Euro. Jnl of Applied Mathematics (2004), *vol.* 15, *pp.* 257–271. © 2004 Cambridge University Press
DOI: 10.1017/S0956792503005369 Printed in the United Kingdom

Roll waves in a non-regular inclined channel

A. BOUDLAL[1] and V. YU. LIAPIDEVSKII[2]

[1]*Laboratoire de Mécanique de Lille, UMR CNRS 8107,*
Université des Sciences et Technologies de Lille, Bâtiment M3 – Mécanique
email: abdelaziz.boudlal@univ-lille1.fr
[2]*Lavrentyev Institute of Hydrodynamics, Novosibirsk, 630090, Russia*
email: liapid@hydro.nsc.ru

(*Received 25 April 2003; revised 18 August 2003*)

For nonlinear hyperbolic equations of a gas dynamic type with flow acceleration and friction terms, the classification of a special class of periodic travelling waves, which are known as roll waves, is given. As an illustration, the shallow water equations for the inclined channels of an arbitrary cross-section are considered. The analysis shows that the flow patterns depend upon the sign of the second derivative of the pressure function. The roll waves in regular channels with the convex pressure have the same structure as the waves described in Dressler [6]. For a nonconvex pressure function, the multi-jump configuration of roll waves is found.

1 Introduction

In the paper we consider the one-dimensional periodic travelling waves governed by the following hyperbolic system:

$$\begin{aligned} h_t + (hu)_x &= 0, \\ (hu)_t + (hu^2 + P(h))_x &= h(\alpha - u^2/R(h)). \end{aligned} \tag{1.1}$$

Equation (1.1) describes nonlinear wave evolution in different areas of continuum mechanics such as gas dynamics, open channel flows, elasticity theory, etc. Here (1.1) is applied to the shallow water flow along an inclined channel. Therefore, t is the time, x is the coordinate along the channel, h and u are the mean depth and the velocity component of the liquid layer. Let the gravity acceleration α along the channel be constant and $R(h)$ be the hydraulic radius. Definition of α and $R(h)$ for the channels of arbitrary cross-section is given in the next section. The source term in (1.1) simulates the interaction between mass forces and turbulent friction.

The nonlinear wave patterns described by (1.1) depend upon the pressure term $P = P(h)$. The system is hyperbolic for $c^2(h) = P'(h) > 0$. For $P(h) = \frac{1}{2}g_n h^2$, where g_n is the component of the gravity acceleration, which is normal to the channel, (1.1) reduce to the shallow water equations investigated in Dressler [6]. Dressler's study can be easily generalized for a convex pressure term $P = P(h)$ in (1.1). For a nonconvex pressure the roll wave patterns are more complicated. The examples of flow with the nonconvex pressure term are given below. The aim of the investigation is to give the classification of periodic travelling waves for the pressure function containing points of inflection.

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

The shallow water theory for the channels of arbitrary cross-section [3, 7, 17] gives examples of (1.1) with a nonconvex function $P(h)$. In these works the structure of regular and inverted roll waves of moderate amplitude have been investigated for one- and two-layer flows. The problem of roll wave generation in liquid-liquid and gas-liquid flows is closely connected with the theoretical prediction of the onset of slugging in near-horizontal channel and pipe flows (see the reviews elsewhere [2, 11]). The instability of steady-state stratified flow is a necessary condition for roll wave generation and their further transition to slug flow. Roll waves generation in a horizontal two-layer flow driven by the pressure gradient has many common features with roll wave generation in open channel flow.

Laboratory experiments in inclined channels [5, 15] show that small perturbations of a steady-state flow lead to development of near periodic finite amplitude wave motion, which can be described by Dressler's theory as the roll waves moving downstream [6]. Nevertheless, Dressler's study cannot predict the parameters of developed roll waves, because the amplitude of roll waves is a free parameter in the theory. Therefore, the question on stability of nonlinear roll waves arises. The problem of stability of periodic roll waves in open channel flow is solved by deriving the modulation equations [4, 13]. The criterion of stability for roll waves of finite amplitude has been formulated by these authors in terms of the hyperbolicity of modulation equations.

2 Examples of flow

We show in this section that (1.1) can describe different types of flow in an inclined channel. Some of flows are governed by the shallow water equations (1.1) with a nonconvex pressure term, therefore in such flows the developed roll waves have a rather complicated internal structure.

2.1 Open channel flow

Consider the shallow water approximation of the free boundary flow in a cylindrical channel of arbitrary cross-section, inclined at an angle φ to the horizontal. The x axis is chosen along the bottom; S and b are the wetted area and the width of the channel at the free surface; t is the time; L_w is the wetted perimeter; u is the stream-wise velocity. The friction factor c_w is taken to be constant. The total pressure $P = \hat{P}(S)$ can be found from the relation

$$\hat{P}'(S) = g_n S/b(S) > 0.$$

The shallow water equations can be written in the form

$$\begin{aligned} S_t + (Su)_x &= 0, \\ (Su)_t + (Su^2 + \hat{P})_x &= S g_n \tan\varphi - c_w L_w u^2. \end{aligned} \tag{2.1}$$

One can derive (1.1) from (2.1) with $h = S/b_0$, $\alpha = g_n \tan\varphi$, $R = S/(c_w L_w)$, where b_0 is the reference width and R is the hydraulic radius. We consider a channel as the regular one if the dependence $P = \hat{P}(S) = P(h)$ is a convex function. A nonconvex function $P = P(h)$ arises in channels of complicated cross-section [7, 17].

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

2.2 Bernoulli effects in gas-liquid flow

Consider the two-dimensional flow of incompressible fluid in an inclined duct. The pressure over the free surface is varying due to co-flowing gas. If the velocity of the gas fraction is high enough, the gas flow can be considered as quasi-stationary. If the friction at the upper lid and the interface is neglected, the shallow water equations with dimensionless variables (the fluid density $\rho_f = 1$, the total depth of the channel $H = h^+ + h \equiv 1$, $g_n = 1$, $c_w = 1$) take the form [2]

$$\begin{gathered} h_t + (hu)_x = 0, \\ u_t + uu_x + h_x + p_x = \alpha - u^2/h, \\ h^+ w = Q_g \equiv \text{const}, \\ \frac{1}{2}\rho w^2 + p = J_g \equiv \text{const}. \end{gathered} \tag{2.2}$$

Here h, h^+ are the thicknesses and u, w are the velocities of the liquid and gas layer, respectively, p is the pressure in the gas fraction, ρ is the gas density ($\rho \ll 1$, $\rho \equiv \text{const}$), $\alpha = \tan\varphi$. Equations (2.2) can be reduced to (1.1) with

$$P(h) = \frac{1}{2}h^2 + \frac{\beta(1-2h)}{2(1-h)^2}, \quad \beta = \rho Q_g^2\,, \quad R(h) = h. \tag{2.3}$$

The model (2.2), (2.3) is hyperbolic for $0 < h < h_m = 1 - \beta^{1/3}$, where $c^2(h) = h(1 - \beta/(1-h)^3) > 0$. Note that the function $P = P(h)$ is nonconvex on the interval $(0, h_m)$.

