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Vectors with orthogonal iterates along IP-sets in unitary group actions

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Abstract. We prove the following result in commutative harmonic analysis.

THEOREM 1. *Suppose μ is a probability measure on the dual $\hat{G}$ of a countable Abelian group G, E is a symmetric subset of G and*

$$\sum_{g\in E-E, g\neq 0} |\widehat{\mu}(g)| < \delta < 1/3.$$

Then there is an $f \in C^+(\hat{G})$ such that $\|f - 1\| < \frac{3}{2}\delta$, $(\widehat{f\mu})(g) = 0$ for all $g \in E \setminus\{0\}$ and $(\widehat{f\mu})(0) = 1$.

This has the following corollary which leads to simple proofs and improvements of many of the results in a recent paper of Bergelson *et al*.

THEOREM 2. *If $g \mapsto U_g$ is a unitary action of a discrete Abelian group G on a Hilbert space $\mathcal{H}$ and $\{g_i\} \subset G$ is a sequence such that both $\{g_i\}$ and $\{2g_i\}$ are mixing for this action then for all $x \in \mathcal{H}$ and $\epsilon > 0$ there are an $x' \overset{\epsilon}{\sim} x$ and an IP-set Γ generated by a subsequence of $\{g_i\}$ such that the vectors $\{U_g x' : g \in \Gamma\}$ are pairwise orthogonal.*

We also show that even in the case of a $\mathbb{Z}$-action (i.e. a single unitary operator U) the hypothesis $\{g_i\}$ mixing alone is not sufficient for the conclusion of Theorem 2. Finally we give conditions on G and on the spectrum of the action under which that assumption is sufficient.

1. *Introduction*

Krengel [**K**] proved the following result. (We use the notation $x' \overset{\epsilon}{\sim} x$ generically, whenever x' and x are elements of a metric space within ϵ of each other.)

THEOREM K. *Suppose U is a unitary operator with continuous spectrum on a Hilbert space $\mathcal{H}$. Then for each $x \in \mathcal{H}$ and $\epsilon > 0$ there exist $x' \overset{\epsilon}{\sim} x$ and an infinite sequence $\{n_i\}$ such that the vectors $\{U_{n_i} x'\}$ are pairwise orthogonal, and this is in fact a necessary and sufficient condition for continuous spectrum.*

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In [**BKM1**] Bergelson *et al* gave a new and simpler proof of Krengel's result, generalizing to $\mathbb{Z}^n$-actions and replacing $\{n_i\}$ by an IP-set. More recently, in [**BKM**], the same authors established various results of the same nature for actions of discrete Abelian groups which are mixing, mildly mixing or weakly mixing (weak mixing is the same as continuous spectrum). The proofs of the results in both these papers use a number of different fixed point arguments which are rather technical.

Our first purpose here is to show that, with one exception, these results all follow rather easily from a single result in commutative harmonic analysis, Theorem 1 below, whose proof is itself quite short and elementary. Moreover, Theorem 1 enables us to improve some of the results of [**BKM**] and to obtain some new results in the same vein. Roughly speaking, as in [**BKM**], these results all hinge on the existence of a sequence $\{g_i\}$ in the group such that $\{g_i\}$ and $\{2g_i\}$ are both mixing for the action in question. Our second purpose is to investigate what happens when $\{2g_i\}$ is not mixing.

We deal throughout with a unitary group action, always denoted $g \mapsto U_g$, of a countable discrete additive Abelian group G on a Hilbert space $\mathcal{H}$. For $x \in \mathcal{H}$, $U_g x$ will sometimes be shortened to gx. The reader is referred to §2 for definitions of unfamiliar terms.

Most of the results in [**BKM**] are of the following type. If the action of G is X-mixing, where X can stand for strong, mild or weak, $x \in \mathcal{H}$ and E is a subset of G having property Y, depending on X, then there is a vector x' close to x and an infinite subset $F \subset G$ such that the vectors

$$\{U_g : g \in FS(F)\}$$

are orthogonal. (Here $FS(F)$ denotes the IP-set generated by F, that is it consists of all finite sums of distinct elements of F.) In some cases $FS(F)$ must be replaced simply by F. In the case of mild mixing, G can be arbitrary, but in the other cases some restriction is placed on G. Whether G can be arbitrary in the weak mixing case remains open. The method of [**BKM1**] gives the weak mixing result for groups G such that $G/2G$ is finite and this also follows from our approach. In [**BKM**] the weak mixing case is also established when G is a direct sum of cyclic groups of bounded orders. This is the one result of that paper which we have not been able to obtain using our approach.

The main technique in [**BKM**] and [**BKM1**] is to write down the desired vector x' as a linear combination of (some of) the vectors $U_g x$, $g \in E$ and then solve for the coefficients using a fixed point theorem. (The exception to this rule is Theorem 2.1 of [**BKM**] which uses a spectral approach, as we do here.) The equations which must be solved in their approach are quadratic, and the equation solving is done on a case by case basis. In our approach all the equation solving is in Theorem 1 and the equations are linear.

To state Theorem 1 we first introduce some terminology and notation. By a *symmetric* subset E of G we mean a subset such that $E = -E$ and $0 \in E$. For such an E, $l^1_h(E)$ will denote the subspace of $l^1(E)$ consisting of *Hermitian symmetric* functions c, that is $c(g) = \overline{c(-g)}$. $h(E)$ denotes the convex subset of $l^1_h(E)$ consisting of functions c such that $c(0) = 1$. $\delta_0 \in h(E)$ denotes the function which is 1 at 0 and 0 elsewhere. For $\alpha > 0$ we let

$$B_\alpha(E) = \{c \in h(E) : \|c - \delta_0\|_1 < \alpha\}.$$

If μ is a probability on the compact dual group $\hat{G}$ then the Fourier transform $\hat{\mu} = \mathcal{F}(\mu)$ is

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a Hermitian symmetric function on G (not generally in l^1) and its restriction to E will be denoted by $\mathcal{F}_E(\mu)$. Finally if E is any subset of G we denote the difference set $E - E$ by $D(E)$, a symmetric subset of G.

THEOREM 1. *Suppose μ is a probability measure on the dual $\hat{G}$ of a countable Abelian group G, E is a symmetric subset of G and*

$$\sum_{g \in D(E), g \neq 0} |\hat{\mu}(g)| < \delta < 1/3.$$

Then:

(a) *there is an $f \in C^+(\hat{G})$ such that $\|f - 1\|_\infty < \frac{3}{2}\delta$ and $\mathcal{F}_E(f\mu) = \delta_0$;*

(b) *given $c \in B_{1/3}$ there is an $f \in C^+(\hat{G})$ such that $\mathcal{F}_E(f\mu) = c$.*

Via the spectral theorem we then deduce the following result. Case (a) of this result is the 'IP-strengthening' of Theorem 4.1 of [**BKM**]. Case (b) is contained in [**BKM**, Theorem 2.1]. We say that a sequence $\{U_i\}$ of unitary operators is *mixing* if $\langle U_i x, y\rangle \to 0$ for all vectors x and y.

THEOREM 2. *Suppose $\{U_i\}$ is a mixing sequence of commuting unitary operators on $\mathcal{H}$ such that either*

(a) *$\{U_i^2\}$ is also mixing, or*

(b) *$U_i^2 = \mathrm{id}$ for all i.*

Then for all $\epsilon > 0$ there exist an $x' \overset{\epsilon}{\sim} x$ and a sequence $\{i_j\}$ such that the set

$$\{Ux : U \in FP\{U_{i_j} : j = 1, 2, \dots\}\}$$

is pairwise orthogonal. Here $FP\{U_{i_j} : j = 1, 2, \dots\}$ denotes the set of finite products of distinct elements of the sequence $\{U_{i_j}\}$.

