = 2$ if and only if H is a BDH graph with at least two vertices.*

Proof. If H is a BDH graph, then by Proposition 4.22 it is simple, connected, and constructed using only pendant vertices and false twins. Thus, $\gamma(H) = 2$ follows from $\gamma(K_2) = 2$ and Corollary 4.17.

If H is a simple connected graph with $\gamma(H) = 2$, we proceed by induction on $|V(H)|$ to show that H is a BDH graph with at least two vertices.

If $|V(H)| \leqslant 1$, then $\gamma(H) \neq 2$. Thus H has at least two vertices. Moreover H is connected by Property 4.5. If $|V(H)| = 2$, and $\gamma(H) = 2$ then, since H is simple and connected, $H = K_2$, a BDH graph. So now assume $|V(H)| = n \geqslant 3$, and pivot on an edge uv of H. Now $2 = \gamma(H) =$

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$\gamma(H-v)+\gamma(H^{uv}-u)$. Since $|V(H)| \geqslant 3$, neither summand can be 1 by Property 4.7, so one must be 0 and the other 2 (neither can be negative by Proposition 2.5).

If $\gamma(H-v)=0$, then, by Property 4.6, v is a cut vertex of H. Since v is a cut vertex, $H-v$ has at least two components, so there are subgraphs H_1 and H_2, each with more than one vertex, such that $H=(H_1)_v\cdot_v(H_2)$, as in Definition 4.18. Hence, by Property 4.7 (with Property 4.5 giving connectedness of both), neither $\gamma(H_1)$ nor $\gamma(H_2)$ is equal to 1. By Proposition 4.19, $2\gamma(H)=\gamma(H_1)\gamma(H_2)$, so $\gamma(H_1)=\gamma(H_2)=2$ and hence by induction H_1 and H_2 are BDH graphs, and by Corollary 4.23 H is a BDH graph.

If $\gamma(H^{uv}-u)=0$, we similarly have that H^{uv} is a BDH graph. Let w be the last vertex added in a construction of H^{uv}, so w is a pendant on, or a false twin of, some vertex in H^{uv}.

There are three cases:

(i) w is equal to u or v,
(ii) w is adjacent to u or v, or possibly both, but is equal to neither u nor v,
(iii) w is not equal to u or v, and is not adjacent to either u or v.

In case (i), suppose $w=v$. Since uv is an edge of H^{uv}, either w is a false twin of some vertex z adjacent to u, or w is pendant on u. If w is a false twin of z, we reverse the roles of w and z in the construction of H^{uv}, so that now w, the last vertex added, is adjacent to u, but equal to neither u nor v. This situation is addressed in case (ii). If $w=v$ is pendant on u, then A_u and A_{uv} are empty, so $H^{uv}=H$. Thus v is a pendant vertex in H. By Corollary 4.17, $2=\gamma(H)=\gamma(H-v)$, so by induction $H-v$ is a BDH graph, and hence so is H.

In case (ii), suppose w is adjacent to u. Then w is pendant on u, is a false twin of v, or is a false twin of some $z\neq v$ with z adjacent to u.

If w is pendant on u in H^{uv}, by Proposition 4.11, w is a false twin of u in H, so by Corollary 4.17, $2=\gamma(H)=\gamma(H-w)$ and H is a BDH graph by induction as above.

Similarly, if w is a false twin of v in H^{uv}, then by Proposition 4.11, w is pendant on v in H, so by Corollary 4.17, $2=\gamma(H)=\gamma(H-w)$ and H is a BDH graph.

If w is a false twin in H^{uv} of some $z\neq v$, with z adjacent to u, then by Proposition 4.13, w is a false twin of z in H, so again $2=\gamma(H)=\gamma(H-w)$ and H is a BDH graph.

In case (iii), w cannot be pendant on, or a false twin of, either u or v. Thus, w is pendant on or a false twin of some vertex $z\notin\{u,v\}$ in H^{uv}. By Proposition 4.13, w is also pendant on, or a false twin of, z in H. Thus, again $2=\gamma(H)=\gamma(H-w)$, and H is a BDH graph. □

For DH graphs that are not necessarily bipartite, we have the following property.

