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Multifractal analysis of ergodic averages in some non-uniformly hyperbolic systems

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Abstract. This article is devoted to the study of the multifractal analysis of ergodic averages in some non-uniformly hyperbolic systems. In particular, our results hold for the robust classes of multidimensional non-uniformly expanding local diffeomorphisms and Viana maps.

1. *Introduction and preliminaries*

The multifractal analysis of dynamical systems is a subfield of the dimension theory of dynamical systems. Roughly speaking, multifractal analysis studies the complexity of the level sets of invariant local quantities obtained from a dynamical system.

Barreira *et al* [**2**] introduced the general concept of multifractal spectrum, as follows.

Fix a metric space X and a set Y and let $\Psi : X \to Y$ be a map. Recently, the following problem, known as multifractal analysis of the map Ψ, has attracted considerable interest. What is the Hausdorff dimension or the topological entropy and so on of the level sets of Ψ, i.e. what is the Hausdorff dimension or the topological entropy and so on of the following so-called multifractal decomposition sets of Ψ:

$$E(t) = \{x \in X : \Psi(x) = t\}, \quad t \in Y? \tag{1.1}$$

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For a topological dynamical system (X, d, T) (or (X, T) for short) consisting of a compact metric space (X, d) and a continuous map $T : X \to X$, let $\Psi(x) = \lim_{n\to\infty} (1/n) \sum_{i=0}^{n-1} \phi(T^i x)$ for some continuous function $\phi : X \to \mathbb{R}$. Then there are fruitful results about the descriptions of the structure (Hausdorff dimension, topological entropy or topological pressure) of the level sets of Ψ in topological dynamical systems. Early studies of the level sets were about their dimensions and topological entropy (see [**3**, **4**, **15**–**17**, **19**, **20**, **25**, **36**, **24**]). Recently, the topological pressures of the level sets have also been investigated (see [**8**, **21**, **26**, **35**]). Olsen [**15**–**17**] and Olsen and Winter [**19**] established a framework which can be viewed as the finer description of multifractal analysis. More precisely, they investigated the following subset:

$$E(I) = \{x \in X : \Psi(x) \subset I\}, \quad I \subset Y. \tag{1.2}$$

One can also see from the previous works [**6**, **35**] that they considered the topological entropy and topological pressure of $E(I)$.

Now, non-uniformly hyperbolic systems attract more and more attention. We refer the reader to [**1**, **7**, **9**–**14**, **32**] and references therein for recent results in non-uniformly hyperbolic systems. It is well known that the specification property plays an important role in some uniformly hyperbolic dynamical systems. The notion of specification is slightly weaker than the one introduced by Bowen that requires any finite sequence of pieces of orbit to be well approximated by periodic orbits. It implies that the dynamical systems have some mixing property. One should mention that other mild forms of specification were introduced by Pfister and Sullivan [**24**] and Thompson [**28**] to the study of multifractal formalism for Birkhoff averages associated to beta-shifts, and by Pfister and Sullivan [**23**], Yamamoto [**33**] and Varandas [**29**] to study large deviations. In this paper we will use the weak form of specification introduced by Varandas [**29**] to study the topological pressure of (1.2). In [**5**], Bomfim and Varandas studied the upper bound of the topological pressure of $E(I)$. This article is mainly devoted to estimates for the lower bound of the topological pressure of $E(I)$ in some non-uniformly hyperbolic systems.

Let (X, T) be a topological dynamical system and $C(X)$ the space of continuous functions from X to $\mathbb{R}$. For $\varphi \in C(X)$ and $n \geq 1$, let $S_n\varphi(x) := \sum_{i=0}^{n-1} \varphi(T^i x)$ and, for $c > 0$, let $\mathrm{Var}(\varphi, c) := \sup\{|\varphi(x) - \varphi(y)| : d(x, y) \leq c\}$. Denote by $M(X)$ and $M(X, T)$ the set of all Borel probability measures on X and the collection of all T-invariant Borel probability measures, respectively. It is well known that $M(X)$ and $M(X, T)$ equipped with weak* topology are both convex, compact spaces. Denote by $A(x_n)$ the limit-point set of $\{x_n\}$.

Definition 1.1. [**29**] We say that (T, m) satisfies the non-uniform specification property with a time lag $p(x, n, \epsilon)$ if there exists $\delta > 0$ such that for m-almost every x and every $0 < \epsilon < \delta$ there exists an integer $p(x, n, \epsilon) \geq 1$ satisfying

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} p(x, n, \epsilon) = 0$$

and so that the following holds: given points $x_1, \ldots, x_k$ in a full m-measure set and positive integers $n_1, \ldots, n_k$, if $p_i \geq p(x_i, n_i, \epsilon)$, then there exists z that ϵ-shadows the

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orbits of each x_i during n_i iterates with a time lag $p(x_i, n_i, \epsilon)$ in between $T^{n_i}(x_i)$ and x_{i+1}, that is,

$$z \in B_{n_1}(x_1, \epsilon) \quad \text{and} \quad T^{n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(z) \in B_{n_i}(x_i, \epsilon)$$

for every $2 \le i \le k$, where

$$B_n(x, \epsilon) = \{y : d_n(x, y) < \epsilon\} := \left\{y : \max_{0 \le i < n} \{d(T^i x, T^i y)\} < \epsilon\right\}.$$

We assume that the shadowing property holds on a set K with a time lag $p(x, n, \epsilon)$ throughout the paper. We may assume that $p(x, n, \epsilon)$ is non-increasing with ϵ, i.e. for any $x \in K$ and $0 < \epsilon_1 < \epsilon_2$, $p(x, n, \epsilon_1) \ge p(x, n, \epsilon_2)$. Hence, for any $x \in K$ and $\epsilon > 0$, $\limsup_{n\to\infty} (p(x, n, \epsilon)/n) = 0$. Let $M(K, T)$ denote the subset of $M(X, T)$ for which the measures μ satisfy $\mu(K) = 1$ and $E(K, T)$ denote those which are ergodic.

Definition 1.2. [**22**] Suppose that $Z \subset X$ is an arbitrary Borel set and $\psi \in C(X)$. Let $\Gamma_n(Z, \epsilon)$ be the collection of all finite or countable covers of Z by sets of the form $B_m(x, \epsilon)$, with $m \ge n$. Let $S_n\psi(x) := \sum_{i=0}^{n-1} \psi(T^i x)$. Set

$$M(Z, t, \psi, n, \epsilon) := \inf_{\mathcal{C} \in \Gamma_n(Z,\epsilon)} \left\{ \sum_{B_m(x,\epsilon) \in \mathcal{C}} \exp\left(-tm + \sup_{y \in B_m(x,\epsilon)} S_m\psi(y)\right)\right\}$$

and

$$M(Z, t, \psi, \epsilon) = \lim_{n\to\infty} M(Z, t, \psi, n, \epsilon).$$

Then there exists a unique number $P(Z, \psi, \epsilon)$ such that

$$P(Z, \psi, \epsilon) = \inf\{t : M(Z, t, \psi, \epsilon) = 0\} = \sup\{t : M(Z, t, \psi, \epsilon) = \infty\}.$$

The number $P(Z, \psi) = \lim_{\epsilon\to 0} P(Z, \psi, \epsilon)$ is called the topological pressure of Z with respect to ψ.