The following is an equivalent formulation of Theorem 2 in terms of actions.

THEOREM 2′. *Suppose $(\mathcal{H}, \{U_g : g \in G\})$ is an action, $x \in \mathcal{H}$, $\epsilon > 0$, $\{g_i\}$ is mixing for $(\mathcal{H}, \{U_g : g \in G\})$ and either*

(a) *$\{2g_i\}$ is also mixing, or*

(b) *$2g_i = 0$ for all i.*

Then for all $\epsilon > 0$ there is a subsequence $\{g_{i_j}\}$ and an $x' \overset{\epsilon}{\sim} x$ such that the set

$$\{gx' : g \in FS(\{g_{i_j}\})\}$$

is orthogonal.

From Theorem 2 we get the following two results rather easily. Theorem 3 is Theorem 3.3 of [**BKM**], although the proof of Theorem 3.3 is omitted, and Theorem 4 is Theorem 3.1 of [**BKM**]. Note that in Theorem 3 we do not get an IP-set but this is by no means ruled out.

THEOREM 3. *An action $(\mathcal{H}, \{U_g : g \in G\})$ is mixing if and only if for all infinite subsets $E \subset G$ and $x \in \mathcal{H}$ there is an infinite subset $F \subset E$ and $x' \overset{\epsilon}{\sim} x$ such that the set $\{gx' : g \in F\}$ is pairwise orthogonal.*

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THEOREM 4. *An action* $(\mathcal{H}, \{U_g : g \in G\})$ *is mildly mixing if and only if for all IP-sets* $\Gamma \subset G, x \in \mathcal{H}$ *and* $\epsilon > 0$ *there exist an IP-subset* $\Gamma' \subset \Gamma$ *and* $x' \overset{\epsilon}{\sim} x$ *such that* $\{U_g x' : g \in \Gamma'\}$ *is pairwise orthogonal.*

In the weak mixing case we have the following partial result which also follows from the methods of [**BKM**], although it is not explicitly stated there. It includes the case $G = Z^d$ covered in [**BKM1**] and also, for example, G any direct sum of finite cyclic groups of odd orders.

THEOREM 5. *Suppose* G *is a group such that* $G/2G$ *is finite. Then* $(\mathcal{H}, \{U_g : g \in G\})$ *is weakly mixing if and only if for all sets* E *of positive density in* $G, x \in \mathcal{H}$ *and* $\epsilon > 0$ *there exist an infinite* $F \subset E$ *and an* $x' \overset{\epsilon}{\sim} x$ *such that* $\{gx' : g \in FS(F)\}$ *is pairwise orthogonal.*

Finally the following result is Theorem 1.1 of [**BKM**].

THEOREM 6. *Let* G *now be the direct sum of countably many copies of* $\mathbb{Z}_p$, p *prime, and for* $E \subset G$ *let* $G(E)$ *denote the subgroup generated by* E. *Then an action* $(\mathcal{H}, \{U_g : g \in G\})$ *is:*

(a) *weakly mixing if and only if for all subsets* $E \subset G$ *of positive density and* $x \in \mathcal{H}$ *there are an infinite subset* $F \subset E$ *and* $x' \overset{\epsilon}{\sim} x$ *such that the set* $\{gx' : g \in G(F)\}$ *is pairwise orthogonal;*

(b) *mildly mixing if and only if for all IP-sets* $\Gamma \subset G$ *and* $x \in \mathcal{H}$ *there are an* $x' \overset{\epsilon}{\sim} x$ *and an infinite subset* $F \subset \Gamma$ *such that the set* $\{gx' : g \in G(F)\}$ *is pairwise orthogonal;*

(c) *mixing if and only if for all infinite sets* $E \subset G$ *and* $x \in \mathcal{H}$ *there are an* $x' \overset{\epsilon}{\sim} x$ *and an infinite subset* $F \subset G$ *such that the set* $\{gx' : g \in G(F)\}$ *is pairwise orthogonal.*

For the proof of Theorem 6 we need the following strengthening of Theorem 2′, whose proof is essentially the same as that of Theorem 2′. $FS_m(F)$ denotes the set of sums of non-empty subsets of F in which each element is permitted to have a multiplicity no larger than m (see §2 for a formal definition).

THEOREM 7. *Suppose* $(\mathcal{H}, \{U_g : g \in G\})$ *is an action,* m *is a positive integer and* $\{g_i\}$ *is a mixing sequence such that either (a) the sequences* $\{qg_i\}$ *are also mixing for* $q \in \{1, \ldots, m\} \cup 2\{1, \ldots, m\}$, *or (b)* $2g_i = 0$ *for all* i *and* $x \in \mathcal{H}$. *Then for all* $\epsilon > 0$ *there exist an* $x' \overset{\epsilon}{\sim} x$ *and a subsequence* $\{g_{i_j}\}$ *such that the set* $\{U_g : g \in FS_m(\{g_{i_j}\})\}$ *is pairwise orthogonal.*

We remark that Theorem 7 also yields strengthenings of Theorems 3, 4 and 5 in which $FS(F)$ is replaced by $FS_m(F)$, provided we assume that G/nG is finite for each $n \in \mathbb{N}$.

The weak mixing counterpart of Theorems 3 and 4 ought to be the following.

CONJECTURE. *An action* $(\mathcal{H}, \{U_g : g \in G\})$ *is weakly mixing if and only if for all subsets* $E \subset G$ *of positive density,* $x \in \mathcal{H}$ *and* $\epsilon > 0$ *there is an infinite subset* $F \subset E$ *and* $x' \overset{\epsilon}{\sim} x$ *such that the set* $\{U_g x' : g \in FS(F)\}$ *is pairwise orthogonal.*

This conjecture remains open as far as we know. In particular it is open when G is the direct sum of copies of $\mathbb{Z}$. This group has weakly mixing actions with the following property: there is a mixing set E of density one, $2E$ contains no mixing sequences but $0 \notin 2E$, so that neither case of Theorem 2 applies.

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Our second purpose in this paper is to ask what happens in Theorem 2 if the assumption that $\{2g_i\}$ is mixing is dropped.

QUESTION A. *If $\{g_i\}$ is mixing, $\epsilon > 0$ and $x \in \mathcal{H}$, must there be $x' \overset{\epsilon}{\sim} x$ and a subsequence $\{g_{i_j}\}$ such that the set $\{U_g x' : g \in FS(\{g_{i_j}\})\}$ is pairwise orthogonal?*

The following example shows that this is far from being true, even in the case of $\mathbb{Z}$-actions.

THEOREM 8. *There is a probability measure μ on the circle group $\mathbb{T}$ and a sequence $\{n_i\}$ such that $\langle z^{n_i} f, g\rangle \to 0$ for all $f, g \in L^2(\mu)$, but $\langle z^{n_i} f, f\rangle \neq 0$ for all non-zero $f \in L^2(\mu)$.*

In view of this example we may then ask for conditions on the group and/or the action which ensure that the answer to the question is yes. For example it is not hard to see that if G is the direct sum of copies of $\mathbb{Z}_p$ then the answer is yes by Theorem 2 (see the proof of Theorem 6). In order to get a positive answer for a fairly general class of groups including $\mathbb{Z}$ we will make a restriction on the action. Let us call a measure μ on $\hat{G}$ *mixing* if its Fourier coefficients go to zero at infinity. We will say a closed subset $K \subset \hat{G}$ is mixing if it supports some mixing measure. Of course any set with positive Haar measure is mixing. In the case $G = \mathbb{Z}$ there is also an interesting class of Cantor subsets of the circle with Lebesgue measure zero which are mixing, described in [**S**, Ch. 4]. Note that Theorem 9 applies, for example, to the direct sum of infinitely many copies of $\mathbb{Z}$.