3 Stability of steady-state flow

Consider a steady-state solution $h = h_0$, $u = u_0$ of (1.1). The uniform flow parameters must satisfy

$$u_0^2 = \alpha R(h_0). \tag{3.1}$$

Two approaches will be applied below to investigate the stability of the uniform flow, namely, the linear analysis and Whitham's approach.

3.1 Linear stability analysis

The linear analysis is used to find the growth rate of small disturbances (primed) superimposed to a uniform flow (subscript 0),

$$u = u_0 + u', \quad h = h_0 + h',$$

with u_0, h_0 satisfying (3.1). The linearized system takes the form

$$\begin{gathered} h'_t + u_0 h'_x + h_0 u'_x = 0, \\ u'_t + u_0 u'_x + c_0^2/h_0 h'_x = \frac{u_0^2}{R(h_0)}\left(\frac{R'_0}{R_0}h' - 2\frac{u'}{u_0}\right), \end{gathered} \tag{3.2}$$

where $R_0 = R(h_0)$, $R'_0 = R'(h_0)$, $c_0 = c(h_0)$. For the solution

$$h' = \hat{h}\exp(ik(x - c_p t))\,, \quad u' = \hat{u}\exp(ik(x - c_p t)),$$

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

(3.2) yields the following dispersion relation between the phase velocity c_p and the wave number k (k is real):

$$ikc_p^2 - 2u_0(ik + 1/R_0)c_p + ik\left(u_0^2 - c_0^2\right) + \frac{u_0^2}{R_0}\left(2 + \frac{R_0' h_0}{R_0}\right) = 0. \tag{3.3}$$

The flow is stable if $c_p = c_p^R + ic_p^J$ satisfies (3.3) with $c_p^J > 0$, $c_p^R \in R$ for $0 < k < \infty$. Standard calculations lead to the following stability condition:

$$\frac{R_0' h_0}{2}\sqrt{\frac{\alpha}{R_0}} < c_0. \tag{3.4}$$

For the flow along the inclined plane ($\alpha = g_n \tan\varphi$, $c^2(h) = g_n h$, $R(h) = h/c_w$), the inequality (3.4) is reduced to the well-known stability condition [10]

$$\mathrm{Fr}^2 = \frac{u_0^2}{g_n h_0} < 4.$$

3.2 Whitham's stability approach

The same criterion (3.4) of transition to unstable flow can be derived by Whitham's method [20]. According to this approach, the instabilities in uniform flow grow, when the kinematic wave velocity of the friction-dominated flow becomes greater than the dynamic velocity of long gravity waves.

Kinematic waves are obtained by neglecting the acceleration terms in the momentum equation in (1.1). This gives $u^2 \equiv \alpha R(h)$, and (1.1) is reduced to one equation:

$$h_t + \lambda_c h_x = 0, \quad \lambda_c = \sqrt{\alpha R}\left(1 + \frac{1}{2}\,\frac{hR'}{R}\right), \tag{3.5}$$

where λ_c is the velocity of kinematic waves.

On the other hand, the shallow water equations (1.1) have the characteristics propagating with dynamic velocities $\lambda_d^{\pm} = u \pm c$. For an equilibrium flow ($u = u_0$, $h = h_0$), satisfying (3.1), the dynamic velocities can be expressed as $\lambda_d^{\pm} = \sqrt{\alpha R} \pm c$. Therefore, the stability conditions

$$\lambda_d^- < \lambda_c < \lambda_d^+$$

give the criterion (3.4).

4 Roll waves in regular channels

We consider here the roll wave structure for (1.1) with a convex pressure. The mathematical theory of roll waves over an incline was developed by Dressler [6]. According to this theory, roll waves are periodic discontinuous solutions depending only on the variable $\xi = x - \mathscr{D}t$, $\mathscr{D} = const$. For regular channels ($P' > 0, P'' > 0$) roll waves travelling downstream ($0 < u < \mathscr{D}$) can be constructed analogously to Dressler's study. Equations (1.1)

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

for travelling waves take the form

$$h(\mathscr{D}-u)=m,$$
$$\tfrac{d}{d\xi}(h(\mathscr{D}-u)^2+P(h))=h(\alpha-u^2/R(h)). \tag{4.1}$$

With the following notations:

$$G(h)=m^2/h+P(h),\quad \Delta(h)=G'(h)=c^2(h)-m^2/h^2,$$
$$c^2(h)=P'(h),\quad F(h)=h(\alpha-R^{-1}(h)(\mathscr{D}-m/h)^2) \tag{4.2}$$

(4.1) are reduced to

$$\frac{dh}{d\xi}=\frac{F(h)}{\Delta(h)}. \tag{4.3}$$

The distinctive feature of roll waves is the existence of the critical depth y in the flow with $\Delta(y)=0$, providing a smooth transition from subcritical flow ($\Delta(h)>0$) downstream of a jump to supercritical flow ($\Delta(h)<0$) upstream of the next jump in a periodic solution. It is necessary for the smooth transition from subcritical flow into a supercritical one that $F(y)=0$. By virtue of (4.2), we have for $u<\mathscr{D}$

$$m=yc(y),\quad \mathscr{D}=c(y)+\sqrt{\alpha/R(y)}. \tag{4.4}$$

For given y, the periodic solution containing a jump can be constructed if the jump amplitude is known. Denoting by subscripts r and l the right and left sides of the discontinuity, one obtains from (1.1) the following jump relations:

$$h_r(\mathscr{D}-u_r)=h_l(\mathscr{D}-u_l)=m,$$
$$G(h_r)=G(h_l). \tag{4.5}$$

For the regular case ($P'(h)>0, P''(h)>0$), the jump is stable for $h_r<y<h_l$. Additional jump stability conditions for (1.1) with a nonconvex pressure term will be discussed below.

