Bull. Aust. Math. Soc. (First published online 2026), page 1 of 12*
doi:10.1017/S0004972726101798
*Provisional—final page numbers to be inserted when paper edition is published

FELL BUNDLE LADDER

S. KALISZEWSKI, JOHN QUIGG and DANA P. WILLIAMS

(Received 29 April 2026 UTC; accepted 24 June 2026 UTC)

Abstract

We use the ladder technique to establish bijections between the ideals of related Fell bundles.

2020 *Mathematics subject classification*: primary 46L55.

Keywords and phrases: groupoid, Fell bundle, ideal, Rieffel correspondence, Morita equivalence, ladder technique.

1. Introduction

In studying the structure of C^*-crossed products associated to groups or groupoids acting on a C^*-algebra, invariant ideals of the algebra play a key role. Such ideals correspond naturally to ideals in the crossed product in such a way that the quotient of the crossed product by the ideal corresponds to a dynamical system for an action on the quotient of the C^*-algebra by the invariant ideal. Since it is profitable to think of the C^*-algebra of a Fell bundle over a groupoid as a powerful generalisation of C^*-crossed products, it is natural to look for an analogous result for Fell bundle C^*-algebras. In [4], Ionescu and the third author showed that one could identify suitably invariant ideals of the C^*-algebra obtained by restricting the Fell bundle to the unit space of the groupoid. Then, it was possible to obtain short exact sequences of C^*-algebras generalising the cases where there is a concrete action. In [5], we showed that these generalised invariant ideals corresponded naturally to a special type of Fell subbundles that we called *ideals* of the Fell bundle itself. Specifically, an invariant ideal I of the C^*-algebra given by $\mathscr{B}|_{G^{(0)}}$ corresponds to a Fell bundle ideal $\mathscr{B}_I$ such that there is a short exact sequence of Fell bundle C^*-algebras

$$0 \longrightarrow C^*(\mathscr{B}_I) \longrightarrow C^*(\mathscr{B}) \longrightarrow C^*(\mathscr{B}^I) \longrightarrow 0$$

for a suitable quotient Fell bundle $\mathscr{B}^I$ [4, Theorem 3.7]. This is a direct generalisation of the classical crossed product result where $\mathscr{B}$ is the Fell bundle associated to a

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dynamical system (A, G, α) and I is a G-invariant ideal in A. Then, $C^*(\mathscr{B}_I) \cong I \rtimes G$ and $C^*(\mathscr{B}^I) \cong A/I \rtimes G$.

The introduction of Fell bundle ideals made the generalisation coming from [4] much easier to work with since the notion of an ideal in a Fell bundle is a purely bundle-theoretic notion which does not involve C^*-algebras or completions. The main result in [5] allowed us to establish an appropriate version of the Rieffel correspondence between ideals of equivalent Fell bundles. At approximately the same time, in [1], together with Gillespie, we introduced what we call the 'ladder technique', and used it to establish a lattice isomorphism between the sets of invariant ideals of a C^*-algebra and the crossed product by an action or coaction of a group.

In this paper, we again employ the ladder technique, this time to prove analogues for Fell bundles of our lattice isomorphism from [1]. More precisely, we have two related theorems: in one, given an action of a groupoid G on a Fell bundle $\mathscr{A}$, we provide a lattice isomorphism between the G-invariant ideals of $\mathscr{A}$ and the ideals of the semidirect product $\mathscr{A} \propto G$, and in the other, we give a lattice isomorphism between the ideals of any Fell bundle $\mathscr{A}$ and the G-invariant ideals of the action product $\mathscr{A} \rtimes G$. (In [3], we introduced these semidirect-product Fell bundles using a different notation: $S(\mathscr{A}, G)$; we always had in mind that this notation was temporary until we decided upon a more appropriate one, which we are introducing here.)

In Section 2, we record our notation and review relevant definitions and results concerning groupoid equivalences, Fell bundle equivalences, ideals of Fell bundles and the ladder technique. Section 5 contains our main results, which use the *ladder technique* from [1], and in fact Theorems 5.1 and 5.2 have roughly the same goal: bijections between sets of ideals, this time for Fell bundles. In Section 4, we build the vertical struts for the ladders.

2. Preliminaries

Our groupoids will all be locally compact Hausdorff with open range and source maps. We refer to [6, Section 2.3] for groupoid equivalences. Recall here that if G and H are groupoids, then a space Z is a (G, H)-equivalence if there are free and proper commuting actions of G and H on Z (G on the left and H on the right) such that the moment map ρ for the G-action factors through a homeomorphism of Z/H with $G^{(0)}$, and the moment map σ for the H-action factors through a homeomorphism of $G\backslash Z$ with $H^{(0)}$. We also require the moment maps to be open, which is actually automatic as we require the range and source maps on groupoids to be open. If the space Z is a (G, H)-equivalence, then there are continuous and open pairings ${}_G[\cdot,\cdot] : Z *_\sigma Z \to G$ and $[\cdot,\cdot]_H : Z *_\rho Z \to H$ which are given by $x = {}_G[x, y] \cdot y$ for all $(x, y) \in Z *_\sigma Z$, and $x \cdot [x, y]_H = y$ for all $(x, y) \in Z *_\rho Z$.