THEOREM 9. *Suppose G is a group such that $\{g : 2g = 0\}$ is finite, $(\mathcal{H}, \{U_g : g \in G\})$ is an action whose spectrum is a mixing subset of $\hat{G}$ and $\{g_i\}$ is mixing for this action. Then for all $x \in \mathcal{H}$ and $\epsilon > 0$ there are an $x' \overset{\epsilon}{\sim} x$ and a subsequence $\{g_{i_j}\}$ such that $\{U_g x' : g \in FS(\{g_{i_j}\})\}$ is pairwise orthogonal.*

Finally, if we do not insist on an IP-set, then we can drop the restriction on G.

THEOREM 10. *Suppose $(\mathcal{H}, \{U_g : g \in G\})$ is an action whose spectrum is a mixing subset of $\hat{G}$ and $\{g_i\}$ is mixing for this action. Then for all $x \in \mathcal{H}$ there are an $x' \overset{\epsilon}{\sim} x$ and a subsequence $\{g_{i_j}\}$ such that*

$$\{U_g x' : g \in \{g_{i_j}\}\}$$

is pairwise orthogonal.

There is a body of analogous results [**K, JRW**] for measure-preserving actions of G on a probability space in which vectors are replaced by finite partitions and orthogonality by (joint) independence. In particular the following is Theorem 5 of [**JRW**].

THEOREM. [**JRW**] *Let $S = \{m_k\}_{k=1}^{\infty}$ be a mixing sequence for a measure-preserving map T of a probability space. Then, the partitions having independent iterates along some IP-set with generators in S are dense in the set of all finite partitions.*

In other words, when $G = \mathbb{Z}$ the answer to the measure-preserving analogue of Question A above is yes. It is interesting to note that in the case of a single operator induced by a measure-preserving map the answer to Question A is also yes, by Theorem 9, since the spectrum of such an operator must be the whole circle.

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2. *Background material*

General references for the mixing notions in this section are [**D, F, Sc**]. We will be dealing throughout with a unitary action of a countable discrete Abelian group G on a Hilbert space $\mathcal{H}$, that is, a homomorphism from G into the group of unitary operators on $\mathcal{H}$. We will denote such an action by $(\mathcal{H}, \{U_g : g \in G\})$ and will sometimes write gx for $U_g x$. By a cyclic subspace we mean the closed subspace $Z(x)$ generated by the G-orbit of any vector $x \in \mathcal{H}$.

The dual group $\hat{G}$ of G is equipped with the topology of pointwise convergence, which is compact and metric. The second dual of G is equipped with the topology of uniform convergence. For $g \in G$, $\hat{g} \in \hat{\hat{G}}$ denotes the character of $\hat{G}$ defined by

$$\hat{g}(\gamma) = \gamma(g).$$

Then the map $g \mapsto \hat{g}$ is an isomorphism from G to $\hat{\hat{G}}$. If μ is a finite Borel measure on $\hat{G}$ its Fourier transform is the function on G defined by

$$\hat{\mu}(g) = \int_{\hat{G}} \hat{g}\, d\mu = \int_{\hat{G}} \gamma(g)\, d\mu(\gamma).$$

If μ is a finite Borel measure on $\hat{G}$ we define the multiplication operator $M_{\hat{g}}$ in the usual way, that is $M_{\hat{g}} f = \hat{g} f$, and we obtain an action of G on $L^2(\hat{G}, \mu)$ by letting $U_g = M_{\hat{g}}$.

Given $x \in \mathcal{H}$, by Bochner's theorem [**Fo**, Theorem 4.18] there is a finite measure $\mu = \mu_x$ on $\hat{G}$ such that

$$\langle U_g x, x \rangle = \hat{\mu}(g).$$

(We remark that Bochner's theorem has quite an elementary proof in the case of a discrete Abelian group.) It follows easily that the map

$$U_g x \mapsto M_{\hat{g}} 1$$

extends uniquely to a unitary equivalence of the actions $(\mathcal{H}, \{U_g : g \in G\})$ restricted to $Z(x)$ and the multiplication action described above. This is an elementary version of the spectral theorem (see [**Fo**, Theorem 4.44]) which will be sufficient for our purposes.

Every $x \in \mathcal{H}$ belongs to a maximal cyclic subspace. If $Z(x)$ and $Z(y)$ are both maximal then μ_x and μ_y are equivalent measures. The equivalence class of μ_x, for any maximal cyclic $Z(x)$ is referred to as the maximal spectral type of the action. The *spectrum* of the action $(\mathcal{H}, \{U_g : g \in G\})$ may be defined as the set of characters $\gamma \in \hat{G}$ such that there is a decreasing sequence $\{\mathcal{H}_n\}$ of closed non-trivial G-invariant subspaces of $\mathcal{H}$ such that

$$\| U_g|_{\mathcal{H}_n} - \gamma(g)\, \mathrm{id}_{\mathcal{H}_n} \| \to 0 \quad \text{for all } g \in G.$$

The spectrum of the action coincides with the support of the maximal spectral type. All of the assertions in this paragraph are well known in the case of $\mathbb{Z}$-actions (see for example the appendix of [**P**]) and the proofs are the same in the more general setting.

A sequence $\{g_i\} \subset G$ is said to be *mixing* (for $(\mathcal{H}, \{U_g : g \in G\})$) if

$$\langle U_{g_i} x, y \rangle \to 0 \quad \text{for all } x, y \in \mathcal{H}.$$

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To avoid worrying about repetitions in a sequence it is also convenient to speak of a mixing subset $E \subset G$; that is, $\{g_i\}$ is mixing whenever $\{g_i\} \subset E$ is a sequence of distinct elements. The action $(\mathcal{H}, \{U_g : g \in G\})$ is said to be mixing if G itself is mixing.

$(\mathcal{H}, \{U_g : g \in G\})$ is *weakly mixing* if it has continuous spectrum, that is, there is no vector x which is simultaneously an eigenvector for all the U_g. Equivalently, if the orbit $\{gx : g \in G\}$ is pre-compact in the norm topology of $\mathcal{H}$ then x must be 0. To see this as a mixing notion we need the concepts of Følner sets and density. A sequence of subsets $\{F_n\}$ of G is called a Følner sequence if

$$|F_n|^{-1}|F_n \cap (g + F_n)| \to 0 \quad \text{for all } g \in G.$$

Every countable discrete Abelian G is *amenable*; that is, has Følner sequences. The density of a subset $E \subset G$, with respect to a given Følner sequence $\{F\}_n$, is defined as

$$d(F) = \lim |F_n|^{-1}|F_n \cap E|,$$

if the limit exists. Then $(\mathcal{H}, \{U_g : g \in G\})$ is weakly mixing if and only if

$$\{g : |\langle U_g x, y\rangle| < \epsilon\}$$

has density 1 for all $x, y \in \mathcal{H}$ and $\epsilon > 0$, with respect to any Følner sequence $\{F_n\}$. We assume henceforth that each G comes equipped with a fixed Følner sequence and densities will always refer to that fixed sequence. Weak mixing is equivalent to the existence of a mixing $E \subset G$ of density one and also to the condition that every subset E of positive density contains a mixing subset. (In fact weak mixing is also equivalent to the existence of any mixing subset whatsoever.)

