103.23 On Pitot's theorem

The most famous necessary and sufficient condition for a convex quadrilateral to have an inscribed circle (a circle inside the quadrilateral that is tangent to all four sides, see Figure 1) is called Pitot's theorem since the necessary condition was proved by Henri Pitot in 1725. In [1] we made some comments on the origin of this important theorem and of the proofs that were known to us at that time before studying other characterisations of quadrilaterals with an inscribed circle, often called *tangential quadrilaterals*. The purpose of the present Note is to update some of that information and to give two new proofs. The sufficient condition was not proved for the first time by Jakob Steiner in 1846 as claimed in [1, p. 65], but as early as 1815 by J.-B. Durrande according to [2, p. 64].

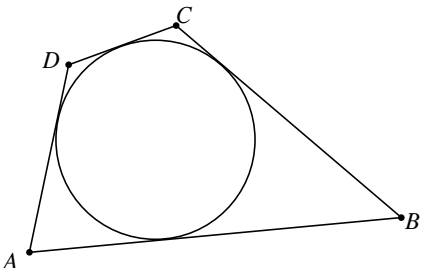


FIGURE 1: A quadrilateral satisfying Pitot's theorem

Before we proceed, let us formulate Pitot's theorem and its converse:

> A convex quadrilateral $ABCD$ can have an inscribed circle if, and only if, its sides satisfy $AB + CD = BC + DA$.

One remarkable thing about Pitot's theorem concerns the number of known proofs. This is the only characterisation in geometry that we know of where there have been considerably more proofs given for the sufficient condition (converse) than for the necessary condition (direct part). Often there are a multitude of known proofs of necessary conditions, with more published from time to time, but hardly any further development on sufficient conditions once they have been proved. But at the time of writing this Note we know of only two proofs of the direct part of Pitot's theorem, whereas there are at least seven different previously published proofs of the converse. The reason for the imbalance between the number of proofs is surely that the direct part is so easily proved using equal tangents to a circle from an external point, that any other argument would likely be longer and more complicated.

Another thing worth mentioning is that the different proofs of the sufficient condition cover all four methods of proving a converse: direct proof, proof by contradiction, equivalence proof and proof by the contrapositive statement. Thus Pitot's theorem is a gold mine in education when working on different proof techniques. In order for the reader to be able to pursue this further, we give references to different proofs in Table 1.

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Of these, numbers 1 and 4 are by far the ones most commonly used when this theorem is proved in textbooks and on websites.

No.	Method of proof	Applies	References
1	Direct	Perpendicular bisectors	[3], [4]
2	Direct	Pythagorean theorem	[5], [6]
3	Direct	Circle property	[7]
4	Contradiction	Triangle inequality	[3], [8]
5	Contradiction	Parallel line segment	[7], [9]
6	Contradiction	AM-GM inequality	[10]
7	Equivalence	Excircle and incircle	[11]
8	Equivalence	Hypotenuse > one leg	This Note
9	Contrapositive	Triangle inequality	This Note

TABLE 1: Known proofs for the converse of Pitot's theorem

We called the third method an 'equivalence proof', which is not as well-known a concept as the names of the other three methods. The principle used here is that a necessary and sufficient condition can be proved at the same time by proving that the new characterisation is equivalent to another previously proved characterisation for the object in the theorem or the definition of that object. For the purpose of our first proof, we state such a characterisation here: *A convex quadrilateral can have an inscribed circle if, and only if, its angle bisectors are concurrent*. This is quite a well-known theorem. It is a direct consequence of the fact that a point is on an angle bisector if, and only if, it is equidistant from the legs of the angle.

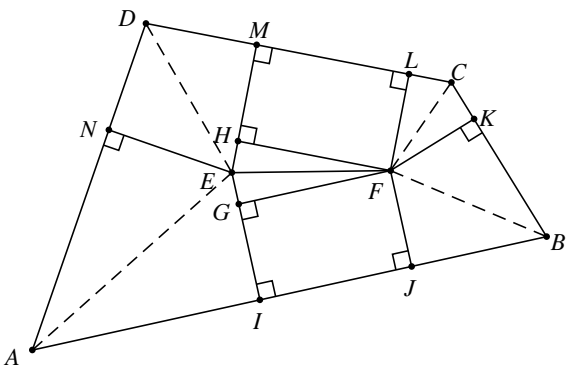


FIGURE 2: Two pairs of intersecting angle bisectors

Now let us start proof number eight, which will prove both Pitot's theorem and its converse at the same time. We begin with a general convex