Theorem 4.25. *If G is a DH graph with at least two vertices, and n true twins are added in some construction sequence of G, then $\gamma(G)=2^{n+1}$.*

Proof. This follows from $\gamma(K_2)=2$ and Corollary 4.17. □

The converse does not hold, however: $\gamma(G)$ being a power of 2 does not necessarily imply that G must be a DH graph. For example, C_6 is not a distance hereditary graph, having no pendant vertices nor twins either true or false. However, from Arratia, Bollobás and Sorkin [3], $q_N(C_6)=4x+10x^2+2x^3$, so $\gamma(C_6)=4$, a power of 2.

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Corollary 4.26. *If G is a DH graph, then all construction sequences for G must have the same number of vertices added as true twins.*

Although Corollary 4.26 may also be shown readily by induction, we include it as an example of how structural information may be encoded by the interlace polynomial.

5. Relation of BDH graphs to series-parallel graphs and polynomial time computability

We now characterize BDH graphs in terms of their relation to series-parallel graphs. Since BDH graphs may be recognized in polynomial time and the Tutte polynomial computed in polynomial time for series-parallel graphs, this characterization, together with the relation between the Tutte and vertex-nullity polynomials, will allow us to conclude that the vertex-nullity polynomial is polynomial time to compute for the class of BDH graphs.

Given a face 2-coloured, 4-regular planar graph, we call the graph constructed by placing a vertex in each black face, and connecting vertices whose faces share a vertex in the original graph, the *black face graph*.

Proposition 5.1. *H is a BDH graph with at least two vertices if and only if it is the circle graph of an Euler circuit in $\vec{G}_m$, where G is a series-parallel graph.*

Proof. We proceed by induction on the number, n, of vertices of a BDH graph H, or correspondingly, the number of edges of a series-parallel graph G. If $n = 2$, and H is a BDH graph, then, by Definition 4.9, H must be K_2. Thus, H is the circle graph of either of the two possible Eulerian circuits of $\vec{G}_m$ when G is a digon (and hence series-parallel). Conversely, if $n = 2$, by Definition 4.1, the only series-parallel graph is a digon, and again the two possible Eulerian circuits of $\vec{G}_m$ give K_2, a BDH graph, as their circle graph.

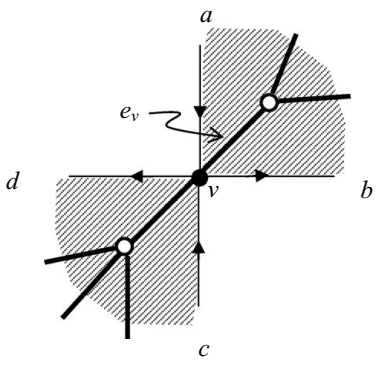
Now suppose the statement holds up to $n-1$, and let H be a BDH graph with n vertices. Suppose v' is the last vertex added in a construction sequence for H, and that v' is either pendant on, or a false twin of, a vertex v. Let $H' = H - v$, and by induction let G' be a series-parallel graph with $n-1$ edges, so that H' is a BDH graph, that is, a circle graph corresponding to some Eulerian circuit of $\vec{G}'_m$. See Figures 7–8, recalling that the same labels index the vertices of the medial graph, the endpoints of chords in the chord diagram, and the vertices of the circle graph. We now add an edge in series or parallel to $\vec{G}'$ to get a series-parallel graph G, and extend the Eulerian circuit of $\vec{G}'_m$ to an Eulerian circuit of $\vec{G}_m$. This will add a chord to the chord diagram of H' that corresponds to adding v' to H' appropriately to achieve H.

Precisely how we do this depends on whether v' is pendant on, or a false twin of, v, and whether the Euler circuit follows the black or white faces incident with v in $\vec{G}'_m$. This gives four cases to consider, and these four cases are illustrated in Figures 9–10.

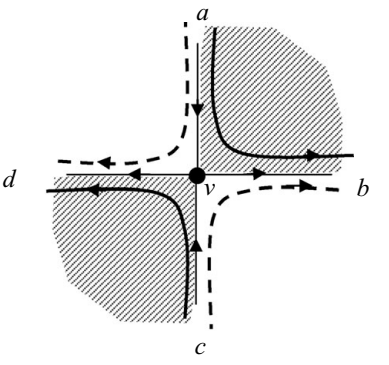
Thus, every BDH graph arises as the circle graph corresponding to an Euler circuit of the medial graph of a series-parallel graph.

