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Dynamics of iterated function systems on the circle close to rotations

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Dedicated to G. Duminy

Abstract. We study the dynamics of iterated function systems generated by a pair of circle diffeomorphisms close to rotations in the $C^{1+\mathrm{bv}}$-topology. We characterize the obstruction to minimality and describe the limit set. In particular, there are no invariant minimal Cantor sets, which can be seen as a Denjoy/Duminy type theorem for iterated systems on the circle.

1. *Introduction*

The rotation number is a classical tool in the study of dynamics of a single diffeomorphism of the circle. When the rotation number is rational, the map has periodic points, and if it is irrational, then either each orbit is dense on the circle or there is an invariant (minimal) Cantor set. Consider now a group of orientation-preserving homeomorphisms of the circle. The above classification can be extended to the group case where the orbit of a point is the group action on this point. Then, there can occur only one of the following three options [**7**, **15**]: existence of a finite orbit, every orbit is dense on the circle, or there exists a unique minimal Cantor set invariant by the group action. Minimal here means that all the points in the invariant set have dense orbits.

For iteration of single diffeomorphisms, the well-known theorem by Denjoy [**4**] implies that in the C^2-topology there cannot exist minimal invariant Cantor sets, while there are counter-examples in the C^1 class. Motivated by the study of co-dimension one foliations, Duminy proved a Denjoy-like theorem for group actions [**6**]. Under the extra assumptions that the group generators are close to rotations in the $C^{1+\mathrm{bv}}$-topology (i.e. the C^1-class with bounded variation derivatives) and *finiteness of periodic points* for one of the generators, the minimal invariant Cantor set does not exist. See [**14**, **15**] for details of the proof.

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In this paper, motivated by the dynamics of partially hyperbolic skew-products [**2**, **3**, **8**, **13**], we study actions of finitely generated semigroups of diffeomorphisms on the circle, which can be viewed as iterated function systems. We extend the theorems of Denjoy and Duminy to the semigroup case and describe the limit set of possible orbits.

In order to state the main results, we first make a couple of definitions.

An *iterated function system* (IFS), generated by a finite family of diffeomorphisms $\Phi = \{\phi_1, \ldots, \phi_k\}$ of the circle $\mathbb{S}^1$, is the set $\mathrm{IFS}(\Phi)$ of all non-empty possible finite compositions of diffeomorphisms $\phi_i \in \Phi$, that is, the semigroup generated by the compositions of $\phi_1, \ldots, \phi_k$.

The *orbit* of x for $\mathrm{IFS}(\Phi)$ is the action of IFS over the point x, i.e.

$$\mathrm{Orb}^+_\Phi(x) \stackrel{\mathrm{def}}{=} \{h(x) : h \in \mathrm{IFS}(\Phi)\} \subset \mathbb{S}^1.$$

Denote by $\mathrm{Per}(\mathrm{IFS}(\Phi))$ the *set of periodic points* of $\mathrm{IFS}(\Phi)$, which is the set of $x \in \mathbb{S}^1$ such that $h(x) = x$ for some $h \in \mathrm{IFS}(\Phi)$. We say the circle $\mathbb{S}^1$ is minimal for $\mathrm{IFS}(\Phi)$ if the orbit of every point is dense.

Let f be a $C^{1+\mathrm{bv}}$-diffeomorphism of the circle. From the Denjoy theorem the following statements are equivalent: $\mathbb{S}^1$ is minimal for $\mathrm{IFS}(f)$ and f does not have periodic points. When the number of generators of the IFS is at least 2, the periodic points are no longer the unique obstruction to minimality. Indeed, this role is now played by the ss-intervals, which are compact intervals whose endpoints are consecutive attracting fixed points of different generators of $\mathrm{IFS}(\Phi)$. See Definition 2.1 for a formal definition.

THEOREM A. (Obstruction to minimality) *There exists $\varepsilon > 0.38$ such that if f_0 and f_1 are diffeomorphisms of the circle with periodic points of period n_0 and n_1 respectively, ε-close to the rotations in the $C^{1+\mathrm{bv}}$-topology and with no periodic points in common, then the following conditions are equivalent:*

- *$\mathbb{S}^1$ is minimal for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$;*
- *$\mathbb{S}^1$ is minimal for $\mathrm{IFS}(f_0^{-n_0}, f_1^{-n_1})$;*
- *there are no ss-intervals for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$.*

When this occurs, both the hyperbolic attracting and the hyperbolic repelling periodic points of $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ are dense in $\mathbb{S}^1$.

Under the assumptions of Theorem A, the first theorem of Duminy [**14**, **15**] can be restated as follows: $\mathbb{S}^1$ is minimal for the group action and the hyperbolic periodic points are dense. See Theorem 5.4 in this article.

If we assume the generators have only hyperbolic periodic points, then Theorem A is robust in the following sense. There exists an open set $\mathcal{U} \subset \mathrm{Diff}^{1+\mathrm{bv}}(\mathbb{S}^1) \times \mathrm{Diff}^{1+\mathrm{bv}}(\mathbb{S}^1)$ with $(f_0, f_1) \in \mathcal{U}$ such that for every pair $(g_0, g_1) \in \mathcal{U}$ it holds that $\mathbb{S}^1$ is minimal for $\mathrm{IFS}(g_0^{n_0}, g_1^{n_1})$ if and only if there are no ss-intervals for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$. Note that examples of robustly minimal IFS on the circle were given in [**8**, **9**, **11**].

Actually, the above robustness of Theorem A is valid for perturbations of (f_0, f_1) in the C^1-topology, i.e. in $\mathrm{Diff}^1(\mathbb{S}^1) \times \mathrm{Diff}^1(\mathbb{S}^1)$. This follows from the work of [**1**] using the fact that the semigroup action is expanding (see Remark 3.11). The action of a semigroup Γ of orientation-preserving diffeomorphisms of the circle is said to be *expanding* if for any point $x \in \mathbb{S}^1$ there exists $g \in \Gamma$ with $Dg^{-1}(x) > 1$. Note that in the same paper [**1**]

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is shown the robust ergodicity of expanding semigroup actions (in the $C^{1+\alpha}$ topology), where the proof is based on a similar result for group actions of the circle [**14**].

The existence of ss-intervals and the non-minimality of the circle leads us to attempt understanding the indecomposable pieces of global dynamics. Thus, we introduce the notion of a limit set for the orbits of an IFS.

The *forward* or *ω-limit of $x \in \mathbb{S}^1$ for* $\mathrm{IFS}(\Phi)$ is the set

$$\omega_\Phi(x) \stackrel{\text{def}}{=} \{y : \exists (h_n)_n \subset \mathrm{IFS}(\Phi) \text{ such that } \lim_{n\to\infty} h_n \circ \cdots \circ h_1(x) = y\},$$

while the *ω-limit of* $\mathrm{IFS}(\Phi)$ is

$$\omega(\mathrm{IFS}(\Phi)) \stackrel{\text{def}}{=} \mathrm{cl}(\{y : \exists x \in M \text{ such that } y \in \omega_\Phi(x)\}),$$

where 'cl' denotes the closure of a set. Similarly, the *backward* or *α-limit of* $\mathrm{IFS}(\Phi)$ is defined as $\alpha(\mathrm{IFS}(\Phi)) \stackrel{\text{def}}{=} \omega(\mathrm{IFS}(\Phi^{-1}))$, where $\Phi^{-1} = \{\phi_1^{-1}, \ldots, \phi_k^{-1}\}$. From the backward and forward limit, we define the *limit set of* $\mathrm{IFS}(\Phi)$ as

$$L(\mathrm{IFS}(\Phi)) \stackrel{\text{def}}{=} \omega(\mathrm{IFS}(\Phi)) \cup \alpha(\mathrm{IFS}(\Phi)).$$

The following result, in particular, shows the spectral decomposition of the limit set of IFS generated by Morse–Smale diffeomorphisms of the circle. That is, the limit set can be written as a finite union of disjoint (closed) maximal transitive sets. A (not necessarily IFS invariant) set $\Lambda \subset \mathbb{S}^1$ is *transitive* for $\mathrm{IFS}(\Phi)$ if

$$\Lambda \subset \overline{\mathrm{Orb}_\Phi^+(x)} \quad \text{for some } x \in \Lambda.$$

The transitive set is *maximal* if there is no other transitive set Ω with $\Lambda \subset \Omega$.

THEOREM B. (Spectral decomposition) *There exists $\varepsilon > 0.30$ such that, under the assumptions of Theorem A and if $\mathbb{S}^1$ is not minimal for* $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$*, then*

$$L(\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})) = \bigcup K_i \quad \textit{and} \quad K_i \subset \overline{\mathrm{Per}(\mathrm{IFS}(f_0^{n_0}, f_1^{n_1}))},$$

where each K_i is a compact, maximal transitive set for $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$*. Moreover, each compact set K_i is either:*

- *a single fixed point of $f_0^{n_0}$ or $f_1^{n_1}$; or*
- *a $**$-interval for* $\mathrm{IFS}(f_0^{n_0}, f_1^{n_1})$ *with $** \in \{ss, su, uu\}$†.*

In particular, if f_0 and f_1 are Morse–Smale diffeomorphisms then the above union is finite and the sets K_i are pairwise disjoint.

Observe that the spectral decomposition of Theorem B is given for the IFS generated by $f_0^{n_0}$ and $f_1^{n_1}$. It remains a question to describe the possible limit set of $\mathrm{IFS}(f_0, f_1)$, and if there is a spectral decomposition, what the pieces are. An extension of the problem is to prove a spectral decomposition type result when the generators are far from rotations. Observe that in this case Cantor sets may appear in the decomposition.

With respect to dynamical decomposition of IFS from the ergodic perspective, let us mention the works of [**12**, **18**], where the existence of finitely many stationary or

† For the precise definition see Definition 2.1.

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Sinai–Ruelle–Bowen (SRB) measures for step skew-products over the interval or the circle is shown. On the contrary, for non-Abelian group actions of the circle in [**5**] the uniqueness of stationary measures is proven.

A set Λ is (forward-)invariant for IFS(Φ) if $\phi_i(\Lambda) \subset \Lambda$ for all $i = 1, \dots, k$. We say that a closed invariant set Λ is *minimal* for IFS(Φ) if every point of Λ has dense orbit in Λ or equivalently if

$$\Lambda = \phi_1(\Lambda) \cup \cdots \cup \phi_k(\Lambda) = \overline{\mathrm{Orb}^+_\Phi(x)} \quad \text{for all } x \in \Lambda.$$

Similarly to the case of group actions, a closed minimal invariant set Λ for IFS has to satisfy one of the following: Λ is a finite orbit, has non-empty interior, or is a Cantor set, see Theorem 5.2. Observe that *a priori* minimal invariant Cantor sets could exist in the pieces of the spectral decomposition of Theorem B. The next result shows that this cannot occur and can be thought of as a Denjoy-type result for the semigroup action.

THEOREM C. (Denjoy for IFS) *There exists $\varepsilon > 0.30$ such that under the assumptions of Theorem A and if $\mathbb{S}^1$ is not minimal for* IFS(f_0, f_1)*, the only minimal invariant closed sets for* IFS($f_0^{n_0}, f_1^{n_1}$) *are the ss-intervals.*

Moreover, there are no minimal invariant Cantor sets for IFS(f_0, f_1).

In the case of iterations of a single diffeomorphism on the circle, the Denjoy examples [**10**, Ch. X] show the existence of minimal invariant Cantor sets in the C^1-topology. Thus one can expect that Theorem C above fails in the C^1-topology. The recent works of Shinohara [**16**, **17**] give examples of IFS(f_0, f_1) with f_0 and f_1 close to the identity in the C^1-topology and forming an ss-interval, such that the minimal set is a Cantor set.

The paper is organized as follows. Firstly, in §2 we state and give a proof of a relevant result of Duminy. This is necessary in order to understand the next section, which is a generalization of Duminy's theorem using a new notion of cycles for IFS and concludes with the proof of Theorem A. In §4 we show a spectral decomposition theorem on the real line, and afterwards prove Theorem B. The trichotomy on the shape of minimal invariant sets is taken up in the next section and finally Theorem C and Duminy's theorem under the assumptions of Theorem A are shown.