If F is a (finite or infinite) subset of G the set of sums of finite subsets of F will be denoted by $FS(F)$, that is $FS(F)$ consists of all sums $\sum_{g\in F} n_f f$ where $n_f \in \{0, 1\}$, $\sum n_f < \infty$. Note that $0 \in G$ always belongs to $FS(F)$. $FS'(F)$ will denote the corresponding set with the added condition that not all n_f are zero, so $FS'(F)$ need not contain 0. An *IP-set* is any set of the form $FS'(F)$ with F infinite. An *IP*-subset* of G is a set which has non-empty intersection with every IP-subset of G. $FS_q(F)$ will denote the set of finite sums of elements in F in which each element is allowed to have multiplicity no greater than q. That is, the definition of $FS_q(F)$ is the same as that of $FS(F)$, except that $n_f \in \{0, 1, \ldots, q\}$. We will also need to consider the symmetric version $FS_q^{\text{symm}}(F)$ of $FS_q(F)$, that is now $n_f \in \{-q, -q-1, \ldots, 0, \ldots, q\}$.

It is useful to have a recursive description of these sets. For example, if $F = \{f_1, f_2, \ldots\}$, let $F_1 = \{0, f_1, 2f_1, \ldots, qf_1\}$ and

$$F_{i+1} = \bigcup_{k=0}^{q} (F_i + kf_{i+1}).$$

Then $\bigcup_i F_i = FS_q(F)$.

For $F \subset G$ we denote the set of differences $F - F$ by $D(F)$. The following lemma, whose proof is an easy exercise, will be used repeatedly.

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LEMMA 2.1. *If F is any subset of G then*

$$D(FS_m(F)) = FS_m^{\text{symm}}(F)$$

and

$$D^2(FS_m(F)) = FS_{2m}^{\text{symm}}(F).$$

A non-zero vector $x \in \mathcal{H}$ is called *rigid* for $(\mathcal{H}, \{U_g : g \in G\})$ if there is a sequence $\{g_i\} \subset G$ such that $g_i \neq 0$ for all i and $g_i x \to x$. The action is called *mild mixing* if it has no rigid vectors. Equivalently, for all $x, y \in \mathcal{H}$ and $\epsilon > 0$ the set

$$\{g \in G : |\langle gx, y\rangle| < \epsilon\}$$

is IP*. This equivalence is proved for $\mathbb{Z}$-actions in [**F1**, Lemma 9.21], and the proof in general is entirely analogous. We will use the following slightly different characterization of mild mixing. For its proof we introduce some terminology: if F is an infinite subset of G and $F' \subset FS(F)$, let us say that F' is disjoint with respect to F if each $g \in F'$ can be written $g = \sum_{f \in F_g} f$ for some $F_g \subset F$, with the F_g, $g \in F'$ pairwise disjoint. Of course this condition implies that $FS(F') \subset FS(F)$.

LEMMA 2.2. *$(\mathcal{H}, \{U_g : g \in G\})$ is mild mixing if and only if every IP-set $\Gamma \subset G$ contains an IP-subset Γ' which is mixing. (Note that this means simply that Γ' is mixing as a set, not as an IP-system in the sense of [**F**], which would be a weaker statement.)*

Proof. The sufficiency of the condition is immediate. For the other direction let $\{x_i\}$ be a countable dense subset of $\mathcal{H}$ and $\{\epsilon_i\}$ a sequence going to zero. Suppose $\Gamma = FS(F)$. Choose $g_1 \in \Gamma$ such that $\langle g_1 x_1, x_1\rangle < \epsilon_1$. Then choose g_2 such that

$$|\langle g_2 x_i, x_j\rangle| < \epsilon_2 \text{ and } |\langle g_2 x_i, (-g_1)x_j\rangle| < \epsilon_2$$

for $i, j \in \{1, 2\}$. This is possible because the intersection of IP* sets is IP*. At the same time observe that g_2 may be chosen disjoint from g_1 with respect to F, since we can replace Γ by the IP-set generated by the generators not appearing in g_1. Continuing in this way we obtain $F' = \{g_1, g_2, \dots\}$ disjoint with respect to F and then $\Gamma' = FS(F')$ will be the desired set. □

Note that we have in fact proved something slightly more precise which we record here for later use.

LEMMA 2.2′. *If $(\mathcal{H}, \{U_g : g \in G\})$ is mildly mixing and F is any infinite subset of G then there is an infinite $F' \subset FS(F)$ which is disjoint with respect to F and such that $FS(F')$ is mixing.*

3. *Main result and application to unitary actions*

In this section we prove Theorem 1 and the results in the introduction which follow from it.

THEOREM 1. *Suppose μ is a probability measure on the dual $\hat{G}$ of a countable Abelian group G, E is a symmetric subset of G and*

$$\sum_{g \in D(E), g \neq 0} \hat{\mu}(g) < \delta < 1/3.$$

Then the following hold:

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(a) *there is an* $f \in C^+(\hat{G})$ *such that* $\|f - 1\| < \frac{3}{2}\delta$ *and* $\mathcal{F}_E(f\mu) = \delta_0$*;*
(b) *given* $c \in B_{1/3}$ *there is an* $f \in C^+(\hat{G})$ *such that* $\mathcal{F}_E(f\mu) = c$.

Proof. Given $b \in l^1_h(E)$ define $f \in C(\hat{G})$ by

$$f(\gamma) = \sum_{g \in E} b(g)\hat{g}(-\gamma).$$

Note that f is real-valued since b is Hermitian symmetric. Moreover $\|1 - f\|_\infty < \|b - \delta_0\|_1$, so that f will be positive if $\|b - \delta_0\|_1 < 1$. Now, for $h \in E$

$$(\widehat{f\mu})(h) = \int_{\hat{G}} \gamma(h) f(\gamma)\, d\mu(\gamma) = \sum_{g \in E} \int \gamma(h - g) b(g)\, d\mu(\gamma) = \sum_{g \in E} \hat{\mu}(h - g) b(g).$$

So for part (a) we want to solve the matrix equation $Mb = \delta_0$ where M is the matrix

$$M(h, g) = (\hat{\mu}(h - g))_{h,g \in E}.$$

Note that M maps $l^1_h(E)$ to itself:

$$(Mb)(-h) = \sum_{g \in E} \hat{\mu}(-h - g) b(g) = \sum_{g \in E} \hat{\mu}(-h + g) b(-g) = \sum_{g \in E} \overline{\hat{\mu}(h - g) b(g)}.$$

Since μ is a probability measure the diagonal entries of M are all 1 so we can write $M = \mathrm{id} + A$, where A is the matrix of off-diagonal elements. The norm of A as an operator on $l^1_h(E)$ is given by

$$\|A\| = \sup_{g \in E} \sum_{h \in E, h \neq g} |\hat{\mu}(h - g)| \leq \sum_{g \in D(E), g \neq 0} |\hat{\mu}(g)| \leq \delta < 1/3,$$

since for each fixed g each difference in $D(E)$ appears at most once among the $h - g$. It follows that $M^{-1} = \mathrm{id} + A + A^2 + \cdots$ exists and

$$\|M^{-1} - \mathrm{id}\| \leq \|A\| \|\mathrm{id} + A + A^2 + \cdots\| \leq \delta(1 - \delta)^{-1} < \delta\tfrac{3}{2}.$$

Thus if we let $b = M^{-1}\delta_0$ then $b \in l^1_h(E)$ and

$$\|b - \delta_0\| \leq \|M^{-1} - \mathrm{id}\| \|\delta_0\| \leq \tfrac{3}{2}\delta < 1$$

as desired.