It is obvious that the following hold:

(1) $P(Z_1, \psi) \le P(Z_2, \psi)$ for any $Z_1 \subset Z_2 \subset X$;

(2) $P(Z, \psi) = \sup_i P(Z_i, \psi)$, where $Z = \bigcup_i Z_i \subset X$.

Let $\varphi : X \to \mathbb{R}$ be a continuous function and $M \subset \mathbb{R}$. We denote

$$X(\varphi, M) := \left\{x \in X : A\left(\frac{1}{n}\sum_{i=0}^{n-1} \varphi(T^i x)\right) \subset M\right\}$$

and

$$X'(\varphi, M) := \left\{x \in X : A\left(\frac{1}{n}\sum_{i=0}^{n-1} \varphi(T^i x)\right) \cap M \ne \emptyset\right\}.$$

Now, we state the main result of this article, as follows.

THEOREM 1.1. *Suppose that (X, T) is a topological dynamical system satisfying the non-uniform specification with a time lag $p(x, n, \epsilon)$. Let $U \subset \mathbb{R}$ be an open set. If φ and ψ are*

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two continuous functions on X, then

$$P(X(\varphi, U), \psi) \le \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(X, T) \text{ and } \int \varphi \, d\nu \in U\right\}$$

and

$$P(X(\varphi, U), \psi) \ge \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(K, T) \text{ and } \int \varphi \, d\nu \in U\right\},$$

where h_ν is the metric entropy of the measure ν and $P(Z, \psi)$ denotes the topological pressure of ψ with respect to the set Z. Furthermore, if $K \supset R(T)$ (the collection of all recurrent points for T), we have

$$P(X(\varphi, U), \psi) = \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(X, T) \text{ and } \int \varphi \, d\nu \in U\right\}.$$

COROLLARY 1.1. *Suppose that (X, T) is a topological dynamical system satisfying the non-uniform specification with a time lag $p(x, n, \epsilon)$. If $\varphi, \psi \in C(X)$ and $I \subset \mathbb{R}$ is a closed convex subset satisfying*

$$\operatorname{int} I \cap \left\{\int \varphi \, d\mu : \mu \in M(K, T)\right\} \ne \emptyset,$$

then

$$P(X(\varphi, I), \psi) \le \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(X, T) \text{ and } \int \varphi \, d\nu \in I\right\}$$

and

$$P(X(\varphi, I), \psi) \ge \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(K, T) \text{ and } \int \varphi \, d\nu \in I\right\}.$$

Furthermore, if $K \supset R(T)$ (the collection of all recurrent points for T), we have

$$P(X(\varphi, I), \psi) = \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(X, T) \text{ and } \int \varphi \, d\nu \in I\right\}.$$

COROLLARY 1.2. *Suppose that (X, T) is a topological dynamical system satisfying the non-uniform specification with a time lag $p(x, n, \epsilon)$. Then, for any $\varphi, \psi \in C(X)$,*

$$\begin{aligned}&\lim_{\delta \to 0} P(X(\varphi, (\alpha - \delta, \alpha + \delta)), \psi)\\ &\quad \ge \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(K, T) \text{ and } \int \varphi \, d\nu = \alpha\right\}.\end{aligned}$$

If the map $\mu \mapsto h_\mu$ is upper semicontinuous, we have

$$\begin{aligned}&\lim_{\delta \to 0} P(X(\varphi, (\alpha - \delta, \alpha + \delta)), \psi)\\ &\quad \le \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(X, T) \text{ and } \int \varphi \, d\nu = \alpha\right\}.\end{aligned}$$

Furthermore, if $K \supset R(T)$ (the collection of all recurrent points for T), we have

$$\begin{aligned}&\lim_{\delta \to 0} P(X(\varphi, (\alpha - \delta, \alpha + \delta)), \psi)\\ &\quad = \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(X, T) \text{ and } \int \varphi \, d\nu = \alpha\right\}.\end{aligned}$$

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COROLLARY 1.3. *Suppose that* (X, T) *is a topological dynamical system satisfying the non-uniform specification with a time lag* $p(x, n, \epsilon)$. *Let* $\varphi, \psi \in C(X)$ *and* $J \subset \mathbb{R}$ *be either an open set or a closed convex set satisfying* $\operatorname{int} J \cap \{\int \varphi \, d\mu : \mu \in M(K, T)\} \neq \emptyset$. *If there exists* $\mu \in M(K, T)$ *such that* μ *is an equilibrium state for* T *with respect to potential* ψ *and* $\int \varphi \, d\mu \in J$, *then* $P(X(\varphi, J), \psi) = P(X, \psi)$.

The paper is organized as follows. In §2, we prove our main results. In §3, we apply the main results to the robust classes of multidimensional non-uniformly expanding local diffeomorphisms and Viana maps.

2. *Proof of main results*

In this section, we will prove our main results. Firstly, we prove Theorem 1.1. The upper bound on $P(X(\varphi, U), \psi)$ is easy to get. However, in order to obtain the lower bound on $P(X(\varphi, U), \psi)$, the dynamical system should be endowed with some mixing property such as specification by Takens and Verbitskiy [**25**] and Thompson [**26**], or g almost product property by Pfister and Sullivan [**23**, **24**] and Pei and Chen [**21**]. Here, we will use the weak specification introduced by Varandas [**29**]. The proof will be divided into the following two subsections.

2.1. *Upper bound on* $P(X(\varphi, U), \psi)$. The upper bound on $P(X(\varphi, U), \psi)$ holds without extra assumptions. Since $U \subset \mathbb{R}$ is a F_σ-set, we write $U = \bigcup_{i=0}^{\infty} K_i$, where $K_i \subset \mathbb{R}$ is closed. By [**35**, Theorem 3.1], we have

$$P(X'(\varphi, K_i), \psi) \leq \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(X, T) \text{ and } \int \varphi \, d\nu \in K_i\right\}.$$

We may assume that $X(\varphi, U) \neq \emptyset$; then we have

$$\begin{aligned}
P(X(\varphi, U), \psi) &\leq P(X'(\varphi, U), \psi) \\
&= P\left(\left\{x \in X : A\left(\frac{1}{n} S_n \varphi(x)\right) \cap \left(\bigcup_{i=0}^{\infty} K_i\right) \neq \emptyset\right\}, \psi\right) \\
&= P\left(\bigcup_{i=0}^{\infty} \left\{x \in X : A\left(\frac{1}{n} S_n \varphi(x)\right) \cap K_i \neq \emptyset\right\}, \psi\right) \\
&= \sup_i P(X'(\varphi, K_i), \psi) \\
&\leq \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(X, T) \text{ and } \int \varphi \, d\nu \in U\right\}.
\end{aligned}$$

2.2. *Lower bound on* $P(X(\varphi, U), \psi)$. To get the lower bound on $P(X(\varphi, U), \psi)$, the dynamical system needs some mild assumption such as non-uniform specification. A key ingredient is the pressure distribution principle.