We consider the case $P'(h)>0$, $P''(h)>0$ as a regular one if $P(h)\to 0$ when $h\to 0$ and $P(h)\to\infty$ when $h\to\infty$. In a regular case, the roll wave structure is similar to the flow over an incline investigated by Dressler [6]. For $P''(h)>0$, the function $G(h)$ is convex too, and there exists a unique critical depth y with $\Delta(y)=0$. It is clear that the function $G(h)$ reaches its minimum at this point (Figure 1).

A periodic discontinuous solution of (4.1) can be constructed in the following way. Let ABEA be a periodic solution which describes the wave travelling with a constant velocity $\mathscr{D}$ ($0<u<\mathscr{D}$ – see Figures 1 and 2). For the convex pressure term, any jump AB should satisfy the stability condition

$$h_r<h_l \tag{4.6}$$

By virtue of (4.6) the flow is supercritical ($\Delta<0$) just upstream of the jump and subcritical ($\Delta>0$) downstream of the jump in a coordinate system moving with the wave velocity $\mathscr{D}$. Therefore, for smooth transition from the subcritical flow into the supercritical flow it is necessary that there exists a critical depth y with $\Delta(y)=0$ at the interval (h_r,h_l) (Figure 1). It follows from Figure 1 that $\Delta(h)<0$ for $h_r<h<y$ and $\Delta>0$ for $y<h<h_l$, so it is

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

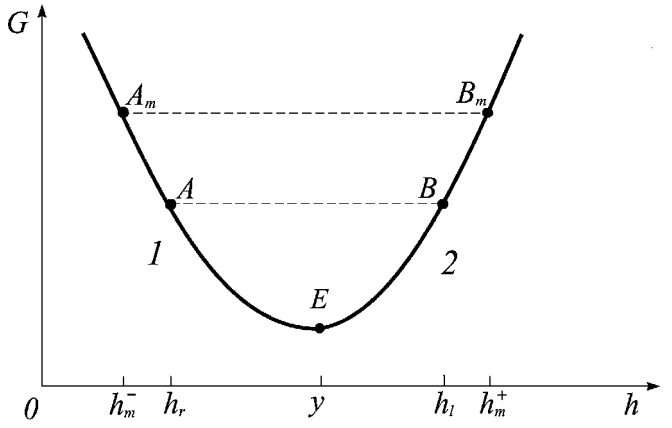


FIGURE 1. Dependence $G(h)$ in a regular case.

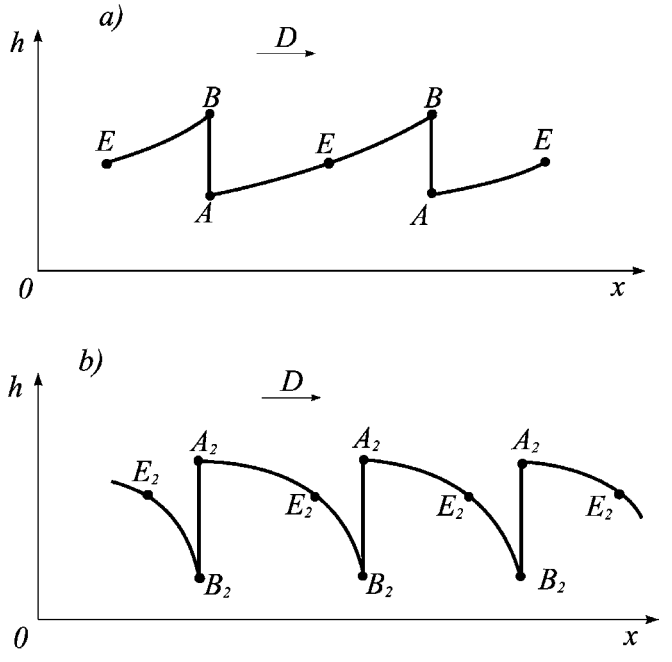


FIGURE 2. (a) Profile of a regular roll wave; (b) profile of an 'inverted' roll wave.

necessary and sufficient for the existence of a smooth solution of (4.3) at the interval (h_r, h_l) that

$$\begin{aligned} F(h) &< 0 \quad \text{for} \quad h_r < h < y, \\ F(h) &> 0 \quad \text{for} \quad y < h < h_l. \end{aligned} \tag{4.7}$$

Note that the linear instability condition

$$F'(y) = \alpha y R'(y)/R(y) - 2c(y)\sqrt{\alpha/R(y)} > 0, \tag{4.8}$$

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

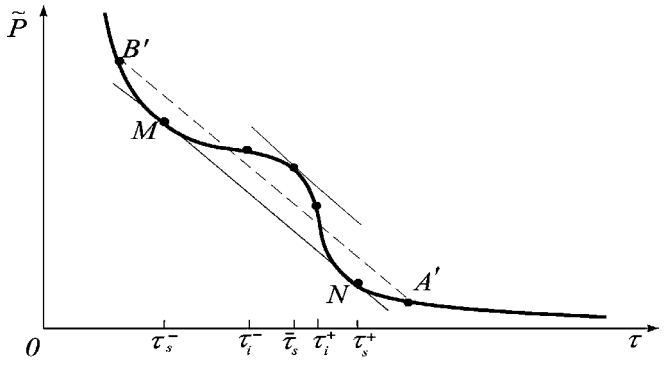


FIGURE 3. The function $\tilde{P}(\tau)$.

which was derived in previous paragraph, is a consequence of (4.7). The condition (4.8) provides the existence of roll waves of infinitesimal amplitude. If (4.7) is satisfied on the interval (h_r, h_l) with $G(h_r) = G(h_l)$ and $F(h_r) \to 0$ for $h \to h_m^-$ or $F(h_l) \to 0$ for $h \to h_m^+$, the length of the roll wave described by (4.3) goes to infinity, since $\Delta(h_m^{\pm}) \neq 0$. In this case, we have the roll wave of maximal amplitude (transition $A_m B_m E A_m$ in Figure 1). It is worth noting that for regular channels, the amplitude of the limiting roll wave is finite. For every critical depth y satisfying (4.8) there exists a value $h_m^- < y$ at which the function $F(h)$ vanishes and $F(h) < 0$ for $h_m^- < h < y$. Therefore, for given h_r $(h_m^- < h_r < y)$ there exists a roll wave with the minimal depth h_r and the maximal depth h_l $(G(h_r) = G(h_l))$ and the two-parameter family of roll waves is governed by the parameters y and h_r.

