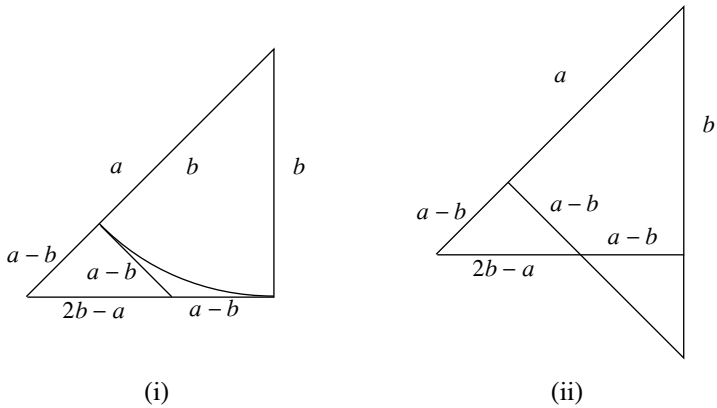
Using A4-sized paper to illustrate that $\sqrt{2}$ is irrational

One familiar proof that $\sqrt{2}$ is irrational is based on the observation that if $\sqrt{2} = \dfrac{a}{b}$ then we also have $\sqrt{2} = \dfrac{2b - a}{a - b}$, (*). Contradiction then ensues by supposing that a, b are whole numbers with (say) b minimal in which case (*) gives the whole numbers $2b - a$, $a - b$ with $a - b < b$. The crux of the matter is thus (*). This may be verified algebraically: $\dfrac{2b - a}{a - b} = \dfrac{2 - \frac{a}{b}}{\frac{a}{b} - 1} = \dfrac{2 - \sqrt{2}}{\sqrt{2} - 1} = \sqrt{2}$. Alternatively (*) may be embedded into the proof by contradiction via the well-ordering principle: this is Estermann's charming proof [1] with follow-ups in [2, 3, 4]. Finally, (*) may be illustrated geometrically as in Apostol's proof [5] based on Figure 1(i): see also [6]. A variation starting from two superimposed triangles is shown in Figure 1(ii).



(i) (ii)

FIGURE 1: All the triangles shown are right-angled isosceles

The challenge implicit in the geometrical proofs is to produce (*) as naturally as possible by exhibiting a triangle or rectangle with sides $2b - a$, $a - b$. Our aim in this Note is to show how A4-sized paper (with a length : width ratio $a : b = \sqrt{2} : 1$) may be used to illustrate (*): we give five possibilities in Figures 2 to 6 below and invite readers to come up with other alternatives. The commentary is deliberately kept to a minimum since, as in Figure 1, these are essentially 'proofs without words'.

Figure 2 shows two overlapping A4 sheets sharing the corner O. With P on the edge of the lower sheet, Q is on the edge of the upper sheet (since a, b are the side-lengths of a right-angled isosceles triangle).

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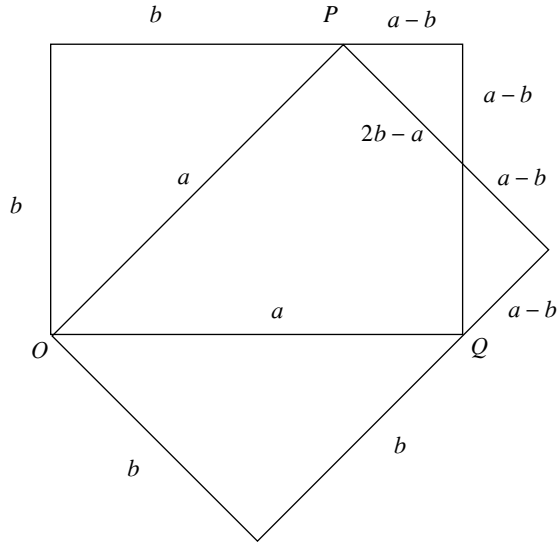


FIGURE 2

Figure 3 shows the top corner P of an A4 sheet folded to P' on the lower edge.

