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On the multiplicity function of generic group extensions with continuous spectrum

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Abstract. A modification of the method of geometric models is proposed and applied to the study of multiplicity functions of group extensions.

It is proved that, for some generic set of the automorphisms T of the Lebesgue space with respect to the standard topology, for any $M \subseteq \mathbb{N} \cup \{\infty\}$ $(1 \in M)$ there exists a generic set of weakly mixing group extensions T' of transformation T with $M(T') = M$, where $M(T)$ denotes the set of essential spectral multiplicities of the unitary operator corresponding to the transformation T.

1. *Introduction*

By a dynamical system we mean a 3-tuple (X, μ, T) consisting of an invertible measure-preserving transformation (automorphism) T defined on a non-atomic Lebesgue space (X, μ), $\mu(X) = 1$. The spectral properties of T are those of the induced unitary operator defined by

$$\tilde{T} f(x) = f(Tx), \quad f \in L^2(X, \mu).$$

Furthermore, for unitary operator $\tilde{T}$, we shall omit the symbol '$\sim$'. Denote by $\mathrm{Aut}(\mu)$ the group of all automorphisms of (X, μ) and by $M(U)$ the set of essential spectral multiplicities of a unitary operator U.

Some publications (see [**1, 2**]) consider the following problem, which is one of the general classical problems of the spectral theory of ergodic measure-preserving transformations: determine the precise conditions on the spectrum of a unitary operator under which it can be realized by an ergodic dynamical system. In order to see one reason why this problem is difficult, let us mention two particular unsolved problems: the existence of a transformation having a simple Lebesgue spectrum (Banach's problem) and the existence of a transformation having absolutely continuous spectrum of finite multiplicity in an orthogonal complement to the space of constant functions $\{f : f \equiv \text{constant}\}^{\perp}$ (see [**2**]).

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It is clear that $M(U) \subseteq \mathbb{N} \cup \{\infty\}$. Note that $M(T)$ can be any subset of $\mathbb{N} \cup \{\infty\}$ if we do not require ergodicity of T. For ergodic transformations T we have $1 \in M(T)$, where 1 arises from the space $\{f \in L_2(\mu) : Tf = f\}$. Papers [**3–6**] and some others were devoted to realizations of some subsets of $\mathbb{N} \cup \{\infty\}$ different from $\{1\}$ and $\{1, \infty\}$ and in the form of $M(T)$ for ergodic transformations T. In [**3**] examples of ergodic transformations with $M(T) = \{1, 2\}$ were constructed. In [**4**] for the each positive integer n transformations T with $M(T) = \{1, n\}$ were constructed. In [**5, 6**] the class of subsets of $\mathbb{N}$ realized as a set $M(T)$ is extended. Note that these constructions essentially used the existence of skew products with continuous group automorphism in its centralizer, giving the following restriction to $M(T)$: $m_1, m_2 \in M(T) \Longrightarrow \operatorname{lcm}(m_1, m_2) \in M(T)$.

The new proposed construction does not have this restriction and it shows that for the typical transformation $T \in \operatorname{Aut}(\mu)$, for each $M \subseteq \mathbb{N} \cup \{\infty\}$ $(1 \in M)$ that T admits an ergodic group extension $\tilde{T}$ with $M(\tilde{T}) = M$. The proof does not use the group automorphisms but is a modification of a method of geometric models used for the study of the spectral properties of the classical von Neumann transformations [**7, 8**]. This construction gives an opportunity to add any subset of $\mathbb{N}$ to a set of essential spectral multiplicities. Constructed transformations, as a rule, have a continuous spectrum (in the subspace $\{f : f \equiv \text{constant}\}^{\perp}$). We remark here that in [**9**] (using another idea) for each $M \subseteq \mathbb{N} \cup \{\infty\}$ $(1 \in M)$ the existence of the ergodic automorphism T with $M(T) = M$ is proved. These examples were constructed as natural factors of transformations from [**6**] .

Note that we usually consider the spectrum of the transformation in $\{f : f \equiv \text{constant}\}^{\perp}$. Special interest in this case appears due to the Rokhlin problem: does there exist an ergodic transformation with $M(T) = \{n\}$, where $n \notin \{1, \infty\}$ (see [**4, 10**]). Note that the author in [**11**] (and V. Ryzhikov [**12**] independently and differently) has solved this problem using the method of [**8**].

2. *Definition of class A*

We consider the piecewise affine maps sending $[0, 1)$ onto $[0, 1)$, where Lebesgue measure μ is invariant for these maps. The transformation T is obtained as a limit of the following process: at stage k we have disjoint half-open intervals $J_1, J_2, \ldots, J_{q_k}, J'_k$ partitioning $[0, 1)$, where J'_k is on the right-hand side. The half-open intervals J_i $(1 \le i \le q_k)$ have the same length and T maps J_i linearly on J_{i+1} $(1 \le i < q_k)$. The transformation T maps $J_{q_k} \cup J'_k$ onto $J'_k \cup J_1$ but it is not defined here at this stage. To describe the transformation at stage $k + 1$, fix positive integer p_k and the sequence of non-negative integers $\{a(k, p) : 1 \le p \le p_k\}$. Divide each J_i into equal length half-open intervals $J_{i,p}$ $(1 \le p \le p_k)$ numbered from left to right. It is clear that T maps $J_{i,p}$ linearly onto $J_{i+1,p}$ $(1 \le i < q_k)$. Divide J'_k into half-open intervals $J'_{k,p,a}$ $(p \in \{p : a(k, p) > 0\}, 1 \le a \le a(k, p))$ of the same length as $J_{i,p}$ and a remaining half-open interval J'_{k+1} on the right-hand side. At stage $k + 1$ we have the intervals

$$\{J'_{k+1}\} \cup \{J_{i,p} : 1 \le i \le q_k, 1 \le p \le p_k\} \cup \{J'_{k,p,a} : p \in \{p : a(k, p) > 0\}, 1 \le a \le a(k, p)\},$$

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partitioning $[0, 1)$ and

$$q_{k+1} = q_k p_k + \sum_p a(k, p).$$

Extend the transformation T by piecewise affine transformations such that it carries intervals onto intervals according to the scheme

$$\begin{aligned}
&J_{q_k,p} \to J_{1,p+1} \quad \text{if } p < p_k \text{ and } a(k, p) = 0,\\
&J_{q_k,p} \to J'_{k,p,1} \to J'_{k,p,2} \to \cdots \to J'_{k,p,a(k,p)} \to J_{1,p+1} \quad \text{if } p < p_k \text{ and } a(k, p) > 0,\\
&J_{q_k,p_k} \to J'_{k,p_k,1} \to \cdots \to J'_{k,p_k,a(k,p_k)} \quad \text{if } a(k, p_k) > 0.
\end{aligned}$$

Clearly, $J_{q_{k+1}}$ is equal to $J'_{k,p_k,a(k,p_k)}$ if $a(k, p_k) > 0$ and is equal to J_{q_k,p_k} otherwise. The transformation T maps $J_{q_{k+1}} \cup J'_{k+1}$ onto $J_1 \cup J'_{k+1}$, but it is not defined at stage $k + 1$.

The considered construction gives the realization of any transformation of rank 1 conveniently for us. We say that an automorphism T described above belongs to class A, if there exists a sequence of positive integers k_n such that

$$\begin{gathered}
k_n \to \infty, \quad p_{k_n} \to \infty, \quad a(k_n, p) = 0 \quad (1 \le p \le p_{k_n}),\\
\frac{\mu(T J_{q_{k_n},p_{k_n}} \cap J_{1,1})}{\mu(J_{1,1})} \to 1 \quad \text{as } n \to \infty. \qquad (1)
\end{gathered}$$

Note that class A is the subset of rigid transformations ($T^{r_n} \to E$ as $r_n \to \infty$, where E is the identity operator) of rank 1.