Notation. In what follows we will use the following notation, where f_0 and f_1 are diffeomorphisms of the circle with periodic points of period n_0 and n_1 respectively:

$$\Phi = (f_0, f_1), \quad \Phi^{-1} = (f_0^{-1}, f_1^{-1}), \quad \Phi^n = (f_0^{n_0}, f_1^{n_1}), \quad \Phi^{-n} = (f_0^{-n_0}, f_1^{-n_1}).$$

For a map f defined on an interval I, we write $f > \mathrm{id}$ in I if $f(x) > x$ for all $x \in I$. Also we denote by $|I|$ the length of this interval.

2. *A result of Duminy*

In this section we will show an important result of Duminy which can be easily deduced from [**14**, **15**]. For the sake of a self-contained paper and to indicate the techniques used to derive the stronger result in §3, the complete proof is given. Before that, the formal definition of $**$-intervals is introduced.

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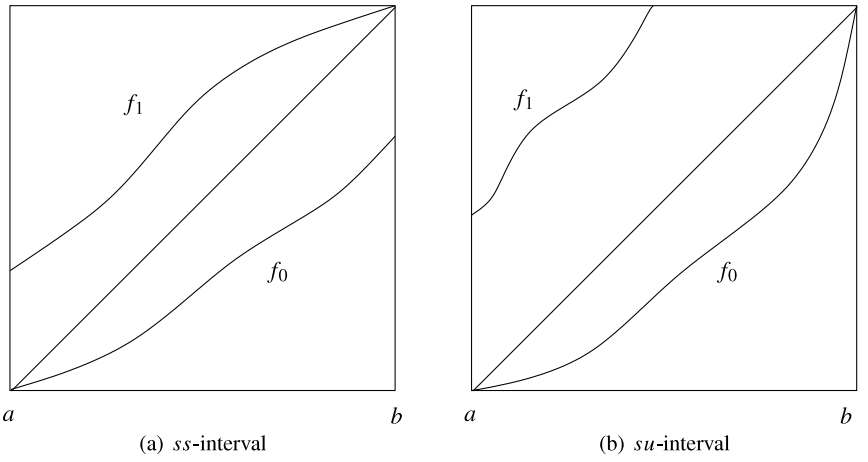


FIGURE 1. Examples of $**$-intervals.

Definition 2.1. ($**$-intervals) Consider f_0, f_1 orientation-preserving homeomorphims on the real line. Let $[a, b]$ be a compact interval such that

$$\mathrm{Fix}(f_i) \cap (a, b) = \emptyset \quad \text{and} \quad [a, b] \subset f_0([a, b]) \cup f_1([a, b]).$$

We say that $[a, b]$ is a $**$-*interval* for IFS(Φ) with $** \in \{ss, su\}$ when a and b satisfy the following additional properties (see Figure 1):

- *ss-interval*: a and b are, respectively, attracting fixed points of the restriction of f_0 and f_1 to $[a, b]$ and $f_0(b) \neq b$, $f_1(a) \neq a$;
- *su-interval*: a and b are either an attractor–repeller or a repeller-attractor pair for the same map restricted to $[a, b]$, say an attractor–repeller pair for f_0. In this case, we ask that $f_1 > \mathrm{id}$ in $[a, b]$ and $f_1([a, b]) \cap (a, b) \neq \emptyset$.

A *uu-interval* for IFS(Φ) is an ss-interval for IFS(Φ^{-1}). An unbounded interval $[a, \infty]$ is:

- *s-interval* for IFS(Φ): if a is an attracting fixed point of the restriction of a map to $[a, \infty)$, say f_0, satisfying $f_0 < \mathrm{id}$ in (a, ∞) and $f_1 > \mathrm{id}$ in $[a, \infty)$.

A *u-interval* for IFS(Φ) is an s-interval for IFS(Φ^{-1}). Analogously, the unbounded case $[-\infty, b]$ is defined in the same manner.

Let I be an interval on the real line or on $\mathbb{S}^1$. Consider an orientation-preserving C^1-map f of I such that $Df(x) \neq 0$ for all $x \in I$. The non-negative number

$$\mathrm{Dist}(f, I) = \sup_{x,y \in I} \log \frac{Df(x)}{Df(y)}$$

is called *distortion constant* of f in I or *maximal variation* of $\log Df$ in I. We say that f belongs to the $C^{1+\mathrm{bv}}$ class on I if it has a bounded distortion constant in I. Observe that an application of the mean value theorem shows that C^2-maps with non-zero first derivative are $C^{1+\mathrm{bv}}$.

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It is not difficult to see that every orientation-preserving $C^{1+\mathrm{bv}}$-map f on the circle satisfies

$$e^{-V_f} \le Df(x) \le e^{V_f} \quad \text{and} \quad |f(x) - (x + \rho(x))| < 1 - e^{-V_f} \quad \text{for all } x \in \mathbb{S}^1,$$

where $\rho(f)$ is the rotation number of f and $V_f = \mathrm{Dist}(f, \mathbb{S}^1)$ (see [**14**] for details). Therefore, if the distortion constant of f is small enough, then f is close to a rotation. We will say that f is *ε-close to rotation in the $C^{1+\mathrm{bv}}$-topology* if $V_f \le \varepsilon$.

Let f_0 and f_1 be orientation-preserving $C^{1+\mathrm{bv}}$-diffeomorphisms on the real line. In what follows, we will study $\mathrm{IFS}(\Phi)$ with f_0 and f_1 restricted to a $**$-interval $K^{**} = [a, b]$ for $** \in \{ss, s, su\}$. In the case of uu-intervals and u-intervals for $\mathrm{IFS}(\Phi)$ the same results hold for $\mathrm{IFS}(\Phi^{-1})$. Observe that

$$K^{**} = f_0(K^{**}) \cup f_1(K^{**}) \quad \text{for } ** \in \{ss, s\}$$

and then, it is an invariant set by the action of $\mathrm{IFS}(\Phi)$. However the above equality does not hold for an su-interval.

We will be interested in stating conditions under which every point in a $**$-interval has a dense orbit for $\mathrm{IFS}(\Phi)$. In the case of an su-interval and an s-interval notice that one of the endpoints of K^{su} and K^{s} can never have a dense orbit.

To unify the notation, we will define a $**$-interval to be *minimal* for $\mathrm{IFS}(\Phi)$ if

$$K^{**} \subset \overline{\mathrm{Orb}^+_\Phi(x)} \quad \text{for all } x \in K^{**}.$$

The details of the following result of Duminy were worked out by Navas [**14**], who reinterpreted the main arguments of the work of Duminy in more dynamical terms, introducing certain return maps and showing they are expanding.

THEOREM 2.2. (Duminy's lemma) *Assume $f_0 < \mathrm{id}$ in (a, b) and suppose that there exists $\varepsilon > 0$ such that*

$$|Df_0(x) - 1| < \varepsilon \quad \textit{for all } x \in (a, f_0^{-1} f_1(a)) \quad \textit{and} \quad (1 - \varepsilon)\varepsilon^{-1} e^{-V} > 1,$$

where $V = (V_0 + V_1)|a - f_0^{-1} f_1(a)|$ with V_0 and V_1 constants satisfying

$$\mathrm{Dist}(f_0, I) \le V_0 |I| \quad \textit{and} \quad \mathrm{Dist}(f_1, J) \le V_1 |J|$$

for all fundamental domains I of f_0 and J of f_1 in $(a, f_0^{-1} f_1(a)]$. Then

$$K^{**} \subset \overline{\mathrm{Per}(\mathrm{IFS}(\Phi))} \quad \textit{and} \quad K^{**} \subset \overline{\mathrm{Orb}^+_\Phi(x)} \quad \textit{for all } x \in K^{**}.$$

Notice that from the definition of $**$-intervals for $\mathrm{IFS}(\Phi)$ with $** \in \{ss, su, s\}$, the *overlap condition* is verified, that is, $f_0(K^{**}) \cap f_1(K^{**}) \neq \emptyset$. This condition implies that

$$A = (f_1(a), f_0^{-1} f_1(a)] \subset [a, b].$$

Next, we will define a first return map $\mathcal{R}$ over the fundamental domain A.

For each $x \in A$ let $m(x) \ge 1$ be the smallest positive number such that $f_1^{-m(x)}(x) \notin A$ and let $n(x)$ be the first time for which $f_1^{-m(x)}(x)$ returns to A by iterations of f_0^{-1}. Then we can define the first return map $\mathcal{R}$ in the following way

$$\mathcal{R}\colon A \to A, \quad \mathcal{R}(x) = f_0^{-n(x)} f_1^{-m(x)}(x).$$

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Note that this map can be written as $\mathcal{R}(x) = F^{n(x)+m(x)}(x)$ for $x \in A$, where $F : [a, b] \to [a, b]$ is defined by $F = f_0^{-1}$ in $[a, f_1(a)]$ and $F = f_1^{-1}$ in $(f_1(a), b]$. Therefore, for every $x \in [a, b]$ there is a smallest non-negative number $k(x) \geq 0$ such that $F^{k(x)}(x) \in A$ and $\mathcal{R}$ can be extended to the whole interval by taking

$$\mathcal{R}\colon (a, b) \to A, \quad \mathcal{R}(x) = F^{n+m+k}(x),$$

where $k = k(x)$, $m = m(F^k(x))$ and $n = n(F^k(x))$.

A point $d \in A$ is said to be a *discontinuity of* $\mathcal{R}$ if $\mathcal{R}(d) = f_0^{-1} f_1(a)$ or equivalently, if $d = f_1^{m(d)} f_0^{n(d)-1} f_1(a)$. These points define a partition on A. In order to describe this partition we have to consider two cases: $f_1^2(a) \notin A$ and $f_1^2(a) \in A$. In the first case, $m(x) = 1$ for all $x \in A$ and we write $I_1 = A$. In the second case, consider $\ell \in \mathbb{N}$ such that $f_1^{\ell+1}(a) \in A$, but $f_1^{\ell+2}(a) \notin A$. Then $f_1^m f_1(a)$ for $m = 1, 2, \ldots, \ell$ defines a partition on A given by

$$I_\ell = (f_1^{\ell+1}(a),\ f_0^{-1} f_1(a)] \quad \text{and}$$
$$I_m = (f_1^m(a),\ f_1^{m+1}(a)] \quad \text{for } 1 \leq m < \ell.$$

Hence, $m(x) = m$ for each $x \in I_m$ and $f_1^{-m(x)}(I_m) \subset (a, f_1(a)]$. Observe that the sequence of points $f_0^n f_1(a)$, $n \geq 0$, creates a partition on the interval $(a, f_1(a)]$. This partition induces a partition in right-closed intervals I_{mn} in each I_m. Namely, $I_{mn} \subset f_1^m f_0^n(A)$ for $1 \leq m \leq \ell$ and $n \geq 1$.

PROPOSITION 2.3. *For every $J = [x, y] \subset I_{mn}$ it holds that*

$$\frac{|\mathcal{R}(J)|}{|J|} \geq (1 - \varepsilon)\varepsilon^{-1} e^{-V} \frac{D(\mathcal{R}(y))}{D(y)},$$

where $D : A \to (0, \infty)$ is given by $D(y) = |f_0(y) - y|$.