Our strategy is inspired by the work of Takens and Verbitskiy [**25**], developed by Chen *et al* [**6**], Pfister and Sullivan [**24**], Pei and Chen [**21**] and Thompson [**27**, **28**]. More precisely, we firstly construct $\Theta_{k,i}$ satisfying the following Lemma 2.4. Since the sequence of functions $p(x, n, \epsilon) : X \to \mathbb{R}$ may be non-measurable, we cannot use

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Egorov's theorem directly. We will construct outer measures to obtain Lemma 2.4. Secondly, we construct the fractal F. We cannot construct the second intermediate families as in Takens and Verbitskiy [**25**] because the sequence of functions $\{p(x, n, \epsilon)/n\}$ is not uniformly convergent. Our construction is motivated by Pfister and Sullivan [**24**] and Thompson [**28**]. Finally, we define a sequence of measures suitable for an application of the pressure distribution principle. Before showing the lower bound, we give some important lemmas, as follows.

LEMMA 2.1. [**26**, **27**] *Let (X, d) be a compact metric space, $T : X \to X$ be a continuous map and μ be an ergodic invariant measure. For $\epsilon > 0$, $\gamma \in (0, 1)$ and $\psi \in C(X)$, define*

$$N^{\mu}(\psi, \gamma, \epsilon, n) = \inf\left\{\sum_{x \in S} \exp\left\{\sum_{i=0}^{n-1} \psi(T^i x)\right\}\right\},$$

where the infimum is taken over all sets S which (n, ϵ) span some set Z with $\mu(Z) \geq 1 - \gamma$. We have

$$h_{\mu} + \int \psi \, d\mu = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N^{\mu}(\psi, \gamma, \epsilon, n).$$

The formula remains true if we replace the lim inf *by* lim sup.

LEMMA 2.2. (Generalized pressure distribution principle [**26**, **27**]) *Let (X, d, T) be a topological dynamical system. Let $Z \subset X$ be an arbitrary Borel set. Suppose that there exist $\epsilon > 0$ and $s \geq 0$ such that one can find a sequence of Borel probability measures μ_k, a constant $K > 0$ and an integer N satisfying*

$$\limsup_{k \to \infty} \mu_k(B_n(x, \epsilon)) \leq K \exp\left(-ns + \sum_{i=0}^{n-1} \psi(T^i x)\right)$$

for every ball $B_n(x, \epsilon)$ such that $B_n(x, \epsilon) \cap Z \neq \emptyset$ and $n \geq N$. Furthermore, assume that at least one limit measure ν of the sequence μ_k satisfies $\nu(Z) > 0$. Then $P(Z, \psi, \epsilon) \geq s$.

Let $\mathbf{P} := \sup\{h_{\nu} + \int \psi \, d\nu : \nu \in M(K, T) \text{ and } \int \varphi \, d\nu \in U\}$. We assume that $\mathbf{P}$ is finite. The case that $\mathbf{P}$ is infinite can be included in our proof. Fix $0 < \gamma < 1$. Let us fix $\mu \in M(K, T)$ satisfying $\int \varphi \, d\mu \in U$ and

$$h_{\mu} + \int \psi \, d\mu \geq \sup\left\{h_{\nu} + \int \psi \, d\nu : \nu \in M(K, T) \text{ and } \int \varphi \, d\nu \in U\right\} - \gamma.$$

Choose $\delta > 0$ such that $(\int \varphi \, d\mu - \delta, \int \varphi \, d\mu + \delta) \subset U$.

LEMMA 2.3. [**29**, **34**] *For $\delta > 0$ and $\gamma > 0$, there exists $\eta \in M(K, T)$ such that $\eta = \sum_{i=1}^{s} \lambda_i \eta_i$, where $\sum_{i=1}^{s} \lambda_i = 1$ and $\eta_i \in E(K, T)$, satisfying $|\int \varphi \, d\mu - \int \varphi \, d\eta| < \delta/4$, $|\int \psi \, d\mu - \int \psi \, d\eta| < \gamma$ and $h_{\eta} > h_{\mu} - \gamma$.*

Assume that $\eta_i \in E(K, T)$, $1 \leq i \leq s$ are ergodic measures obtained from Lemma 2.3. For $1 \leq i \leq s$, let η_i^* be the outer measure induced from η_i. Then η_i^*, $1 \leq i \leq s$, are regular outer measures. For $\epsilon > 0$, we can choose a strictly decreasing sequence $\delta_k \to 0$ and a strictly increasing sequence $l_k \to \infty$ so that the set

$$Y_{k,i} := \left\{x \in K : \frac{p(x, n, \epsilon)}{n} < \delta_k, \left|\frac{1}{n} S_n \varphi(x) - \int \varphi \, d\eta_i\right| < \delta_k \forall n \geq l_k\right\}$$

satisfies $\eta_i^*(Y_{k,i}) > 1 - \gamma$ for every $k \in \mathbb{N}$, $i \in \{1, \ldots, s\}$.

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LEMMA 2.4. *For any sufficiently small $\epsilon > 0$, there exist a sequence $\widehat{n}_k \to \infty$ and finite sets $\Theta_{k,i}$ so that each $\Theta_{k,i}$ is a $([\lambda_i \widehat{n}_k], 4\epsilon)$ separated set for $Y_{k,i}$ and*

$$M_{k,i} := \sum_{x \in \Theta_{k,i}} \exp\left\{ \sum_{j=0}^{[\lambda_i \widehat{n}_k]-1} \psi(T^j x) \right\}$$

satisfies

$$M_{k,i} \geq \exp\left\{ [\lambda_i \widehat{n}_k] \left(h_{\eta_i} + \int \psi \, d\eta_i - 3\gamma \right) \right\}.$$

Furthermore, the sequence $\widehat{n}_k$ can be chosen so that $[\lambda_i \widehat{n}_k] \geq l_k$, where $i \in \{1, \ldots, s\}$. Then, for every $k \in \mathbb{N}$ and $i \in \{1, \ldots, s\}$, we have $\max_{x \in \Theta_{k,i}} p(x, [\lambda_i \widehat{n}_k], \epsilon)/[\lambda_i \widehat{n}_k] < \delta_k$.