3. *Construction of skew products*

We fix $M = \{1, n_1, n_2, \dots\} \subseteq \mathbb{N}$ and T from class A. We denote

$$\begin{gathered}
\tilde{N} = \sum_j n_j, \quad (\tilde{N} \le \infty), \quad l = \#\{n_1, n_2, \dots\} \quad (l \le \infty),\\
\tilde{N} - l = \sum_j (n_j - 1), \quad \Omega = \mathbb{T}^l \times C(T)^{\tilde{N}-l},
\end{gathered}$$

where $\mathbb{T}$ is the circle $\{c \in \mathbb{C} : |c| = 1\}$ identified with $[0, 1)$ in a standard way, and $C(T) = \{S \in \mathrm{Aut}(\mu) : TS = ST\}$. Denote

$$\omega_\gamma(x) = \begin{cases} -1, & 0 \le x < \gamma \\ 1, & \gamma \le x < 1 \end{cases} \quad (\gamma, x \in \mathbb{T} \equiv [0, 1)).$$

Henceforth, it is convenient to use the numbering of the coordinates of $\tilde{\beta} \in \Omega$ according to the structure of M. We define a vector function

$$\omega_{\tilde{\beta}}(x) = (\omega_{\beta_1}(x), \omega_{\beta_1}(S_{1,1}x), \dots, \omega_{\beta_1}(S_{1,n_1-1}x), \omega_{\beta_2}(x), \omega_{\beta_2}(S_{2,1}x), \dots)$$

and the transformation $T_{\omega_{\tilde{\beta}}}$, acting on the group $\mathbb{T} \times \mathbb{Z}_2^{\tilde{N}} \equiv [0, 1) \times \{-1, 1\}^{\tilde{N}}$ with Haar measure $m = \mu \times l \times l \times \cdots$, $(l\{i\} = 0.5, i \in \{-1, 1\})$ by the formula

$$T_{\omega_{\tilde{\beta}}}(x, y) = (Tx, \omega_{\tilde{\beta}}(x) * y),$$

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where $*$ is the coordinate-wise multiplication. Then $T_{\omega_{\tilde\beta}}$ corresponds to $\tilde\beta \in \Omega$, where

$$\tilde\beta = (\beta_1, \beta_2, \dots, S_{1,1}, S_{1,2}, \dots, S_{1,n_1-1}, S_{2,1}, S_{2,2}, \dots) \quad (S_{i,j} \in C(T), \beta_i \in \mathbb{T})$$

Let us remark that $C(T)$ as a closed subset of $\mathrm{Aut}(\mu)$ relative to the weak topology is a Polish (complete, separable and metric) space. Therefore Ω is a Polish space relative to the product topology.

We say that a property holds for a generic element from topological space D if the set of elements from D with this property contains a dense G_δ subset of D.

THEOREM 1. *For each automorphism T from class A for a generic $\tilde\beta \in \Omega$ the transformations $T_{\omega_{\tilde\beta}}$ are ergodic, $M(T_{\omega_{\tilde\beta}}) = M$ and any eigenfunction of $T_{\omega_{\tilde\beta}}$ is independent of y.*

4. *Geometrical representations*

The new variant, approximation of automorphisms of Lebesgue space (X, μ) by periodic transformations (see other variants in [**9**]), is more convenient for us.

We say that an automorphism T admits an approximation with flat stacks in pairs (a.f.s.p.), if there exists a sequence $\{\xi_n\}$ of partitions of X such that

$$\xi_k = \{B'(k), B_0(k), TB_0(k), \dots, T^{q_k-1}B_0(k)\}, \quad \xi_k \to \varepsilon,$$
$$\xi_{2k} \le \xi_{2k+1} \ (k = 0, 1, \dots), \quad \frac{q_{2k+1}}{q_{2k}} = p_k \in \mathbb{N},$$
$$\mu(B_0(2k) | T^{nq_{2k}} B_0(2k+1)) = 1, \quad (0 \le n < p_k),$$
$$\mu(B_0(2k) | T^{p_k q_{2k}} B_0(2k+1)) \to 1 \quad \text{as } k \to \infty,$$
$$\mu(B_0(2k)) = p_k \mu(B_0(2k+1)), \quad p_k \to \infty \text{ as } k \to \infty.$$

It follows that

$$\mu(B_0(2k+1) | T^{p_k q_{2k}} B_0(2k+1)) \to 1 \quad \text{as } k \to \infty.$$

LEMMA 1. *For a generic $T \in \mathrm{Aut}(\mu)$ with respect to weak topology, the transformation T admits an a.f.s.p.*

LEMMA 2. *If T admits an a.f.s.p., then T is isomorphic to some $\tilde T$ from class A.*

It is clear that from Lemmas 1 and 2 using Theorem 1 we have immediately the following general result.

THEOREM 2. *For a generic $T \in \mathrm{Aut}(\mu)$ with respect to weak topology and for a generic $\tilde\beta \in \Omega = \mathbb{T}^l \times C(\varphi_T T \varphi_T^{-1})$, any eigenfunction of $\tilde T$ is independent of y and*

$$M(\tilde T) = M,$$

where $\tilde T(x, y) = (Tx, \omega_{\tilde\beta}(\varphi_T(x)) * y)$ acts on $X \times \mathbb{Z}_2^{\tilde N}$, $\varphi_T(x) : X \to \mathbb{T}$ is the isomorphism from Lemma 2.

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Remark 1. It is easily shown that for each transformation $\tilde{T}$ there exists a transformation S of the form $\tilde{T} \times S_1$ such that any eigenfunction of S is dependent only on x and y and

$$M(S) = M(\tilde{T}) \cup \{\infty\}.$$

Remark 2. For a generic vector function $\omega(x)$ with respect to weak topology and for a generic transformation T we have $M(\tilde{T}) = \{1\}$ (consequently $\tilde{T}$ is ergodic), where $\tilde{T}(x, y) = (Tx, \omega(x) * y)$.

It is clear that there exists an automorphism admitting an a.f.s.p. such that this automorphism does not have a continuous spectrum (in $\{f : f \equiv \text{constant}\}^{\perp}$), but the set of the transformations having continuous spectrum is generic in $\mathrm{Aut}(\mu)$. Finally, we have the following.

COROLLARY. *For a generic $T \in \mathrm{Aut}(\mu)$ and for each $M \subseteq \mathbb{N} \cup \{\infty\}$ $(1 \in M)$ there exists a group extension $\tilde{T}$ of T such that*

$$M(\tilde{T}) = M$$

and $\tilde{T}$ has a continuous spectrum in $\{f : f \equiv \text{constant}\}^{\perp}$.

5. *Generalized geometric models and metrics ρ_k*

We fix T from class A and a collection of half-open intervals $J_1, \ldots, J_{q_k'}, J_k'$ at stage k. We define the injective map $\varphi_k : [0, 1) \to \mathbb{R}^2$ as follows. For any $x \in [0, 1)$ there exists a number $t \in \mathbb{Z}_+$ such that $T^{-t}x \in J_1$.

Let

$$\varphi_k(x) = \left(\frac{T^{-t(x)}x - a_k}{|J_1|}, \frac{t(x)}{q_k} \right) \quad (x \in [0, 1)),$$

where $J_1 = [a_k, b_k)$ and

$$t(x) = \min\{t \in \mathbb{Z}_+ : T^{-t}x \in J_1\}.$$

Obviously, for any x and for sufficiently large k, we have $\varphi_k(x) \in [0, 1)^2$. The Euclidean metric on $\mathbb{R}^2$ defines a metric ρ_k on $[0, 1)$ by the formula

$$\rho_k(x, y) = \rho(\varphi_k(x), \varphi_k(y)).$$

6. *The equivalence classes of the characters*

We consider the group $\hat{\mathbb{Z}}_2^{\tilde{N}}$ of all characters of the group $\mathbb{Z}_2^{\tilde{N}} \equiv \{-1, 1\}^{\tilde{N}}$. Every character $\chi(y)$ depends on the finite coordinates. Therefore, for any $y = (y_{11}, y_{12}, \ldots) \in \{-1, 1\}^{\tilde{N}}$ we have

$$\chi(y) = y_{11}^{i_{11}} y_{12}^{i_{12}} \cdots,$$

where $i_{kl} \in \{0, 1\}$ and $i_{kl} = 0$ for sufficiently large k. Henceforth we shall identify the character χ and the corresponding collection $(i_{11}, i_{12}, \ldots)$ from $\{0, 1\}^{\tilde{N}}$ with finite support.

We define an equivalence relation on $\hat{\mathbb{Z}}_2^{\tilde{N}}$ by the following conditions.

