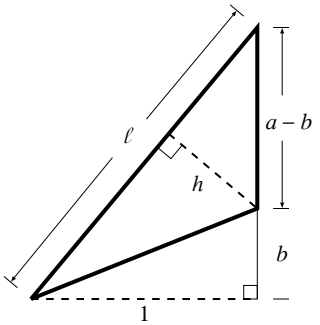
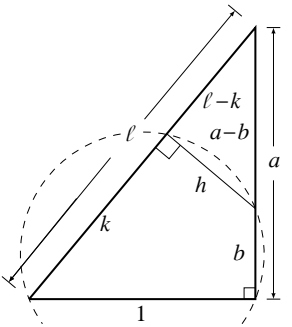
## 103.18 Trigonometric addition and subtraction formulas, a unified visual approach

*Subtraction formulas*




$$\ell h = a - b \quad (*)$$

FIGURE 1




$$\ell^2 = 1 + a^2$$

$$[-] \quad \frac{\ell(\ell - k) = a(a - b)}{\ell k = 1 + ab} \quad (**)$$

FIGURE 2

Equation $(*)$ in Figure 1 (and in Figures 4 and 5 below) follows from computing the area of the thick triangle in two ways. Pythagoras' Theorem and the Intersecting Secants theorem are invoked in deriving $(**)$ in Figures 2, 4 and 5. (Alternatively we may argue by similar triangles.)

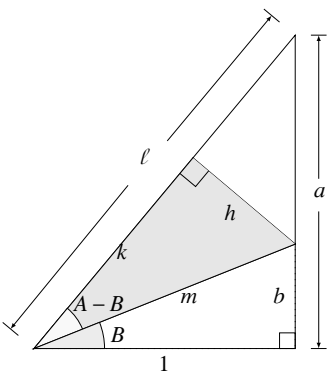



FIGURE 3

From $(*)$, $(**)$ and Figure 3 we have:

- $\tan(A - B)$

$$= \frac{h}{k} = \frac{\ell h}{\ell k} = \frac{a - b}{1 + ab}$$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

- $\sin(A - B)$

$$= \frac{h}{m} = \frac{\ell h}{\ell m} = \frac{a - b}{\ell m}$$

$$= \frac{a}{\ell}\frac{1}{m} - \frac{1}{\ell}\frac{b}{m}$$

$$= \sin A \cos B - \cos A \sin B$$

- $\cos(A - B)$

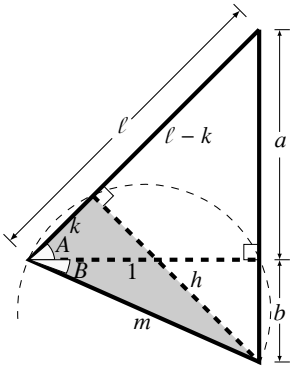
$$= \frac{k}{m} = \frac{\ell k}{\ell m} = \frac{1 + ab}{\ell m}$$

$$= \frac{1}{\ell}\frac{1}{m} + \frac{a}{\ell}\frac{b}{m}$$

$$= \cos A \cos B + \sin A \sin B$$

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*Addition formulas*



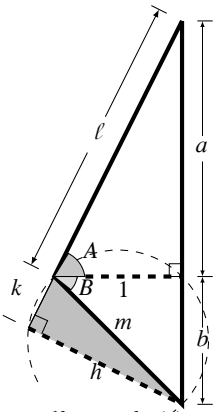

$$\ell h = a + b \quad (*)$$

$$\ell^2 = 1 + a^2$$

$$[-] \quad \ell(\ell - k) = a(a + b)$$

$$\ell k = 1 - ab \quad (**)$$

FIGURE 4: $\left(A + B < \frac{\pi}{2}\right)$




$$\ell h = a + b \quad (*)$$

$$\ell^2 = 1 + a^2$$

$$[-] \quad \ell(\ell + k) = a(a + b)$$

$$-\ell k = 1 - ab \quad (**)$$

FIGURE 5: $\left(A + B > \frac{\pi}{2}\right)$

From Figures 4 and 5 the addition formulas follow in the same way the subtraction formulas are derived above. In fact, when $A + B$ is acute, it suffices to swap the + and − signs throughout the part dealing with subtraction. If $A + B$ is obtuse, using Figure 5, the same formulas are obtained. For example,

$$\tan(A + B) = -\frac{h}{k} = \frac{\ell h}{-\ell k} = \frac{a + b}{1 - ab} = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

For more proofs of the addition or subtraction formulas see [1], [2] and [3].

*References*

1. Roger Nelsen, *Proofs without words II, more exercises in visual thinking*, The Mathematical Association of America (2000).
2. Eric W. Weisstein, *Trigonometric addition formulas*. http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html
3. Derek Colwell and Jack Gillett, The addition formula for sine. A compilation of methods, *Math. Gaz.* **73** (Oct.1989) pp. 197-206.

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DAVID GAU
*Department of Math. and Stat., California State University, Long Beach, CA, 90630, USA*
e-mail: *david.gau@csulb.edu*

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