For part (b) we set $b = M^{-1}c$ and we just need to check that $\|b - \delta_0\| < 1$:

$$\begin{aligned} \|b - \delta_0\| &\leq \|M^{-1}c - c\| + \|c - \delta_0\| \\ &\leq \|M^{-1} - \mathrm{id}\| \|c\| + \tfrac{1}{3} \leq \tfrac{3}{2}\delta\tfrac{4}{3} + \tfrac{1}{3} < \tfrac{2}{3} + \tfrac{1}{3} = 1. \end{aligned}$$

□

THEOREM 2. *Suppose* $\{U_i\}$ *is a mixing sequence of commuting unitary operators on a Hilbert space* $\mathcal{H}$ *such that either*
(a) U_i^2 *is also mixing, or*
(b) $U_i^2 = \mathrm{id}$ *for all* i.
Then for all $x \in \mathcal{H}$ *and* $\epsilon > 0$ *there is an* $x' \overset{\epsilon}{\sim} x$ *and a subsequence* $\{U_{i_j}\}$ *such that the vectors* $\{Ux' : U \in FP(\{U_{i_j}\})\}$ *are pairwise orthogonal. Here* $FP(E)$ *denotes the set of finite products of elements of* E*, analogous to the finite sum sets.*

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Proof. We assume without loss of generality that $\|x\| = 1$. For case (a) let G be the direct sum of countably many copies of $\mathbb{Z}$, so $\hat{G} = \mathbb{T}^{\mathbb{N}}$. Letting the ith generator g_i of G act by $U_{g_i} = U_i$ we obtain a unitary action of G on $\mathcal{H}$. We will find the vector x' in the cyclic subspace $Z(x)$ generated by x under this action. By the spectral theorem we may assume that $\mathcal{H} = L_2(\mu)$ for some probability measure μ on $\hat{G}$, $x = 1 \in L_2(\mu)$ and U_{g_i} is the multiplication by $\hat{g}_i$. We will look for a unit vector $x' = \phi \overset{\epsilon}{\sim} 1$ in $L_2(\mu)$ and a subsequence $\{h_j = g_{i_j}\}$ such that

$$\langle \hat{g}\phi, \hat{h}\phi \rangle = 0 \quad \text{for all } g, h \in FS\{h_j\},\ g \neq h,$$

that is

$$\mathcal{F}_{D(FS\{h_j\})}(|\phi|^2\mu) = \delta_0.$$

We are going to apply Theorem 1(a) to the set $E = D(FS\{h_j\})$ so recall that $D(E) = FS_2^{\text{symm}}(\{h_j\})$ (Lemma 2.1).

Now, since the sequences $\{qg_i\}$ are mixing for $q = \pm 1, \pm 2$ we can inductively choose $\{h_j\}$ such that

$$\sum_{g \in FS_2^{\text{symm}}(\{h_j\}) \setminus \{0\}} |\hat{\mu}(g)| < \epsilon/2.$$

To do this, once $h_1, \ldots, h_n$ have been chosen, just choose h_{n+1} such that the sum of the moduli of all the inner products $\langle U_{qh_{n+1}} 1, U_g 1 \rangle$ for $q = \pm 1, \pm 2$ and $g \in FS_2^{\text{symm}}(\{h_1, \ldots, h_n\})$ is less than $\epsilon 2^{-(n+1)}$. Having chosen the h_j we apply Theorem 1 to obtain $f \in C^+(\hat{G})$ such that $\|f - 1\|_\infty < \epsilon$ and

$$\mathcal{F}_{D(FS(\{h_j\}))}(f\mu) = \delta_0.$$

Taking $\phi = \sqrt{f}$ we have $|1 - \phi| = |1 - f|/|1 + \sqrt{f}| \leq |1 - f| < \epsilon$, so $\|1 - \phi\|_2 < \epsilon$, as desired.

In case (b) rather than an action of the sum of copies of $\mathbb{Z}$ we obtain an action of the sum of copies of $\mathbb{Z}_2$, so in this case

$$D(E) = FS_2^{\text{symm}}(\{h_j\}) = FS(\{h_j\}).$$

This means that in this situation mixing of the sequence $\{g_i\}$ by itself suffices to choose $\{h_j\}$ so that

$$\sum_{g \in D(E) \setminus \{0\}} |\hat{\mu}(g)| < \epsilon/2. \qquad \square$$

THEOREM 3. *A unitary G-action is mixing if and only if for all infinite $E \subset G$, $x \in \mathcal{H}$ and $\epsilon > 0$ there exist an infinite $F \subset E$ and $x' \overset{\epsilon}{\sim} x$ such that the set $\{U_g x' : g \in F\}$ is pairwise orthogonal.*

Proof. The sufficiency of the condition is straightforward. To prove the other direction suppose first that $2E$ is infinite. We may then choose a sequence $\{g_i\} \subset E$ such that the sequence $\{2g_i\}$ contains no repetitions and hence must be mixing. The result then follows immediately from Theorem 2(a), indeed in this case we get the fact that $\{U_g x' : g \in FS(F)\}$ is pairwise orthogonal.

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Now suppose that $2E$ is finite. By passing to a subset we may assume that $2E = \{2g_0\}$ for some $g_0 \in E$, so $2(E - g_0) = 0$. Since $E - g_0$ is also mixing Theorem 2(b) gives us an infinite $F \subset E$ such that

$$\{U_g x' : g \in FS(F - g_0)\}$$

is pairwise orthogonal. Unfortunately $FS(F-g_0)$ is not the same as $FS(F)-g_0$ but we do at least get that $\{U_g x' : g \in F-g_0\}$ is pairwise orthogonal, and hence so is $\{U_g x' : g \in F\}$. □

For the proof of Theorem 4 we will use the following lemma from [**BKM**], whose proof we include for the reader's convenience.

LEMMA 3.1. *Suppose* Γ *is an IP-subset of* G *and* 2Γ *is finite. Then there is an IP-subset* $\Gamma' \subset \Gamma$ *such that* $2\Gamma' = \{0\}$.

Proof. If we set $\Gamma_h = \{g \in \Gamma : 2g = h\}$ then we have $\Gamma = \cup\{\Gamma_h : h \in 2\Gamma\}$. Therefore by Hindman's theorem [**F1**, Theorem 8.11] one of the sets Γ_h, say Γ_{h_0}, contains an IP-set Γ', so $2\Gamma' = \{h_0\}$. If g_1 and g_2 are two of the generators of Γ' then we have

$$2g_1 = 2g_2 = 2(g_1 + g_2) = h_0,$$

so $h_0 = 0$. □

THEOREM 4. *An action* $(\mathcal{H}, \{U_g : g \in G\})$ *is mildly mixing if and only if for all IP-sets* $\Gamma \subset G, x \in \mathcal{H}$ *and* $\epsilon > 0$ *there exist an IP-subset* $\Gamma' \subset \Gamma$ *and* $x' \overset{\epsilon}{\sim} x$ *such that* $\{U_g x' : g \in \Gamma'\}$ *is pairwise orthogonal.*

Proof. The sufficiency of the condition is an exercise. For the other direction, suppose first that there exists an IP-subset $\tilde{\Gamma} \subset \Gamma$ such that $2\tilde{\Gamma}$ is finite. By Lemma 3.1, we may as well assume that $2\tilde{\Gamma} = \{0\}$. By Lemma 2.2, $\tilde{\Gamma}$ must contain an infinite subset F such that $FS'(F)$ is mixing and contained in $\tilde{\Gamma}$. Applying Theorem 2(b) to $F = \{g_i\}$ finishes the proof since $FS'(\{g_{i_j}\}) \subset FS'(F) \subset \tilde{\Gamma} \subset \Gamma$.