5 Roll waves with nonconvex pressure term

5.1 Gas dynamic analogy

To construct a piecewise smooth periodic solution of (4.1) with arbitrary pressure term, it is convenient to use the gas dynamic analogy for (1.1). Let $\tau = 1/h$, $\tilde{P}(\tau) = P(h)$, $\tilde{P}'(\tau) < 0$. If the function $\tilde{P}(\tau)$ is not convex, the roll wave patterns are more complicated. When considering the wave structure in non-regular case, attention should principally be paid to the jump stability conditions.

Consider the function $\tilde{P}(\tau)$, which is not convex on some interval (τ_*^-, τ_*^+), $0 < \tau_*^- < \tau_*^+ < \infty$. This means that, as in the previous section, $\tilde{P}'(\tau) < 0$, $(0 < \tau < \infty)$; $\tilde{P}(\tau) \to \infty$ when $\tau \to 0$, $\tilde{P}(\tau) \to 0$ when $\tau \to \infty$, and there exists only a finite number of points of inflection for the function $\tilde{P}(\tau)$ on the interval (τ_*^-, τ_*^+). Let us next consider the pressure term $\tilde{P}(\tau)$ having only two points of inflection τ_i^- and τ_i^+ $(\tau_i^- < \tau_i^+)$. It follows from asymptotics of $\tilde{P}(\tau)$ that $\tilde{P}''(\tau) < 0$ for (τ_i^-, τ_i^+) and $\tilde{P}''(\tau) > 0$ for $0 < \tau < \tau_i^-$ and $\tau_i^+ < \tau < \infty$ (see Figure 3). The behaviour of $\tilde{P}'(\tau)$ is shown in Figure 4. We choose $\tau_*^{\pm}$ from the relations $\tilde{P}'(\tau_*^-) = \tilde{P}'(\tau_i^+)$ and $\tilde{P}'(\tau_*^+) = \tilde{P}'(\tau_i^-)$ (see Figure 4). For the pressure term considered above, the roll wave structure depends on the position of the inverse critical depth $\tau_y = 1/y$ with respect to the characteristic points $\tau_i^{\pm}$, $\tau_*^{\pm}$, $\tau_s^{\pm}$,$\bar{\tau}_s$ of the function $\tilde{P}(\tau)$. The characteristic points $\tau_s^{\pm}$, $\bar{\tau}_s$ are chosen as follows. At the points $\tau_s^{\pm}$ the chord

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

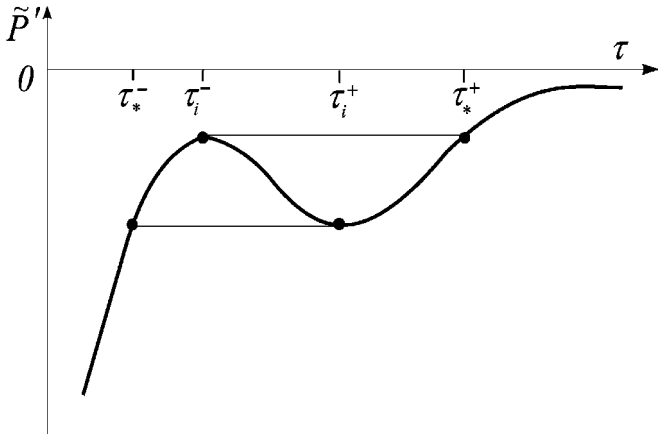


FIGURE 4. The function $\tilde{P}'(\tau)$.

MN is tangent to the curve $(\tau, \tilde{P}(\tau))$ and at the point $\bar{\tau}_s$ the tangent to this curve is parallel to the chord MN (see Figure 3).

For a given critical depth y $(0 < y < \infty)$ the travelling waves of (1.1) are described by (4.2)–(4.4). In particular, the velocity of a travelling wave $\mathscr{D} = \mathscr{D}(y)$ can be found from (4.4) and all functions in (4.2) such as $G(h)$, $\Delta(h)$, $F(h)$ depend upon y as a parameter. Note that the following relation:

$$\Delta(h) = G'(h) = -\tilde{P}'\left(\tfrac{1}{h}\right)/h^2 - m^2/h^2 = 0, \tag{5.1}$$

is fulfilled when and only when

$$\tilde{P}'(\tau) = -m^2 = \tilde{P}'(\tau_y), \quad \tau = 1/h, \quad \tau_y = 1/y. \tag{5.2}$$

It follows from (5.1) and (5.2) that for $\tau_y < \tau_*^-$ and $\tau_y > \tau_*^+$ we have $\Delta(h) \neq 0$ when $h \neq y$. Therefore, there exists only one extremum of the function $G(h)$. In this case the dependence $G(h)$ is analogous to the regular case shown in Figure 1, and the roll waves can be constructed in the way as in previous section. Now flow patterns can be found for $\tau_y \in (\tau_*^-, \tau_*^+)$. It is clear from Figure 4 that for $\tau_y \neq \tau_i^{\pm}$ we have two more points $\tau \in (\tau_*^-, \tau_*^+)$, with $\tilde{P}'(\tau) = \tilde{P}'(\tau_y)$, and due to (5.1) and (5.2), the function $G(h)$ has exactly three local extrema at the depths $h = y_i$, $i = 1, 2, 3$, which may be ordered $y_1 < y_2 < y_3$, and y coincides with one of these extrema. Note that $G(y_1) < G(y_3)$ when and only when

$$\left(\tilde{P}\left(\tau_{y_1}\right) - \tilde{P}\left(\tau_{y_3}\right)\right)/\left(\tau_{y_3} - \tau_{y_1}\right) > m^2 = -\tilde{P}'(\tau_y). \tag{5.3}$$

The possible configurations of the graphs of the function $G(h)$ with $1/\tau_*^+ < y < 1/\tau_*^-$ are shown in Figures 5(a) and (b). The classification of roll waves depending on the relative position of y in the interval $(1/\tau_*^+, 1/\tau_*^-)$ will be given below.