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(1) $\forall \chi \in \hat{\mathbb{Z}}_2^{\tilde{N}}$, $\chi \sim \chi$.
(2) If $\chi_1 \neq \chi_2$, then $\chi_1 \sim \chi_2 \Longleftrightarrow (i_{11}, i_{12}, \dots) \sim (j_{11}, j_{12}, \dots) \Longleftrightarrow \exists k\ \exists m_1, m_2 \leq n_k$ ($m_1 \neq m_2$ and $i_{km_1} = \sum_{rl} i_{rl} = \sum_{rl} j_{rl} = j_{km_2} = 1$).

The Hilbert space $L_2(\mathbb{T} \times \{-1, 1\}^{\tilde{N}}, \mu \times l^{\tilde{N}})$ can be represented as the direct sum of subspaces H_χ invariant with respect to $T_{\omega_{\tilde{\beta}}}$, where

$$H_\chi = \{f(x) \cdot \chi(y) : f \in L_2(\mathbb{T}, \mu), \chi \in \hat{\mathbb{Z}}_2^{\tilde{N}}\}.$$

The operator $T_{\omega_{\tilde{\beta}}}$ on the subspace H_χ is spectrally isomorphic to the operator

$$\omega_\chi(x)T : f(x) \to \omega_\chi(x) f(Tx),$$

acting on $L_2(\mathbb{T}, \mu)$, where

$$\omega_\chi(x) \equiv \omega_{\beta_1}^{i_{11}}(x)\omega_{\beta_1}^{i_{12}}(S_{11}x) \cdots \omega_{\beta_1}^{i_{1n_1}}(S_{1n_1-1}x)\omega_{\beta_2}^{i_{21}}(x) \cdots .$$

We denote $V_\chi = \omega_\chi(x)T$, σ_χ is the measure of maximal spectral type of the operator V_χ.

7. *Theorem 1 as a corollary of Theorem 3*

THEOREM 3. *For each transformation T from class A and for a generic $\tilde{\beta} \in \Omega$ we have:*
(1) *for each $\chi \in \hat{\mathbb{Z}}_2^{\tilde{N}}$ the operator V_χ has a simple spectrum;*
(2) *for each $\chi_1, \chi_2 \in \hat{\mathbb{Z}}_2^{\tilde{N}}$, $\chi_1 \sim \chi_2 \Longrightarrow \sigma_{\chi_1} \sim \sigma_{\chi_2}$;*
(3) *for each $\chi_1, \chi_2 \in \hat{\mathbb{Z}}_2^{\tilde{N}}$, $\chi_1 \not\sim \chi_2 \Longrightarrow \sigma_{\chi_1} \perp \sigma_{\chi_2}$;*
(4) *for each χ ($\chi \not\equiv \chi_0 \equiv 1$), σ_χ is a continuous measure.*

Proof of Theorem 1. Let
(1) $L_2(m) = \bigoplus_i C(h_i)$,
(2) $k_i = \#\{j : \sigma_{h_i} \sim \sigma_{h_j}\}$,
(3) for each i, j either $\sigma_{h_i} \sim \sigma_{h_j}$ or $\sigma_{h_i} \perp \sigma_{h_j}$,
where $C(h) = C_{T_{\omega_{\tilde{\beta}}}}(h)$ is the closed linear span of the vectors $T_{\omega_{\tilde{\beta}}}^n h$, $n \in \mathbb{Z}$ (cyclic space with respect to $T_{\omega_{\tilde{\beta}}}$ for vector h), σ_h is the spectral measure corresponding to h; that is the unique Borel measure on $\mathbb{T} \equiv [0, 1)$ for which

$$\hat{\sigma}_h(n) = \int_0^1 e^{2\pi i n\lambda}\, d\sigma_h(\lambda) = (T_{\omega_{\tilde{\beta}}}^n h, h),$$

then

$$M(T_{\omega_{\tilde{\beta}}}) = \{k_1, k_2, \dots\}.$$

We write down all the cardinalities of the equivalence classes of the set $\hat{\mathbb{Z}}_2^{\tilde{N}}$:

$$1, 1, 1, \dots, n_1, n_2, \dots.$$

Now, by Theorem 3(1),

$$L_2(m) = \bigoplus_i C(\chi_i).$$

Combining this with Theorem 3(2) and 3(3), we obtain

$$M(T_{\omega_{\tilde{\beta}}}) = \{1, n_1, n_2, \dots\} = M.$$

The ergodicity of $T_{\omega_{\tilde{\beta}}}$ and the independence of all eigenfunctions of $T_{\omega_{\tilde{\beta}}}$ from y follows immediately from Theorem 3(4), since T is ergodic. □

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8. *Topological lemma*

Fix $\varepsilon > 0$, $\tilde{N} < \infty$, $l \le \tilde{N}$ and the sequences $r_i(k) \in \mathbb{N}$ $(l < i \le \tilde{N})$ and $\tilde{\beta}(k) \in \Omega$, where

$$\tilde{\beta}(k) = (\beta_1(k), \dots, \beta_l(k), T^{s_1(k)}, T^{s_2(k)}, \dots, T^{s_{\tilde{N}-l}(k)}).$$

We denote

$$\tilde{U}_i(k) = \{\beta \in \mathbb{T} : \rho_k(\beta, \beta_i(k)) < \varepsilon\} \quad (1 \le i \le l)$$
$$\tilde{U}_i(k) = \{T^{s_i(k)+j} : |j| < r_i(k)\} \quad (l < i \le \tilde{N})$$
$$U_\varepsilon(k) = \bigotimes_{i=1}^{\tilde{N}} \tilde{U}_i(k) \subseteq \Omega.$$

LEMMA 3. *Let $\varepsilon > 0$ and $k_n \to \infty$ as $n \to \infty$. Suppose that*

(1) $\forall i\ \varphi_{k_n}(\beta_i(k_n)) \in [0, 1)^2$;

(2) $\forall i\ r_i(k_n) \to +\infty$ *as* $n \to \infty$;

(3) $\forall i\ T^{s_i(k_n)} \to E$ *as* $n \to \infty$.

Then the set

$$\bigcap_m \bigcup_{n>m} U_\varepsilon(k_n)$$

is an everywhere dense subset in Ω with respect to the product topology.

Proof of Lemma 3. Every neighborhood of Ω contains a set of the following form:

$$\bigotimes_{i=1}^{\tilde{N}} O_i,$$

where $O_i = (a_i, b_i)$ $(1 \le i \le l)$, O_i is a open subset in $C(T)$ $(l < i \le \tilde{N})$. It suffices to prove that for any i and for sufficiently large n,

$$\tilde{U}_i(k_n) \cap O_i \ne \emptyset. \tag{2}$$

We use the following fact.

LEMMA 4. [7] *If a sequence of functions h_n is bounded in norm, U is a unitary operator on separable Hilbert space and $h_n - Uh_n$ converges weakly to 0, then $h_n - P_U h_n$ converges weakly to 0, where P_U is the operator of orthogonal projection onto $H_{inv} = \{h : h = Uh\}$.*

We consider the sets $\varphi_{k_n}(\tilde{U}_i(k_n)) \subseteq \mathbb{R}^2$ $(1 \le i \le l)$. It is not hard to see that

$$\liminf_n \mu((\tilde{U}_i(k_n))) > c > 0, \tag{3}$$
$$\|Tf_n - f_n\|_{L_2} \to 0 \quad \text{as } n \to \infty,$$

where

$$f_n(x) = \begin{cases} 1, & x \in \tilde{U}_i(k_n), \\ 0, & x \notin \tilde{U}_i(k_n). \end{cases}$$

Using the ergodicity property of T, for operator T from Lemma 4 we have

$$\forall A \quad \mu(A \cap \tilde{U}_i(k_n)) - \mu(\tilde{U}_i(k_n))\mu(A) \to 0 \quad \text{as } n \to \infty.$$

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Therefore, by (3), we get (2) ($1 \le i \le l$).