Conversely, let G be a series-parallel graph with n edges, with edge e' added last in a construction sequence for G. Let G' be G with e' contracted if it was added in series or deleted if it was added in parallel. We note that any Eulerian circuit of $\vec{G}_m$ must extend an Eulerian circuit of $\vec{G}'_m$ as in Figures 9–10. Thus, the chord diagram corresponding to the Eulerian circuit of G_m is precisely the chord diagram of H' with one more chord added. Since this chord is a small arc

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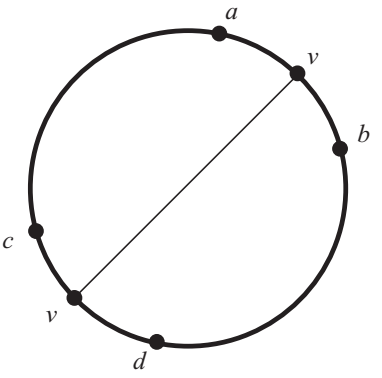


A neighbourhood of v in $\vec{G}'_m$ with the graph of G' superimposed in bold, showing the edge e_v corresponding to v. Labels a, b, c, d index the neighbours of v not shown in the figure.

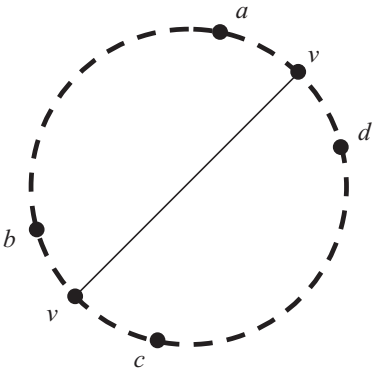


A neighbourhood of v in $\vec{G}'_m$ with the two possible ways an Eulerian circuit could pass through v. As an expository convenience, we traverse each circuit so that the upper arc in the figure is traversed before the lower.

Figure 7. A neighbourhood of v in the medial graph of G'.



Chord diagram corresponding to the bold line circuit following the black faces incident with v.



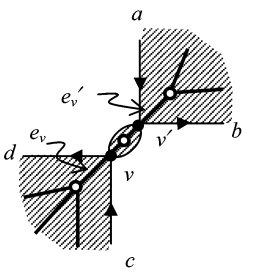
Chord diagram corresponding to the dashed line circuit following the white faces incident with v.

Figure 8. Chord diagrams corresponding to the circuits through v.

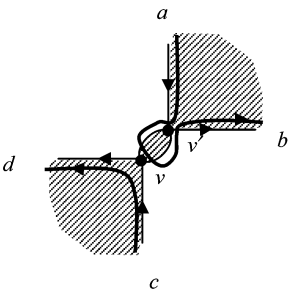
perpendicular to an existing chord or is parallel to an existing chord, the new vertex added to H' is either pendant on, or is a false twin of, a vertex of H'. By induction, the circle graph H' arising from any Eulerian circuit of G'_m is a BDH graph. Thus, the resulting circle graph corresponding to the Eulerian circuit of $\vec{G}_m$ is again a BDH graph. □

The motivation for connecting BDH and series-parallel graphs arises from the desire for a class of graphs on which the vertex-nullity interlace polynomial would be tractable. Theorem 3.1 gives a relation between the vertex-nullity interlace polynomial and the Tutte polynomial via a medial

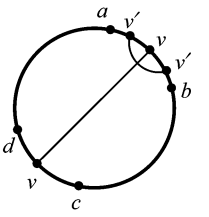
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A neighbourhood of v and v' in $\vec{G}_m$, with G superimposed in bold, showing edges e_v and e_v' in series.

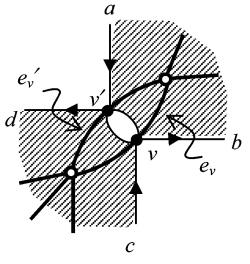


The Eulerian circuit of $\vec{G}_m'$ extended to an Eulerian circuit of $\vec{G}_m$.