Proof. Notice that $\mathcal{R}(J) = f_0^{-n} f_1^{-m}(J)$ and hence

$$\frac{|\mathcal{R}(J)|}{|J|} = \frac{|f_0^{-n} f_1^{-m}(J)|}{|f_1^{-m}(J)|} \cdot \frac{|f_1^{-m}(J)|}{|J|}. \tag{1}$$

In order to estimate the first factor, observe that $f_1^{-m}(J) \subset [f_0 f_1^{-m}(y),\ f_1^{-m}(y)]$ and so

$$\begin{aligned}\operatorname{Dist}(f_0^{-n}, [f_0 f_1^{-m}(y),\ f_1^{-m}(y)]) &= \operatorname{Dist}(f_0^n,\ f_0^{-n}([f_0 f_1^{-m}(y),\ f_1^{-m}(y)])) \\ &\leq \sum_{i=0}^{n-1} \operatorname{Dist}(f_0,\ f_0^{i-n}([f_0 f_1^{-m}(y),\ f_1^{-m}(y)])) \leq V_0 |a - f_0^{-1} f_1(a)|.\end{aligned}$$

Thus,

$$\frac{|f_0^{-n} f_1^{-m}(J)|}{|f_0^{-n} f_0 f_1^{-m}(y) - f_0^{-n} f_1^{-m}(y)|} \geq e^{-V_0 |a - f_0^{-1} f_1(a)|} \frac{|f_1^{-m}(J)|}{|f_0 f_1^{-m}(y) - f_1^{-m}(y)|}. \tag{2}$$

Let us denote $t = f_1^{-m}(y)$. Using the mean value theorem, there is some $\xi \in (a, t)$ such that $f_0(t) - a = f_0(t) - f_0(a) = Df_0(\xi)(t - a)$. Hence

$$f_0(t) - t = \frac{Df_0(\xi) - 1}{Df_0(\xi)} (f_0(t) - a) \leq \frac{\varepsilon}{1 - \varepsilon} (f_0(t) - a).$$

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Substituting in (2) one has that

$$\frac{|f_0^{-n} f_1^{-m}(J)|}{|f_1^{-m}(J)|} \geq (1-\varepsilon)\varepsilon^{-1} e^{-V_0|a-f_0^{-1}f_1(a)|} \frac{|f_0(\mathcal{R}(y)) - \mathcal{R}(y)|}{|f_0 f_1^{-m}(y) - a|}. \tag{3}$$

Now we will estimate the second factor in (1). Observe that $f_1^{-m}(J)$ and $[a,\, f_1^{-m}(y)]$ are contained in the fundamental domain $[a,\, f_1(a)]$ of f_1. Thus,

$$\mathrm{Dist}(f_1^m,\, [a,\, f_1(a)]) \leq \sum_{i=0}^{m-1} \mathrm{Dist}(f_1,\, f_1^i([a,\, f_1(a)])) \leq V_1|a - f_0^{-1} f_1(a)|$$

and so

$$\frac{|J|}{|f_1^m(a) - y|} = \frac{|f_1^m f_1^{-m}(J)|}{|f_1^m([a,\, f_1^{-m}(y)])|} \leq e^{V_1|a-f_0^{-1}f_1(a)|} \frac{|f_1^{-m}(J)|}{|f_1^{-m}(y) - a|}.$$

As $a < f_0 f_1^{-m}(y) < f_1^{-m}(y)$ and since $y \in I_{mn}$, by the construction of the partition I_{mn} of I_m, one has that $f_0(y) \leq f_1(a) \leq f_1^m(a) < y$. Then the above inequality implies

$$\frac{|f_1^{-m}(J)|}{|J|} \geq e^{-V_1|a-f_0^{-1}f_1(a)|} \frac{|f_0 f_1^{-m}(y) - a|}{|f_0(y) - y|}. \tag{4}$$

Putting together (3) and (4) in equation (1), one deduces the desired inequality and concludes the proof of the lemma. □

The following lemma shows that some iterate of the first return map $\mathcal{R}$ is expanding.

LEMMA 2.4. *There exists $N \in \mathbb{N}$ such that $D\mathcal{R}^N(x) > 1$ for all $x \in A$.*

Proof. Let us denote $C(f_0) = \inf\{D(x) : x \in A\}/\sup\{D(x) : x \in A\} > 0$. Fix $\kappa > 1$ and let $N = N(f_0)$ be a natural number such that

$$((1-\varepsilon)\varepsilon^{-1} e^{-V})^N C(f_0) > \kappa.$$

Suppose $x \in I_{mn}$ and consider $\delta > 0$ small enough such that $J = [x - \delta,\, x] \subset I_{mn}$. Hence, by Proposition 2.3 it follows that

$$\begin{aligned}\frac{|\mathcal{R}^N(J)|}{|J|} &= \frac{|\mathcal{R}^N(J)|}{|\mathcal{R}^{N-1}(J)|} \cdots \frac{|\mathcal{R}(J)|}{|J|} \\ &\geq ((1-\varepsilon)\varepsilon^{-1} e^{-V})^N \frac{D(\mathcal{R}^N(x))}{D(x)} \\ &\geq ((1-\varepsilon)\varepsilon^{-1} e^{-V})^N C(f_0) > \kappa.\end{aligned}$$

Since this inequality holds for every $\delta > 0$ small enough, this allows to conclude that $D\mathcal{R}^N(x) \geq \kappa > 1$. □

The first simplification to prove the minimality and the density of periodic points in a $**$-interval is to note that it suffices to show that the orbit of the global attractor a for $f_0|_{K^{**}}$, with respect to $\mathrm{IFS}(\Phi)$ is dense in the interval $K^{**} = [a, b]$.

LEMMA 2.5. *If $K^{**} \subset \overline{\mathrm{Orb}_\Phi^+(a)}$ then*

$$K^{**} \subset \overline{\mathrm{Per}(\mathrm{IFS}(\Phi))} \quad \textit{and} \quad K^{**} \subset \overline{\mathrm{Orb}_\Phi^+(x)} \quad \textit{for all } x \in K^{**}.$$

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Proof. Consider $x \in K^{**}$ and let U be any open set in K^{**}. From the density of the orbit of a for IFS(Φ) in K^{**}, there is $h \in \mathrm{IFS}(\Phi)$ such that $h(a) \in U$. Since h is a continuous map and a is a global attractor of f_0 in K^{**}, there exists $\ell \in \mathbb{N}$ such that $f_0^\ell(x)$ is close enough to 0, so that $h \circ f_0^\ell(x) \in U$. Therefore, the orbit of x for IFS(Φ) is dense in K^{**}.

Now, given $x \in K^{**}$ we will show that $x \in \overline{\mathrm{Per}(\mathrm{IFS}(\Phi))}$. Let I be any open interval such that $x \in I$. From the density of the orbit of a, there is $h \in \mathrm{IFS}(\Phi)$ such that $h(a) \in I$. Since h is a continuous map there is $\delta > 0$ such that $h((a-\delta, a+\delta)) \subset I$. Using that a is a global attractor of f_0, there is $\ell \geq 0$ such that $f_0^\ell(I) \subset (a-\delta, a+\delta)$. Then the closure of $h \circ f_0^\ell(I)$ is strictly contained in I. By the intermediate value theorem, $h \circ f_0^\ell$ has a fixed point in I and thus $I \cap \mathrm{Per}(\mathrm{IFS}(\Phi)) \neq \emptyset$. This implies that x belongs to the closure of the periodic points of IFS(Φ) and we conclude the proof of the lemma. □

Now, we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Recall that the return map $\mathcal{R} : A \to A$ can be extended to the interval (a, b). In particular, this implies that for any open interval $I \subset K^{**}$, there exists $h \in \mathrm{IFS}(\Phi)$ such that $h^{-1}(I) \cap A \neq \emptyset$. From Lemma 2.4, the return map $\mathcal{R}^N$ is an expanding map in A. Thus, there is $\ell \in \mathbb{N}$ such that $\mathcal{R}^{\ell N}(h^{-1}(I) \cap A)$ contains some discontinuity of $\mathcal{R}^N$.

Recall that the discontinuities $d \in A$ are points in the orbit of a for IFS(Φ), i.e., applying the appropriate inverse branch, $d = \mathcal{R}^{-N}(f_0^{-1} f_1(a)) = L \circ f_1(a)$ for some $L \in \mathrm{IFS}(\Phi)$. Then, one has $h \circ L \circ f_1(a) \in I$. Therefore, the orbit of a is dense in K^{**} with respect to IFS(Φ). Finally, from Lemma 2.5 we conclude the proof of the theorem. □

3. *Cycles for IFS and proof of Theorem A*

Now we will give a proof of Theorem A. Observe that an ss-interval is a forward-invariant set with respect to IFS($f_0^{n_0}$, $f_1^{n_1}$). Thus, the existence of an ss-interval immediately implies that IFS($f_0^{n_0}$, $f_1^{n_1}$) is not minimal. From now on assume that there are no ss-intervals for IFS($f_0^{n_0}$, $f_1^{n_1}$), and so we have to show minimality of $\mathbb{S}^1$.

We will first show the result when the rotation numbers of f_0 and f_1 are zero, that is, both of the diffeomorphisms have fixed points. The proof will require the following notion of a cycle for an IFS of two diffeomorphisms on the circle. Similarly to Duminy's result (Theorem 2.2), using the cycle we will then construct a return map and afterwards estimate the derivative. Here the expanding return map is of a global character, while Duminy's lemma can be thought of as a local version.

3.1. *Cycles.*

Since f_0 and f_1 do not have fixed points in common, we can (without ambiguity) denote by $\{s_i\}$ the set of fixed points of f_0 and f_1 with a non-empty basin of attraction. This set includes the fixed points which may attract only from one side (semi-attracting). Let $B(s_i)$ represent the basin of attraction of s_i with respect to the relevant map f_0 or f_1, which does not include the point s_i (it is an open set).

Definition 3.1. (Cycle for IFS) Define a partial order on the (semi)attracting fixed points by $s_i \prec s_j$ if and only if s_i belongs to $B(s_j)$. A sequence of (semi)attractors is said to be a

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cycle of length n for IFS(Φ) if

$$s_{i_1} \prec s_{i_2} \prec \cdots \prec s_{i_n} \prec s_{i_{n+1}} \quad \text{and} \quad s_{i_1} = s_{i_{n+1}}.$$

Notice that:

- since the (semi)attracting points s_i of the cycle alternate with respect to the maps f_0 and f_1, the length of a cycle is always even;
- an ss-interval is a cycle of length 2.

The following result shows the existence of a cycle under the assumption of no periodic points in common.

PROPOSITION 3.2. *Let f_0, f_1 be circle diffeomorphisms with zero rotation number and no fixed points in common. Then there exists at least one cycle for* IFS(Φ).

Proof. Since f_0 and f_1 have no fixed points in common, given $x \in \mathbb{S}^1$, then $x \in B(s_i)$ for some s_i. By compactness we can take a finite sub-covering and re-ordering the indices, we may suppose that $\mathbb{S}^1 = B(s_1) \cup \cdots \cup B(s_n)$.

For any s_i, there exists $s_{i_2} \in \{s_1, \ldots, s_n\}$ with $s_i \prec s_{i_2}$. Proceeding inductively we obtain the sequence $(s_i = s_{i_1})$, $s_{i_1} \prec s_{i_2} \prec \cdots \prec s_{i_n} \prec s_{i_{n+1}}$. Then necessarily $s_{i_{n+1}} = s_{i_j}$ with $j \in \{1, \ldots, n\}$, and so we have the cycle, $s_{i_j} \prec s_{i_2} \prec \cdots \prec s_{i_n} \prec s_{i_j}$. □

By the above proposition we may assume that there exists a cycle for IFS(Φ). Let n be the length of the cycle; after re-numbering the indices i_j of the (semi)attractors if necessary, the cycle can be written as

$$s_n \prec s_{n-1} \prec \cdots \prec s_2 \prec s_1 \prec s_0 \quad \text{with } s_0 = s_n,$$

where s_0 is a (semi)attracting fixed point of f_0. Note that s_k is a (semi)attracting fixed point for $f_{k \bmod 2}$: if k is an even number then s_k is a fixed point of f_0 and if k is an odd number then of f_1.

Notation. For the rest of the section f_k will stand for $f_{k \bmod 2}$.