For any $S \in C(T)$ there exists a sequence of integers t_n ($n \in \mathbb{N}$) such that $T^{t_n} \to S$ as $n \to \infty$ [**13**]. Therefore for any sequence $m_n \to \infty$ and for any i we have

$$T^{s_i(k_n)+t_{m_n}} \to S \quad \text{as } n \to \infty. \tag{4}$$

We choose m_n so that

$$|t_{m_n}| < r_i(k_n).$$

Therefore, by (3), we get (2) ($l < i \le \tilde{N}$). □

9. *Principal points of the proof of Theorem 3*

Theorem 3(2) follows from the definition of a unitary operator U with $UV_{\chi_1} = V_{\chi_2}U$, where $\chi_1 \sim \chi_2$.

For each character $\chi \in \hat{\mathbb{Z}}_2^{\tilde{N}}$ we choose open sets $U'_\varepsilon(k) \subseteq \Omega$, where $U_\varepsilon(k) \subseteq U'_\varepsilon(k)$ and $U_\varepsilon(k)$ come from the topological lemma, such that the proof of Theorem 3(1) for V_χ and $\tilde{\beta} \in \bigcap_m \bigcup_{n>m} U'_\varepsilon(k_n)$ essentially follows the proof of [**8**, Theorem 4]. Here the sets on which the functions $\omega_\chi(x)$ have constant sign are determined by the analogous sets for the functions $\omega_\beta(S_{ij}\chi)$.

As before, for each pair χ_1, χ_2 ($\chi_1 \not\sim \chi_2$) we also choose its open set $U'_\varepsilon(k)$. The proof of Theorem 3(3) is based on the following assertions.

LEMMA 5. *Suppose that T is ergodic and $\varphi \in L_2(X, \mu)$, where $|\varphi(x)| = 1$ for almost all x. If T^{k_i} converges weakly to E, then the operator $U_\varphi^{k_i} - (U_\varphi^{k_i}\mathbb{1}, \mathbb{1})E$ converges weakly to 0, where*

$$U_\varphi : f(x) \to \varphi(x) f(Tx)$$

denotes a unitary operator on $L_2(X, \mu)$.

LEMMA 6. [**6**] *Let U_i ($i = 1, 2$) be a unitary operator on a separable Hilbert space. Let σ_i denote the measure of maximal spectral type of U_i. Assume that $k_m \to \infty$ as $m \to \infty$ and $c_1 \ne c_2$. If the sequence $U_i^{k_m}$ converges weakly to $c_i E$ ($i = 1, 2$) then*

$$\sigma_1 \perp \sigma_2.$$

Henceforth for $\tilde{\beta} \in \bigcap_m \bigcup_{n>m} U'_\varepsilon(k_n)$ there exists $\delta > 0$ such that for any n

$$|(V_{\chi_1}^{q_{k_n}}\mathbb{1}, \mathbb{1}) - (V_{\chi_2}^{q_{k_n}}\mathbb{1}, \mathbb{1})| > \delta.$$

Combining this with $T^{q_{k_n}} \to E$ as $n \to \infty$ we prove the Theorem 3(3).

Henceforth we choose $U'_\varepsilon(k_n)$ so that for sufficiently large n

$$|(V_{\chi_1}^{q_{k_n}}\mathbb{1}, \mathbb{1})| < 1 - \varepsilon.$$

Theorem 3(4) for $\tilde{\beta} \in \bigcap_m \bigcup_{n>m} U'_\varepsilon(k_n)$ follows from Lemma 5 and from the next assertion, which is very simple.

PROPOSITION 1. *Let $c \in \mathbb{C}$ and $|c| < 1$. If a sequence of unitary operators U^{r_n} converges weakly to cE as $n \to \infty$, then the operator U has a continuous spectrum.*

Finally, the generic set in Theorem 3 is obviously the intersection of the generic sets considered above.

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General notation. We shall use the following notation:

$$\omega_\gamma(x)\cdot\omega_\gamma(Tx)\cdots\omega_\gamma(T^{q_{k_n}-1}x)=\omega_\gamma^{(n)}(x)\quad(\gamma\in\mathbb{T}),$$
$$\omega_\chi(x)\cdot\omega_\chi(Tx)\cdots\omega_\chi(T^{q_{k_n}-1}x)=\omega_\chi^{(n)}(x)\quad(\chi\in\hat{\mathbb{Z}}_2^{\tilde{N}}),$$
$$\mathrm{Mom}_n(\chi,\tilde{\beta})=|(V_\chi^{q_{k_n}}\mathbb{1},\mathbb{1})|=\left|\int_{\mathbb{T}}\omega_\chi^{(n)}(x)\,d\mu\right|.$$

Note that

$$\omega_\chi^{(n)}(x)=(\omega_{\beta_1}^{(n)}(x))^{i_{11}}\cdot(\omega_{\beta_1}^{(n)}(S_{1,1}x))^{i_{12}}\cdots,$$

where $\chi(y)=y_{11}^{i_{11}}y_{12}^{i_{12}}\cdots$.

Proof of Lemma 5. Let $f,g\in L_2(\chi,\mu)$ and

$$\varphi_i(x)=\varphi(x)\varphi(Tx)\cdots\varphi(T^{k_i-1}x).$$

Using

$$\|T\varphi_i-\varphi_i\|_{L_2}\to 0\quad\text{as } i\to\infty$$

and an equality $H_{inv}=\mathbb{C}\mathbb{1}$, we have, by Lemma 4,

$$(\varphi_i\cdot f,g)-(\varphi_i\cdot\mathbb{1},\mathbb{1})\cdot(f,g)\to 0\quad\text{as } i\to\infty.$$

Using

$$\|T^{k_i}f-f\|_{L_2}\to 0\quad\text{as } i\to\infty$$

and an equality $(\varphi_i\cdot\mathbb{1},\mathbb{1})=(U_\varphi^{k_i}\mathbb{1},\mathbb{1})$ we consequently obtain,

$$(U_\varphi^{k_i}f,g)-(\varphi_i f,g)\to 0\quad\text{as } i\to\infty,$$
$$(U_\varphi^{k_i}f,g)-(U_\varphi\mathbb{1},\mathbb{1})\cdot(f,g)\to 0\quad\text{as } i\to\infty.$$ □

10. *The sets on which the functions $\omega_\beta^{(n)}(x)$ have constant sign*

We consider generalized geometric models of T from class A.

PROPOSITION 2. *If $z_1(n,\beta)<1-1/p_{k_n}$, then*

$$\omega_\beta^{(n)}(x)=\begin{cases}\omega_\beta^{(n)}(\beta) & \text{if } z_1(n,\beta)\le z_1(n,x)<1-1/p_{k_n},\\ -\omega_\beta^{(n)}(\beta) & \text{if } z_1(n,x)<z_1(n,\beta)-1/p_{k_n},\end{cases}$$

where $z_i(n,x)$ $(i=1,2)$ are the coordinates of the point $\varphi_{k_n}(x)\in\mathbb{R}^2$.

The proof of Proposition 2 follows immediately from the following:

(1) if $z_2(n,x)=z_2(n,x)\ne z_2(n,\beta)$, then $\omega_\beta(x_1)=\omega_\beta(x_2)$;

(2) if $z_2(n,x)=z_2(n,\beta)$, then

$$\omega_\beta(x)=\begin{cases}\omega_\beta(\beta) & \text{if } z_1(n,\beta)\le z_1(n,x),\\ -\omega_\beta(\beta) & \text{otherwise;}\end{cases}$$

(3) if $z_2(n,x)<1-1/q_{k_n}$, then $(z_1(n,Tx),z_2(n,Tx))=(z_1(n,x),z_2(n,x)+1/q_{k_n})$;

(4) if $z_1(n,x)<1-1/p_{k_n}$, then

$$(z_1(n,T^{q_{k_n}}x),z_2(n,T^{q_{k_n}}x))=(z_1(n,x)+1/p_{k_n},z_2(n,x)).$$

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11. *Four technical lemmas*

LEMMA 7. *There exist a number $\varepsilon > 0$ and an index n_0 such that for any index n greater that n_0 there exist $\gamma_i(n) \in \{x \in \mathbb{T} : z_1(n,x) < 1 - 1/p_{k_n}\}$, $s_i(n) \in \mathbb{Z}$ $(1 \le i \le 3)$ such that the following inequalitys hold:*

$$1 - \left| \int_{\mathbb{T}} \omega_{\gamma_1}^{(n)}(x)\, d\mu \right| > \varepsilon, \tag{5}$$

$$1 - \left| \int_{\mathbb{T}} \omega_{\gamma_2}^{(n)}(x)\omega_{\gamma_2}^{(n)}(T^{s_1}x)\, d\mu \right| > \varepsilon, \tag{6}$$

$$\left| \left| \int_{\mathbb{T}} \omega_{\gamma_3}^{(n)}(x)\omega_{\gamma_3}^{(n)}(T^{s_2}x)\omega_{\gamma_3}^{(n)}(T^{s_3}x)\, d\mu \right| - \left| \int_{\mathbb{T}} \omega_{\gamma_3}^{(n)}(x)\, d\mu \right| \right| > \varepsilon, \tag{7}$$

$$T^{s_i(n)} \to E \quad \text{as } n \to \infty \ (i = 1, 2, 3).$$

Proof of Lemma 7. We consider generalized geometric models of T. Part 1 (inequality (5)) follows directly from Proposition 2.

