Econometric Theory, **27**, 2011, 907–911.
doi:10.1017/S0266466610000538

COMMENT ON "WEAK CONVERGENCE TO A MATRIX STOCHASTIC INTEGRAL WITH STABLE PROCESSES"

VYGANTAS PAULAUSKAS
Vilnius University

SVETLOZAR T. RACHEV
University of Karlsruhe and KIT
University of California, Santa Barbara and FinAnalytica

FRANK J. FABOZZI
Yale School of Management

In this comment we identify a lacuna in a proof in the paper by M. Caner published in 1997 in *Econometric Theory* concerning the weak limit behavior of various expressions involving heavy-tailed multivariate vectors and the convergence of stochastic integrals. In a later paper (Caner, 1998) the results for these limit relations are used to formulate tests for cointegration with infinite variance errors.

Caner (1997) proved several important results (see Thms. 3, 4, and 5) on the limit behavior of various expressions constructed by means of independent, identically distributed (i.i.d.) random vectors in the domain of attraction of a multivariate stable law with an index $0 < \alpha < 2$. In a later paper (Caner, 1998) important results on cointegration are based on these limit relations. There is, however, a lacuna in the proof of Theorem 2 in Caner (1997) and, as a consequence, the above-mentioned results based on this theorem cannot be considered as strictly proven.

The main difficulty is the convergence of stochastic integrals. This difficulty is described in Paulauskas and Rachev (1998) (PR hereafter). In that paper, Theorem 2.7 from Kurtz and Protter (1991) (KP hereafter), in which the so-called uniform tightness (UT) condition is the main condition, is used to prove the convergence of stochastic integrals. In Caner (1997), Theorem 2 is proven, yielding the same result as Theorem 2.7 in KP but without the UT condition. Roughly speaking, in Theorem 2 in Caner (1997) it is assumed that semi-martingale P_n is written as

The authors are grateful to the three anonymous referees and the editor for fruitful discussions that led to a substantial improvement of this comment. Address correspondence to Frank J. Fabozzi, Yale School of Management, 135 Prospect Street, Box 208200 New Haven, CT 06520-8200, USA; e-mail: frank.fabozzi@yale.edu.

© Cambridge University Press 2011

https://doi.org/10.1017/S0266466610000538 Published online by Cambridge University Press

a sum of two components $P_n^{(1)}$ and $P_n^{(2)}$, where $P_n^{(1)}$ represents a jump process and $P_n^{(2)}$ is a locally square-integrable martingale. That is, unlike in KP, the decomposition in Caner has no component of finite variation, A_n in the notation of KP. This can be seen from the beginning of the proof of Theorem 2, where Caner (1997, p. 518) states, "This is a special case of Theorem 2.7 in Kurtz and Protter (1991), where $A_n = 0$." Then he claims that the weak convergence of $P_n^{(2)}$ with a trivial stopping time $\tau_n^a = 2a$ implies the UT condition of Theorem 2.7 from KP, namely, that $\sup_n E[P_n^{(2)}]_{t\wedge 2a}$ is finite, where $[X]_t$ denotes the quadratic variation of X and $a > 0$ is an arbitrary number. But this is not true. Intuitively, it is clear that boundedness in L_2 does not follow from weak convergence, and it is not difficult to construct a counterexample. Here we provide the sketch of the construction of a counterexample.

Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a probability space with a Brownian motion defined on it, and let $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$ be a filtration generated by a Brownian motion. Let us take a sequence of $\mathcal{F}_1$-measurable random variables X_n and X such that the following properties hold:

Property 1. $X_n \xrightarrow{L_1} X$.

Property 2. $EX_n = EX = 0,\ EX_n^2 < \infty,\ EX^2 < \infty$.

Property 3. $\sup_n EX_n^2 = \infty$.

Here for simplicity we assume that $2a = 1$ (the stopping time mentioned above), otherwise we should take X_n and X measurable with respect to $\mathcal{F}_{2a}$. Let us take

$$P_n^{(2)}(t) = E(X_n|\mathcal{F}_t), \qquad P^{(2)}(t) = E(X|\mathcal{F}_t).$$

Due to Property 2, $P_n^{(2)}(t)$ and $P^{(2)}(t)$ will be square-integrable martingales. Moreover, due to the well-known representation of martingales generated by Brownian filtration (see, for example, Øksendal (2000, Thm. 4.3.4), these martingales are continuous. Now, applying Doob's inequality and Property 1, we have that for any $\varepsilon > 0$

$$\mathbb{P}\Big\{\sup_{0\leq t\leq 1} |P_n^{(2)}(t) - P^{(2)}(t)| > \varepsilon\Big\}$$

$$\leq \varepsilon^{-1} E|P_n^{(2)}(1) - P^{(2)}(1)| = \varepsilon^{-1} E|X_n - X| \to 0,$$

therefore $P_n^{(2)} \xrightarrow{d} P^{(2)}$. Recalling that all variables X_n are $\mathcal{F}_1$-measurable, we have

$$E[P_n^{(2)}(1)]^2 = EX_n^2.$$

Hence, from Property 3 we see that the UT condition with a stopping time, used by Caner, cannot be satisfied.