We consider $\mathbb{S}^1$ parameterized by $[s_0, s_0 + 1]$ mod 1, and so $s_0 < s_k < s_0 + 1 = s_n$ for $k = 1, \ldots, n-1$ with respect to the real order on the interval $[s_0, s_0 + 1]$. We denote by s_k^- and s_k^+ (when they exist) the fixed points of the lift of f_k on $[s_0, s_0 + 1]$ closest to s_k from the left and right, respectively. If f_k has more than one fixed point and since s_k is a (semi)attractor, necessarily one of the points, s_k^- or s_k^+, exists and is a (semi)repeller. With respect to the order on the interval $[s_0, s_0 + 1]$, two (not mutually excluding) cases are possible: $s_1 \in (s_0, s_0^+)$ or $s_1 \in (s_n^-, s_n)$. We will assume the former case, the latter being analogous. The next lemma on the order of the points is crucial for the creation of the return map.

LEMMA 3.3. *If there are no ss-intervals for* IFS(Φ) *then on the interval $[s_0, s_0 + 1]$ the cycle has the order*

$$s_0 < s_1 < \cdots < s_{n-1} < s_0 + 1 = s_n,$$

where $s_{k+1} \prec s_k$ for $k = 0, \ldots, n-1$. Moreover,

$$s_k < f_{k+1}^{-1}(s_k) < s_{k+1} < s_k^+ \quad \text{for } k = 0, \ldots, n-1.$$

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Proof. We will show that $s_{k+1} \in (s_k, s_k^+) \subset B(s_k)$ for every $k = 0, \ldots, n-1$, which proves the first part of the lemma. The second part will follow as the fixed point of f_{k+1} closest to s_k from the right has to be a repeller from the left (otherwise an ss-interval is created) and so $s_k < f_{k+1}^{-1}(s_k) < s_{k+1} < s_k^+$.

First, we will assume that f_0 or f_1 has a unique fixed point. In this case, the cycle has length 2 and since there are no ss-intervals then $s_2 = s_0 + 1$. Thus $s_0 < s_1 < s_0^+ \leq s_0 + 1 = s_2$, where $s_2 \prec s_1 \prec s_0$. In particular, $s_1 \in (s_0, s_0^+) \subset B(s_0)$ and we have $s_2 \in (s_1, s_1^+) \subset B(s_1)$.

Now, we will assume that f_0 and f_1 have more than one fixed point. By the initial hypothesis we have $s_1 \in (s_0, s_0^+) \subset B(s_0)$. Let us show that $s_2 \in (s_1, s_1^+) \subset B(s_1)$. Since $s_2 \prec s_1$, $s_2 \in B(s_1) \subset (s_1^-, s_1) \cup (s_1, s_1^+)$ and so either $s_2 \in (s_1^-, s_1)$ or $s_2 \in (s_1, s_1^+)$. It is enough to prove that the first case cannot occur.

Suppose that $s_2 \in (s_1^-, s_1)$. If $s_0 < s_1^-$ then $s_0 < s_2 < s_1 < s_0^+$, which is a contradiction since s_0^+ is the closest fixed point of f_0 to s_0 from the right. If $s_0 > s_1^-$, a similar argument concludes that $s_0 = s_2$. Since there are no ss-intervals and $s_2 \prec s_1$ then $B(s_1) = (s_1, s_1^+)$ and $s_0 + 1 = s_2 + 1 < s_1^+$. Thus $s_1^- < s_0 < s_0 + 1 < s_1^+$, which implies that f_1 has a unique fixed point, a contradiction. Therefore, s_2 does not belong to (s_1^-, s_1) and so it is contained in (s_1, s_1^+).

Suppose by the inductive hypothesis that $s_i \in (s_{i-1}, s_{i-1}^+) \subset B(s_{i-1})$ for all $i \leq k$. To prove that $s_{k+1} \in (s_k, s_k^+) \subset B(s_k)$ the idea is basically the same as above. If $s_{k+1} \in (s_k^-, s_k)$, then either $s_k^- \in (s_{k-1}, s_k)$ or $s_k^- < s_{k-1}$. The first case cannot occur because s_{k+1} and s_{k-1} are fixed points of the same map $f_{k+1} = f_{k-1}$ and $s_{k+1} \in (s_k^-, s_k)$. Thus $s_k^- < s_{k-1}$ and so $s_{k+1} \leq s_{k-1}$. This fact together with $s_{k+1} \prec s_k$ implies that $(s_k^-, s_k) \subset B(s_k)$ and hence $[s_{k-1}, s_k]$ is an ss-interval, and thus we get a contradiction. Therefore, $s_k < s_{k+1} < s_k^+$ and so $(s_k, s_k^+) \subset B(s_k)$, which completes the induction and concludes the proof of the lemma. □

3.2. *Creating a return map.* In the previous section the existence of a cycle was shown that can be ordered in the following manner:

$$s_n \prec s_{n-1} \prec \cdots \prec s_2 \prec s_1 \prec s_0.$$

Let us now use this cycle to create a return map. Recall that f_k stands for $f_{k \bmod 2}$.

LEMMA 3.4. *There exist families of right-closed pairwise disjoint intervals* $I_{i_1 \ldots i_n} \subset \mathbb{S}^1$ *and maps* $h_{i_i \ldots i_n} \in \mathrm{IFS}(\Phi)$ *with* $i_j \geq 0$ *for* $j = 1, \ldots, n$ *such that:*

(i) $A = (s_0, f_1^{-1}(s_0)] = \bigcup I_{i_1 \ldots i_n}$;

(ii) $h_{i_1 \ldots i_n}^{-1}(I_{i_1 \ldots i_n}) \subset A$ *if* $i_n = 0$ *and* $h_{i_1 \ldots i_n}^{-1}(I_{i_1 \ldots i_n}) = A$ *if* $i_n > 0$;

(iii) *if* $c \in A \backslash \{f_1^{-1}(s_0)\}$ *is an endpoint of* $I_{i_1 \ldots i_n}$ *then there exist a map* $h \in \mathrm{IFS}(\Phi)$ *and a point* $s \in \mathrm{Orb}_\Phi^+(s_1) \cup \cdots \cup \mathrm{Orb}_\Phi^+(s_n)$ *such that* $h(s) = c$. *That is, the point* c *is in the orbit of the cycle for* $\mathrm{IFS}(\Phi)$.

Proof. We can define a non-empty fundamental domain of f_{k+1},

$$A_k = (s_k, f_{k+1}^{-1}(s_k)] \quad \text{for } k = 0, \ldots, n.$$

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Since on the circle $s_0 = s_n$, we identify on $\mathbb{S}^1$ the intervals in the real line $A_0 = (s_0,\ f_1^{-1}(s_0)]$ and $A_n = (s_n,\ f_1^{-1}(s_n)]$, denoting this interval by A. In order to create the expanding return map we will divide this fundamental domain inductively.

By Lemma 3.3, $s_0 < f_1^{-1}(s_0) < s_1$ and since $s_1 \prec s_0$, there exists $j \in \mathbb{N}$ such that $s_0 < f_0^j(s_1) < f_1^{-1}(s_0) \le f_0^{j-1}(s_1) < s_1$. Then

$$A = (s_0,\ f_1^{-1}(s_0)] = \bigcup_{i_1=0}^{\infty} I_{i_1},$$

with

$$I_0 = (f_0^j(s_1),\ f_1^{-1}(s_0)] \quad \text{and} \quad I_{i_1} = (f_0^{j+i_1}(s_1),\ f_0^{j+i_1-1}(s_1)] \quad \text{if } i_1 > 0.$$

Let $h_{i_1} = f_0^{j+i_1}$. We have that $h_{i_1}^{-1}(I_{i_1}) = (s_1, c_{i_1}]$, where

$$c_0 = f_0^{-j} f_1^{-1}(s_0) \in (s_1,\ f_0^{-1}(s_1)] \quad \text{and} \quad c_{i_1} = f_0^{-1}(s_1) \quad \text{if } i_1 > 0.$$

Therefore, $h_0^{-1}(I_0) \subset A_1$ and $h_{i_1}^{-1}(I_1) = A_1$ if $i_1 > 0$.

Let $c \in A \backslash \{f_1^{-1}(s_0)\}$ be an endpoint of I_{i_1}. Then either $c = f_0^j(s_1)$ if $i_1 = 0$, or $c \in \{f_0^{j+i_1}(s_1),\ f_0^{j+i_1-1}(s_1)\}$ if $i_1 > 0$. In any case, c belongs to the orbit of the cycle. This completes the first step of the induction and now we proceed with the inductive hypothesis.

Suppose that we have families of right-closed pairwise disjoint intervals $I_{i_1\dots i_k} \subset \mathbb{S}^1$ and maps $h_{i_1\dots i_k} \in \mathrm{IFS}(\Phi)$ with $i_j \ge 0$ for $j = 1, \dots, k$ such that:

(i) $A = \bigcup I_{i_1\dots i_k}$;

(ii) $h_{i_1\dots i_k}^{-1}(I_{i_1\dots i_k}) = (s_k, c_{i_1\dots i_k}]$, where

$$c_{i_1\dots i_{k-1}0} \in (s_k,\ f_{k+1}^{-1}(s_k)] \quad \text{and} \quad c_{i_1\dots i_k} = f_{k+1}^{-1}(s_k) \quad \text{if } i_k > 0.$$

In particular, $h_{i_1\dots i_{k-1}0}^{-1}(I_{i_1\dots i_{k-1}0}) \subset A_k$ and $h_{i_1\dots i_k}^{-1}(I_{i_1\dots i_k}) = A_k$ if $i_k > 0$;

(iii) if $c \in A \backslash \{f_1^{-1}(s_0)\}$ is an endpoint of $I_{i_1\dots i_k}$ then there exist a map $h \in \mathrm{IFS}(\Phi)$ and a point $s \in \mathrm{Orb}_\Phi^+(s_0) \cup \dots \cup \mathrm{Orb}_\Phi^+(s_n)$ such that $h(s) = c$.

By Lemma 3.3, $s_k < f_{k+1}^{-1}(s_k) < s_{k+1} < s_k^+$. Hence, from the inductive hypothesis we obtain

$$s_k < c_{i_1\dots i_k} \le f_{k+1}^{-1}(s_k) < s_{k+1} < s_k^+.$$

Now, since $s_{k+1} \prec s_k$, for each multi-index $i_1 \dots i_k$ there exists $j_{i_1\dots i_k} \in \mathbb{N}$ such that

$$s_k < f_k^{j_{i_1\dots i_k}}(s_{k+1}) < c_{i_1\dots i_k} \le f_k^{j_{i_1\dots i_k}-1}(s_{k+1}) < s_{k+1}.$$

Then

$$h_{i_1\dots i_k}^{-1}(I_{i_1\dots i_k}) = (s_k, c_{i_1\dots i_k}] = \bigcup_{\ell=0}^{\infty} J_{i_1\dots i_k\ell}$$

with

$$J_{i_1\dots i_k 0} = (f_k^{j_{i_1\dots i_k}}(s_{k+1}),\ c_{i_1\dots i_k}] \quad \text{and}$$
$$J_{i_1\dots i_k\ell} = (f_k^{j_{i_1\dots i_k}+\ell}(s_{k+1}),\ f_k^{j_{i_1\dots i_k}+(\ell-1)}(s_{k+1})] \quad \text{if } \ell > 0.$$

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Notice that the intervals $J_{i_1\dots i_k\ell}$, $\ell \geq 0$ are pairwise disjoint. Define $I_{i_1\dots i_k\ell} \subset I_{i_1\dots i_k}$ by

$$I_{i_1\dots i_k\ell} = h_{i_1\dots i_k}(J_{i_1\dots i_k\ell}).$$

By definition, for a fixed multi-index $i_1 \dots i_k$, the intervals $I_{i_i\dots i_k\ell}$, $\ell \geq 0$, are also pairwise disjoint. Hence, by means of the induction hypothesis on the intervals $I_{i_1\dots i_k}$ and since $I_{i_1\dots i_k\ell} \subset I_{i_1\dots i_k}$ it follows that $\{I_{i_1\dots i_{k+1}} : i_j \geq 0 \text{ for } j = 1, \dots, k+1\}$ is a family of right-closed pairwise disjoint intervals. Note that each right-closed interval $I_{i_1\dots i_k}$ is the union of the intervals $I_{i_1\dots i_k\ell}$, $\ell \geq 0$. Then, by the induction hypothesis it also follows that $A = \bigcup I_{i_1\dots i_{k+1}}$.