We prove part 2 (inequality (6)). Let $s_1 = s_1(n) = -[p_{k_n}/4] \cdot q_{k_n}$. We choose $\gamma_2 = \gamma_2(n)$ so that $z_1(n,\gamma) = 0.25$. For any interval $I \subset [0,1)$ we denote $I(n) = \{x \in \mathbb{T} : z_1(n,x) \in I\}$. Using the definition of class A we have

$$\sup_{\substack{0 \le i \le p_{k_n} \\ 0 \le j \le q_{k_n}}} \mu(T^{iq_{k_n}+j} I(n) \triangle (I(n) + i/p_{k_n})) \to 0 \quad \text{as } n \to \infty, \tag{8}$$

where $I(n) + c = \{x \in \mathbb{T} : \exists y \in I(n) : z_1(n,x) = z_1(n,y) + c \bmod 1\}$.

Using Proposition 2 we have

$$\mu(\{x : \omega_{\gamma_2(n)}^{(n)}(x)\omega_{\gamma_2(n)}^{(n)}(T^{s_1(n)}x) = -1\} \triangle \{x : z_1(n,x) < 0.5\}) \to 0 \quad \text{as } n \to \infty.$$

Using

$$\mu\{x : z_1(n,x) < c\} \to c \quad \text{as } n \to \infty \quad (0 \le c \le 1), \tag{9}$$

for sufficiently large n we have (6).

We prove part 3 (inequality (7)). Let $s_i(n) = -[p_{k_n} \cdot (i-1)/5] g_{k_n}$ $(i = 2, 3)$. We choose $\gamma_3 = \gamma_3(n)$ so that $z_1(n, \gamma_3) = 0.2$. Using Proposition 2 and (9), we have

$$\left| \int_{\mathbb{T}} \omega_{\gamma_3}^{(n)}(x)\, d\mu \right| \to 0.6 \quad \text{as } n \to \infty.$$

Using (8) and Proposition 2, we have for some $c \in \{-1, 1\}$,

$$\mu(\{x : \omega_{\gamma_3}^{(n)}(x)\omega_{\gamma_3}^{(n)}(T^{s_2(n)}x)\omega_{\gamma_3}^{(n)}(T^{s_3(n)}x) = c\} \triangle \{x : z_1(n,x) < 0.6\}) \to 0$$

as $n \to \infty$. Therefore,

$$\left| \int_{\mathbb{T}} \omega_{\gamma_3}^{(n)}(x)\omega_{\gamma_3}^{(n)}(T^{s_2(n)}x)\omega_{\gamma_3}^{(n)}(T^{s_3(n)}x)\, d\mu \right| \to 0.2 \quad \text{as } n \to \infty.$$

From (1), we have

$$\mu(T^{-s_i(n)} \tilde{I}(n) | \tilde{I}(n)) \to 0 \quad \text{as } n \to \infty,$$

where $\tilde{I}(n) = \{x \in \mathbb{T} : z_2(n,x) = 0\}$. Therefore $T^{-s_i(n)} \to E$ as $n \to \infty$, and part 4 is proved. □

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LEMMA 8. *If $\chi_1 \not\sim \chi_2$, then there exist a number $\varepsilon > 0$ and an index n_0 such that for any $n > n_0$ there exists $\tilde{\beta} = (\beta_1, \beta_2, \dots, T^{l_1}, T^{l_2}, \dots) \in \Omega$ such that*

$$\forall i \quad T^{l_i} \to E \text{ as } n \to \infty, \quad z_1(n, \beta_i) < 1 - 1/p_{k_n},$$
$$\mathrm{Mom}_n = |\mathrm{Mom}_n(\chi_1, \tilde{\beta}) - \mathrm{Mom}_n(\chi_2, \tilde{\beta})| > \varepsilon.$$

Proof of Lemma 8. We use the notation $\tilde{\beta} = (\beta_1, \beta_2, \dots, S_{1,1}, S_{1,2}, \dots)$.

I. We consider the special case $i_{rt} = j_{rt} = 0$ $(2 \le r, 1 \le t \le n_r)$, where $\chi_1 = y_{11}^{i_{11}} y_{12}^{i_{12}} \cdots$, $\chi_2 = y_{11}^{j_{11}} y_{12}^{j_{12}} \cdots$. If

$$\sum_t i_{1t} = \sum_t j_{1t} \pmod 2 \tag{10}$$

is false, then we denote $\beta_1 = \gamma_1$, $S_{1,1} = S_{1,2} = \cdots = E$ and γ_1 from Lemma 7 (we choose β_i $(2 \le i)$ so that $z_1(n, \beta_i) < 1 - 1/p_{k_n}$). Using (5), for sufficiently large n we have

$$\mathrm{Mom}_n = 1 - \left| \int_{\mathbb{T}} \omega_{\beta_1}^{(n)}(x)\, d\mu \right| > \varepsilon.$$

If (10) is true, then from $\chi_1 \not\sim \chi_2$ it is not hard to see that substituting i for j and j for i (if necessary) at least one of the following alternatives holds:

(1) $\exists m_1$ $(1 < m_1, i_{11} = i_{1m_1} = 1, j_{11} = j_{1m_1} = 0, \sum_r i_{1r} = 0 \pmod 2)$;
(2) $\exists m_1, m_2$ $(1 < m_1 < m_2, i_{1m_p} = 1, j_{1m_p} = 0, p = 1, 2, \sum_r i_{1r} = 0 \pmod 2)$;
(3) $\exists m_1, m_2$ $(1 < m_1 < m_2, i_{1m_p} = 1, j_{1m_p} = 0, p = 1, 2, \sum_r i_{1r} = 1 \pmod 2)$;
(4) $\exists m_2, m_3$ $(1 < m_p,\ p = 2, 3,\ i_{11} = i_{1m_2} = i_{1m_3} = j_{1m_3} = 1,\ j_{11} = j_{1m_2} = 0,$ $\sum_r i_{1r} = 1 \pmod 2)$;
(5) $\exists m_1, m_2$ $(i_{11} = j_{11} = i_{1m_1} = j_{1m_2} = 1, j_{1m_1} = i_{1m_2} = 0, \sum_r i_{1r} = 0 \pmod 2)$;
(6) $\exists m_2, m_3$ $(i_{11} = i_{1m_3} = j_{1m_2} = j_{1m_3} = 1, j_{11} = i_{1m_2} = 0, \sum_r i_{1r} = 0 \pmod 2)$;
(7) $\exists m_1, m_2, m_3$ $(1 < m_p,\ p = 1, 2, 3,\ i_{1m_1} = j_{1m_2} = i_{1m_3} = j_{1m_3} = 1,$ $j_{1m_1} = i_{1m_2} = 0, \sum_r i_{1r} = 0 \pmod 2)$;
(8) $\exists m_1, m_2, m_3$ $(m_1 \ne m_3,\ i_{11} = j_{1m_2} = i_{1m_3} = j_{1m_3} = i_{1m_1} = j_{1m_1} = 1,$ $j_{11} = i_{1m_2} = 0, \sum_r i_{1r} = 1 \pmod 2)$;
(9) $\exists m_1, m_2, m_3$ $(m_3 \ne 1,\ i_{11} = j_{11} = i_{1m_1} = j_{1m_2} = i_{1m_3} = j_{1m_3} = 1,$ $j_{1m_1} = i_{1m_2} = 0, \sum_r i_{1r} = 1 \pmod 2)$;
(10) $\exists m_1, m_2, m_3$ $(1 < m_p,\ p = 1, 2, 3,\ i_{1m_1} = j_{1m_2} = i_{1m_3} = j_{1m_3} = 1,$ $j_{1m_1} = i_{1m_2} = 0, \sum_r i_{1r} = 1 \pmod 2)$.