5.2 Admissible jumps

To construct a periodic roll wave containing discontinuities, we can use only stable jumps. Properties of finite amplitude discontinuities in channels of arbitrary cross section were

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

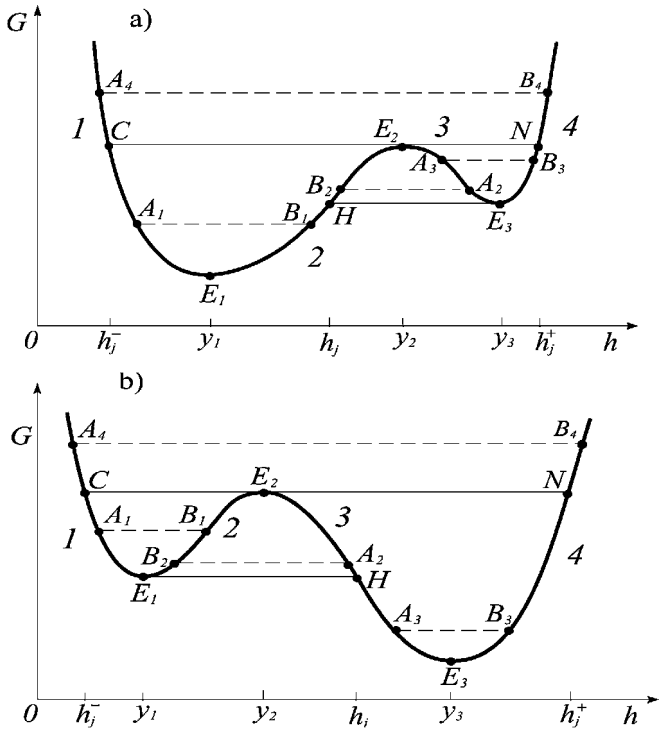


FIGURE 5. Dependence $G(h)$ in a nonregular case.

studied in Dyment [8]. Stability conditions of jumps or shocks in discontinuous solutions of (1.1) and for more general case of gas dynamics with a nonconvex pressure term were investigated by Galin [9], Sidorenko [18] and Wendroff [19]. For regular case the Lax stability conditions [12] express the well-known result that the discontinuity of a solution of (1.1) is stable when and only when the flow ahead of the shock (jump) is supersonic (supercritical) and behind the jump is subsonic (subcritical) in the coordinate system moving with the jump velocity. In gas dynamics this result is known as Zemplen's theorem. It means that for a convex function $\tilde{P}(\tau)$ $(\tilde{P}''(\tau) > 0)$ the curve 1 in Figure 1 with $\Delta < 0$ corresponds to any state before the jump and the curve 2 with $\Delta > 0$ corresponds to a state just behind the jump. Therefore, any transition AB, which is parallel to the abscissa, describes the stable jump, as considered in the previous paragraph. For a nonconvex pressure term with just two points of inflection, we have for $y \in (1/\tau_*^+, 1/\tau_*^-)$ four branches of the graph $G = G(h)$ shown in Figure 5: the branches 1, 3 with $\Delta < 0$ and the branches 2, 4 with $\Delta > 0$. The Lax stability condition lets us choose the branches 1, 3 for the states before a jump and the branches 2,4 for states behind it. Nevertheless, the Lax condition is only the necessary condition for jump stability. It does not provide the uniqueness of the Riemann problem for the homogeneous system (1.1). As for the problem considered here the jump $A'B'$, which connects the curves 1 and 4, satisfies the Lax conditions, but it is unstable as it is shown in Wendroff [19]. If we consider the states on the left and on the right sides of this jump as initial data for the Riemann problem

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

for the homogeneous system (1.1), we shall have as a stable solution of this problem the nonstationary multi-jump configuration consisting of the two admissible jumps with the simple centred wave between them. Moreover, the jump $A'B'$ should be eliminated from consideration, because it cannot be realized by the method of vanishing viscosity [19]. The stability condition for discontinuities, which provides the correctness of the Riemann problem, was introduced for an arbitrary nonlinear scalar equation by Oleinik [16], then the corresponding conditions were developed for gas dynamic equations with anomalous thermodynamics by Galin [9], Sidorenko [18] and Wendroff [19]. For the case considered in this paper, the stability condition of a jump connecting a state τ_0 before the jump ($\Delta<0$) and a state τ_1 behind the jump ($\Delta>0$) can be expressed as the following inequality:

$$\frac{\tilde{P}(\tau_1)-\tilde{P}(\tau_0)}{\tau_0-\tau_1}\geqslant\frac{\tilde{P}(\tau)-\tilde{P}(\tau_0)}{\tau_0-\tau} \tag{5.4}$$

for $(\tau_0-\tau)(\tau_1-\tau)<0$. It follows from (5.4) that the jump $A'B'$, which is shown in Figures 3 and 5 as broken lines, satisfies the Lax conditions, but it is inadmissible since the chord $A'B'$ intersects the graph of the function $\tilde{P}(\tau)$ four times. The jump admissibility criterion (5.4) can be easily formulated for the dependence $G(h)$ shown in Figure 5. Every segment AB, which is parallel to the abscissa and which connects points on the curves 1 or 3 with points on curves 2 or 4, represents the admissible jump if it does not intersect another branches. Therefore, curves 1 and curve 4 can be connected by an admissible jump, then and only then when the segment A_4B_4 lies over the point E_2 (Figures 5(a,b)). Below the segment CN only the jumps A_iB_i, $(i=1,2,3)$, which connects adjacent branches are admissible.

5.3 Roll wave patterns

We consider first regular roll waves, moving to the right, i.e. the periodic travelling waves of (1.1) with $0<u<\mathscr{D}$, containing just one admissible jump on the period. It is supposed that on the period there exists a critical depth y, at which the functions $\Delta(h)$ and $F(h)$ vanish. To construct roll waves of finite amplitude, it is convenient to use the dependence $G=G(h)$. The behaviour of the function $G(h)$ is shown in Figures 1 and 5 for different positions of the critical depth y. For $y<1/\tau_*^+$ and for $y>1/\tau_*^-$ there exists only one minimum of $G(h)$ at $h=y$ (Figure 1). For given conjugate depths h_l and h_r ($G(h_l)=G(h_r)$, $h_r<y<h_l$), the periodic solution $AEBA$ (Figure 1) can be constructed by the procedure described in § 4 for the regular case, if the conditions (4.7) are satisfied. The roll wave profile in this case is shown in Figure 2(a).