https://doi.org/10.1017/S0266466610000538 Published online by Cambridge University Press

Therefore, the application of Theorem 2.7 from KP is not possible, and Theorem 2 in Caner (1997), which is subsequently used in Theorems 3, 4, and 5 and then applied in Caner (1998), is not proven. Moreover, because KP, as well as other authors who have investigated the convergence of stochastic integrals, have shown that the UT condition in some sense is necessary for the convergence of stochastic integrals (see PR, where it is discussed[1]), one is bound to believe that Theorem 2 in Caner (1997) is false. We admit that we did not try to construct a counterexample to Theorem 2 since our experience of dealing with the UT condition is very limited. On the other hand, the results of Theorem 3 in Caner (1997) in which innovations are i.i.d. random variables, although based on Theorem 2 (which as we argued above is not proven and most probably false), are true. The reason is that PR verifies the UT condition even in a more general situation. In Caner (1997) strong assumptions are made of (1) symmetricity, (2) independence of coordinates of innovations, and (3) the same index of stability for all coordinates. All conditions of Theorem 2.7 in KP are verified without these assumptions in PR; therefore, coordinates of a limit stable vector are allowed to have different exponents, and normalization by a diagonal matrix is used, providing flexibility in the application of this important result. For the same reason it is possible to say that Theorem 4 in Caner (1997) (using Theorem 3 in its proof) is true. We did not analyze, however, the proof of Theorems 5 and 6, where innovations are taken as linear processes, that is, dependent random variables. The difficulties arising in this case are mentioned at the end of this comment.

The principal purpose of this comment, in addition to pointing out the gap in the proof of Theorem 2 in Caner (1997), is to warn researchers working in econometrics about the difficulties associated with the convergence problems for stochastic integrals and the possible misuse of the continuous mapping theorem. In many cases of finite variance and continuous limiting Brownian motion, the stochastic integrals under consideration were continuous functionals in an appropriate functional space, and convergence was obtained by the standard continuous mapping theorem. The situation changed when econometricians began dealing with heavy-tailed distributions and Lévy processes as limits. In probability theory during the 1970s, it was observed that stochastic integrals with integrator and integrand having jumps can be discontinuous functionals and that convergence of such stochastic integrals requires special attention. Powerful tools to cope with this problem were introduced in the works of T. Kurtz, P. Protter, A. Jakubowski, J. Mémin, and other researchers working in the area of stochastic analysis and stochastic differential equations.

It seems that the first correct application of these tools in econometrics was in PR, which was published in an applied probability journal. We looked at recent papers in econometrics that deal with similar problems and that also cite Caner (1997).[2] For example, Zarepour and Roknossadati (2008), using results from PR and from Mittnik, Paulauskas, and Rachev (2001), improve Theorem 3 in Caner (1997) in the above-mentioned direction: As in PR, dependence between coordinates and different exponents for coordinates are allowed, and matrix

https://doi.org/10.1017/S0266466610000538 Published online by Cambridge University Press

normalization is used. Callegari, Cappuccio, and Lubian (2003) and Cappuccio and Lubian (2007) also cite Caner (1997), although neither paper clearly states which theorems from Caner (1997) are used. A careful reading of these papers reveals that Theorem 3 from Caner (1997) is applied.

It is worth mentioning that in Ibragimov and Phillips (2008) general results on the convergence of semi-martingales are used within an econometric context; the existence of second- and higher-order moments is assumed, however. A recent doctoral dissertation by Ferstl (2009) should also be mentioned in which cointegration in discrete and continuous time is considered, and the results and proofs from PR are substantially used.