(If $\exists m_1, m_2$ $(i_{1m_p} = 1, j_{1m_p} = 0, p = 1, 2, (m_1 \ne m_2))$ then we have at least one of the alternatives (1)–(4). Otherwise, at least one of the alternatives (5)–(10) is valid.)

We determine β_1 and $S_{1,j}$ $(1 \le j \le n_1 - 1)$ in any variant respectively.

(1)–(2) $\beta_1 = \gamma_2$, $S_{1,m_1-1} = T^{s_1}$, $S_{1,j} = E$ $(j \ne m_1 - 1)$.
(3) $\beta_1 = \gamma_3$, $S_{1,m_1-1} = T^{s_2}$, $S_{1,m_2-1} = T^{s_3}$, $S_{1,j} = E$ $(j \notin \{m_1 - 1, m_2 - 1\})$.
(4) $\beta_1 = \gamma_3$, $S_{1,m_2-1} = T^{s_2}$, $S_{1,j} = T^{s_3}$ $(j \ne m_2 - 1)$.
(5) $\beta_1 = \gamma_2$, $S_{1,m_1-1} = T^{s_1}$, $S_{1,j} = E$ $(j \ne m_1 - 1)$.
(6)–(7) $\beta_1 = \gamma_2$, $S_{1,m_2-1} = T^{s_1}$, $S_{1,j} = E$ $(j \ne m_2 - 1)$.
(8)–(10) $\beta_1 = \gamma_3$, $S_{1,m_1-1} = T^{s_2}$, $S_{1,m_3-1} = S_{1,m_2-1} = T^{s_3}$, $S_{1,j} = E$ $(j \notin \{m_1 - 1, m_2 - 1, m_3 - 1\})$,

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where γ_i, s_i $(1 \le i \le 3)$ from Lemma 7. In every case we get that Mom_n is a left-hand side of (6) or (7). Lemma 8 in case I is proved.

II. The general case is a simple consequence of I. Indeed, for any character $\chi = y_{11}^{i_{11}} y_{12}^{i_{12}} \cdots$ we consider the character $\chi(t) = y_{11}^{i_{11}(t)} y_{12}^{i_{12}(t)} \cdots$, where

$$i_{lr}^{(t)} = \begin{cases} i_{lr} & \text{if } l = t \\ 0 & \text{otherwise.} \end{cases}$$

If for some $t_0 \chi_1(t_0) \not\sim \chi_2(t_0)$ $(\chi_1 \not\sim \chi_2)$, then we determine $\beta_{t_0}, S_{t_0,1}, S_{t_0,2}, \ldots$ as in case I. Let $\beta_t = 0$, $S_{t,j} = E$ $(t \ne t_0)$. Then we have what is required.

From the definition of class A it follows that $z_1(n, 0) = 0 < 1 - 1/p_{k_n}$ for sufficiently large n.

If for each t $\chi_1(t) \sim \chi_2(t)$, then there exist $t_1, t_2, m_1, m_2, m_3, m_4$ $(m_1 \ne m_2, t_1 \ne t_2, m_1, m_2 \le n_{t_1}, m_3, m_4 \le n_{t_2})$ such that

$$i_{t_1,r} = 1 \Longleftrightarrow r = m_1, \quad i_{t_2,r} = 1 \Longleftrightarrow r = m_3,$$
$$j_{t_1,r} = 1 \Longleftrightarrow r = m_2, \quad j_{t_2,r} = 1 \Longleftrightarrow r = m_4.$$

(It is possible that $m_3 = m_4$.) To be definite, assume that $m_1 \ne 1$. We choose $\beta_{t_1} = \beta_{t_2}$, $\beta_t = 0$ $(t \notin \{t_1, t_2\})$, $S_{t,j} = E$ $(t \ne t_1)$ and, if $m_2 \ne 1$ then $S_{t_1,m_2-1} = E$. We have

$$\mathrm{Mom}_n = 1 - \left| \int_{\mathbb{T}} \omega_{\beta_{t_1}}^{(n)}(S_{t_1,m_1-1}x)\omega_{\beta_{t_1}}^{(n)}(x)\, d\mu \right| > \varepsilon.$$

The last inequality is true for $\beta_{t_1} = \gamma_2$ and $S_{t_1,m_1-1} = T^{s_1}$, where $\gamma_2(n)$ and $s_1(n)$ are from Lemma 7. □

LEMMA 9. *If $\chi \ne \chi_0$, then there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for any $n > n_0$ there exists $\tilde{\beta} = (\beta_1, \beta_2, \ldots, T^{l_1}, T^{l_2}, \ldots) \in \Omega$ such that*

$$\forall i \quad z_1(n, \beta_i) < 1 - p_{k_n}, \quad T^{l_i} \to E \text{ as } n \to \infty,$$
$$\mathrm{Mom}_n(\chi, \tilde{\beta}) < 1 - \varepsilon.$$

Proof of Lemma 9. Fix n, $\chi = y_{11}^{i_{11}} y_{12}^{i_{12}} \cdots$ and r_0, j_0 with $i_{r_0,j_0} = 1$. Determine $\tilde{\beta} = (\beta_1, \beta_2, \ldots, S_{1,1}, S_{1,2}, \ldots)$. Let $\beta_p = 0$, $S_{p,j} = E$ $(p \ne r_0)$.

If

$$\sum_j i_{r_0 j} = 1 \pmod 2 \tag{11}$$

then we choose $\beta_{r_0} = \gamma_1$, $S_{r_0 j} = E$, where $\gamma_1(n)$ is from Lemma 7.

If (11) is false, then there exists j_1 $(j_1 \ne j_0)$ such that $i_{r_0 j_1} = 1$. In this case we choose $\beta_{r_0} = \gamma_2$, $S_{r_0,j_{t_0}-1} = T^{s_1}$, $S_{r_0,j} = E$ $(j \ne j_{t_0} - 1)$, where for $t_0 \in \{0, 1\}$ the condition $j_{t_0} \ne 1$ is true, and $\gamma_2(n), s_1(n)$ are from Lemma 7.

Using (5) and (6) respectively, for any $n > n_0$ we have

$$\mathrm{Mom}_n(\chi, \beta(n)) < 1 - \varepsilon.$$ □

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Fix a character $\chi = y_{11}^{i_{11}} y_{12}^{i_{12}} \cdots$. We choose j_1 and j_2, so that

$$\max\{j : \exists r\ i_{jr} = 1\} \le j_1 < \infty, \quad j_2 = \sum_{r=1}^{j_1} (n_r - 1).$$

LEMMA 10. *For any $\varepsilon > 0$ there exist $\delta > 0$ and n_0 such that for any $n > n_0$ and any $\tilde\beta^{(i)} = (\beta_1^{(i)}, \beta_2^{(i)}, \dots, S_1^{(i)}, S_2^{(i)}, \dots) \in \Omega$, if $z_1(n, \beta_j^{(i)}) < 1 - 1/p_{k_n}$, $\rho_{k_n}(\beta_j^{(1)}, \beta_j^{(2)}) < \delta$ $(1 \le j \le j_1, i = 1, 2)$, $S_j^{(1)} = T^{t_j} S_j^{(2)}$, $|t_j| < \delta q_{k_n} (1 \le j \le j_2)$, then*

$$|\mathrm{Mom}_n(\chi, \tilde\beta^{(1)}) - \mathrm{Mom}_n(\chi, \tilde\beta^{(2)})| < \varepsilon.$$

Proof of Lemma 10. This follows directly from the next proposition. □

PROPOSITION 3. *For any $\varepsilon > 0$ there exist $\delta > 0$ and n_0 such that for any $n > n_0$ we have:*