Now suppose there is no $\tilde{\Gamma} \subset \Gamma$ such that $2\tilde{\Gamma}$ is finite. By Lemma 2.2 there is a mixing IP-set $\tilde{\Gamma} = FS'(F) \subset \Gamma$. If $2F$ is infinite we may assume, by dropping to a subset of F, that the map $f \mapsto 2f$ is one-to-one on F, so that $2\tilde{\Gamma} = FS'(2F)$ is an IP-set. By Lemma 2.2′ we may now find a subset F' of $\tilde{\Gamma}$ which is disjoint with respect to F and such that $2F'$ is mixing. Now both F' and $2F'$ are mixing so we may apply Theorem 2′ to obtain $F'' \subset F'$ and $x' \overset{\epsilon}{\sim} x$ such that the set $\{gx' : g \in FS'(F'')\}$ is pairwise orthogonal. Since $FS'(F'') \subset FS'(F) \subset \tilde{\Gamma} \subset \Gamma$ we are done.

Finally, if $2F$ is finite we must nonetheless have $2\tilde{\Gamma}$ infinite. Let F' denote the set of elements of F having infinite order and $F'' = F \backslash F'$. Then

$$2FS'(F) \subset 2FS'(F') + 2FS'(F'') \subset 2FS'(F') + G(2F'').$$

Since $2F''$ is finite and consists of elements of finite order $G(2F'')$ is finite so we must have F' infinite. Replacing F by F' we now have that each element of F has infinite order. Since $2F$ is finite, by dropping to a further subset we may assume that $2F$ is a singleton, say $\{h\}$. It follows that

$$2\tilde{\Gamma} = \{nh : n \in \mathbb{N}\} = FS'(\{2^i h : i \in \mathbb{N}\})$$

is an IP-set, so the proof can now be finished exactly as in the last case. □

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THEOREM 5. *Suppose G is a group such that $G/2G$ is finite. Then $(\mathcal{H}, \{U_g : g \in G\})$ is weakly mixing if and only if for all sets E of positive density in G, $x \in \mathcal{H}$ and $\epsilon > 0$ there are an infinite $F \subset E$ and an $x' \overset{\epsilon}{\sim} x$ such that $\{U_g x' : g \in FS(F)\}$ is pairwise orthogonal.*

Proof. The sufficiency of the condition is again an exercise. For the other direction first observe that the action $g \mapsto U_{2g}$ is weakly mixing, by the hypothesis that $G/2G$ is finite and the characterization of weak mixing as absence of pre-compact orbits. Intersecting mixing sets of density one for the actions $g \mapsto U_g$ and $g \mapsto U_{2g}$ respectively we obtain a set D of density one which is mixing for both actions. Then $D \cap E$ has positive density so we may extract from it a sequence $\{g_i\}$ without repetitions. This means both $\{g_i\}$ and $\{2g_i\}$ are mixing so we can apply Theorem $2'$. □

THEOREM 7. *Suppose $(\mathcal{H}, \{U_g : g \in G\})$ is an action, $m \geq 2$ is a positive integer and $\{g_i\}$ is a sequence such that for each $q \in \{1, \ldots, m\} \cup 2\{1, \ldots, m\}$ the sequence $\{qg_i\}$ is mixing. Then for all $x \in \mathcal{H}$ and $\epsilon > 0$ there exist an $x' \overset{\epsilon}{\sim} x$ and a subsequence $\{g_{i_j}\}$ such that the set*

$$\{U_g x' : g \in FS_m(\{g_{i_j}\})\}$$

is pairwise orthogonal.

Proof. This is almost the same as Theorem 2. Recall that

$$D^2(FS_m(\{g_{i_j}\})) = FS_{2m}^{\text{symm}}(\{g_{i_j}\})$$

so the mixing we have assumed is precisely what is needed to inductively choose $\{g_{i_j}\}$ so that the sum of the appropriate Fourier coefficients over the set $D^2(FS_m(\{g_{i_j}\}))\backslash\{0\}$ is small and then apply Theorem 1. □

THEOREM 6. *Suppose G is the direct sum of countably many copies of $\mathbb{Z}_p$, p prime and for $E \subset G$ let $G(E)$ denote the subgroup generated by E. Then an action $(\mathcal{H}, \{U_g : g \in G\})$ is:*

(a) *weakly mixing if and only if for all subsets $F \subset G$ of positive density and $x \in \mathcal{H}$ there are an infinite subset $F' \subset F$ and $x' \overset{\epsilon}{\sim} x$ such that the set $\{U_g x' : g \in G(F')\}$ is pairwise orthogonal;*

(b) *mildly mixing if and only if for all IP-sets $F \subset G$ and $x \in \mathcal{H}$ there are an $x' \overset{\epsilon}{\sim} x$ and an infinite $F' \subset \Gamma$ such that the set $\{U_g x' : g \in G(F')\}$ is pairwise orthogonal;*

(c) *mixing if and only if for all infinite sets $F \subset G$ and $x \in \mathcal{H}$ there are an $x' \overset{\epsilon}{\sim} x$ and an infinite subset $F' \subset F$ such that the set $\{U_g x' : g \in G(F')\}$ is pairwise orthogonal.*

Proof. Evidently $G(F) = FS_{p-1}(F)$. Thus, in all cases, the result will follow from Theorem 7 if we can find a sequence $\{g_i\} \subset F$ such that $\{qg_i\}$ is mixing for all

$$q \in \{1, \ldots, p-1\} \cup 2\{1, \ldots, p-1\}.$$

First let us suppose $p \neq 2$. Then $2\{1, \ldots, p-1\} = \{1, \ldots, p-1\}$ in $\mathbb{Z}_p$ so it suffices to have $\{qg_i\}$ mixing for all $q \in \{1, \ldots, p-1\}$. In the mixing case we can simply take $\{g_i\}$ to be any infinite sequence in E, since the maps $g \mapsto qg$ are all one-to-one.

In the mild mixing case, since the maps $g \mapsto qg$ are group automorphisms we see that the actions $g \mapsto U_{qg}$ are all mild mixing. Thus we may successively choose IP-sets

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$F \supset \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_{p-1}$ such that Γ_q is mixing for $g \mapsto U_{qg}$, so Γ_{p-1} is mixing for all of them. We then take $\{g_i\}$ to be any sequence in Γ_{p-1} having no repetitions.

The proof in the weakly mixing case is exactly the same, with sets of positive density replacing IP-sets.

When $p = 2$ we have $G(F) = FS(F)$ so in all cases the result follows directly from Theorem 2(b). □

4. *The specification property and proofs of Theorems 8, 9 and 10*

THEOREM 8. *There is a probability measure μ on the circle group $\mathbb{T}$ and a sequence $\{n_i\}$ such that $\langle z^{n_i} f, g\rangle \to 0$ for all $f, g \in L^2(\mu)$ but $\langle z^{n_i} f, f\rangle \neq 0$ for all non-zero $f \in L^2(\mu)$.*

Proof. We begin with the observation that if μ is a probability measure and $f \in L^2(\mu)$ with $\|f\|_2 = 1$ then

$$\langle z^n f, f\rangle = \int_{\mathbb{T}} z^n |f|^2 \, d\mu$$

is the centroid of the probability measure $\tau_n(|f|^2\mu)$, where τ_n denotes the map $z \mapsto z^n$. Let us denote the centroid of a probability measure ν on $\mathbb{T}$ by $C(\nu)$. By the polarization identity in order to show that $\langle z^{n_i} f, g\rangle \to 0$ for all $f, g \in L^2(\mu)$ it suffices to show that $C(\tau_{n_i}\nu) \to 0$ for all probability measures $\nu \ll \mu$.