Proof. By Lemma 2.1, we can choose ϵ sufficiently small so that

$$\liminf_{n \to \infty} \frac{1}{n} \log N^{\eta_i}(\psi, \gamma, 4\epsilon, n) \geq h_{\eta_i} + \int \psi \, d\eta_i - \gamma$$

for $i = 1, 2, \ldots, s$. For every $k \in \mathbb{N}$ and $i \in \{1, \ldots, s\}$, let

$$Q_n(Y_{k,i}, \psi, 4\epsilon) = \inf\left\{ \sum_{x \in S} \exp\left\{ \sum_{j=0}^{n-1} \psi(T^j x) \right\} : S \text{ is an } (n, 4\epsilon) \text{ spanning set for } Y_{k,i} \right\},$$

$$P_n(Y_{k,i}, \psi, 4\epsilon) = \sup\left\{ \sum_{x \in S} \exp\left\{ \sum_{j=0}^{n-1} \psi(T^j x) \right\} : S \text{ is an } (n, 4\epsilon) \text{ separated set for } Y_{k,i} \right\}.$$

Since $\eta_i^*(Y_{k,i}) > 1 - \gamma$, we have

$$P_n(Y_{k,i}, \psi, 4\epsilon) \geq Q_n(Y_{k,i}, \psi, 4\epsilon) \geq N^{\eta_i}(\psi, \gamma, 4\epsilon, n).$$

Let $M_{k,i}^n = P_n(Y_{k,i}, \psi, 4\epsilon)$. We obtain

$$\liminf_{n \to \infty} \frac{1}{n} \log M_{k,i}^n \geq \liminf_{n \to \infty} \frac{1}{n} \log N^{\eta_i}(\psi, \gamma, 4\epsilon, n) \geq h_{\eta_i} + \int \psi \, d\eta_i - \gamma.$$

We choose a sequence $\widehat{n}_k \to \infty$ such that $[\lambda_i \widehat{n}_k] \geq l_k$ and

$$\frac{1}{[\lambda_i \widehat{n}_k]} \log M_{k,i}^{[\lambda_i \widehat{n}_k]} \geq h_{\eta_i} + \int \psi \, d\eta_i - 2\gamma \quad \text{for } i = 1, 2, \ldots, s.$$

For every $k \in \mathbb{N}$ and $i \in \{1, \ldots, s\}$, let $\Theta_{k,i}$ be a $([\lambda_i \widehat{n}_k], 4\epsilon)$ separated set for $Y_{k,i}$ which satisfies

$$\frac{1}{[\lambda_i \widehat{n}_k]} \log \left\{ \sum_{x \in \Theta_{k,i}} \exp\left\{ \sum_{j=0}^{[\lambda_i \widehat{n}_k]-1} \psi(T^j x) \right\} \right\} \geq \frac{1}{[\lambda_i \widehat{n}_k]} \log M_{k,i}^{[\lambda_i \widehat{n}_k]} - \gamma.$$

Let $M_{k,i} := \sum_{x \in \Theta_{k,i}} \exp\{\sum_{j=0}^{[\lambda_i \widehat{n}_k]-1} \psi(T^j x)\}$; then

$$\frac{1}{[\lambda_i \widehat{n}_k]} \log M_{k,i} \geq \frac{1}{[\lambda_i \widehat{n}_k]} \log M_{k,i}^{[\lambda_i \widehat{n}_k]} - \gamma \geq h_{\eta_i} + \int \psi \, d\eta_i - 3\gamma$$

and the desired result follows. □

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We choose ϵ so small that the lemma applies and $\mathrm{Var}(\psi, \epsilon) < \gamma$, $\mathrm{Var}(\varphi, \epsilon) < \delta/8$. Let

$$p(k, \epsilon) := \max_{1 \le i \le s} \max\{p(x, [\lambda_i \widehat{n}_k], \epsilon) : x \in \Theta_{k,i}\}.$$

Let

$$\Theta_k := \Theta_{k,1} \times \cdots \times \Theta_{k,s},$$

$$M_k := M_{k,1} \cdots M_{k,s} \quad \text{and} \quad n_k = \sum_{i=1}^{s} ([\lambda_i \widehat{n}_k] + p(k, \epsilon)).$$

Then we have

$$\frac{n_k}{\widehat{n}_k} \to 1, \quad \frac{p(k, \epsilon)}{n_k} \to 0 \tag{2.3}$$

as $k \to \infty$.

2.2.1. *Construction of the fractal F.* Let $n_0 = 0$. Let us choose a strictly increasing sequence N_k with $N_0 = 0$ so that

$$n_{k+1} \le \delta_k \sum_{j=1}^{k} n_j N_j \tag{2.4}$$

and

$$\lim_{k \to \infty} \frac{N_1 n_1 + \cdots + N_k n_k}{N_{k+1}} = 0. \tag{2.5}$$

Let $\underline{x}_i = (x_1^i, x_2^i, \ldots, x_{N_i}^i) \in \Theta_i^{N_i}$, where $x_l^i = (x_l^{i,1}, x_l^{i,2}, \ldots, x_l^{i,s}) \in \Theta_{i,1} \times \Theta_{i,2} \times \cdots \times \Theta_{i,s}$ for $l = 1, 2, \ldots, N_i$. For any $(\underline{x}_1, \ldots, \underline{x}_k) \in \Theta_1^{N_1} \times \cdots \times \Theta_k^{N_k}$, by the non-uniform specification property, we have

$$\begin{aligned} B(\underline{x}_1, \ldots, \underline{x}_k) = \bigcap_{i=1}^{k} \bigcap_{j=1}^{N_i} (&T^{-\sum_{l=0}^{i-1} N_l n_l - (j-1)n_i} B_{[\lambda_1 \widehat{n}_i]}(x_j^{i,1}, \epsilon) \\ &\cap T^{-\sum_{l=0}^{i-1} N_l n_l - (j-1)n_i - [\lambda_1 \widehat{n}_i] - p(i,\epsilon)} B_{[\lambda_2 \widehat{n}_i]}(x_j^{i,2}, \epsilon) \\ &\cap \cdots \cap T^{-\sum_{l=0}^{i-1} N_l n_l - (j-1)n_i - \sum_{l=1}^{s-1} ([\lambda_l \widehat{n}_i] + p(i,\epsilon))} B_{[\lambda_s \widehat{n}_i]}(x_j^{i,s}, \epsilon)) \ne \emptyset. \end{aligned}$$

We define F_k by

$$F_k = \bigcup \{\overline{B(\underline{x}_1, \ldots, \underline{x}_k)} : (\underline{x}_1, \ldots, \underline{x}_k) \in \Theta_1^{N_1} \times \cdots \times \Theta_k^{N_k}\}.$$

Obviously, F_k is compact and $F_{k+1} \subset F_k$. Define $F = \bigcap_{k=1}^{\infty} F_k$.