(1) *$(\rho_{k_n}(\beta_1, \beta_2) < \delta$ and $z_1(n, \beta_i) < 1 - 1/p_{k_n}) \Longrightarrow \exists j$ such that $\mu\{x \in \mathbb{T} : \omega_{\beta_1}^{(n)}(x) \ne (-1)^j \omega_{\beta_2}^{(n)}(x)\} < \varepsilon$;*

(2) *$(|t| < \delta q_{k_n}$ and $z_1(n, \beta_1) < 1 - 1/p_{k_n}) \Longrightarrow \exists j$ such that $\mu\{x \in \mathbb{T} : \omega_{\beta_1}^{(n)}(x) \ne (-1)^j \omega_{\beta_1}^{(n)}(T^t x)\} < \varepsilon$.*

Proof of Proposition 3. From the condition $\rho_{k_n}(\beta_1, \beta_2) < \delta$ and Proposition 2 we have consequently

$$|z_1(n, \beta_1) - z_1(n, \beta_2)| < \delta,$$

$$\mu(\{x \in \mathbb{T} : \omega_{\beta_1}^{(n)}(x)\omega_{\beta_2}^{(n)}(x) \ne \omega_{\beta_1}^{(n)}(\beta_1)\omega_{\beta_2}^{(n)}(\beta_2)\} \triangle \{x : z_1(n, x) \in [z_1(n, \beta_i), z_1(n, \beta_{3-i}))\}) \to 0 \quad \text{as } n \to \infty.$$

Taking into account (9), we obtain the proof of Proposition 3(1). Proposition 3(2) follows immediately from (8) and Proposition 2. □

Proof of Theorem 3. We fix the characters χ_1, χ_2, where $\chi_1 \sim \chi_2$. We can assume, without loss of generality, that $\chi_1(y) = y_{11}$, $\chi_2(y) = y_{12}$, $n_1 > 1$. Theorem 3(2) follows from the equality

$$V_{\chi_2} S_{1,1} = S_{1,1} V_{\chi_1}.$$

Fix a character $\chi = y_{11}^{i_{11}} y_{12}^{i_{12}} \cdots$. We shall construct some dense G_δ subset K of Ω, where V_χ has a simple spectrum if $\tilde\beta \in K$. Let $\varepsilon = 0.01$, $l_1 = \max\{j : \exists r : i_{j,r} = 1\}$,

$$\tilde U_i'(k_n) = \{\beta \in \mathbb{T} : \rho_{k_n}(\beta, \beta(n)) < \varepsilon\} \quad (1 \le i \le l_1),$$

$$\tilde U_i'(k_n) = \mathbb{T} \quad (l_1 < i \le l),$$

$$\tilde U_i''(k_n) = \{S \in C(T) : \exists j : \mu(SA_n \triangle T^j A_n) < \delta(n) \text{ and } |j| < \varepsilon q_{k_n}\} \quad (1 \le i \le l_2),$$

$$\tilde U_i''(k_n) = C(T) \quad (l_2 < i),$$

$$\tilde U_i'(k_n) = \bigotimes_{i=1}^{l} \tilde U_i'(k_n) \times \bigotimes_{i=1}^{\tilde N - l} \tilde U_i''(k_n),$$

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where $l_2 = \sum_{i=1}^{l_1}(n_r - 1)$, $\beta(n)$ is defined by $\varphi_{k_n}(\beta(n)) = (0.15, 0)$, and

$$A_n = \{x \in \mathbb{T} : z_2(n,x) = 1 - 1/q_{k_n} \text{ and } 0.3 \le z_1(n,x) < 0.99\},$$
$$\delta(n) = \frac{0.2}{l_2 q_{k_n}} \tag{12}$$

It follows from $\tilde{U}_i''(k_n) \supseteq \tilde{U}_i(k_n) = \{T^j : |j| < \varepsilon q_{k_n}\}$ $(1 < i \le l_2)$ and the topological lemma that the set $\bigcap_m \bigcup_{n>m} U_\varepsilon'(k_n)$ is a dense G_δ subset of Ω.

Let $\tilde{\beta} \in \bigcap_m \bigcup_{n>m} U_\varepsilon'(k_n)$. Fix $n, j_{11}, j_{12}, \dots$, where $\beta = (\beta_1, \dots, S_{11}, S_{12}, \dots) \in U_\varepsilon'(k_n)$ and $\mu(T^{j_{mr}} A_n \triangle S_{mr} A_n) < \delta(n)$ $(m \le l_1)$. We denote

$$\tilde{A}_n = A_n \cap \bigcap_{\substack{m \le l_1 \\ 1 \le r \le n_m - 1}} T^{-j_{mr}} S_{mr} A_n,$$
$$\tilde{A}_n' = \bigcup_{j=0}^{q_{k_n}-1} T^{-j} \tilde{A}_n,\ h_n(x) = \chi_{\tilde{A}_n},$$

where

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

From (12) and (9), for sufficiently large n we have

$$\mu(\tilde{A}_n') \ge q_{k_n} \left(\mu(A_n) - \frac{\delta(n) l_2}{2} \right) \to 0.59. \tag{13}$$

Since $\|\chi_{\tilde{A}_n'} - T\chi_{\tilde{A}_n'}\| \to 0$ as $n \to \infty$ and T is ergodic, by Lemma 4, we have

$$\int_{\tilde{A}_n'} f \, d\mu - \int_{\mathbb{T}} f \, d\mu \cdot \mu(\tilde{A}_n') \to 0 \quad \text{as } n \to \infty, \tag{14}$$

where $f \in L_2(\mathbb{T}, \mu)$.

We use the following fact which is a consequence of a proposition in [**14**, Ch. 3].

PROPOSITION 4. *If there exist a number $\Theta > 0$ and a sequence of vectors h_n from $L_2(\mathbb{T}, \mu)$ with the following property*

$$\forall h \ (\|h\| = 1) \quad \limsup_n d^2(C_U(h_n), h) \le \tfrac{1}{2} - \Theta, \tag{15}$$

then the unitary operator U has a simple spectrum, where $C_U(h)$ is the cyclic space for vector h relative to U and

$$d(C_U(h), g) = \min_{f \in C_U(h)} \|g - f\|_{L_2}.$$

It is sufficient to prove property (15) for an everywhere dense set of vectors h. Therefore we consider only a function h constant on all half-open intervals in generalized geometric models of T, beginning with some k. It is clear that $h(x)$ is constant on sets $T^{-j}\tilde{A}_n$ $(0 \le j < q_{k_n})$. It follows from definition of $\varepsilon, \delta(n), \beta(n)$ and A_n that for sufficiently large n,

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$\omega_\chi(x)$ is constant on sets $T^{-j}\tilde{A}_n$ $(0 \le j < q_{k_n})$. This implies that $V_\chi^j \chi_{\tilde{A}_n} = d(j) \cdot \chi_{T^{-j}\tilde{A}_n}$ $(0 \le j < q_{k_n})$, where $d(j)$ is independent of x. Therefore,

$$\chi_{T^{-j}\tilde{A}_n} \in C_{V_\chi}(h_n).$$

Hence,

$$h(x)\chi_{\tilde{A}'_n} = \sum_j h(x)\chi_{T^{-j}\tilde{A}_n} \in C_{V_\chi}(h_n).$$

Combining this with (13) and (14), we get

$$\begin{aligned} d^2(C_{V_\chi}(h_n), h) \le \|h \cdot \chi_{\tilde{A}'_n} - h\|^2 &= \int_{\mathbb{T}\setminus\tilde{A}'_n} |h|^2 \, d\mu \\ &= (1 - \mu(\tilde{A}'_n)) + \left(\mu(\tilde{A}'_n) \cdot \int_{\mathbb{T}} |h|^2 d\mu - \int_{\tilde{A}'_n} |h|^2 \, d\mu\right) < 0.42 \end{aligned}$$

for sufficiently large n. Using Proposition 4, we have that the proof of part 1 is complete.