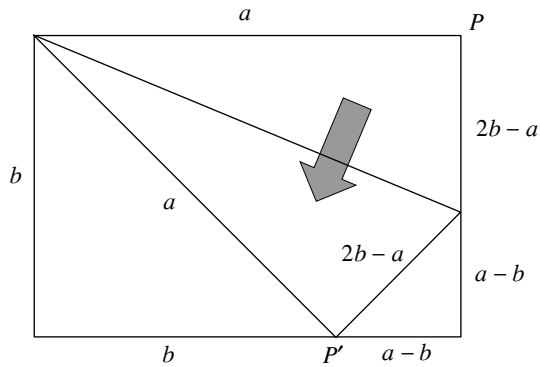
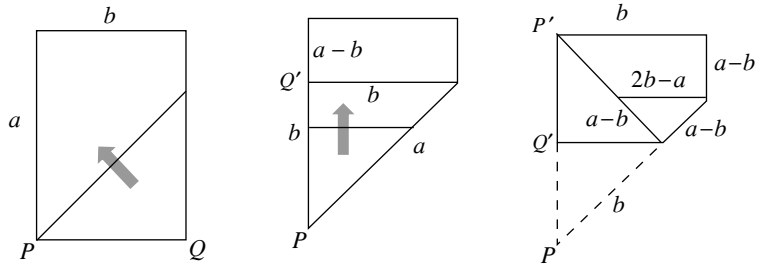


FIGURE 3

Figure 4 involves two folds of an A4 sheet: notice that $PP' = a$ is the hypotenuse of a right-angled isosceles triangle with legs b.

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(i) First fold (ii) Second fold (iii) Resulting triangle

FIGURE 4

Our last two proofs make use of the fact that A3, A4, A5 rectangles are all similar and that folding each one in half generates its successor.

In Figure 5, the three small rectangles are A4; the large one is A5 (and so with dimensions $2b \times a$).

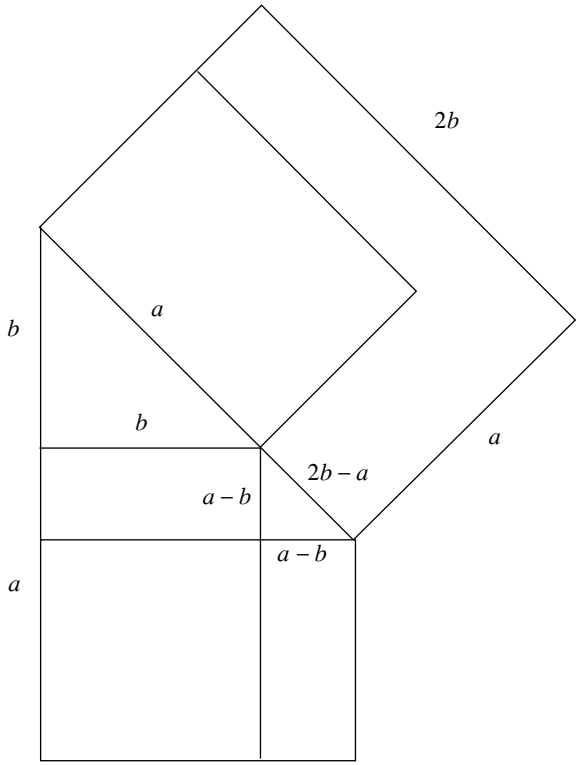
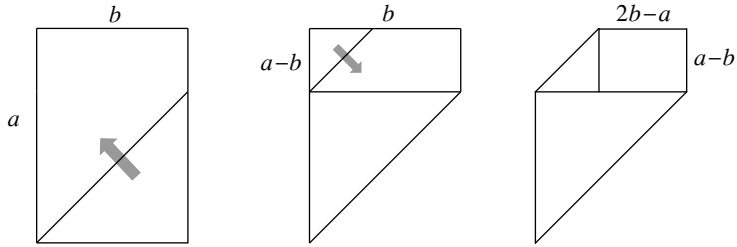


FIGURE 5

In Figure 6, because A4 and A5 are similar rectangles, $\frac{a}{b} = \frac{b}{\frac{1}{2}a} = \frac{2b}{a}$,

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thus $\frac{2b - a}{a - b} = \frac{a}{b}$ so that the resulting rectangle in (iii) is similar to the initial A4 rectangle in (i).



(i) First fold (ii) Second fold (iii) Resulting rectangle

FIGURE 6

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10.1017/mag.2017.24

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