For $q \in \mathbb{N}$, we define the sequence M_q as follows:

$$M_q = \begin{cases} qn_1 & \text{if } 0 < q \le N_1, \\ N_1 n_1 + (q - N_1)n_2 & \text{if } N_1 < q \le N_1 + N_2, \\ \cdots & \\ \displaystyle\sum_{i=1}^{k} N_i n_i + \left(q - \sum_{i=1}^{k} N_i\right) n_{k+1} & \text{if } \displaystyle\sum_{i=1}^{k} N_i < q \le \sum_{i=1}^{k+1} N_i, \\ \cdots & \end{cases}$$

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LEMMA 2.5. *For any $y \in F$, $A((1/n)\sum_{i=0}^{n-1} \varphi(T^i y)) \subset (\int \varphi\, d\mu - \delta, \int \varphi\, d\mu + \delta) \subset U$.*

Proof. Because of (2.4), it is sufficient to show that

$$A\left(\frac{1}{M_q}\sum_{i=0}^{M_q-1}\varphi(T^i y)\right) \subset \left(\int \varphi\, d\mu - \delta, \int \varphi\, d\mu + \delta\right) \subset U.$$

We assume that $M_q = \sum_{k=1}^{j} N_k n_k + t n_{j+1}$, where $0 < t = q - \sum_{k=1}^{j} N_k \le N_{j+1}$. For $y \in F$, there exists $(\underline{x}_1, \ldots, \underline{x}_{j+1}) \in \Theta_1^{N_1} \times \cdots \times \Theta_{j+1}^{N_{j+1}}$ such that $y \in \overline{B(\underline{x}_1, \ldots, \underline{x}_{j+1})}$. For $1 \le k \le j$, let $y_k = T^{\sum_{i=0}^{k-1} N_i n_i} y$. Let $a_m = (m-1)n_k$, $b_i = \sum_{l=1}^{i}([\lambda_l \widehat{n}_k] + p(k,\epsilon))$ and $b_0 = 0$. We have

$$\begin{aligned}
&\left|S_{N_k n_k}\varphi(y_k) - N_k n_k \int \varphi\, d\mu\right| \\
&\quad\le \left|S_{N_k n_k}\varphi(y_k) - N_k n_k \int \varphi\, d\eta\right| + \left|N_k n_k \int \varphi\, d\eta - N_k n_k \int \varphi\, d\mu\right| \\
&\quad\le \left|\sum_{m=1}^{N_k}\sum_{i=1}^{s} S_{[\lambda_i \widehat{n}_k]}\varphi(T^{a_m+b_{i-1}} y_k) - N_k n_k \int \varphi\, d\eta\right| + N_k s p(k,\epsilon)\|\varphi\| + N_k n_k \frac{\delta}{4} \\
&\quad\le \left|\sum_{m=1}^{N_k}\sum_{i=1}^{s} S_{[\lambda_i \widehat{n}_k]}\varphi(T^{a_m+b_{i-1}} y_k) - \sum_{m=1}^{N_k}\sum_{i=1}^{s} S_{[\lambda_i \widehat{n}_k]}\varphi(x_m^{k,i})\right| \\
&\qquad + \left|\sum_{m=1}^{N_k}\sum_{i=1}^{s} S_{[\lambda_i \widehat{n}_k]}\varphi(x_m^{k,i}) - N_k n_k \int \varphi\, d\eta\right| + N_k s p(k,\epsilon)\|\varphi\| + N_k n_k \frac{\delta}{4} \\
&\quad\le \sum_{m=1}^{N_k}\left|\sum_{i=1}^{s} S_{[\lambda_i \widehat{n}_k]}\varphi(x_m^{k,i}) - n_k \int \varphi\, d\eta\right| + N_k n_k \operatorname{Var}(\varphi,\epsilon) \\
&\qquad + N_k s p(k,\epsilon)\|\varphi\| + N_k n_k \frac{\delta}{4}.
\end{aligned}$$

Since

$$\begin{aligned}
&\left|\sum_{i=1}^{s} S_{[\lambda_i \widehat{n}_k]}\varphi(x_m^{k,i}) - n_k \int \varphi\, d\eta\right| \\
&\quad\le \left|\sum_{i=1}^{s} S_{[\lambda_i \widehat{n}_k]}\varphi(x_m^{k,i}) - \widehat{n}_k \int \varphi\, d\eta\right| + \left|\widehat{n}_k \int \varphi\, d\eta - n_k \int \varphi\, d\eta\right| \\
&\quad\le \sum_{i=1}^{s}\left|S_{[\lambda_i \widehat{n}_k]}\varphi(x_m^{k,i}) - [\lambda_i \widehat{n}_k]\int \varphi\, d\eta_i\right| + \left|(\widehat{n}_k - n_k)\int \varphi\, d\eta\right| + s\|\varphi\| \\
&\quad\le n_k \delta_k + s p(k,\epsilon)\|\varphi\| + s\|\varphi\|,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\left|S_{N_k n_k}\varphi(y_k) - N_k n_k \int \varphi\, d\mu\right| \\
&\quad\le N_k n_k \operatorname{Var}(\varphi,\epsilon) + N_k s(2p(k,\epsilon)+1)\|\varphi\| + N_k n_k \delta_k + N_k n_k \frac{\delta}{4}.
\end{aligned}$$

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Let $y_{j+1} = T^{\sum_{i=0}^{j} N_i n_i} y$. Similarly, we have

$$\left|S_{tn_{j+1}}\varphi(y_{j+1}) - tn_{j+1}\int \varphi\, d\mu\right| \le tn_{j+1}\,\mathrm{Var}(\varphi,\epsilon) + ts(2p(j+1,\epsilon)+1)\|\varphi\| + tn_{j+1}\delta_{j+1} + tn_{j+1}\frac{\delta}{4}.$$

Thus, we have

$$\begin{aligned}\left|S_{M_q}\varphi(y) - M_q\int \varphi\, d\mu\right| &\le \sum_{k=1}^{j}\left|S_{N_k n_k}\varphi(y_k) - N_k n_k\int \varphi\, d\mu\right| + \left|S_{tn_{j+1}}\varphi(y_{j+1}) - tn_{j+1}\int \varphi\, d\mu\right| \\ &\le M_q\,\mathrm{Var}(\varphi,\epsilon) + \sum_{k=1}^{j} N_k s(2p(k,\epsilon)+1)\|\varphi\| + ts(2p(j+1,\epsilon)+1)\|\varphi\| \\ &\quad + \left(\sum_{k=1}^{j} N_k n_k\delta_k + tn_{j+1}\delta_{j+1}\right) + M_q\frac{\delta}{4}.\end{aligned}$$

From (2.5) and the facts that

$$\mathrm{Var}(\varphi,\epsilon) < \frac{\delta}{8}, \quad \lim_{k\to\infty}\frac{p(k,\epsilon)}{n_k} \to 0 \text{ and } \lim_{k\to\infty}\delta_k = 0,$$

we obtain, for sufficiently large q,

$$\left|\frac{1}{M_q}S_{M_q}\varphi(y) - \int \varphi\, d\mu\right| < \frac{\delta}{2}.$$

Thus, the desired result follows. □

2.2.2. *Construction of a special sequence of measures* μ_k. For each $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_k) \in \Theta_1^{N_1} \times \cdots \times \Theta_k^{N_k}$, we choose one point $z = z(\underline{x})$ such that $z \in B(\underline{x}_1, \ldots, \underline{x}_k)$. Let L_k be the set of all points constructed in this way. For $k \in \mathbb{N}$, let $t_k = \sum_{i=0}^{k} N_i n_i$. The following lemma shows that $\#L_k = \#\Theta_1^{N_1} \times \cdots \times \#\Theta_k^{N_k}$.