Let Δ denote the set of finite sequences of zeroes and ones. We will construct a Cantor set $C \subset \mathbb{T}$ by defining for each δ in Δ, a closed arc I_δ of positive length such that $I_{\delta 0}$ and $I_{\delta 1}$ are disjoint and contained in I_δ and the length of I_δ goes to zero as the length of δ goes to infinity. This defines in a standard way an injective continuous map $\theta : \{0,1\}^{\mathbb{N}} \to \mathbb{T}$. C will be $\theta(\{0,1\}^{\mathbb{N}})$ and μ will be the standard Cantor measure on C, that is μ is the image under θ of $(1/2, 1/2)$-product measure on $\{0,1\}^{\mathbb{N}}$.

For $s, t \in [0,1]$, $E[s,t]$ will denote the arc which is the image of the interval $[s,t]$ under the exponential map. The definition of the arcs I_δ will proceed by induction on the length of δ. We start the induction by letting I, the arc corresponding to the empty sequence, be $\mathbb{T}$. Now suppose that I_δ has been defined for all δ of length less than or equal to i.

Obviously if J is an arc of positive length then for all sufficiently large n, $\tau_n J = \mathbb{T}$. It follows that we can find n_i and, for each δ of length i, disjoint closed arcs of positive length $I_{\delta 0}$ and $I_{\delta 1}$ contained in I_δ such that

$$\tau_{n_i} I_{\delta 0} \subset E[0, 1/i]$$
$$\tau_{n_i} I_{\delta 1} \subset E[1/2 - 1/i, 1/2].$$

To check that $C(\tau_{n_i}\nu) \to 0$ for all probabilities $\nu \ll \mu$ it suffices to check that

$$C(\tau_{n_i}\mu_{I_\delta}) \to 0 \quad \text{for all } \delta \in \Delta.$$

Now our construction ensures that for δ of length i, $\tau_{n_i}(\mu_{I_\delta})$ has mass $1/2$ in $E[0, 1/i]$ and $1/2$ in $E[1/2 - 1/i, 1/2]$, so its centroid is certainly within $2\pi/i$ of zero. For any extension δ' of δ of length $j > i$ we similarly have that

$$C(\tau_{n_j}(\mu_{I_{\delta'}})) \overset{2\pi/j}{\sim} 0.$$

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But $C(\tau_{n_j}(\mu_{I_\delta}))$ is just the average of the $C(\tau_{n_j}(\mu_{I_{\delta'}}))$, where δ' runs over the extensions of length j of δ, so it is also within $2\pi/j$ of 0. This gives the desired convergence. On the other hand $\tau_{n_j}\mu$ is supported in the open upper half-circle; so for any probability $\nu \ll \mu$ the centroid of $\tau_{n_j}\nu$ is in the open upper half-plane, so it is not zero. □

For the proof of Theorem 9 we will need a new definition and several lemmas.

Given a probability measure μ on $\hat{G}$ we denote by $P(\mu)$ the set of probability measures absolutely continuous with respect to μ. Given such a μ and a symmetric subset $E \subset G$ we will say μ has *α-specification* on E if for every $c \in B_\alpha E$ there is a measure $\mu_1 \in P(\mu)$ such that $\mathcal{F}_E(\mu_1) = c$. We will say μ has *finite α-specification* on E if for every finite symmetric $F \subset E$ and $c \in B_\alpha(F)$ there is a measure $\mu_1 \in P(\mu)$ such that $\mathcal{F}_F(\mu_1) = c$. Note that the conclusion of Theorem 1(b) says that μ has $1/3$-specification on E. The following result is an easy consequence of Theorem 1.

PROPOSITION 4.1. *Suppose μ is mixing and E is an infinite subset of G such that $2E$ is also infinite. Then there is an infinite subset F of E such that μ has $1/3$-specification on $D(FS(F))$.*

Proof. Since μ is mixing we can inductively construct, in the manner of Theorem 2, an infinite $F \subset E$ such that

$$\sum_{g \in D^2(FS(F)), g \neq 0} |\hat{\mu}(g)| < 1/3,$$

so the result follows from Theorem 1(b). □

We shall see that finite specification is a property of the support of μ and not of μ itself. For this we need several lemmas about Borel measures on a compact metric space X. By a partition of X we mean a finite partition $P = \{P_1, \ldots, P_n\}$ of X into Borel sets and by the diameter of P we mean the maximum of the diameters of the P_i.

LEMMA 4.2. *If X is compact metric and $\epsilon > 0$ then there is a partition $P = \{P_1, \ldots, P_n\}$ with diameter less than ϵ such that each P_i has non-empty interior.*

Proof. Cover X by open sets $U_1, \ldots, U_n$ of diameter less than ϵ, let $C_i = \overline{U_1} \cup \cdots \cup \overline{U_i}$ and $P_i = C_i \backslash C_{i-1}$. Then each P_i is either empty or has non-empty interior and so we need only throw out those P_i which happen to be empty. □

LEMMA 4.3. *Suppose $\{P^n\}$ is a sequence of partitions of X whose diameters go to zero, ν is a Borel probability measure on X, and $\{\mu_n\}$ is a sequence of Borel probability measures on X such that*

$$\mu_n(A) = \nu(A) \quad \text{for all } A \in P_n.$$

Then $\mu_n \to \nu$ vaguely, that is, in the weak- topology.*

Proof. If $f \in C(X)$ then, for all sufficiently large n, f is within ϵ in uniform norm of a bounded measurable function f_n which is constant on the atoms of P_n. Thus for large n we have

$$\mu_n(f) \overset{\epsilon}{\sim} \mu_n(f_n) = \nu(f_n) \overset{\epsilon}{\sim} \nu(f).$$ □

We denote the support of a Borel measure μ on X by $\operatorname{supp}\mu$

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LEMMA 4.4. *Suppose μ and ν are probability measures on a compact metric space X and* $\operatorname{supp}\nu \supset \operatorname{supp}\mu$. *Then there is a sequence $\{\nu_n\} \in P(\nu)$ such that $\nu_n \to \mu$ vaguely.*

Proof. By restricting to $\operatorname{supp}\mu$ we may assume that $\operatorname{supp}\mu = \operatorname{supp}\nu = X$. Choose a sequence $\{P^n\}$ of partitions whose atoms have non-empty interiors and whose diameters go to zero. Let $\nu_n = f_n\nu$ where f_n is constant on each atom A of P^n with value $\mu(A)/\nu(A)$. This means that $\mu_n(A) = \nu(A)$ so the result follows from Lemma 4.2. □

PROPOSITION 4.5. *If μ and ν are probabilities on $\hat{G}$ such that* $\operatorname{supp}\nu \supset \operatorname{supp}\mu$ *and μ has finite α-specification on a symmetric $E \subset G$, then so does ν.*

Proof. Suppose F is a finite symmetric subset of E and $c \in B_\alpha(F)$, so there is a $\lambda \in P(\mu)$ such that $\mathcal{F}_F(\lambda) = c$. Use Lemma 4.4 to find probabilities $\nu_n \ll \nu$ such that $\nu_n \to \lambda$. Then $\mathcal{F}_F(\nu_n) \to \mathcal{F}_F(\lambda) = c$. This shows that the convex set

$$\{\mathcal{F}_F(\nu') : \nu' \in P(\nu)\}$$

is dense in $B_\alpha F$. The result then follows from Lemma 4.6 below, whose proof is geometrically clear. □

LEMMA 4.6. *If a convex subset C of a finite-dimensional vector space is dense in an open subset O then $C \supset O$.*

The following result is an immediate consequence of Propositions 4.1 and 4.5.