Consider now the case $1/\tau_*^+<y<1/\tau_*^-$ (see Figure 5). As was shown above, there are three points y_i, $i=1,\ 2,\ 3$ ($y_1<y_2<y_3$) of the local extremum of the function $G(h)$ with y_1 and y_3 being the points of minimum. The critical depth y coincides with one of the points y_i, $i=1,\ 2,\ 3$. The depth h_r before the jump in a periodic solution of (1.1) corresponds to a supercritical flow ($\Delta(h_r)<0$) and the flow in the conjugate state h_l ($G(h_l)=G(h_r)$) is subcritical ($\Delta(h_l)>0$). Therefore, for $y=y_1$ the roll wave structure is regular for any transition $A_1B_1E_1A_1$ if the conditions (4.7) are satisfied and $h_j^-<h_r<y_1<h_l<y_2$, where $G(h_j^-)=G(y_2)$ (see Figure 5). The roll waves, which are not regular, include more than one jump in the period. Let $G(y_1)<G(y_2)$ (Figure 5(a)) and $h_r<h_j^-$. The roll wave containing a

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

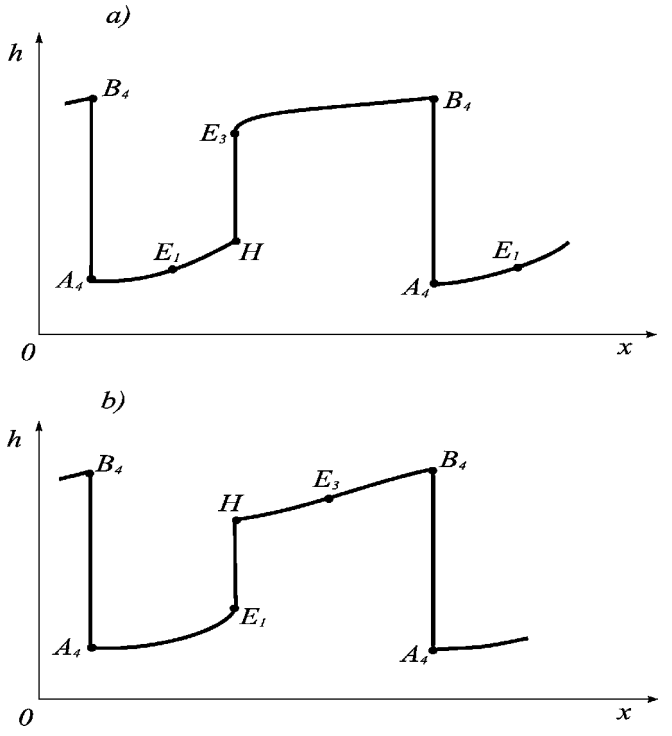


FIGURE 6. Profile of a multi-jump roll wave.

jump A_4B_4 with the conjugate depths h_r and h_l ($h_r < h_l$, $G(h_l) = G(h_r)$) can be constructed if the following modification of condition (4.7) is satisfied:

$$\begin{aligned} &F(h) > 0 \quad \text{for} \quad y_3 < h < h_l \quad \text{and} \quad y_1 < h < h_j, \\ &F(h) < 0 \quad \text{for} \quad h_r < h < y_1. \end{aligned} \tag{5.5}$$

Here h_j is chosen so that $G(h_j) = G(y_3)$ (Figure 5(a)). The wave structure is described by the transition $A_4B_4E_3HE_1A_4$. The wave consists of the stable jump A_4B_4, of the 'sonic' jump E_3H as well as of the supercritical part of flow E_1A_4 and the subcritical parts B_4E_3 and HE_1. The term 'sonic' jump is used by analogy with the term 'sonic' shock in gas dynamics, since the flow conditions just before the jump E_3H are critical ($\Delta(y_3) = 0$). The 'sonic' jump E_3H satisfies (5.4), so it can be considered as a stable jump too. The profile of the roll wave is shown in Figure 6(a). This configuration is possible up to limiting case of the jump A_mB_m corresponding to conjugate points h_m^-, h_m^+, $h_m^- < h_m^+$, $G(h_m^-) = G(h_m^+)$ with $F(h_m^-) = 0$ or $F(h_m^+) = 0$. In this case the length of the wave of limiting amplitude goes to infinity.

For $y = y_2$ a stable jump A_2B_2 connects the upstream state on the curve 3 with downstream state on curve 2. In this case the transition $A_2B_2E_2A_2$ describes the inverted roll wave. (Figures 2(b) and 5). It is necessary and sufficient for the roll wave existence that for the conjugate depths h_r, h_l ($h_r > h_l$, $G(h_r) = G(h_l)$), which correspond to the points

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

A_2 and B_2, the following conditions analogous to (4.7) are satisfied:

$$\begin{aligned} F(h) > 0 \quad &\text{for} \quad y_2 < h < h_r, \\ F(h) < 0 \quad &\text{for} \quad h_l < h < y_2. \end{aligned} \tag{5.6}$$

In contrast to the case $y = y_1$, the inverted wave of maximal amplitude is realized for the transition $E_3HE_2E_3$, in Figure 5(a) and for the transition HE_1E_2H in Figure 5(b), if the conditions (5.6) are satisfied for the conjugate depths h_j, y_3 and y_1, h_j, respectively. There are no inverted roll waves, which amplitude exceed these limiting waves.

For $y = y_3$ regular roll waves can be described by the transition $A_3B_3E_3A_3$ analogously to the case $y = y_1$ considered above, if $y_2 < h_r < y_3 < h_l < h_j^+$ and conditions (4.7) are fulfilled. Here the depth h_j^+ is chosen so that $G(y_2) = G(h_j^+)$ (Figures 5(a, b)). For the case $G(y_1) < G(y_3)$ shown in Figure 5(a), there is no roll wave exceeding in amplitude the limiting roll wave $E_2NE_3E_2$. Nevertheless, for the case $G(y_1) > G(y_3)$ shown in Figure 5(b) it is possible to construct the nonregular roll wave with the amplitude higher than the maximal regular wave $E_2NE_3E_2$ in the following way. Let h_r and h_l be conjugate depths with $h_r < h_j^- < h_j^+ < h_l$ and the following modification of the conditions (4.7) be satisfied:

$$\begin{aligned} F(h) > 0 \quad &\text{for} \quad y_3 < h < h_l, \\ F(h) < 0 \quad &\text{for} \quad h_r < h < y_1 \quad \text{and} \quad h_j < h < y_3. \end{aligned} \tag{5.7}$$

Here $G(h_j) = G(y_1)$ (Figure 5(b)). Then the transition $A_4B_4E_3HE_1A_4$ describes the roll wave containing two admissible jumps A_4B_4 and HE_1. The profile of the wave is shown in Figure 6(b).