Set $h_{i_1\dots i_k i_{k+1}} = h_{i_1\dots i_k} \circ f_k^{j_{i_1\dots i_k}+i_{k+1}}$. By construction

$$h^{-1}_{i_1\dots i_{k+1}}(I_{i_1,\dots i_{k+1}}) = f_k^{-j_{i_1\dots i_k}-i_{k+1}}(J_{i_1\dots i_k i_{k+1}}) = (s_{k+1}, c_{i_1\dots i_{k+1}}],$$

where $c_{i_1\dots i_k i_{k+1}} = f_k^{-1}(s_{k+1})$ if $i_{k+1} > 0$ and

$$c_{i_1\dots i_k 0} = h^{-1}_{i_1\dots i_k 0} \circ h_{i_1\dots i_k}(c_{i_1\dots i_k}) = f_k^{-j_{i_1\dots i_k}}(c_{i_1\dots i_k}) \in (s_{k+1},\ f_k^{-1}(s_{k+1})].$$

Therefore, $h^{-1}_{i_1\dots i_k 0}(I_{i_1\dots i_k 0}) \subset A_{k+1}$ and $h^{-1}_{i_1\dots i_{k+1}}(I_{i_1\dots i_{k+1}}) = A_{k+1}$ if $i_{k+1} > 0$.

In order to prove the third item, we fix an interval $I_{i_1\dots i_k\ell}$. For every $\ell \geq 0$ the left endpoint of this interval is $h_{i_1\dots i_k} \circ f_k^{j_{i_1\dots i_k}+\ell}(s_{k+1})$ and so it belongs to the orbit of the cycle. The right endpoint is

$$h_{i_1\dots i_k}(c_{i_1\dots i_k}) \text{ if } \ell = 0 \quad \text{and} \quad h_{i_1\dots i_k} \circ f_k^{j_{i_1\dots i_k}+\ell-1}(s_{k+1}) \text{ if } \ell > 0.$$

Note that $h_{i_1\dots i_k}(c_{i_1\dots i_k})$ is an endpoint of $I_{i_1\dots i_k}$. Therefore, by the inductive hypothesis, either $h_{i_1\dots i_k}(c_{i_1\dots i_k}) = f_1^{-1}(s_0)$ or $h_{i_1\dots i_k}(c_{i_1\dots i_k}) = h(s)$ for some $h \in \mathrm{IFS}(\Phi)$ and $s \in \bigcup_{i=0}^{n} \mathrm{Orb}^+_\Phi(s_i)$.

Going through the n steps of the cycle we conclude the lemma. □

From the above lemma we define the return map over $A = (s_0,\ f_1^{-1}(s_0)]$ as

$$\mathcal{R}\colon A \to A, \quad \mathcal{R}|_{I_{i_1\dots i_n}} = h^{-1}_{i_1\dots i_n}.$$

The endpoints of the intervals $I_{i_1\dots i_n}$ are called *discontinuities* of $\mathcal{R}$ and if s is a discontinuity then $s \in \mathrm{Orb}^+_\Phi(s_0) \cup \dots \cup \mathrm{Orb}^+_\Phi(s_n)$. We summarize some of the points from the inductive process in the construction of the return map in Lemma 3.4. This notation will be required for the next section.

ADDENDUM 3.5. *For each $k = 1, \dots, n$, there exists a family*

$$\{(I_{i_1\dots i_k}, h_{i_1\dots i_k}, m_{i_1\dots i_k}) : i_j \geq 0,\ j = 1, \dots, k\}$$

with $I_{i_1\dots i_k}$ pairwise disjoint right-closed intervals of $\mathbb{S}^1$, $h_{i_1\dots i_k} \in \mathrm{IFS}(\Phi)$ and $m_{i_1\dots i_k}$ natural numbers such that $A = (s_0,\ f_1^{-1}(s_0)] = \bigcup I_{i_1\dots i_k}$, and for $k = 1, \dots, n-1$:

(i) *$I_{i_1\dots i_k i_{k+1}} \subset I_{i_1\dots i_k}$ and $I_{i_1}, h_{i_1\dots i_k}(I_{i_1\dots i_{k+1}})$ are respectively contained in a fundamental domain of f_0 and f_k;*

(ii) *$m_{i_1} = j + i_1$, $m_{i_1\dots i_k} = j_{i_1\dots i_k} + i_{k+1}$, and $h_{i_1}^{-1} = f_0^{-m_{i_1}}$,*

$$h^{-1}_{i_1\dots i_k i_{k+1}} = f_k^{-m_{i_1\dots i_{k+1}}} \circ h^{-1}_{i_1\dots i_k} = f_k^{-m_{i_1\dots i_{k+1}}} f_{k-1}^{-m_{i_1\dots i_k}} \cdots f_1^{-m_{i_1 i_2}} f_0^{-m_{i_1}};$$

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(iii) $h^{-1}_{i_1\dots i_{k-1}0}(I_{i_1\dots i_{k-1}0}) = (s_k, c_{i_1\dots i_k}]$ *and* $h^{-1}_{i_1\dots i_k}(I_{i_1\dots i_k}) = A_k$ *if* $i_k > 0$ *with*

$$s_k < c_{i_1\dots i_k} \le f^{-1}_{k+1}(s_k) \quad \text{and} \quad A_k = (s_k,\, f^{-1}_{k+1}(s_k)];$$

(iv) $m_{i_1\dots i_k+1} = m_{i_1\dots i_k} + 1$ *and* $m_{i_1\dots i_{k-1}0} \ge 1$ *satisfies that*

$$f_k^{m_{i_1\dots i_{k-1}0}}(s_{k+1}) < c_{i_1\dots c_{i_k}} \le f_k^{m_{i_1\dots i_{k-1}0}-1}(s_{k+1}).$$

3.3. *Estimation of the derivative for the return map.* We will prove the following proposition which can be compared to Proposition 2.3.

PROPOSITION 3.6. *For every* $J = [x,\, y] \subset I_{i_1\dots i_n}$ *it holds that*

$$\frac{|\mathcal{R}(J)|}{|J|} \ge \left(\frac{e^{-(V_0+V_1)}}{e^{V_1}-1}\right)^{n/2} \cdot \frac{D(\mathcal{R}(y))}{D(y)},$$

where $D : A \to (0, \infty)$ *is given by* $D(y) = |f_0(y) - y|$, n *is the length of the cycle (always even), and* V_0, V_1 *are the distortion constants of* f_0, f_1 *respectively.*

Define by $\mathcal{R}_j = h^{-1}_{i_1\dots i_j}$, $1 \le j \le n$, and then $\mathcal{R}_n = \mathcal{R}$. We adopt the convention $\mathcal{R}_0 = \mathrm{id}$. To obtain Proposition 3.6 we need the following two-by-two term estimate.

LEMMA 3.7. *For any* $0 \le j \le n-2$ *even,*

$$\frac{|\mathcal{R}_{j+2}(J)|}{|\mathcal{R}_j(J)|} \ge \frac{e^{-(V_0+V_1)}}{e^{V_1}-1} \cdot \frac{D(\mathcal{R}_{j+2}(y))}{D(\mathcal{R}_j(y))}.$$

This lemma concludes immediately Proposition 3.6. Indeed, we have that

$$\frac{|\mathcal{R}(J)|}{|J|} = \frac{|\mathcal{R}_n(J)|}{|\mathcal{R}_{n-2}(J)|} \cdot \frac{|\mathcal{R}_{n-2}(J)|}{|\mathcal{R}_{n-4}(J)|} \cdots \frac{|\mathcal{R}_2(J)|}{|J|}$$

and applying Lemma 3.7 to each of the terms one gets the estimate required in Proposition 3.6.

Proof of Lemma 3.7. Since

$$\frac{|\mathcal{R}_{j+2}(J)|}{|\mathcal{R}_j(J)|} = \frac{|\mathcal{R}_{j+2}(J)|}{|\mathcal{R}_{j+1}(J)|} \cdot \frac{|\mathcal{R}_{j+1}(J)|}{|\mathcal{R}_j(J)|},$$

the lemma will be proved by estimating each term.

Observe that since the cycle begins with the map f_0, necessarily $j+2 \pmod 2 = 0$ and $j+1 \pmod 2 = 1$. Since s_j is a fixed point of f_0 and $\mathcal{R}_j(J)$ is contained in a fundamental domain of f_0 (see Addendum 3.5) then

$$s_j < f_0(\mathcal{R}_j(y)) < \mathcal{R}_j(x) < \mathcal{R}_j(y).$$

Applying $f_0^{-m_{i_1\dots i_{j+1}}}$ to the fundamental domain $[f_0(\mathcal{R}_j(y)), \mathcal{R}_j(y)]$, the usual distortion estimate gives

$$\frac{|f_0(\mathcal{R}_j(y)) - \mathcal{R}_j(y)|}{|\mathcal{R}_j(x) - \mathcal{R}_j(y)|} \ge e^{-V_0} \frac{|f_0^{-m_{i_1\dots i_{j+1}}}(f_0(\mathcal{R}_j(y))) - f_0^{-m_{i_1\dots i_{j+1}}}(\mathcal{R}_j(y))|}{|f_0^{-m_{i_1\dots i_{j+1}}}(\mathcal{R}_j(y)) - f_0^{-m_{i_1\dots i_{j+1}}}(\mathcal{R}_j(x))|}.$$

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Observe that

$$f_0^{-m_{i_1\dots i_{j+1}}+1}(\mathcal{R}_j(y)) < s_{j+1} < f_0^{-m_{i_1\dots i_{j+1}}}(\mathcal{R}_j(y)) \quad \text{and} \quad \mathcal{R}_{j+1} = f_0^{-m_{i_1\dots i_{j+1}}} \circ \mathcal{R}_j.$$

Using this information and rearranging the terms one obtains,

$$\frac{|\mathcal{R}_{j+1}(J)|}{|\mathcal{R}_j(J)|} \geq e^{-V_0} \frac{|s_{j+1} - \mathcal{R}_{j+1}(y)|}{D(\mathcal{R}_j(y))}.$$

To estimate the other term, since s_{j+2} is again a fixed point of f_0 and $\mathcal{R}_{j+2}(J)$ is contained in a fundamental domain of f_0, it follows that

$$s_{j+2} < f_0(\mathcal{R}_{j+2}(y)) < \mathcal{R}_{j+2}(x) < \mathcal{R}_{j+2}(y).$$

By construction of the return map (see Addendum 3.5), $[s_{j+2}, \mathcal{R}_{j+2}(y)] \subset A_{j+2}$, which is a fundamental domain of f_1.