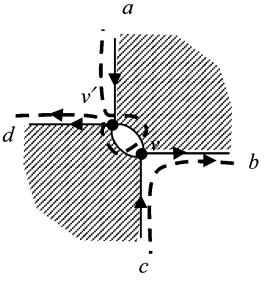


The chord diagram of the extended circuit. Vertex v' is now pendant on v in H' to give H.

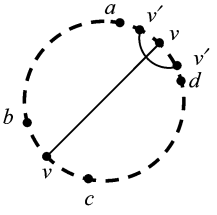
(a) The case where the Eulerian circuit in $\vec{G}_m'$ giving rise to $H' = H - v'$ follows the black faces incident with v. In this case we add an edge e_v' in series with the edge e_v'.



A neighbourhood of v and v' in $\vec{G}_m$, with G superimposed in bold, showing edges e_v and e_v' in parallel.



The Eulerian circuit of $\vec{G}_m'$ extended to an Eulerian circuit of $\vec{G}_m$.



The chord diagram of the extended circuit. Vertex v' is now pendant on v in H' to give H.

(b) The case where the Eulerian circuit in $\vec{G}_m'$ giving rise to $H' = H - v'$ follows the white faces adjacent to v. In this case we add an edge e_v' in parallel with the edge e_v'.

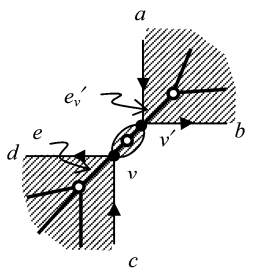
Figure 9. Configurations when v' is pendant on v.

graph construction, and Oxley and Welsh [59] show that the Tutte polynomial is polynomial time to compute for series-parallel graphs. Thus, we consider graphs that arise as circle graphs of Euler circuits in the oriented medial graphs of series-parallel graphs.

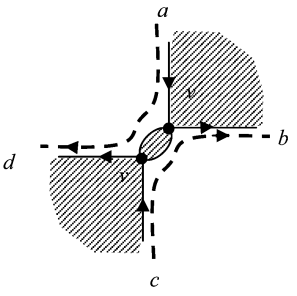
Recalling that K_4 is the excluded minor for series-parallel graphs (see Duffin [36] and Oxley [58]), observe that the only Euler circuits of the oriented medial graph of K_4 give rise to C_6 and C_6 with a single antipodal chord as circle graphs. This suggests that these two graphs should be prohibited, leading us to the bipartite $(6, 2)$-chordal graphs, and hence BDH graphs.

Readers familiar with 4-closure will note that BDH graphs are 4-closed. The notion of k-closure was introduced in [62] and used in [63] in the more general context of matroids. Graphs that are 4-closed are characterized by their closed sets of rank up to four (see [62]).

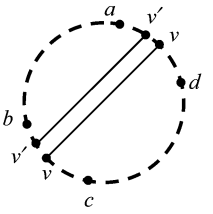
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A neighbourhood of v and v' in $\vec{G}_m$, with G superimposed in bold, showing edges e_v and e'_v in series.

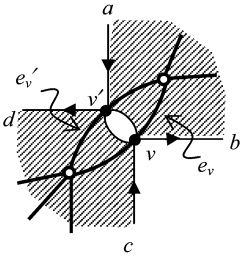


The Eulerian circuit of $\vec{G}'_m$ extended to an Eulerian circuit of $\vec{G}_m$.

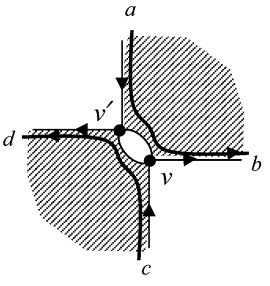


The chord diagram of the extended circuit. Vertex v' is now a false twin of v in H' to give H.

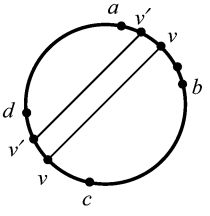
(a) The case where the Eulerian circuit in $\vec{G}'_m$ giving rise to $H' = H - v'$ follows the white faces incident with v. In this case we add an edge e'_v in series with the edge e'_v.



A neighbourhood of v and v' in $\vec{G}_m$, with G superimposed in bold, showing edges e_v and e'_v in parallel.