We prove part 3. We fix $\chi_m = y_{11}^{i_{11}^{(m)}} y_{12}^{i_{12}^{(m)}} \cdots (m = 1, 2, \chi_1 \not\sim \chi_2)$, take $\varepsilon > 0$, n sufficiently large and $\tilde{\beta} = (\beta_1, \beta_2 \ldots, T^{l_1}, T^{l_2}, \ldots) \in \Omega$ from Lemma 8. Let

$$j_1 = \max\{j : \exists r, m : i_{jr}^{(m)} = 1\}, \quad j_2 = \sum_{r=1}^{j_1} (n_r - 1).$$

We shall construct the set $U'_\varepsilon(k_n)$. Using the continuity of $\mathrm{Mom}_n(\chi, \tilde{\beta})$ with respect to the coordinates $S_i \in C(T)$ it is not hard to determine the neighborhoods $U_i \subseteq C(T)$ of T^{l_i} $(i \le j_2)$ such that

$$|\mathrm{Mom}_n(\chi_m, \tilde{\beta}) - \mathrm{Mom}_n(\chi_m, \tilde{\tilde{\beta}})| < 0.1 \cdot \varepsilon, \tag{16}$$

where $\tilde{\tilde{\beta}} = (\beta_1, \beta_2, \ldots, S_1, S_2, \ldots)$, $S_i \in U_i$ $(i \le j_2)$.

We choose $\delta > 0$ (see Lemma 10) such that

$$|\mathrm{Mom}_n(\chi_m, \tilde{\beta}^{(1)}) - \mathrm{Mom}_n(\chi_m, \tilde{\beta}^{(2)})| < 0.1 \cdot \varepsilon \quad (m = 1, 2) \tag{17}$$

if $\rho_{k_n}(\beta_j^{(1)}, \beta_j^{(2)}) < \delta$, $z_1(n, \beta_j^{(m)}) < 1 - 1/p_{k_n}$ $(j \le j_1)$ and $S_j^{(1)} = T^{t_j} S_j^{(2)}$, $|t_j| < \delta \cdot q_{k_n}$ $(j \le j_2)$, where $\tilde{\beta}^{(m)} = (\beta_1^{(m)}, \beta_2^{(m)}, \ldots, S_1^{(m)}, S_2^{(m)}, \ldots)$.

Let

$$\begin{aligned} \tilde{U}'_j(k_n) &= \{\beta \in \mathbb{T} : \rho_{k_n}(\beta, \beta_j) < \delta \text{ and } z_1(n, \beta) < 1 - 1/p_{k_n}\} \quad (1 \le j \le j_1), \\ \tilde{U}'_j(k_n) &= \mathbb{T} \quad (j_1 < j \le l), \\ \tilde{U}''_j(k_n) &= \bigcup_{|r| < \delta q_{k_n}} T^r U_j \quad (1 \le j \le j_2), \\ \tilde{U}''_j(k_n) &= C(T) \quad (j_2 < j), \\ U'_\varepsilon(k_n) &= \bigotimes_j \tilde{U}'_j(k_n) \times \bigotimes_j \tilde{U}''_j(k_n). \end{aligned}$$

Using the same arguments, as above, we have that the set $\bigcap_m \bigcup_{n>m} U'_\varepsilon(k_n)$ is a dense G_δ subset of Ω.

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Suppose that $\tilde{\beta}' \in U_\varepsilon(k_n)$ ($\tilde{\beta}'$ from $\bigcap_m \bigcup_{n>m} U'_\varepsilon(k_n)$). By (16) and (17) the choice of $\tilde{\beta}$, we have

$$|\mathrm{Mom}_n(\chi_1, \tilde{\beta}') - \mathrm{Mom}_n(\chi_2, \tilde{\beta}')| > 0.6\varepsilon. \tag{18}$$

From (18) and Lemma 5 it follows that there exists a subsequence $s(p)$ such that

$$V_{\chi_i}^{q_{k_{s(p)}}} \to c_i E \quad \text{as } p \to \infty \ (i = 1, 2),$$
$$|c_1 - c_2| \geq 0.6 \cdot \varepsilon.$$

Using Lemma 6, we have that

$$\sigma_{\chi_1} \perp \sigma_{\chi_2}.$$

The proof of part 4 is completely similar to part 3, where we replace Lemma 8 by Lemma 9 and Lemma 6 by Proposition 1. □

Proof of Lemma 1. We fix the sequences of integers $q_{2n} \to \infty$, $p_n \to \infty$. Let ξ_{2n} and ζ_{2n+1} be a partition of $X = [0, 1)$ into q_{2n} and $q_{2n} \cdot p_n$ half-open interval of the same length, respectively. We define a set O_n of automorphisms T_n by the following conditions:

(1) $T_n \xi_n = \xi_n$;

(2) $\forall j \ (0 < j \leq q_n) \ T_n^i B_j(n) = B_j(n) \Longleftrightarrow i = 0 \bmod q_n$,

where $\xi_n = \{B_1(n), B_2(n), \ldots, B_{q_n}(n)\}$.

Let

$$C = \bigcap_{k=1} \bigcap_{n>k} \bigcup_{T_n \in O_{2n} \cap O_{2n+1}} \Bigg\{ T \in \mathrm{Aut}(\mu) : \sum_i \mu(T B_i(2n+1) \triangle T_n B_i(2n+1)) < \frac{1}{n q_{2n+1}} \Bigg\}.$$

We fix $T \in C$, n and T_n, where

$$\sum_i \mu(T B_i(2n+1) \triangle T_n B_i(2n+1)) < \frac{1}{n q_{2n+1}}.$$

Let

$$\tilde{B}_1(2n+1) = \{x \in B_1(2n+1) : T^i x \in T_n^i B_1(2n+1) \quad (0 \leq i \leq q_{2n+1})\},$$
$$\eta_{2n+1} = \{B'_{2n+1}, \tilde{B}_1(2n+1), T\tilde{B}_1(2n+1), \ldots, T^{q_{2n+1}-1}\tilde{B}_1(2n+1)\},$$
$$A_n = \tilde{B}_1(2n+1) \cup T^{q_{2n}}\tilde{B}_1(2n+1) \cup \cdots \cup T^{q_{2n}(p_n-1)}\tilde{B}_1(2n+1),$$
$$\eta_{2n} = \{A'_n, A_n, TA_n, \ldots, T^{q_{2n}-1}A_n\}.$$

It is clear that

$$\eta_{2n} \leq \eta_{2n+1}, \mu(B_1(2n+1) \setminus \tilde{B}_1(2n+1)) < \frac{\mu(B_1(2n+1))}{n},$$

and therefore, $\eta_{2n} \to \varepsilon$. Finally, we obtain that T admits an a.f.s.p.

To conclude the proof, it remains to note that C is a dense G_δ subset. But the latter is obvious, since

$$C \supset \bigcap_{k=1} \bigcup_{n>k} (O_{2n} \cap O_{2n+1}),$$
$$\overline{\bigcup_{n>k} (O_{2n} \cap O_{2n+1})} = \overline{\bigcup_{n>k} O_{2n}} = \mathrm{Aut}(\mu).$$ □

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Proof of Lemma 2. This lemma can be easily proved according to the following sketch and we omit the small details. Let T admit an a.f.s.p. for a sequence of partitions ξ_n.

Step 1. Change ξ_n to ξ'_n such that T admits an (a.f.s.p.) for ξ'_n and $\xi'_n \leq \xi'_{n+1}$ $(n = 1, 2, \dots)$.

The proof is the same as in [**15**].

Step 2. We construct isomorphism φ between (X, μ) and $([0, 1), l)$, where l is Lebesgue measure such that $\tilde{T} = \varphi T \varphi^{-1}$ maps J_i linearly onto J_{i+1} $(i < q_n)$, where $\varphi\xi'_n = \{J_1, \dots, J_{q_n}, J'_n\}$ consists of half-open intervals.

It is sufficient to construct φ as an isomorphism between σ-algebras and, moreover, only to define it on an everywhere dense set of all elements of partitions ξ'_n $(n = 1, 2, \dots)$.

Step 3. Change $\varphi\xi'_n$ to ξ''_n such that $\tilde{T}$ admits an a.f.s.p. for ξ''_n, maps $J_i(n)$ onto $J_{i+1}(n)$ $(i < q_n)$, where $\xi''_n = \{J_1(n), \dots, J_{q_n}(n), J''_n\}$ consists of half-open intervals and $J_1(n+1) \subseteq J_1(n)$ $(n = 1, 2, \dots)$.

The proof is left to the reader. □

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