Now all basic configurations of roll waves are considered. Note that the function $F(h)$ may vanish at different points of extremum for the dependence $G(h)$ simultaneously. In this case more complicated roll waves can be constructed if the necessary conditions of their existence such as (4.7), (5.5)–(5.7) are fulfilled. They will represent some combinations of roll waves considered above. Such solutions are not included in the roll wave classification, which is given in the next section.

5.4 Roll wave classification

Every roll wave considered above is uniquely defined by two parameters, y and h_r, and the roll wave patterns depend on the position of the point $\tau_y = 1/y$ relative to the characteristic points $\tau_*^\pm$, $\tau_s^\pm$, $\tau_i^\pm$, $\bar{\tau}_s$ ($\tau_*^- < \tau_s^- < \tau_i^- < \bar{\tau}_s < \tau_i^+ < \tau_s^+ < \tau_*^+$) shown in Figures 3–4 for the pressure term $\tilde{P}(\tau)$.

Consider the following cases:

(1) $y < 1/\tau_*^+$ (Figure 1). There are only regular roll waves (Figure 2a) if (4.7) is satisfied.

(2) $1/\tau_*^+ < y < 1/\tau_s^+$ (Figure 5(a)). Here $y = y_1$ and $G(y_1) < G(y_3)$ due to (5.3). Regular waves shown in Figure 2(a) exist if $h_j^- < h_r < y_1 < h_l < y_2$ and (4.7) is fulfilled. Roll waves with two stable jumps on the period are possible if $h_r < h_j^- < h_j^+ < h_l$ and (5.5) is satisfied (Figure 6(a)).

(3) $1/\tau_s^+ < y < 1/\tau_i^+$ (Figure 5(b)). We have $y = y_1$, $G(y_1) > G(y_3)$. Regular waves up to the limiting wave CE_2E_1C are the same as described above. In contrast to

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

the previous case, there are no roll waves with $h_r < h_j^-$. Nevertheless, as discussed in the next section, the roll wave of maximal amplitude can transform into an 'anomalous' wave. The transition CE_2HE_1C describes a periodic travelling wave containing stable jumps, but this wave is no more the roll wave, because the necessary condition $F(y_1)=0$ or $F(y_2)=0$ can be no longer satisfied.

(4) $1/\tau_i^+ < y < 1/\bar{\tau}_s$ (Figure 5(b)). Here $y = y_2$ and $G(y_1) > G(y_3)$. The inverted roll waves $A_2B_2E_2A_2$ exist if (5.6) is satisfied. The wave profile is shown in Figure 2(b). As in the previous case, the inverted roll wave of maximal amplitude HE_1E_2H can transform in the 'anomalous' wave HE_1CE_2H.

(5) $1/\bar{\tau}_s < y < 1/\tau_i^-$ (Figure 5(a)). In this case, $y = y_2$, $G(y_1) < G(y_3)$. The structure of inverted roll waves $A_2B_2E_2A_2$ is analogous to the previous consideration. There are no roll waves exceeding the limiting wave $E_3HE_2E_3$.

(6) $1/\bar{\tau}_i^- < y < 1/\tau_s^-$ (Figure 5(a)). Here $y = y_3$ and $G(y_1) < G(y_3)$. Roll waves $A_3B_3E_3A_3$ satisfying (4.7) are shown in Figure 2(a). The wave $E_2NE_3E_2$ is the roll wave of maximal amplitude, if (4.7) is satisfied for $h_r = y_2$ and $h_l = h_j^+$.

(7) $1/\tau_s^- < y < 1/\tau_*^-$ (Figure 5(b)). We have $y = y_3$, $G(y_1) > G(y_3)$. Roll waves $A_3B_3E_3A_3$ up to the limiting case $E_2NE_3E_2$ have the wave profile shown in Figure 2(a). For $h_r < h_j^- < h_j^+ < h_l$ the wave can be described by the transition $A_4B_4E_3HE_1A_4$, if the conditions (5.7) are satisfied. The profile of the wave is shown in Figure 6(b).

(8) $y > 1/\tau_*^-$ (Figure 1). There are only regular roll waves (Figure 2(a)) satisfying (4.7).

5.5 'Anomalous' waves

Roll waves of maximal amplitude considered for the cases 3 and 4 in previous section can be used for the construction of periodic travelling waves of (1.1). For $1/\tau_s^+ < y < 1/\bar{\tau}_s$ the transition CE_2HE_1C is possible, if

$$\begin{aligned} F(h) > 0 \quad &\text{for} \quad y_2 < h < h_j \\ F(h) < 0 \quad &\text{for} \quad h_j^- < h < y_1. \end{aligned} \tag{5.8}$$

The wave consists of the two smooth supercritical parts of flow (E_1C and E_2H) and of the two 'sonic' jumps CE_2 and HE_1. This type of wave belongs to the so called 'anomalous' waves which were investigated in Baines [1] and Liapidevskii [14] for two-layer flow over an obstacle. Note that the necessary condition for roll wave existence is $F(y_1)=0$ or $F(y_2)=0$. For the existence of the wave CE_2HE_1C, these conditions are not necessary, because there is no transition from supercritical into subcritical flow in such wave. This wave is anomalous since for 'sonic' jumps CE_2 and HE_1 the derivative $dh/d\xi$ is not bounded when $F(y_1) \neq 0$ and $F(y_2) \neq 0$. Nevertheless, such 'sonic' jumps (in gas dynamics they are known as 'sonic' shocks) can be easily realized experimentally and in numerical calculations [1, 14]. The profile of the wave CE_2HE_1C is shown in Figure 7. It differs considerably from the shape of roll waves. Therefore, we consider the transition to this type of flow as a new flow regime corresponding to so called slug (or plug) flow regime in 'liquid–liquid' flow [14]. Note that for slug flow the wave amplitude is a function of the critical depth y only, but the wave velocity $\mathscr{D}$ (or the critical velocity u_c) is now a free