The Eulerian circuit of $\vec{G}'_m$ extended to an Eulerian circuit of $\vec{G}_m$.



The chord diagram of the extended circuit. Vertex v' is now a false twin of v in H' to give H.

(b) The case where the Eulerian circuit in $\vec{G}'_m$ giving rise to $H' = H - v'$ follows the black faces incident with v. In this case we add an edge e'_v in parallel with the edge e'_v.

Figure 10. Configurations when v' is a false twin of v.

Proposition 5.2. *If G is a BDH graph and v is a pendant or false twin vertex, then $G - v$ is also a BDH graph.*

Proof. By Proposition 4.22, G is bipartite $(6, 2)$-chordal graph, and clearly if v is a pendant vertex then $G - v$ is still a bipartite $(6, 2)$-chordal, and hence BDH, graph. If v is a false twin of u in G, and C_n with $n \geqslant 6$, is a cycle of $G - v$, then it is a cycle in G, and hence has two chords. The graph $G - v$ clearly remains bipartite, and is connected since v as a twin vertex cannot be a cut vertex. Thus, $G - v$ is still a bipartite $(6, 2)$-chordal graph, and hence a BDH graph. □

Recall from Bandelt and Mulder [9, Corollary 1] that every DH graph G with at least 4 vertices has at least two disjoint twin pairs, or a twin pair and a pendant vertex, or at least two pendant vertices. Also note that all connected graphs on 3 or fewer vertices are DH graphs.

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We now give the following elementary greedy algorithm for recognizing BDH graphs in polynomial time. There are certainly more sophisticated and general recognition algorithms for BDH graphs (see work by Cicerone and Di Stefano in [32] and [33], for example), but we use the following simplistic approach in order to leverage computability properties of series-parallel graphs.

Corollary 5.3. *A BDH graph may be recognized, and a construction sequence found, in polynomial time.*

Proof. We can identify pendant vertices in $O(n^2)$ steps by examining each vertex to determine if it is adjacent to exactly one other vertex. (In the case that the graph information is stored in an adjacency list rather than matrix, this can be done in $O(n)$ steps.) We can identify duplicate vertices in $O(n^3)$ steps by comparing the neighbours of each of the $O(n^2)$ pairs of vertices. We use this to successively find and remove pendant or twin vertices. The original graph is a BDH graph if and only if the graph that remains at the end of this process is a single vertex. If the resulting graph is a single vertex, then reversing the order of vertex removals gives a BDH construction sequence for the original graph. □

We now show that BDH graphs form a tractable class of graph for the vertex-nullity interlace polynomial.

Theorem 5.4. *Let H be a BDH graph. Then the vertex-nullity interlace polynomial $q_N(H;x)$ of H can be calculated in polynomial time.*

Proof. Oxley and Welsh [59] have shown that the Tutte polynomial $t(G;x,y)$ of a series-parallel graph G can be calculated in polynomial time. In fact, Noble [55] has shown that it can be calculated in a linear number of multiplications involving $O(|V|)$ factors. By Proposition 5.1, H is the circle graph of an Eulerian circuit of $\vec{G}_m$, where G is a series-parallel graph. By Corollary 5.3 we can find a construction sequence for H in polynomial time. Construct G using the construction sequence of H as in Proposition 5.1. By Theorem 3.1, $q_N(H;x) = t(G;x,x)$. Therefore $q_N(H;x)$ can be calculated in polynomial time. □

There has been continued exploration of the interlace polynomial and distance hereditary graphs since this paper was initially submitted. In particular, we refer the reader to work of Oum [56] and Makowsky, Rotics, Averbouch and Godlin [51]. The former shows that a graph has rank-width 1 if and only if it is distance hereditary, and the latter gives complexity results (polynomial time algorithm) for the interlace polynomial on classes of bounded clique-width. Oum and Seymour [57] showed that a class is of bounded clique-width if and only if it is of bounded rank-width, so [56] and [51] may be combined to give an alternative proof of Theorem 5.4. Also, an alternative, less constructive, proof of Proposition 5.1 can be deduced from [56] by noting that bipartite distance hereditary graphs are the fundamental graphs of matroids of branch-width 2 corresponding to series-parallel graphs.