This time let us apply $f_1^{m_{i_1\dots i_{j+2}}}$ to $[f_0(\mathcal{R}_{j+2}(y)), \mathcal{R}_{j+2}(y)]$, contained in a fundamental domain of f_1. Then the distortion estimate becomes

$$\frac{|\mathcal{R}_{j+2}(x) - \mathcal{R}_{j+2}(y)|}{|f_0(\mathcal{R}_{j+2}(y)) - \mathcal{R}_{j+2}(y)|} \geq e^{-V_1} \frac{|f_1^{m_{i_1\dots i_{j+2}}}(\mathcal{R}_{j+2}(y)) - f_1^{m_{i_1\dots i_{j+2}}}(\mathcal{R}_{j+2}(x))|}{|f_1^{m_{i_1\dots i_{j+2}}}(f_0(\mathcal{R}_{j+2}(y))) - f_1^{m_{i_1\dots i_{j+2}}}(\mathcal{R}_{j+2}(y))|}.$$

Since A_{j+2} is a fundamental domain of f_1,

$$f_1(\mathcal{R}_{j+2}(y)) < f_0(\mathcal{R}_{j+2}(y)) < \mathcal{R}_{j+2}(y).$$

Therefore,

$$\begin{aligned} &|f_1^{m_{i_1\dots i_{j+2}}}(f_0(\mathcal{R}_{j+2}(y))) - f_1^{m_{i_1\dots i_{j+2}}}(\mathcal{R}_{j+2}(y))| \\ &\quad < |f_1^{m_{i_1\dots i_{j+2}}}(f_1(\mathcal{R}_{j+2}(y))) - f_1^{m_{i_1\dots i_{j+2}}}(\mathcal{R}_{j+2}(y))|. \end{aligned}$$

As $f_1^{m_{i_1\dots i_{j+2}}} \circ \mathcal{R}_{j+2} = \mathcal{R}_{j+1}$, using the above inequality and rearranging the terms,

$$\frac{|\mathcal{R}_{j+2}(J)|}{|\mathcal{R}_{j+1}(J)|} \geq e^{-V_1} \frac{D(\mathcal{R}_{j+2}(y))}{|f_1(\mathcal{R}_{j+1}(y)) - \mathcal{R}_{j+1}(y)|}.$$

Thus,

$$\frac{|\mathcal{R}_{j+2}(J)|}{|\mathcal{R}_j(J)|} \geq e^{-(V_0+V_1)} \cdot \frac{D(\mathcal{R}_{j+2}(y))}{D(\mathcal{R}_j(y))} \cdot \frac{|s_{j+1} - \mathcal{R}_{j+1}(y)|}{|f_1(\mathcal{R}_{j+1}(y)) - \mathcal{R}_{j+1}(y)|}.$$

Finally let us estimate the last term. Applying the mean value theorem to the function $f_1 - \mathrm{id}$ and using that s_{j+1} is a fixed point of f_1,

$$\frac{|s_{j+1} - \mathcal{R}_{j+1}(y)|}{|f_1(\mathcal{R}_{j+1}(y)) - \mathcal{R}_{j+1}(y)|} = \frac{1}{|Df_1(z) - 1|} > \frac{1}{e^{V_1} - 1},$$

which proves the lemma. □

Assume the maps f_0, f_1 are close enough to rotations in the $C^{1+\mathrm{bv}}$-topology so that

$$\frac{e^{-(V_0+V_1)}}{e^{V_1} - 1} > 1.$$

Notice that, if the distortion constants satisfy $V_0 \leq \varepsilon$, $V_1 \leq \varepsilon$, the above condition is fulfilled for every positive $\varepsilon \leq 0.38$. Then also $((e^{V_1} - 1)e^{(V_0+V_1)})^{-n/2} > 1$. By the same proof as Lemma 2.4, we conclude the following analogous result.

LEMMA 3.8. *For any $\lambda > 1$, there exists $N \in \mathbb{N}$ such that $D\mathcal{R}^N(x) > \lambda$ for all $x \in A$.*

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3.4. *Proof of Theorem A.* Let us continue assuming that the maps f_i both have fixed points. We will need the next two lemmas.

LEMMA 3.9. *There is an ss-interval for* IFS(Φ) *if and only if there is a uu-interval.*

Proof. By contradiction, suppose there exists a uu-interval but no ss-intervals (the other case is similar). Then, Lemma 3.3 implies that there is a cycle of length n with the order $s_0 < s_1 < \cdots < s_0 + 1 = s_n$ on the interval $[s_0, s_0 + 1]$ identified with the circle.

Consider the maps f_i as maps of the real line (in particular, of the interval $[s_0, s_0 + 1]$), and the uu-interval as the interval $[a, b] \subset [s_0, s_0 + 1]$. By the definition of the uu-interval, if $x \leq a$, then $f_i(x) \leq a$ and if $x \geq b$, then $f_i(x) \geq b$. And so if $x > b$, then any finite composition of the maps f_0 and f_1, denoted by h, will satisfy that $h(x) > b$.

Take $x > s_{n-1}$ and $x > b$. Using the cycle we can inductively attract x to s_{n-2}, then to s_{n-3}, until s_0, such that in the end we will obtain a map h, a finite composition of f_0 and f_1, with $h(x) < a$. This contradicts the previous observation. □

LEMMA 3.10. *Suppose that there are no ss-intervals for* IFS(Φ) *and for some interval* I, $f_i(I) \cap I \neq \emptyset$, $i = 0, 1$. *Then for every* $x \in \mathbb{S}^1$, *there exist* $h_1, h_2 \in$ IFS(Φ) *such that* $h_1(x) \in I$ *and* $h_2^{-1}(x) \in I$.

Proof. Since there are no ss-intervals, Lemma 3.3 implies that there is a cycle with the order $s_0 < s_1 < \cdots < s_n + 1$ in the interval $[s_0, s_0 + 1]$. We identify the interval I on the circle $\mathbb{S}^1$ with an interval in $[s_0, s_0 + 1]$ that has the endpoints $a < b$.

Assume first that on the interval $[s_0, s_0 + 1]$, the point x satisfies $b < x \leq s_0 + 1$. Considering f_i as maps of the interval and arguing as in the proof of the above lemma, by using the attractors of the cycle we can inductively start attracting x to $s_0 = s_n$. Then, there exists h a finite composition of f_0 and f_1 of the form $h = f_i \circ h_1$ ($i = 0$ or 1), such that $h(x) \geq a$ and $h(x) < a$. Since $f_i(I) \cap I \neq \emptyset$, necessarily $h_1(x) \in I$, as required.

For the case when $x \leq a$, suppose $I \subset [s_{i-1}, s_i]$. By means of the cycle but now looking on the circle $\mathbb{S}^1$, we can attract x so s_i so that on the interval $[s_0, s_0 + 1]$, $x > s_i$. Then this is similar to the situation above, applied not to x but to $h(x)$, with $h(x) > s_i \geq b$.

Since by Lemma 3.9 having no ss-intervals is equivalent to having no uu-intervals, the same argument can be repeated with IFS(Φ^{-1}) to prove the existence of h_2^{-1} with $h_2^{-1}(x) \in I$. □

3.4.1. *Proof of minimality.* Let us show that IFS(Φ) is minimal. The minimality of IFS(Φ^{-1}) follows analogously since by Lemma 3.9, IFS(Φ^{-1}) has no ss-intervals.

Consider a point $x \in \mathbb{S}^1$ and an open interval J. Since there are no ss-intervals, by Lemma 3.8, there exists an expanding return map $\mathcal{R}^N$ defined in the domain A and associated with a cycle. Since $A = (s_0, f^{-1}(s_0))$, which is a fundamental domain of f_1 and is in the basin of attraction of s_0, then $f_i(A) \cap A \neq \emptyset$. By Lemma 3.10, there exists $h_1 \in$ IFS(Φ) with $h_1^{-1}(J) \cap A \neq \emptyset$. We can assume J is small enough so that $h_1^{-1}(J)$ is contained in one of the domains of the map $\mathcal{R}^N$. Since $D\mathcal{R}^N > \lambda > 1$, then $|\mathcal{R}^{kN}(J)| > \lambda^k |J|$. Iterating sufficiently many times $\mathcal{R}^{kN}(J)$ eventually must contain a discontinuity s of $\mathcal{R}^N$.

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By Lemma 3.4, $s \in \mathrm{Orb}^+_\Phi(s_0) \cup \cdots \cup \mathrm{Orb}^+_\Phi(s_n)$. Without loss of generality, we assume that s is in the orbit of s_0. Then there exists $h_2 \in \mathrm{IFS}(\Phi)$ with $h_2(s_0) = s$ and so

$$h_2(s_0) \in \mathcal{R}^{kN} \circ h_1^{-1}(J).$$

Writing $\mathcal{R}^{kN}$ as h_3^{-1} with $h_3 \in \mathrm{IFS}(\Phi)$, we get $h_1 \circ h_3 \circ h_2(s_0) \in J$.

Having the order $s_0 < s_1 < \cdots < s_0 + 1 = s_n$, given a point x we may attract x arbitrary close to s_0 by inductively attracting to each s_i of the cycle. Then there exists h_4 with $h_4(x)$ close enough to s_0. It can be concluded that $h_4 \circ h_1 \circ h_3 \circ h_2(x) \in J$, and since x and J were arbitrary, this proves minimality of $\mathrm{IFS}(\Phi)$.

Remark 3.11. The action of the semigroup $\mathrm{IFS}(\Phi)$ is expanding: for any x in the circle, there exists $g \in \mathrm{IFS}(\Phi)$ such that $Dg^{-1}(x) > 1$.

To prove this, let us use the fact that $\mathcal{R}$ is an expanding return map on some interval A. The branches of this expanding return map are maps in $\mathrm{IFS}(\Phi^{-1})$. Since $\mathrm{IFS}(\Phi^{-1})$ is minimal, there is $h \in \mathrm{IFS}(\Phi)$ such that $h^{-1}(x) \in A$. Then, for n sufficiently large, $\mathcal{R}^n \circ h^{-1}(x)$ will have derivative larger than 1.

3.4.2. *Density of hyperbolic periodic points.* With respect to the return map $\mathcal{R}$, defined on the interval $A = (s_0, f_1^{-1}(s_0)]$, consider one of the branches $I_{i_1 \ldots i_n}$, such that $h^{-1}_{i_1 \ldots i_n}(I_{i_1 \ldots i_n}) = A$. Since $h^{-1}_{i_1 \ldots i_n}$ is expanding, there exists exactly one repelling fixed point, say s, of $h^{-1}_{i_1 \ldots i_n}$. Then s is an attracting fixed point of $h_{i_1 \ldots i_n}$, whose basin of attraction includes A.

Now given an interval J, by minimality of $\mathrm{IFS}(\Phi)$ and assuming that J is small enough, there exists $h_1 \in \mathrm{IFS}(\Phi)$ such that $h_1(J) \subset A$. Again by minimality of $\mathrm{IFS}(\Phi)$, there exists $h_2 \in \mathrm{IFS}(\Phi)$ such that $h_2(s) \in J$.

Since $h_1(J)$ is contained in the basin of attraction of s, we may assume that $h_1(J)$ is attracted sufficiently close to s, so that $h_2 \circ h_1(J) \subset J$ and $D(h_2 \circ h_1)|_J < \beta < 1$. This implies that there is a hyperbolic attractor in J. The density of hyperbolic repellers for $\mathrm{IFS}(\Phi)$ follows from the density of hyperbolic attractors for $\mathrm{IFS}(\Phi^{-1})$.

3.4.3. *The case of generators with periodic points.* When the generators f_i have non-zero rotation number, consider the next lemma, whose proof is left to the reader.

LEMMA 3.12. *Let f be an orientation-preserving $C^{1+\mathrm{bv}}$-diffeomorphism of the circle with periodic points of period n, and consider B the immediate basin of attraction of some periodic point. Then*

$$e^{-V_f} \le \frac{Df^n(x)}{Df^n(y)} \le e^{V_f} \quad \textit{for all } x, y \in B,$$

where V_f is the distortion of f on $\mathbb{S}^1$. That is, $\mathrm{Dist}(f^n, B) \le V_f$.

Since in the immediate basin there is always a point with derivative 1,

$$e^{-V_f} \le Df^n(x) \le e^{V_f} \quad \textit{for all } x \in B.$$

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The lemma states that the the distortion of $f_i^{n_i}$ is the same as that of f_i in the relevant regions and the derivative of $f_i^{n_i}$ is bounded by the distortion and thus will still be close to 1 if f_i is close to rotations. Namely, $|Df_i^n(x) - 1| \leq e^{V_i} - 1$, where V_i is the distortion constant of f_i. Therefore, the same proof as in the case of fixed points applies. Observe that also in Proposition 3.6 we have to ask again that $e^{-(V_0+V_1)}/(e^{V_1} - 1) > 1$. Since $V_0 \leq \varepsilon$, $V_1 \leq \varepsilon$, the above condition is fulfilled for all $\varepsilon \leq 0.38$.

This completes the proof of Theorem A.

4. *Spectral decomposition*

4.1. *Spectral decomposition on the real line.*

The next theorem gives a complete description of the global topological dynamics of an IFS generated by a pair of diffeomorphisms on the real line.