LEMMA 2.6. *Let* $\underline{x}$ *and* $\underline{y}$ *be distinct elements of* $\Theta_1^{N_1} \times \cdots \times \Theta_k^{N_k}$. *Then* $z_1 = z(\underline{x})$ *and* $z_2 = z(\underline{y})$ *are* $(t_k, 2\epsilon)$ *separated points.*

Proof. Since $\underline{x} \neq \underline{y}$, there exist i, j, l such that $x_j^{i,l} \neq y_j^{i,l}$. Let $a_j = (j-1)n_i$, $b_0 = 0$ and $b_l = \sum_{m=1}^{l}([\lambda_m \widehat{n}_i] + p(i,\epsilon))$. We have

$$\begin{aligned} d_{[\lambda_l \widehat{n}_i]}(x_j^{i,l}, T^{t_{i-1}+a_j+b_{l-1}} z_1) &< \epsilon, \\ d_{[\lambda_l \widehat{n}_i]}(y_j^{i,l}, T^{t_{i-1}+a_j+b_{l-1}} z_2) &< \epsilon. \end{aligned}$$

Together with $d_{[\lambda_l \widehat{n}_i]}(x_j^{i,l}, y_j^{i,l}) > 4\epsilon$, we have

$$\begin{aligned} d_{t_k}(z_1, z_2) &\ge d_{[\lambda_l \widehat{n}_i]}(T^{t_{i-1}+a_j+b_{l-1}} z_1, T^{t_{i-1}+a_j+b_{l-1}} z_2) \\ &\ge d_{[\lambda_l \widehat{n}_i]}(x_j^{i,l}, y_j^{i,l}) - d_{[\lambda_l \widehat{n}_i]}(x_j^{i,l}, T^{t_{i-1}+a_j+b_{l-1}} z_1) \\ &\quad - d_{[\lambda_l \widehat{n}_i]}(y_j^{i,l}, T^{t_{i-1}+a_j+b_{l-1}} z_2) \\ &> 4\epsilon - \epsilon - \epsilon = 2\epsilon. \end{aligned}$$

□

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We now define the measures on F which yield the required estimates for the pressure distribution principle. For each $z \in L_k$, we associate a number $\mathcal{L}_k(z) \in (0, \infty)$. Using these numbers as weights, we define, for each k, an atomic measure centered on L_k. Precisely, if $z = z(\underline{x}_1, \ldots, \underline{x}_k)$, we define

$$\mathcal{L}_k(z) := \mathcal{L}(\underline{x}_1) \cdots \mathcal{L}(\underline{x}_k),$$

where, if $\underline{x}_i = (x_1^i, \ldots, x_{N_i}^i) = ((x_1^{i,1}, \ldots, x_1^{i,s}), \ldots, (x_{N_i}^{i,1}, \ldots, x_{N_i}^{i,s})) \in \Theta_i^{N_i}$, then

$$\mathcal{L}(\underline{x}_i) := \prod_{l=1}^{N_i} \prod_{m=1}^{s} \exp S_{[\lambda_m \widehat{n}_i]} \psi(x_l^{i,m}).$$

We define $\nu_k := \sum_{z \in L_k} \delta_z \mathcal{L}_k(z)$. We normalize ν_k to obtain a sequence of probability measures μ_k. More precisely, we let $\mu_k := (1/\kappa_k)\nu_k$, where κ_k is the normalizing constant

$$\kappa_k := \sum_{z \in L_k} \mathcal{L}_k(z) = \sum_{\underline{x}_1 \in \Theta_1^{N_1}} \cdots \sum_{\underline{x}_k \in \Theta_k^{N_k}} \mathcal{L}(\underline{x}_1) \cdots \mathcal{L}(\underline{x}_k) = M_1^{N_1} \cdots M_k^{N_k}.$$

In order to prove the main results of this article, we present some lemmas.

LEMMA 2.7. *Suppose that ν is a limit measure of the sequence of probability measures μ_k. Then $\nu(F) = 1$.*

Proof. Suppose that $\nu = \lim_{k \to \infty} \mu_{l_k}$ for $l_k \to \infty$. For any fixed l and all $p \geq 0$, $\mu_{l+p}(F_l) = 1$ since $F_{l+p} \subset F_l$. Thus, $\nu(F_l) \geq \limsup_{k \to \infty} \mu_{l_k}(F_l) = 1$. It follows that $\nu(F) = \lim_{l \to \infty} \nu(F_l) = 1$. □

Let $\mathcal{B} = B_n(q, \epsilon)$ be an arbitrary ball which intersects F. Let k be the unique number which satisfies $t_k \leq n < t_{k+1}$. Let $j \in \{0, \ldots, N_{k+1} - 1\}$ be the unique number so that

$$t_k + n_{k+1} j \leq n < t_k + n_{k+1}(j+1).$$

We assume that $j \geq 1$; the simpler case $j = 0$ is similar.

LEMMA 2.8. *For any $p \geq 1$, suppose that $\mu_{k+p}(\mathcal{B}) > 0$; we have*

$$\mu_{k+p}(\mathcal{B}) \leq \frac{1}{\kappa_k M_{k+1}^j} \exp\Bigg\{ S_n \psi(q) + 2n \operatorname{Var}(\psi, \epsilon) + \|\psi\| \Bigg(\sum_{i=1}^{k} N_i sp(i, \epsilon) + sp(k+1, \epsilon) j + n_{k+1} \Bigg) \Bigg\}.$$

Proof. Case $p = 1$. Suppose that $\mu_{k+1}(\mathcal{B}) > 0$; then $L_{k+1} \cap \mathcal{B} \neq \emptyset$. Let $z = z(\underline{x}, \underline{x}_{k+1}) \in L_{k+1} \cap \mathcal{B}$, where $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_k) \in \Theta_1^{N_1} \times \cdots \times \Theta_k^{N_k}$ and $\underline{x}_{k+1} = (x_1^{k+1}, \ldots, x_{N_{k+1}}^{k+1}) \in \Theta_{k+1}^{N_{k+1}}$. Let

$$\mathcal{A}_{\underline{x}; x_1^{k+1}, \ldots, x_j^{k+1}} = \{ z(\underline{x}, (y_1^{k+1}, \ldots, y_{N_{k+1}}^{k+1})) \in L_{k+1} : y_1^{k+1} = x_1^{k+1}, \ldots, y_j^{k+1} = x_j^{k+1} \}.$$