In closing we observe that there are a number of graph classes closely related to DH graphs (see Brandstädt, Le and Spinrad [27] and McKee and McMorris [54]), many with construction

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methods similar to those for DH graphs, and further investigation of these classes in relation to the interlace polynomial may well prove a fruitful area of research.

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References

[1] Aigner, M. and van der Holst, H. (2004) Interlace polynomials. *Linear Algebra Appl.* **377** 11–30.
[2] Arratia, R., Bollobás, B., Coppersmith, D. and Sorkin, G. (2000) Euler circuits and DNA sequencing by hybridization. *Discrete Appl. Math.* (special issue on combinatorial molecular biology) **104** 63–96.
[3] Arratia, R., Bollobás, B. and Sorkin, G. (2000) The interlace polynomial: A new graph polynomial. In *Proc. 11th Annual ACM–SIAM Symposium on Discrete Algorithms: San Francisco 2000*, pp. 237–245.
[4] Arratia, R., Bollobás, B. and Sorkin, G. (2004) The interlace polynomial of a graph. *J. Combin. Theory Ser. B* **92** 199–233.
[5] Arratia, R., Bollobás, B. and Sorkin, G. (2004) A two-variable interlace polynomial. *Combinatorica* **24** 567–584.
[6] Ausiello, G., D'Atri, A. and Moscarini, M. (1986) Chordality properties on graphs and minimal conceptual connections in semantic data models. *J. Comput. Syst. Sci.* **33** 179–202.
[7] Balister, P. N., Bollobás, B., Cutler, J. and Pebody, L. (2002) The interlace polynomial of graphs at −1. *Europ. J. Combin.* **23** 761–767.
[8] Balister, P. N., Bollobás, B., Riordan, O. M. and Scott, A. D. (2001) Alternating knot diagrams, Euler circuits and the interlace polynomial. *Europ. J. Combin.* **22** 1–4.
[9] Bandelt, H. J. and Mulder, H. M. (1986) Distance-hereditary graphs. *J. Combin. Theory Ser. B* **41** 182–208.
[10] Benashki, J., Martin, R., Moore, J. and Traldi, L. (1995) On the β-invariant for graphs. *Congr. Numer.* **109** 211–221.
[11] Bollobás, B. (1998) *Modern Graph Theory*. *Graduate Texts in Mathematics*, Springer.
[12] Bollobás, B. (2002) Evaluations of the circuit partition polynomial. *J. Combin. Theory Ser. B* **85** 261–268.
[13] Bollobás, B. and Riordan, O. (2001) A polynomial invariant of graphs on orientable surfaces. *Proc. London Math. Soc.* (3) **83** 513–531.
[14] Bollobás, B. and Riordan, O. (2002) A polynomial of graphs on surfaces. *Math. Ann.* **323** 81–96.
[15] Bouchet, A. (1986) Characterizing and recognizing circle graphs. In *Graph Theory: Dubrovnik 1985*, pp. 57–69,
[16] Bouchet, A. (1987) Isotropic systems. *Europ. J. Combin.* **8** 231–244.
[17] Bouchet, A. (1987) Reducing prime graphs and recognizing circle graphs. *Combinatorica* **7** 243–254.
[18] Bouchet, A. (1987) Unimodularity and circle graphs. *Discrete Math.* **66** 203–208.
[19] Bouchet, A. (1988) Graphic presentations of isotropic systems. *J. Combin. Theory Ser. B* **45** 58–76.
[20] Bouchet, A. (1989) Connectivity of isotropic systems. In *Combinatorial Mathematics: Proc. 3rd International Conference, New York 1985*, Vol. 555 of *Ann. NY Acad. Sci.*, pp. 81–93.
[21] Bouchet, A. (1991) Tutte–Martin polynomials and orienting vectors of isotropic systems. *Graphs Combin.* **7** 235–252.
[22] Bouchet, A. (1993) Compatible Euler tours and supplementary Eulerian vectors. *Europ. J. Combin.* **14** 513–520.
[23] Bouchet, A. (1994) Circle graph obstructions. *J. Combin. Theory Ser. B* **60** 107–144.

https://doi.org/10.1017/S0963548307008723 Published online by Cambridge University Press