THEOREM 4.1. (Spectral decomposition on the real line) *There is $\varepsilon > 0.38$ such that if f_0 and f_1 are diffeomorphisms on the real line, one of them having at least one fixed point, with no fixed points in common and ε-close to the identity in the $C^{1+\mathrm{bv}}$-topology, then*

$$L(\mathrm{IFS}(\Phi))\backslash\{\pm\infty\} = \bigcup K_i \quad \textit{and} \quad K_i \subset \overline{\mathrm{Per}(\mathrm{IFS}(\Phi))},$$

where each K_i is a compact, maximal transitive set for IFS(Φ). *Moreover, each compact set K_i is either:*

- *a single fixed point of f_0 or f_1; or*
- *a $**$-interval for* IFS(Φ) *with $** \in \{ss, su, uu, s, u\}$.*

In particular, if f_0 and f_1 have only hyperbolic fixed points, then the above union of the sets K_i is countable and they are pairwise disjoint.

Proof. Consider $z \in L(\mathrm{IFS}(\Phi))\backslash\{\pm\infty\}$. We can assume that $z \in \omega(\mathrm{IFS}(\Phi))$ since the situation for the α-limit of IFS(Φ) is settled by a similar argument. Then, by definition of ω-limit of IFS(Φ), the point z is approximated by points of the form $y_k \in \omega_\Phi(x_k)$. For each k, there is a sequence $(h_{kn})_n \subset \mathrm{IFS}(\Phi)$ such that each point y_k is again approximated by points of the form $h_{kn} \circ \cdots \circ h_{k1}(x_k)$.

CLAIM 4.1.1. *If $y \in \omega_\Phi(x)$, then either y belongs to some $**$-interval for* IFS(Φ) *with $** \in \{ss, su, uu, s, u\}$ or it is a fixed point of f_0 or f_1.*

This claim concludes that the limit set of IFS(Φ) is contained in the union of $**$-intervals and the set of fixed points of f_0 and f_1. Indeed, from Claim 4.1.1, either z is a fixed point of f_0 or f_1, or, for k_0 large enough, y_k belongs in the same $**$-interval for all $k \geq k_0$ and thus z is also in this $**$-interval.

Proof of Claim 4.1.1. Since $y \in \omega_\Phi(x)$, there is $(h_n)_n \subset \mathrm{IFS}(\Phi)$ such that $y = \lim_{n\to\infty} h_n \circ \cdots \circ h_1(x)$. If y is an accumulation point of fixed points then, obviously, y is a fixed point of f_0 or f_1. Thus, we can assume that y is not of this type. Also, without loss of generality, we may suppose that $y \geq 0$.

Since at least one of the generators has a fixed point, we can assume by the following argument that $y \in [p, q]$, where $p, q \in \mathrm{Fix}(f_0) \cup \mathrm{Fix}(f_1)$ and $(p, q) \cap \mathrm{Fix}(f_i) = \emptyset$ for $i = 0, 1$. Otherwise, there is a fixed point P such that any other fixed point is less then

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P and $y \in [P, \infty)$. If $y \neq P$, then it is not hard to check via the geometry of the functions, that it is not possible that $f_0, f_1 > \mathrm{id}$ in (P, ∞), since then $y = \infty$. In the other case, $[P, \infty)$ is an s- or u-interval or $y = P$.

Suppose p and q are both attractors or repellers but for different maps restricted to $[p, q]$. Since f_0 and f_1 are close to the identity, this closed interval is an ss- or uu-interval. So, we can consider the final case when p and q are an attractor–repeller pair for the same map, say f_0 (the repeller-atractor case is analogous). Note that then $f_0 < \mathrm{id}$ in (p, q). We have two options: either $f_1 < \mathrm{id}$ or $f_1 > \mathrm{id}$ in $[p, q]$. In the first case, both maps are below the identity and hence this geometry implies that the unique possible ω-limit points for $\mathrm{IFS}(\Phi)$ in $[p, q]$ are p and q. For the second case, we have again two options:

$$f_1([p, q]) \cap [p, q] \neq \emptyset \quad \text{or} \quad f_1([p, q]) \cap [p, q] = \emptyset.$$

In the first case $[p, q]$ is an su-interval for $\mathrm{IFS}(\Phi)$, and for the other one it follows as before that y is either p or q.

Therefore, we have shown that in every possible case, the point $y \in \omega_\Phi(x)$ belongs to a $**$-interval for $\mathrm{IFS}(\Phi)$ with $** \in \{ss, su, uu, s, u\}$ or it is a fixed point of f_0 or f_1. □

To conclude the proof of the theorem we need to show that every set K_i is contained in the limit set of $\mathrm{IFS}(\Phi)$. Recall that K_i denotes both, a $**$-interval or a fixed point of f_0 or f_1. It is clear that every fixed point is contained in the limit set and so we only need to prove that every $**$-interval for $\mathrm{IFS}(\Phi)$ is recurrent.

We will show that there exists $\varepsilon > 0$ such that if f_0 and f_1 are ε-close to the identity in the $C^{1+\mathrm{bv}}$-topology, then any $**$-interval for $\mathrm{IFS}(\Phi)$ with $** \in \{ss, su, s\}$ satisfies the hypothesis of Theorem 2.2. This proximity implies that $|f_i(x) - x| < \varepsilon$, $|Df_i(x) - 1| < \varepsilon$ and $\mathrm{Dist}(f_i, \mathbb{R}) < \varepsilon$ for $i = 0, 1$, and we only need to show that $(1 - \varepsilon)\varepsilon^{-1}e^{-V} > 1$.

Notice that $V \leq 2\varepsilon|a - f_i^{-1} f_j(a)|$, where $i \neq j$ and the a is the endpoint of the $**$-interval. Suppose a is the left endpoint and $f_0 < \mathrm{id}$ in $(a, f_0^{-1} f_1(a)]$. From the proximity to the identity, $f_1(a) \leq a + \varepsilon$. Since

$$f_0(x) \geq (1 - \varepsilon)(x - a) + a \quad \text{for all } x \in (a, f_0^{-1} f_1(a)],$$

it follows that $f_0^{-1} f_1(a) - a \leq \varepsilon(1 - \varepsilon)^{-1}$. Thus,

$$(1 - \varepsilon)\varepsilon^{-1}e^{-V} \geq (1 - \varepsilon)\varepsilon^{-1}e^{-2\varepsilon^2(1-\varepsilon)^{-1}} > 1$$

for all $\varepsilon \leq 0.38$.

Finally by Theorem 2.2, ss-, su- and s-intervals are minimal with a dense set of periodic points. In particular, these kinds of intervals are transitive sets for $\mathrm{IFS}(\Phi)$. The same properties are obtained for uu-intervals and u-intervals for $\mathrm{IFS}(\Phi^{-1})$.

LEMMA 4.2. *If A is minimal for* $\mathrm{IFS}(\Phi^{-1})$ *then A is transitive for* $\mathrm{IFS}(\Phi)$.

This lemma follows from the facts that minimality implies transitivity and that if A is transitive for $\mathrm{IFS}(\Phi^{-1})$ then A is transitive for $\mathrm{IFS}(\Phi)$.

Applying the above lemma we obtain that uu- and u-intervals are also transitive sets for the $\mathrm{IFS}(\Phi)$. This basically concludes the proof of the theorem. The last part is immediately obtained since if all the fixed points are hyperbolic, then they are isolated and so there exists only a countable number of the fixed points and the $**$-intervals. □

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Remark 4.3. Theorem 4.1 holds also for IFS generated by a pair of diffeomorphisms of the compact interval I under the same assumptions.

To see this, we view the compact interval I like the compactified real line $[-\infty, \infty]$ and the K^s or K^u intervals in I are identified with the relevant intervals on the real line.

4.2. *Spectral decomposition on the circle: Proof of Theorem B.* The following result is a version of Theorem 2.2 for diffeomorphisms of the circle with periodic points.

PROPOSITION 4.4. *There is $\varepsilon > 0.30$ such that if f_0 and f_1 are circle diffeomorphisms with periodic points of period n_0 and n_1 respectively, ε-close to the rotation in the $C^{1+\mathrm{bv}}$-topology then for every $**$-interval K^{**}, $** \in \{ss, su\}$, with respect to* $\mathrm{IFS}(\Phi^n)$ *it holds that*

$$K^{**} \subset \overline{\mathrm{Per}(\mathrm{IFS}(\Phi^n))} \quad \text{and} \quad K^{**} \subset \overline{\mathrm{Orb}^+_{\Phi^n}(x)} \quad \text{for all } x \in K^{**}.$$

Proof. We will apply Theorem 2.2 for $\mathrm{IFS}(\Phi^n)$ for the $**$-interval $K^{**} = [a, b]$. Without loss of generality, we assume that $f_0^{n_0} < \mathrm{id}$ in (a, b). According to Lemma 3.12,

$$e^{-V_0} \le Df_0^{n_0}(x) \le e^{V_0} \quad \text{for all } x \in \mathbb{S}^1,$$

where V_0 is the distortion constant of f_0. This implies that $|Df_0^{n_0}(x) - 1| < e^{V_0} - 1$ for all $x \in \mathbb{S}^1$. Similarly we denote by V_1 the maximal variation of f_1 and then it also follows

$$\mathrm{Dist}(f_0^{n_0}, I) \le V_0 \quad \text{and} \quad \mathrm{Dist}(f_1^{n_1}, J) \le V_1$$

for all fundamental domains I of $f_0^{n_0}$ and J of $f_1^{n_1}$ in $(a, f_0^{-n_0} f_1^{n_1}(a)]$.

Since $(a, f_0^{-n_0} f_1^{n_1}(a)] \subset \mathbb{S}^1$, its length is less than 1. Let us remember the necessary condition in Theorem 2.2,

$$(1 - (e^{V_0} - 1))(e^{V_0} - 1)^{-1} e^{-(V_0+V_1)|a - f_0^{-n_0} f_1^{n_1}(a)|} > 1.$$

As $V_0, V_1 \le \varepsilon$, it suffices to take $\varepsilon \le 0.30$ to guarantee that the above condition is fulfilled, which concludes the proof of the proposition. □

Proof of Theorem B. This result is immediately deduced from Theorem 4.1, Proposition 4.4 and Theorem A. Indeed, consider $\tilde{f}_0^{n_0}$ and $\tilde{f}_1^{n_1}$ as the lifts on the real line of $f_0^{n_0}$ and $f_1^{n_1}$. Note that $\tilde{f}_0^{n_0}$ and $\tilde{f}_1^{n_1}$ are periodic functions. Arguing as in Theorem 4.1, there is a decomposition into compacts sets on the real line of $L(\mathrm{IFS}(\tilde{f}_0^{n_0}, \tilde{f}_1^{n_1}))\backslash\{\pm\infty\}$. From the periodicity of $\tilde{f}_i^{n_i}$, it follows that each of these compact sets projects on the circle as either a periodic point of f_i, or as a $**$-interval for $** \in \{ss, su, uu\}$. By Proposition 4.4 these intervals are transitive sets for $\mathrm{IFS}(\Phi^n)$.

What is left is to study the limit set of a point whose ω-limit (or α-limit) contains $\pm\infty$ on the real line. This can only happen if there is a cycle for $\mathrm{IFS}(\Phi^n)$ different from an ss-interval. Then Theorem A implies that $\mathbb{S}^1$ is minimal for $\mathrm{IFS}(\Phi^n)$ and $\mathrm{IFS}(\Phi^{-n})$, a contradiction. Therefore, we obtain a decomposition of the limit set, as required. □

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5. *A Denjoy type theorem for IFS*

5.1. *A trichotomy for IFS.* The next lemma shows some relations between the ω-limit sets and the orbits of an IFS. These properties will be necessary afterwards for the proof of the trichotomy result. We will use the following notation: given a set A on a manifold M, we denote by A' the set of accumulation points of A, i.e. the set of points y such that there exists a sequence $(x_n)_n \subset A$ converging to y with $x_n \neq y$ for all $n \in \mathbb{N}$.