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Suppose that $z' = z(\underline{y}, \underline{y}_{k+1}) \in L_{k+1} \cap \mathcal{B}$. Since $d_n(z, z') < 2\epsilon$, we have $\underline{y} = \underline{x}$ and $y_l^{k+1} = x_l^{k+1}$ for $l \in \{1, \ldots, j\}$. Thus, we have

$$\begin{aligned}\nu_{k+1}(\mathcal{B}) &\le \sum_{z \in \mathcal{A}_{\underline{x}; x_1^{k+1}, \ldots, x_j^{k+1}}} \mathcal{L}_{k+1}(z) \\ &= \mathcal{L}(\underline{x}_1) \cdots \mathcal{L}(\underline{x}_k) \prod_{l=1}^{j} \prod_{m=1}^{s} \exp S_{[\lambda_m \widehat{n}_{k+1}]} \psi(x_l^{k+1,m}) M_{k+1}^{N_{k+1}-j}.\end{aligned}$$

Case $p > 1$. Similarly, we have

$$\nu_{k+p}(\mathcal{B}) \le \mathcal{L}(\underline{x}_1) \cdots \mathcal{L}(\underline{x}_k) \prod_{l=1}^{j} \prod_{m=1}^{s} \exp S_{[\lambda_m \widehat{n}_{k+1}]} \psi(x_l^{k+1,m}) M_{k+1}^{N_{k+1}-j} M_{k+2}^{N_{k+2}} \cdots M_{k+p}^{N_{k+p}}.$$

Combining these with the fact that $d_n(z, q) < \epsilon$, one can readily verify that

$$\begin{aligned}&\mathcal{L}(\underline{x}_1) \cdots \mathcal{L}(\underline{x}_k) \prod_{l=1}^{j} \prod_{m=1}^{s} \exp S_{[\lambda_m \widehat{n}_{k+1}]} \psi(x_l^{k+1,m}) \\ &\le \exp\left\{ S_n \psi(q) + 2n \operatorname{Var}(\psi, \epsilon) + \|\psi\| \left(\sum_{i=1}^{k} N_i sp(i, \epsilon) + sp(k+1, \epsilon) j + n_{k+1} \right) \right\}.\end{aligned}$$

Since $\mu_{k+p} = (1/\kappa_{k+p}) \nu_{k+p}$ and $\kappa_{k+p} = \kappa_k M_{k+1}^{N_{k+1}} \cdots M_{k+p}^{N_{k+p}}$, the desired result follows. □

Let $\mathbf{C} := h_\mu + \int \psi \, d\mu$.

LEMMA 2.9. *For sufficiently large n,*

$$\limsup_{l \to \infty} \mu_l(B_n(q, \epsilon)) \le \exp\{-n(\mathbf{C} - 2 \operatorname{Var}(\psi, \epsilon) - 8\gamma) + S_n \psi(q)\}.$$

Proof. Let $c = \max_{1 \le l \le s} \{|h_{\eta_l} + \int \psi \, d\eta_l - 3\gamma|\}$. From Lemma 2.3, we have $h_\eta + \int \psi \, d\eta > h_\mu + \int \psi \, d\mu - 2\gamma$. Then

$$\begin{aligned}M_i &= M_{i,1} \cdots M_{i,s} \ge \exp\left\{ \sum_{l=1}^{s} [\lambda_l \widehat{n}_i] \left(h_{\eta_l} + \int \psi \, d\eta_l - 3\gamma \right) \right\} \\ &\ge \exp\left\{ \sum_{l=1}^{s} \left(\lambda_l \widehat{n}_i \left(h_{\eta_l} + \int \psi \, d\eta_l - 3\gamma \right) \right) - sc \right\} \\ &\ge \exp\left\{ n_i \left(h_\mu + \int \psi \, d\mu - 5\gamma \right) - (n_i - \widehat{n}_i) \left(h_\mu + \int \psi \, d\mu - 5\gamma \right) - sc \right\}.\end{aligned}$$

Since $s(p(i, \epsilon) - 1) \le n_i - \widehat{n}_i \le sp(i, \epsilon)$, we have

$$M_i \ge \exp\left\{ n_i \left(h_\mu + \int \psi \, d\mu - 5\gamma \right) - A_i \right\} = \exp\{n_i(\mathbf{C} - 5\gamma) - A_i\},$$

where $A_i = sp(i, \epsilon)|h_\mu + \int \psi \, d\mu - 5\gamma| + sc$. Thus, we have

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$$\begin{aligned}\kappa_k M_{k+1}^j &= M_1^{N_1} \cdots M_k^{N_k} M_{k+1}^j \\ &\geq \exp\{(\mathbf{C} - 5\gamma)(N_1 n_1 + N_2 n_2 + \cdots + N_k n_k + j n_{k+1}) - D\} \\ &= \exp\{(\mathbf{C} - 5\gamma)n - D - E\},\end{aligned}$$

where

$$D = N_1 A_1 + \cdots + N_k A_k + j A_{k+1}, \quad E = (\mathbf{C} - 5\gamma)(n - t_k - j n_{k+1}).$$

It follows from (2.3)–(2.5) that we obtain, for sufficiently large n, $|D/n| < \gamma$, $|E/n| < \gamma$.

By Lemma 2.8, for sufficiently large n and any $p \geq 1$, we have

$$\begin{aligned}\mu_{k+p}(\mathcal{B}) &\leq \frac{1}{\kappa_k M_{k+1}^j} \exp\{S_n\psi(q) + n(2\,\mathrm{Var}(\psi, \epsilon) + \gamma)\} \\ &\leq \exp\{-n(\mathbf{C} - 2\,\mathrm{Var}(\psi, \epsilon) - 8\gamma) + S_n\psi(q)\}.\end{aligned}$$

Hence, the desired result follows. □

Applying the generalized pressure distribution principle, we have

$$P(X(\varphi, U), \psi, \epsilon) \geq P(F, \psi, \epsilon) \geq \mathbf{C} - 2\,\mathrm{Var}(\psi, \epsilon) - 8\gamma.$$

Recall that $\mathrm{Var}(\psi, \epsilon) < \gamma$; we have

$$P(X(\varphi, U), \psi, \epsilon) \geq P(F, \psi, \epsilon) \geq \mathbf{C} - 10\gamma.$$

Since ϵ and γ were arbitrary, the proof of Theorem 1.1 is complete.

Next, we prove Corollaries 1.1 and 1.2. We need the following lemmas.