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quadrilateral $ABCD$. Draw the four angle bisectors, which intersect in pairs at E and F, and draw the normals from these points to the sides. We label other points as in Figure 2, with GF parallel to AB and HF parallel to DC. Then, since the hypotenuse of a right triangle is longer than either of its legs, we have

$$\begin{aligned} 2EF &\geqslant GF + HF \\ &= IJ + ML \\ &= AB - AI - JB + CD - CL - MD \\ &= AB - NA - BK + CD - KC - DN \\ &= AB + CD - BC - DA. \end{aligned}$$

There is another case where all four terms in the final expression have reversed signs, so in general

$$2EF \geqslant |AB + CD - BC - DA|.$$

The quadrilateral $ABCD$ can have an inscribed circle if, and only if, the angle bisectors are concurrent, which is equivalent to $EF = 0$. This in turn is equivalent to $AB + CD = BC + DA$. Hence both Pitot's theorem and its converse are proved at the same time. (This is the third known proof of the direct theorem.) The inequality we derived is not original. It has been discussed at [12] (without any comment on the connection to Pitot). With this proof we also solved a problem discussed at [13], which was to prove that, if Pitot's equality holds, then all the angle bisectors of the quadrilateral are concurrent. The only difference is that we approached it from the reverse direction, which is permissible, since we have equivalence in the steps connecting these two statements.

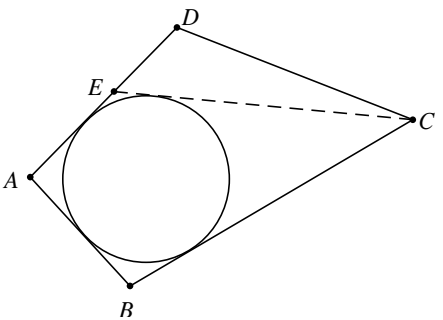


FIGURE 3: A circle tangent to three of the sides

The methods of proof by contradiction and proof by the contrapositive statement are closely related, but nevertheless considered as different [14]. (They are both indirect proof techniques.) We have been unable to locate any reference to a proof of the converse to Pitot's theorem that uses the

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contrapositive statement, so we conclude by giving such an argument here. Instead of proving the implication that in a convex quadrilateral where $AB + CD = BC + DA$, there can be an inscribed circle, the contrapositive statement is that if a convex quadrilateral can *not* have an inscribed circle, then $AB + CD \neq BC + DA$. Consider the quadrilateral $ABCD$ in Figure 3 with a circle tangent to three of the sides but not the fourth. (The centre of this circle is at the intersection of the angle bisectors to vertex angles A and B.) Draw CE tangent to this circle. Applying the direct part of Pitot's theorem to the tangential quadrilateral $ABCE$ yields

$$\begin{aligned} & AB + CE = BC + DA - ED \\ \Rightarrow\ & AB + CE + ED = BC + DA \\ \Rightarrow\ & AB + CD < AB + CE + ED = BC + DA \\ \Rightarrow\ & AB + CD \neq BC + DA \end{aligned}$$

where we used the triangle inequality in triangle CDE in the third step. The other case when the circle that is tangent to three sides intersects the fourth side is similar (the inequality being reversed), completing this ninth proof for the converse of Pitot's theorem.

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MARTIN JOSEFSSON
Västergatan 25d, 285 37 Markaryd, Sweden
e-mail: *martin.markaryd@hotmail.com*

103.24 Partitions, geometric progressions and a Putnam problem

Introduction: A 1983 Putnam problem

Our starting point is the following problem in the 1983 Putnam Mathematical Competition [1]:

Problem 1983: B2

For positive integers n, let $C(n)$ be the number of representations of n as a sum of nonincreasing powers of 2, where no power can be used more than three times. For example, $C(8) = 5$ since the representations for 8 are:

$$8,\ 4+4,\ 4+2+2,\ 4+2+1+1 \text{ and } 2+2+2+1+1.$$

Prove or disprove that there is a polynomial $P(x)$ such that $C(n) = \lfloor P(n) \rfloor$ for all positive integers n; here $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u.

Before giving a solution to this Putnam problem, we would like to introduce some terminology in the theory of partitions, the study of which can be dated back to the time of Euler.

A *partition* of a positive integer n is a non-increasing sequence of positive integers (which are called the *parts* of this partition) whose sum equals n. Conventionally, we use $p(n)$ to count the number of partitions of n. We also set $p(0) = 1$. This is helpful when we deal with the generating function of $p(n)$:

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