LEMMA 5.1. *Consider $\Phi = (\phi_1, \ldots, \phi_k) \in \mathrm{Hom}(M)^k$, a non-empty closed subset K of a manifold M. Then it holds that:*

(i) $\omega_\Phi(h(x)) \subset \omega_\Phi(x) \subset \overline{\mathrm{Orb}^+_\Phi(x)}$ *for all $x \in M$ and for $h \in \mathrm{IFS}(\Phi)$;*

(ii) *if $K = \overline{\mathrm{Orb}^+_\Phi(x)}$ for all $x \in K$ then*

$$K = \phi_1(K) \cup \cdots \cup \phi_k(K) = \omega_\Phi(x) \quad \text{for all } x \in K;$$

(iii) $\mathrm{Orb}^+_\Phi(x)' = \phi_1(\mathrm{Orb}^+_\Phi(x)') \cup \cdots \cup \phi_k(\mathrm{Orb}^+_\Phi(x)')$ *for all $x \in M$.*

Proof. By definition of the ω-limit set of a point $x \in M$, $\omega_\Phi(h(x)) \subset \omega_\Phi(x)$ for all $h \in \mathrm{IFS}(\Phi)$. On the other hand, since

$$\overline{\mathrm{Orb}^+_\Phi(x)} = \left\{ y : \text{there exists } (g_n)_n \subset \mathrm{IFS}(\Phi) \text{ such that } y = \lim_{n\to\infty} g_n(x) \right\}$$

then $\omega_\Phi(x)$ is a subset of the closure of the orbit of x. Therefore, we conclude (i).

According to the first item, to obtain (ii) it suffices to prove that

$$K \subset \bigcup_{i=1}^{k} \phi_i(K) \quad \text{and} \quad K \subset \omega_\Phi(x) \quad \text{for all } x \in K.$$

The first inclusion follows from the minimality of K. Indeed, for any $y \in K$ there is $(g_n)_n \subset \mathrm{IFS}(\Phi)$ such that $x = \lim_{n\to\infty} g_n(y)$. Since we are working with a finitely generated semigroup, taking a subsequence if necessary, we can assume that $g_n = \phi_i \circ \tilde{g}_n$ for some $i \in \{1, \ldots, k\}$ and $\tilde{g}_n \subset \mathrm{IFS}(\Phi)$. Thus, $\phi_i^{-1}(x) \in K$ as it is the limit of $\tilde{g}_n(y)$ and K is an invariant closed set. Therefore, $x = \phi_i(\phi_i^{-1}(x)) \in \phi_i(K)$, showing the first inclusion.

In order to prove the second inclusion, we fix $x, y \in K$ and consider a sequence of positive real numbers $\varepsilon_n = 1/n \to 0$. Let us construct by induction a sequence $(h_n)_n \subset \mathrm{IFS}(\Phi)$, such that the distance between y and $h_n \circ \cdots \circ h_1(x)$ is less than ε_n. Since the orbit of x is dense in K, there is $h_1 \in \mathrm{IFS}(\Phi)$ such that $d(y, h_1(x)) < \varepsilon_1$. Similarly, since $h_1(x) \in \mathrm{Orb}^+_\Phi(x) \subset K$ then the orbit of $h_1(x)$ is dense in K and there exists h_2 such that $d(y, h_2 \circ h_1(x)) < \varepsilon_2$. Inductively we obtain the desired sequence $(h_n)_n \subset \mathrm{IFS}(\Phi)$. Hence, $y = \lim_{n\to\infty} h_n \circ \cdots \circ h_1(x)$ and thus $y \in \omega_\Phi(x)$ for all x, $y \in K$, concluding (ii).

Now we will show that $\mathrm{Orb}^+_\Phi(x)'$ is a self-similar set, that is, we prove the last item. Note that $\phi_i(\mathrm{Orb}^+_\Phi(x)') \subset \mathrm{Orb}^+_\Phi(x)'$ for all $i = 1, \ldots, k$. Indeed, if y is an accumulation point of the orbit of x, then $\phi_i(y)$ is approximated by points of the form $\phi_i \circ g_n(x) \in \mathrm{Orb}^+_\Phi(x)$, where $y = \lim_{n\to\infty} g_n(x)$ and $g_n(x) \neq y$ for all $n \in \mathbb{N}$. This implies that $\phi_i(y)$ is also an accumulation point of $\mathrm{Orb}^+_\Phi(x)$ and so we conclude one of the inclusions.

To obtain the other inclusion

$$\mathrm{Orb}^+_\Phi(x)' \subset \phi_1(\mathrm{Orb}^+_\Phi(x)') \cup \cdots \cup \phi_k(\mathrm{Orb}^+_\Phi(x)'),$$

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we fix any $y \in \mathrm{Orb}_\Phi(x)'$. Then there exists $(g_n) \subset \mathrm{IFS}(\Phi)$ such that $y = \lim_{n\to\infty} g_n(x)$ and $g_n(x) \neq y$ for all $n \in \mathbb{N}$. Since the semigroup $\mathrm{IFS}(\Phi)$ is finitely generated, taking a subsequence if necessary, we can assume that for some fixed $i \in \{1, \ldots, k\}$ we have $g_n = \phi_i \circ \tilde{g}_n$ with $\tilde{g}_n \in \mathrm{IFS}(\Phi)$. Hence, $\phi_i^{-1}(y) = \lim_{n\to\infty} \tilde{g}_n(x)$ and $\tilde{g}_n(y) \neq \phi_i^{-1}(y)$. Since the accumulation set is a closed set, it holds that $\phi_i^{-1}(y) \in \mathrm{Orb}_\Phi^+(x)'$. This implies that $y \in \phi_i(\mathrm{Orb}_\Phi^+(x)')$ proving the desired inclusion and therefore (iii). □

The next trichotomy theorem is the IFS counterpart of the similar result for group actions in [**7**]. An interval I is forward-invariant for $\mathrm{IFS}(\Phi)$ if $f_i(I) \subset I$ for all I.

THEOREM 5.2. (Trichotomy for IFS) *Consider* $\mathrm{IFS}(\Phi)$ *generated by* $\Phi = (f_1, \ldots, f_k) \in \mathrm{Hom}(I)^k$, *where* I *is* $\mathbb{S}^1$ *or a forward-invariant compact interval. Then there exists a non-empty closed subset* K *of* M *such that*

$$K = \bigcup_{i=1}^{k} f_i(K) \quad \textit{and} \quad K = \overline{\mathrm{Orb}_\Phi^+(x)} = w_\Phi(x) \quad \textit{for all } x \in K.$$

Moreover, one (and only one) of the following possibilities occurs:

(i) K *is a finite orbit for* $\mathrm{IFS}(\Phi)$*;*

(ii) K *has non-empty interior;*

(iii) K *is a Cantor set.*

Proof. A family of non-empty closed subsets of M such that each member Λ satisfies $\Lambda = f_1(\Lambda) \cup \cdots \cup f_N(\Lambda)$ can be ordered by inclusion. Since the intersection of nested compact sets is compact and non-empty, Zorn's lemma allows to conclude the existence of a minimal (regarding the inclusion) non-empty closed set K such that $K = \bigcup f_i(K)$. We will show that K is an invariant minimal set for $\mathrm{IFS}(\Phi)$,

$$K = \overline{\mathrm{Orb}_\Phi^+(x)} \quad \text{for all } x \in K. \tag{5}$$

Then, according to item (ii) in Lemma 5.1, we have $K = \overline{\mathrm{Orb}_\Phi^+(x)} = \omega_\Phi(x)$ for all $x \in K$.

In order to prove (5), since K is a closed invariant set, $\mathrm{Orb}_\Phi^+(x)' \subset \overline{\mathrm{Orb}_\Phi^+(x)} \subset K$ for all $x \in K$. On the other hand, according to (iii) in Lemma 5.1, for each $x \in K$ the set of accumulation points $\mathrm{Orb}_\Phi^+(x)'$ is a closed self-similar set. Since K is minimal (regarding the inclusion) then either $\mathrm{Orb}_\Phi^+(x)'$ is an empty set or $K = \mathrm{Orb}_\Phi^+(x)'$. We have two possibilities:

(i) there is $x \in K$ such that $\mathrm{Orb}_\Phi^+(x)'$ is empty;

(ii) for all $x \in K$, it holds that $K = \mathrm{Orb}_\Phi^+(x)'$.

In the first case, it follows that $\mathrm{Orb}_\Phi^+(x)$ is a finite set, and therefore it is a non-empty closed self-similar set contained in K. This implies, via Zorn's Lemma, that $K = \mathrm{Orb}_\Phi^+(x) = \omega_\Phi(x)$.

In the second case, we obtain that K is an invariant minimal set for $\mathrm{IFS}(\Phi)$ and so $K' = K$. Moreover, we have two options: K has non-empty interior or the interior of K is empty and thus, since M is a one-dimensional manifold, K is a Cantor set. This concludes the proof of the theorem. □

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As an attractor for IFS is generated by a forward-invariant set, the theorem above describes the shape of possible attractors of an IFS.

Remark 5.3. In the case when I is an ss-interval, which is forward-invariant, the endpoints are global (with respect to I) attractors and are necessarily part of any minimal set in I. Thus the minimal set is unique. See also [**16**].

5.2. *Proof of Theorem C.* From Theorem B it follows that any minimal invariant closed set, Λ, for IFS(Φ^n) is contained in some piece of the spectral decomposition of the limit set. By the hypothesis of the theorem this piece cannot be a single point because the generators do not have fixed points in common. It also cannot be neither an su- nor a uu-interval because they are not forward-invariant and any point can be made to leave the interval. Thus necessarily the piece has to be an ss-interval K^{ss}, in which case by Remark 5.3, the only minimal invariant set is K^{ss} and so $\Lambda = K^{ss}$.

It is left to show that there cannot exist invariant Cantor sets for IFS(Φ). Let us suppose that Λ is an minimal invariant Cantor set for IFS(Φ) and take any point $p \in \Lambda$. Since f_0 and f_1 have no periodic points in common, the basins of all attracting periodic points of f_0 and f_1 cover $\mathbb{S}^1$. By hypothesis $\mathbb{S}^1$ is not minimal and so from Theorem A there is at least one ss-interval for IFS(Φ^n) whose endpoints are attractors. The cover of $\mathbb{S}^1$ by the basins of attractors allows p to be attracted inside some ss-interval K^{ss} for IFS(Φ^n). Since Λ is a minimal invariant set for IFS(Φ), then $K^{ss} \subset \Lambda$, contradicting that it has empty interior.

5.3. *A theorem of Duminy on group actions of the circle.* Finally, we will reprove Duminy's theorem [**15**] under the assumption of Theorem A. We denote by G(Φ^n) the group generated by $\Phi^n = (f_0^{n_0}, f_1^{n_1})$.

THEOREM 5.4. (Duminy) *There exists* $\varepsilon > 0.30$ *such that if* f_0 *and* f_1 *are diffeomorphisms on the circle with periodic points of period* n_0 *and* n_1 *respectively,* ε*-close to the rotations in the* $C^{1+\mathrm{bv}}$*-topology and with no periodic points in common then:*

- $\mathbb{S}^1$ *is minimal for* G(Φ^n)*; and*
- *the hyperbolic periodic points of* G(Φ^n) *are dense in* $\mathbb{S}^1$.

Proof. Since there are no periodic points in common and we are working with a group, then considering the inverses if necessary, there always exists an ss-interval and a cycle different from the ss-interval for the group G(Φ^n). Moreover, this cycle has the same order as in Lemma 3.3. By Proposition 4.4 every point in the ss-interval has a dense orbit for the action of G(Φ^n). Now similarly to the proof of minimality in Theorem B, the cycle allows to bring any point and interval to the ss-interval and this implies minimality of $\mathbb{S}^1$ for G(Φ^n).

The density of hyperbolic periodic points follows from the argument described in Lemma 2.5 (see also §3.4.2). This concludes the proof of the result. □

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