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

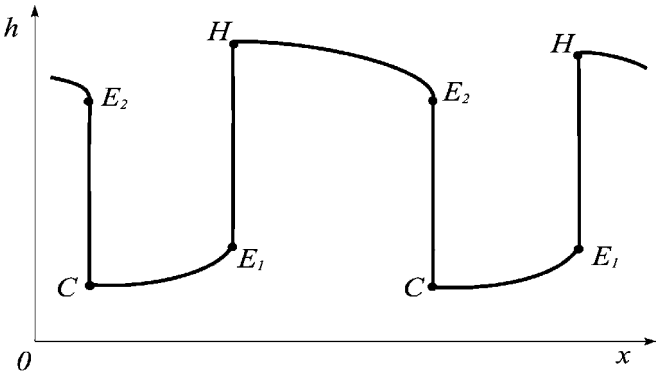


FIGURE 7. Profile of an 'anomalous' wave.

parameter. Therefore, a slug CE_2HE_1C can be described by two parameters, say y and $\mathscr{D}$, and we have again the two-parameter family of periodic travelling waves, corresponding to a slug flow regime. The investigation of this regime is beyond the scope of this paper.

6 Conclusion

Roll wave generation as the nonlinear stage of flow instability is a common feature of many nonstationary processes in different fields of continuum mechanics. The approach, developed in this paper to reveal the pattern of periodic discontinuous travelling waves in inclined channel flow, can be easily applied to more general types of flow. The system (1.1) represents a variant of nonlinear acoustic equations which can be used in various applications. The necessary and sufficient conditions for nonlinear roll wave existence (4.7), (5.5)–(5.8) are formulated for an arbitrary right-hand term in (1.1). It can be concluded from the considerations in section 5 that the roll wave structure depends mainly on the pressure function behaviour. Even for a nonconvex pressure containing only two points of inflection the multi-jump configuration of the roll wave is possible. Non-regular roll waves include so-called 'sonic' jumps. A new feature of roll waves of (1.1) is the transformation of regular wave for the cases 3, 4 considered in the section 5.4 into an 'anomalous' regime with two 'sonic' jumps on the period. Such periodic waves are not roll waves, because there is no transition from subcritical to supercritical flow in the wave. The profile of 'anomalous' waves differs essentially from the profile of roll waves, so equations (1.1) with nonconvex pressure term can describe different flow regimes analogously to the two-layer shallow water equations [14]. Nonlinear stability of the roll waves constructed above has been investigated in Boudlal *et al.* [4] and Liapidevskii [13] for the regular channels only. Stability of travelling periodic waves of finite amplitude in the non-regular case (roll waves and slugs) is the subject of our future investigations.

Acknowledgements

The research reported in this paper was supported in part by the Russian Foundation for Basic Research under Grant 01-01-00767 and by INTAS (project 01-0460).

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press

References

[1] Baines, P. G. (1984) A unified description of two-layer flow over topography. *J. Fluid Mech.* **146**, 127–167.

[2] Barnea, D. & Taitel, Y. (1994) Interfacial and structural stability. *Int. Multiphase Flow*, **20**, 387–414.

[3] Boudlal, A. & Dyment, A. (1996) Weakly nonlinear interfacial waves in a duct of arbitrary cross section. *Euro. J. Mech. B/Fluids*, **15**(3), 331–366.

[4] Boudlal, A. & Liapidevskii, V. Yu. (2002) Stability of roll waves in open channel flows. *C. R. Mécanique*, **303**, 291–295.

[5] Brock, R. R. (1969) Development of roll wave trains in open channels. *J. Hydr. Div.* **95**(4), 1401–1428.

[6] Dressler, R. F. (1949) Mathematical solution of the problem of roll-waves in inclined open channels. *Com. Pure. Appl. Math.*, **2**, 149–194.

[7] Dyment, A. (1981) Phenomènes exceptionnels de propagation d'ondes de pesanteur dans un canal. *J. de Mécanique*, **2**, 59–78.

[8] Dyment, A. (1998) Discontinuités d'amplitude finie dans les conduites de forme quelconque partiellement remplies de liquide. *C. R. Acad. Sci. Paris, SérieII b*, **386**, 179–184.

[9] Galin, G. Ja. (1958) Shock waves in media wit arbitrary equations of state. *Dokl. Akad. Nauk. SSSR*, **119**, 1106–1109. (English Translation in *Soviet Phys. Dokl.* **3**.)

[10] Jeffreys, H. (1925) The flow of water in an inclined channel of rectangular section. *Phil. Mag.* **49**(6), 793–807.

[11] Kordyban, E. (1990) Horizontal slug flow: a comparison of existing theories. *Trans. ASME. J. Fluids Eng.* **112**, 74–83.

[12] Lax, P. D. (1957) Hyperbolic systems of conservation laws II. *Com. Pure Appl. Math.* **XX**, 537–566.

[13] Liapidevskii, V. Yu. (1998) Stability of roll waves. *Dokl. Phys.* **43**(11), 703–705.

[14] Liapidevskii, V. Yu. (2000) The structure of roll waves in two-layer flows. *J. Appl. Math. Mech.* **64**, 937–943.

[15] Mayer, P. J. (1981) Roll waves and slug flows in inclined open channels. *Trans. ASCE*, **3158**, 505–535

[16] Oleinik, O. A. (1957) Discontinuous solutions of nonlinear differential equations. *Upskhi. Math. Nauk.* **12**, 3–73. (English translation in *Am. Math. Soc. Transl. Ser.* **2**(26), 95–172.)

[17] Peregrine, D. H. (1968) Long waves in a uniform channel of arbitrary cross section. *J. Fluid Mech.* **22**, 353–365.

[18] Sidorenko, A. D. (1968) Wave adiabats for media with an arbitrary equation of state. *Dokl. Akad. Nauk. SSSR*, **178**, 818–821 (English translation in *Soviet Phys. Dokl.* **13**.)

[19] Wendroff, B. (1972) The Riemann problem for materials with nonconvex equation of state. *J. Math. Anal. Appl.* **38**, 454–466.

[20] Whitham, G. B. (1974) *Linear and Nonlinear Waves*. Wiley.

https://doi.org/10.1017/S0956792503005369 Published online by Cambridge University Press