PROPOSITION 4.7. *Suppose* $\operatorname{supp}\mu$ *is mixing and E is an infinite subset of G such that $2E$ is also infinite. Then there is an infinite subset F of E such that μ has finite $1/3$-specification on $D(FS(F))$.*

The next result says that finite specification implies a property which might be called stable finite specification. (The same is true for specification, but we will not need this.)

PROPOSITION 4.8. *Suppose that μ has finite α-specification on F, E is a finite symmetric subset of F, $c \in B_{\alpha/2}(E)$, $\nu \in P(\mu)$ and $\mathcal{F}_E(\nu) \overset{\alpha\epsilon/4}{\sim} c$. Then there is a $\nu' \in P(\mu)$ such that $\|\nu' - \nu\| < \epsilon$ and $\mathcal{F}_E(\nu') = c$.*

Proof. Define $b \in l^1_h(E)$ by the equation

$$c = \left(1 - \frac{\epsilon}{2}\right)\mathcal{F}_E(\nu) + \frac{\epsilon}{2}b.$$

Note that our assumptions on c and ν ensure that

$$\|b - c\| = \frac{1 - \epsilon/2}{\epsilon/2}\|c - \mathcal{F}_E(\nu)\| \le \frac{\alpha}{2}.$$

Since $c \in B_{\alpha/2}(E)$ we get $b \in B_\alpha(E)$. Thus we can find $\rho \in P(\mu)$ such that $\mathcal{F}_E(\rho) = b$. Let

$$\nu' = \left(1 - \frac{\epsilon}{2}\right)\nu + \frac{\epsilon}{2}\rho.$$

Then we have $\mathcal{F}_E(\nu') = c$ and

$$\|\nu - \nu'\| < \frac{\epsilon}{2}\|\nu\| + \frac{\epsilon}{2}\|\rho\| = \epsilon.$$ □

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THEOREM 9. *Suppose G is a group such that $\{g : 2g = 0\}$ is finite, $(\mathcal{H}, \{U_g : g \in G\})$ is an action whose spectrum is a mixing subset of $\hat{G}$ and $\{g_i\}$ is mixing for this action. Then for all $x \in \mathcal{H}$ and $\epsilon > 0$ there are an $x' \overset{\epsilon}{\sim} x$ and a subsequence $\{g_{i_j}\}$ such that $\{U_g x' : g \in FS(\{g_{i_j}\})\}$ is pairwise orthogonal.*

Proof. $x \in Z(y)$ for some maximal cyclic $Z(y)$. Let $\mu = \mu_y$ and note that $\operatorname{supp}\mu$ is a mixing subset of $\hat{G}$, since it coincides with the spectrum of the action. Using the spectral theorem we may assume that U_g is $M_{\hat{g}}$ on $L^2(\mu)$ and x is a function $\phi \in L^2(\mu)$. There is no harm in assuming also that $\|\phi\|_2 = 1$. The assumption on G ensures that $\{2g_i\}$ is infinite so we can apply Proposition 4.7 to find an infinite subset F of $\{g_i\}$ such that μ has finite $1/3$- specification on $D(FS(F))$.

We now inductively define the sequence $\{g_{i_j}\} = \{h_j\} \subset F$ together with a sequence of functions $\{f_j\} \subset L^1_+(\mu)$ such that $f_0 = |\phi^2|$ and $\int f_j \, d\mu = 1$. Setting

$$D_j = D(FS(\{h_1, \ldots, h_j\})) = FS^{\text{symm}}(\{h_1, \ldots, h_j\})$$

and $D_\infty = D(FS(\{h_1, h_2, \ldots\}))$ we will require that

$$\|f_{j+1} - f_j\|_1 < \delta/2^j,$$

for a δ to be specified later, and

$$\mathcal{F}_{D_j}(f_j \mu) = 0.$$

Suppose that $h_1, \ldots, h_j$ and $f_1, \ldots, f_j$ have already been defined. Since F is a mixing set we can find $h_{j+1} \in F$ such that the Fourier coefficients

$$\widehat{f_j \mu}(\pm h_{j+1} - g), \quad g \in D_j$$

are all as small as we please. Then applying Proposition 4.8 we obtain $f_{j+1} \overset{\delta/2^j}{\sim} f_j$ such that $\mathcal{F}_{D_{j+1}}(f_{j+1}\mu) = 0$.

Then the f_j converge in L^1 to a limit $f \overset{\delta}{\sim} f_0$ for which $\mathcal{F}_{D_j}(f\mu) = 0$ for all j. Setting $x' = \phi' = f^{1/2}\operatorname{sgn}\phi$ we will then have that all the required inner products are zero so we need only show that if δ is sufficiently small then $\|\phi - \phi'\|_2 < \epsilon$.

Since

$$\|\phi - \phi'\|_2^2 = \|f_0^{1/2} - f^{1/2}\|_2^2 = 2 - 2\int f_0^{1/2} f^{1/2} \, d\mu$$

and $\int f_0^{1/2} f^{1/2} \, d\mu \leq 1$, it suffices to show that $\int f_0^{1/2} f^{1/2} \, d\mu$ is almost 1 for δ sufficiently small.

Setting $X^* = \{|f_0 - f| < \sqrt{\delta}\}$ we have $\mu(X^*) > 1 - \sqrt{\delta}$ and

$$\begin{aligned} \int f_0^{1/2} f^{1/2} \, d\mu &\geq \int_{X^*} \min(f_0, f) \, d\mu \\ &\geq \int_{X^*} f_0 \, d\mu - \sqrt{\delta}. \end{aligned}$$

Even though X^* depends on f, by absolute continuity if δ is sufficiently small then $\mu(X \backslash X^*) < \sqrt{\delta}$ will be small enough to ensure that $\int_{X^*} f_0 \, d\mu$ is close to 1. □

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THEOREM 10. *Suppose* $(\mathcal{H}, \{U_g : g \in G\})$ *is an action whose spectrum is a mixing subset of* $\hat{G}$ *and* $\{g_i\}$ *is mixing for this action. Then for all* $x \in \mathcal{H}$ *there are an* $x' \overset{\epsilon}{\sim} x$ *and a subsequence* $\{g_{i_j}\}$ *such that*

$$\{U_g x' : g \in \{g_{i_j}\}\}$$

is pairwise orthogonal.

Proof. If $\{2g_i\}$ is infinite then the proof of Theorem 9 applies. Otherwise we may assume that for all i, $2(g_i - g) = 0$ for some $g \in G$ and then Theorem 2(b) gives $\{g_{i_j}\}$ such that

$$\{U_g x : g \in FS(\{g_{i_j} - g\})\}$$

is pairwise orthogonal. It follows that $\{U_g x : g \in \{(g_{i_j} - g)\}\}$, and hence also $\{U_g : g \in \{g_{i_j}\}\}$, is pairwise orthogonal. □

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