We conclude with one more difficulty encountered with heavy-tailed distributions when considering the convergence of sums $a_n^{-1}\sum_{j=1}^{[nt]} X_j$ formed by linear processes $X_j = \sum_{i=0}^{\infty} c_i \varepsilon_{j-i}$. It was demonstrated in Avram and Taqqu (1992), in the case of linear processes generated by heavy-tailed i.i.d. innovations, that the preceding sum generally does not converge in the usual Skorohod topology J_1 and, therefore, another topology M_1 should be used. (It is worth mentioning here that A.V. Skorohod (1956), in his famous paper on the space of cadlag functions, introduced four different topologies J_1, J_2, M_1, M_2 adapted for convergence of various types of stochastic processes, but the main topology used in the literature is J_1.) Taking into account that in KP the traditional Skorohod topology J_1 is used, one must deal with the case of linear processes very carefully (as mentioned in Phillips and Solo, 1992). It is worth noting that for statistical applications, instead of proving weak convergence in some functional space, it is sometimes sufficient to prove only convergence of finite-dimensional distributions.

NOTES

1. Prigent (1999) also emphasizes this condition in a different context.

2. We also reviewed the literature citing either Caner (1997) or Caner (1998). Several papers mention these two papers in their review of the literature but do nothing more. Two papers refer to Caner (1998) to support an empirical position. In Boswijk, Lucas, and Taylor (2000), a discussion of the empirical work refers to the proof of the critical values for the Gaussian cointegration test with infinite variance errors presented in Caner (1998) compared to the critical values found by other researchers. Falk and Wang (2003) use the results of Caner (1998) to perform statistical tests. Because these papers are empirical, we cannot comment on the reported results.

REFERENCES

Avram, F. & M.S. Taqqu (1992) Weak convergence of sums of moving averages in the α-stable domain of attraction. *Annals of Probability* 20, 483–503.

Boswijk, H.P., A. Lucas, & N. Taylor (2000) A comparison of parametric, semi-nonparametric, adaptive, and nonparametric cointegration tests. In T.B. Fomby & R.C. Hill (eds.), *Advances in Econometrics: Applying Kernel and Nonparametric Estimation to Economic Topics*, vol. 14, pp. 25–47. Cambridge University Press.

Callegari, F., N. Cappuccio, & D. Lubian (2003) Asymptotic inference in time series regressions with a unit root and infinite variance errors. *Journal of Statistical Planning and Inference* 116, 277–303.

https://doi.org/10.1017/S0266466610000538 Published online by Cambridge University Press

Caner, M. (1997) Weak convergence to a matrix stochastic integral with stable processes. *Econometric Theory* 13, 506–528.

Caner, M. (1998) Tests for cointegration with infinite variance errors. *Journal of Econometrics* 86, 155–175.

Cappuccio, N. & D. Lubian (2007) Asymptotic null distributions of stationarity and nonstationarity tests under local-to-finite variance errors. *Annals of the Institute of Statistical Mathematics* 59, 403–423.

Falk, B. & C.-H. Wang (2003) Testing long run PPP with infinite variance returns. *Journal of Applied Econometrics* 18, 471–484.

Ferstl, Ch. (2009) Cointegration in Discrete and Continuous Time. Ph.D. thesis. Technical University of Munich, http://www-m4.ma.tum.de/Diplarb/DA-Ferstl.pdf.

Ibragimov, R. & P.C.B. Phillips (2008) Regression asymptotics using martingale convergence methods. *Econometric Theory* 24, 888–947.

Kurtz, T.G. & P. Protter (1991) Weak limit theorems for stochastic integrals and stochastic differential equations. *Annals of Probability* 19, 1035–1070.

Mittnik, S., V. Paulauskas, & S.T. Rachev (2001) Statistical inference in regression with heavy tailed integrated variables. *Mathematical and Computer Modeling* 34, 1145–1158.

Øksendal, B. (2000) *Stochastic Differential Equations*, Springer.

Paulauskas, V. & S.T. Rachev (1998) Cointegrated processes with infinite variance innovations. *Annals of Applied Probability* 8, 775–792.

Phillips, P.C.B. & V. Solo (1992) Asymptotics for linear processes. *Annals of Statistics* 20, 971–1001.

Prigent, J.-L. (1999) Incomplete markets: Convergence of options values under the minimal martingale measure. *Advances in Applied Probability* 31, 1058–1077.

Skorohod, A.V. (1956) Limit theorems for stochastic processes. *Theory of Probability and Its Applications* 1, 261–290.

Zarepour, M. & S.M. Roknossadati (2008) Multivariate autoregression of order one with infinite variance innovations. *Econometric Theory* 24, 677–695.

https://doi.org/10.1017/S0266466610000538 Published online by Cambridge University Press