LEMMA 2.10. [**18**] *Suppose that (X, T) is a topological dynamical system and $K \subset X$. Let Y be a normed vector space. Let $\Gamma : M(K, T) \to Y$ be continuous and affine and let $\Delta : M(K, T) \to \mathbb{R}$ be continuous and affine with $\Delta(\mu) \neq 0$ for all $\mu \in M(K, T)$. Define $U : M(X, T) \to Y$ by $U = \Gamma/\Delta$. Let C be a closed and convex subset of Y and assume that*

$$\operatorname{int} C \cap U(M(K, T)) \neq \emptyset.$$

Then, for $\psi \in C(X)$,

$$\sup_{\substack{\mu \in M(K,T) \\ U\mu \in C}} \left(h_\mu + \int \psi \, d\mu\right) = \sup_{\substack{\mu \in M(K,T) \\ U\mu \in \operatorname{int} C}} \left(h_\mu + \int \psi \, d\mu\right).$$

LEMMA 2.11. [**18**] *Let X be a metric space and let $F : X \to \mathbb{R}$ be an upper semi-continuous function. Let $K_1, K_2, \ldots \subset X$ be non-empty compact subsets of X with $K_1 \supset K_2 \supset \cdots$. Then*

$$\inf_n \sup_{x \in K_n} F(x) = \sup_{x \in \bigcap_n K_n} F(x).$$

Proof of Corollary 1.1. We may assume that $X(\varphi, I) \neq \emptyset$. By [**35**, Theorem 3.1], we have

$$\begin{aligned}P(X(\varphi, I), \psi) &\leq P(X'(\varphi, I), \psi) \\ &\leq \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(X, T) \text{ and } \int \varphi \, d\nu \in I\right\}.\end{aligned}$$

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Let $Y=\mathbb{R}$ and $\Delta=1$. We define $\Gamma: M(K,T)\to\mathbb{R}$ by $\Gamma(\mu)=\int\varphi\,d\mu$. From Lemma 2.10, we have

$$\sup_{\substack{\nu\in M(K,T)\\ \int\varphi\,d\nu\in I}}\left(h_\nu+\int\psi\,d\nu\right)=\sup_{\substack{\nu\in M(K,T)\\ \int\varphi\,d\nu\in \operatorname{int} I}}\left(h_\nu+\int\psi\,d\nu\right).$$

Thus, we obtain

$$\begin{aligned}P(X(\varphi,I),\psi)&\geq P(X(\varphi,\operatorname{int} I),\psi)\\&\geq \sup\left\{h_\nu+\int\psi\,d\nu:\nu\in M(K,T)\text{ and }\int\varphi\,d\nu\in\operatorname{int} I\right\}\\&=\sup\left\{h_\nu+\int\psi\,d\nu:\nu\in M(K,T)\text{ and }\int\varphi\,d\nu\in I\right\}.\end{aligned}$$

Proof of Corollary 1.2. It is obvious that

$$\begin{aligned}&\lim_{\delta\to0}P(X(\varphi,(\alpha-\delta,\alpha+\delta)),\psi)\\&\quad\geq\sup\left\{h_\nu+\int\psi\,d\nu:\nu\in M(K,T)\text{ and }\int\varphi\,d\nu=\alpha\right\}.\end{aligned}$$

From Theorem 1.1 and Lemma 2.11, we have

$$\begin{aligned}&\lim_{\delta\to0}P(X(\varphi,(\alpha-\delta,\alpha+\delta)),\psi)\\&\quad\leq\lim_{\delta\to0}\sup\left\{h_\nu+\int\psi\,d\nu:\nu\in M(X,T)\text{ and }\int\varphi\,d\nu\in[\alpha-\delta,\alpha+\delta]\right\}\\&\quad=\sup\left\{h_\nu+\int\psi\,d\nu:\nu\in M(X,T)\text{ and }\int\varphi\,d\nu=\alpha\right\}.\end{aligned}$$

3. *Some applications*

In this section, by the work of Varandas [**29**], Theorem 1.1 can be applied to multidimensional local diffeomorphisms and Viana maps.

Example 1. (Multidimensional local diffeomorphisms) Let T_0 be an expanding map in $\mathbb{T}^n$ and take a periodic point p for T_0. Let T be a C^1-local diffeomorphism obtained from T_0 by a bifurcation in a small neighborhood U of p in such a way that:

(1) every point $x\in\mathbb{T}^n$ has some preimage outside U;
(2) $\|DT(x)^{-1}\|\leq\sigma^{-1}$ for every $x\in\mathbb{T}^n\backslash U$, and $\|DT(x)^{-1}\|\leq L$ for every $x\in\mathbb{T}^n$, where $\sigma>1$ is large enough or $L>0$ is sufficiently close to 1;
(3) T is topologically exact: for every open set U, there is $N\geq1$ for which $T^N(U)=\mathbb{T}^n$.

From [**29**, **30**], we know that T has a unique (ergodic) equilibrium m for a Hölder continuous potential $-\log|\det DT|$ and (T,m) satisfies the non-uniform specification property.

COROLLARY 3.1. *Let m be the unique ergodic equilibrium state for a Hölder continuous potential $-\log|\det DT|$ in multidimensional local diffeomorphisms. If $\varphi,\psi\in C(\mathbb{T}^n)$ and*

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$J \subset \mathbb{R}$ is either an open set or a closed convex set with $\operatorname{int} J \cap \{\int \varphi \, d\mu : \mu \in M(K, T)\} \neq \emptyset$, *then*

$$P(\mathbb{T}^n(\varphi, J), \psi) \leq \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(\mathbb{T}^n, T) \text{ and } \int \varphi \, d\nu \in J\right\}$$

and

$$P(\mathbb{T}^n(\varphi, J), \psi) \geq \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(K, T) \text{ and } \int \varphi \, d\nu \in J\right\}.$$

Furthermore, if $\int \varphi \, dm \in J$, *then*

$$P(\mathbb{T}^n(\varphi, J), -\log|\det DT|) = P(\mathbb{T}^n, -\log|\det DT|).$$

Example 2. (Viana maps) In Viana [**31**], the author introduced Viana maps, which are obtained as C^3 small perturbations of the skew product ϕ_α of the cylinder $S^1 \times I$ given by

$$\phi_\alpha(\theta, x) = (d\theta(\mathrm{mod}1), 1 - ax^2 + \alpha \cos(2\pi\theta)),$$

where $d \geq 16$ is an integer, a is a Misiurewicz parameter for the quadratic family and α is small.

Varandas [**29**] proved that when m is a SRB measure (see [**1**, p. 417] for a definition) for ϕ_α, then (ϕ_α, m) satisfies non-uniform specification.

COROLLARY 3.2. *Let m be a SRB measure for a Viana map ϕ_α. If $\varphi, \psi \in C(S^1 \times I)$ and $J \subset \mathbb{R}$ is either an open set or a closed convex set with* $\operatorname{int} J \cap \{\int \varphi \, d\mu : \mu \in M(K, T)\} \neq \emptyset$, *then*

$$P(S^1 \times I(\varphi, J), \psi) \leq \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(S^1 \times I, T) \text{ and } \int \varphi \, d\nu \in J\right\}$$

and

$$P(S^1 \times I(\varphi, J), \psi) \geq \sup\left\{h_\nu + \int \psi \, d\nu : \nu \in M(K, T) \text{ and } \int \varphi \, d\nu \in J\right\}.$$

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