[24] Bouchet, A. (2001) Multimatroids III: Tightness and fundamental graphs. In *Combinatorial Geometries: Luming 1999*. *Europ. J. Combin.* **22** 657–677.
[25] Bouchet, A. (2005) Graph polynomials derived from Tutte–Martin polynomials. *Discrete Math.* **302** 32–38.
[26] Bouchet, A. and Ghier, L. (1996) Connectivity and β invariants of isotropic systems and 4-regular graphs. *Discrete Math.* **161** 25–44.
[27] Brandstädt, A., Le, V. B. and Spinrad, J. P. (1999) *Graph Classes: A Survey*. *SIAM Monographs on Discrete Mathematics and Applications*, SIAM, Philadelphia, PA.
[28] Brylawski, T. (1971) A combinatorial model for series-parallel networks. *Trans. Amer. Math. Soc.* **154** 1–22.
[29] Brylawski, T. (1980) The Tutte polynomial. In *Proc. 3rd International Mathematical Summer Centre*, pp. 125–275.
[30] Brylawski, T. and Oxley, J. (1992) The Tutte polynomial and its applications. In *Matroid Applications*, Vol. 40 of *Encyclopedia Math. Appl.*, Cambridge University Press, pp. 123–225.
[31] Burlet, M. and Uhry, J. P. (1982) Parity graphs. *Ann. Discrete Math.* **16** 1–26.
[32] Cicerone, S. and Di Stefano, G. (1999) Graph classes between parity and distance-hereditary graphs. In *Proc. Conference on Optimal Discrete Structures and Algorithms: ODSA '97, Rostock*. *Discrete Appl. Math.* **95** 197–216.
[33] Cicerone, S. and Di Stefano, G. (1999) On the extension of bipartite to parity graphs. In *Proc. Conference on Optimal Discrete Structures and Algorithms: ODSA '97, Rostock*. *Discrete Appl. Math.* **95** 181–195.
[34] Crapo, H. H. (1967) A higher invariant for matroids. *J. Combin. Theory* **2** 406–417.
[35] Czemerinski, H., Durán, G. and Gravano, A. (2002) Bouchet graphs: A generalization of circle graphs. *Congr. Numer.* **155** 95–108.
[36] Duffin, R. J. (1965) Topology of series-parallel networks. *J. Math. Anal. Appl.* **10** 303–318.
[37] Durán, G. (2003) Some new results on circle graphs. In *Latin-American Workshop on Cliques in Graphs: Rio de Janeiro 2002*. *Mat. Contemp.* **25** 91–106.
[38] Ellis-Monaghan, J. A. (1998) New results for the Martin polynomial. *J. Combin. Theory Ser. B* **74** 326–352.
[39] Ellis-Monaghan, J. A. (2004) Exploring the Tutte–Martin connection. *Discrete Math.* **281** 173–187.
[40] Ellis-Monaghan, J. A. (2004) Identities for circuit partition polynomials, with applications to the Tutte polynomial. *Adv. Appl. Math.* **32** 188–197.
[41] Ellis-Monaghan, J. A. and Sarmiento, I. (2002) Generalized transition polynomials. *Congr. Numer.* **155** 57–69.
[42] de Fraysseix, H. (1984) A characterization of circle graphs. *Europ. J. Combin.* **5** 223–238.
[43] Gasse, E. (1997) A proof of a circle graph characterization. *Discrete Math.* **173** 277–283.
[44] Howorka, E. (1977) A characterization of distance-hereditary graphs. *Quart. J. Math. Oxford Ser. 2* **26** 417–420.
[45] Howorka, E. (1977) A characterization of Ptolemaic graphs: Survey of results. In *Proc. 8th SE Conference on Combinatorics, Graph Theory, and Computing*, pp. 355–361.
[46] Jaeger, F., Vertigan, D. L. and Welsh, D. J. A. (1990) On the computational complexity of the Jones and Tutte polynomials. *Math. Proc. Cambridge Philos. Soc.* **108** 35–53.
[47] Las Vergnas, M. (1979) On Eulerian partitions of graphs. In *Graph Theory and Combinatorics* (R. J. Wilson, ed.), Vol. 34 of *Research Notes in Mathematics*, Pitman Advanced Publishing Program, pp. 62–65.
[48] Las Vergnas, M. (1981) Eulerian circuits of 4-valent graphs imbedded in surfaces. In *Algebraic Methods in Graph Theory, Vols I, II: Szeged 1978*, Vol. 25 of *Colloq. Math. Soc. János Bolyai*, North-Holland, pp. 451–477.
[49] Las Vergnas, M. (1983) Le polynôme de Martin d'un graphe Eulérien. In *Combinatorial Mathematics: Marseille–Luminy 1981*, Vol. 75 of *North-Holland Math. Stud.*, North-Holland, pp. 397–411.

https://doi.org/10.1017/S0963548307008723 Published online by Cambridge University Press

[50] Las Vergnas, M. (1988) On the evaluation at (3,3) of the Tutte polynomial of a graph. *J. Combin. Theory Ser. B* **45** 367–372.
[51] Makowsky, J. A., Rotics, U., Averbouch, I. and Godlin, B. (2006) Computing graph polynomials on graphs of bounded clique-width. In *Graph-Theoretic Concepts in Computer Science: 32nd International Workshop, WG 2006, Bergen 2006* (F. V. Fomin, ed.), *Lecture Notes in Computer Science*, Springer, pp. 191–204.
[52] Martin, P. (1977) Enumerations Eulériennes dans le multigraphs et invariants de Tutte–Grothendieck. Thesis, Grenoble.
[53] Martin, P. (1978) Remarkable valuation of the dichromatic polynomial of planar multigraphs. *J. Combin. Theory Ser. B* **24** 318–324.
[54] McKee, T. A. and McMorris, F. R. (1999) *Topics in Intersection Graph Theory*. *SIAM Monograms on Discrete Mathematics and Applications.*
[55] Noble, S. D. (1998) Evaluating the Tutte polynomial for graphs of bounded tree-width. *Combin. Probab. Comput.* **7** 307–321.
[56] Oum, S. (2005) Rank-width and vertex-minors. *J. Combin. Theory Ser. B* **95** 79–100.
[57] Oum, S. and Seymour, P. (2006) Approximating clique-width and branch-width. *J. Combin. Theory Ser. B* **96** 514–528.
[58] Oxley, J. (1982) On Crapo's beta invariant for matroids. *Stud. Appl. Math.* **66** 267–277.
[59] Oxley, J. G. and Welsh, D. J. A. (1992) Tutte polynomials computable in polynomial time. *Discrete Math.* **109** 185–192.
[60] Read, R. C. and Rosenstiehl, P. (1978) On the Gauss crossing problem. In *Combinatorics II*, Vol. 18 of *Colloq. Math. Soc. János Bolyai*, pp. 843–876.
[61] Read, R. C. and Rosenstiehl, P. (1978) On the principal edge tripartition of a graph. *Ann. Discrete Math.* **3** 195–226.
[62] Sarmiento, I. (1998) Algebraic problems in matroid theory. D.Phil thesis, Oxford.
[63] Sarmiento, I. (1999) A characterisation of jointless Dowling geometries. *Discrete Math.* **197–198** 713–731.
[64] Tutte, W. T. (1947) A ring in graph theory. *Proc. Cambridge Philos. Soc.* **43** 26–40.
[65] Tutte, W. T. (1954) A contribution to the theory of chromatic polynomials. *Canad. J. Math.* **6** 80–91.
[66] Tutte, W. T. (1967) On dichromatic polynomials. *J. Combin. Theory* **2** 301–320.
[67] Tutte, W. T. (1979) All the king's horses: A guide to reconstruction. In *Graph Theory and Related Topics: Proc. Conference at the University of Waterloo, Ont., 1977*, Academic Press, pp. 15–33.
[68] Tuza, Z. (1997) Graph colorings with local constraints: A survey. *Discuss. Math. J. Graph Theory* **17** 161–228.
[69] Welsh, D. J. A. (1993) *Complexity: Knots, Colorings and Counting*. Vol. 186 of *London Math. Soc. Lecture Notes Series*, Cambridge University Press.
[70] Wessel, W. and Pöschel, R. (1985) On circle graphs. In *Graphs, Hypergraphs and Applications: Eyba 1984*, Vol. 73 of *Teubner-Texte Math.*, Teubner, pp. 207–210.

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