We refer to [5, Section 2] for the required background on Banach bundles, Fell bundles and Fell bundle equivalence. In particular, our Fell bundles will all be saturated (so that each fibre is an imprimitivity bimodule). If $p : \mathscr{B} \to G$ is a Fell bundle, $q : \mathscr{E} \to T$ is a Banach bundle and T is a left G-space, then $\mathscr{B}$ acts on the left of $\mathscr{E}$ if there is a continuous pairing,

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$$(b,e)\mapsto b\cdot e \text{ from } \mathscr{B}*\mathscr{E}=\{(b,e)\in\mathscr{B}\times\mathscr{E}: s(p(b))=\rho(q(e))\} \text{ to } \mathscr{E},$$

that covers the pairing $(\gamma,x)\mapsto\gamma\cdot x$ from $G*T$ to T, that is, $q(b\cdot e)=\ p(b)\cdot q(e)$ is associative in the sense that $b\cdot(c\cdot e)=(bc)\cdot e$ and is submultiplicative for the norms in the sense that $\|b\cdot e\|\le\|b\|\,\|e\|$. Right actions are defined similarly. If $\mathscr{B}\to G$ and $\mathscr{C}\to H$ are Fell bundles, then a Banach bundle $\mathscr{E}\to T$ over a (G,H)-equivalence T is a $\mathscr{B}$–$\mathscr{C}$ equivalence if there are continuous commuting actions of $\mathscr{B}$ and $\mathscr{C}$ on $\mathscr{E}$ ($\mathscr{B}$ on the left and $\mathscr{C}$ on the right) such that the bimodule actions are associative in the sense that $(b\cdot e)\cdot c=b\cdot(e\cdot c)$, there is a continuous fibre-wise sesquilinear pairing ${}_L\langle\cdot\,,\cdot\rangle:\mathscr{E}*_\sigma\mathscr{E}\to\mathscr{B}$ that covers the pairing ${}_G[\cdot,\cdot]$ in the sense that $p_{\mathscr{B}}({}_L\langle e\,,f\rangle)={}_G[q(e),q(f)]$ commutes with the pairing $\mathscr{B}*\mathscr{E}\to\mathscr{B}$ in the sense that ${}_L\langle b\cdot e\,,f\rangle=b\,{}_L\langle e\,,f\rangle$, and similarly for the right action, and each E_x is a $B_{\rho(x)}$–$C_{\sigma(x)}$ imprimitivity bimodule.

We now summarise from [2, 3] the definitions of semidirect product Fell bundles and action Fell bundles, although we change the notation for semidirect products; our original notation was always meant to be temporary, and we have now settled on what seems to us to be an acceptable choice. If G and H are locally compact groupoids such that G acts on H by isomorphisms, then the *semidirect product groupoid* [2, Definition 5.1] is the set

$$H\propto G=\{(h,x)\in H\times G:\rho(h)=r(x)\},$$

with source and range

$$s(h,x)=(x^{-1}\cdot s_H(h),s_G(x))\quad\text{and}\quad r(h,x)=(r_H(h),r_G(x)),$$

and operations

$$(h,x)(k,y)=(h(x\cdot k),xy)\quad\text{and}\quad(h,x)^{-1}=(x^{-1}\cdot h^{-1},x^{-1}).$$

It is shown in [2, Lemma 5.4] that if H has an invariant Haar system $\lambda_H=\{\lambda_H^v\}_{v\in H^{(0)}}$ in the sense that

$$\int_H f(x\cdot t)\,d\lambda_H^v(t)=\int_H f(t)\,d\lambda_H^{x\cdot v}(t),$$

then we get a Haar system on the semidirect product $H\propto G$ by

$$d\lambda^{(v,w)}(t,x)=d\lambda_H^v(t)\,d\lambda_H^w(x).$$

If $p:\mathscr{A}\to H$ is a Fell bundle and G acts on $\mathscr{A}$ by isomorphisms, we get a Banach bundle over $H\propto G$ via the pull back $\pi_1^*\mathscr{A}=\{(a,x)\in A\times(H\propto G):\rho(p(a))=r(x)\}$. We define Fell bundle operations by

$$(a,x)(b,y)=(a(x\cdot b),xy)\quad\text{and}\quad(a,x)^*=(x^{-1}\cdot a^*,x^{-1}),$$

getting the *semidirect product* Fell bundle $\mathscr{A}\propto G$ with bundle projection $p'(a,x)=(p(x),x)$ onto $H\propto G$. We get an action α of G on $\mathscr{A}$ via

$$\alpha_x(f)(h)=x\cdot f(x^{-1}\cdot h)\quad\text{for } f\in\Gamma_c(H_{s(x)},\mathscr{A}),$$

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and, provided H has an invariant Haar system, α implements an action of G on the C^*-algebra of $\mathscr{A}$. Moreover, the C^*-algebra of the semidirect product Fell bundle is isomorphic to the crossed product of the C^*-algebra of $\mathscr{A}$ by the action α [2, Theorem 6.3].

Now, we turn to action Fell bundles, defined following [3, Remark 4.12]. Let $\overline{p} : \mathscr{B} \to M$ be a Fell bundle, assume that M acts on a space T and let $M \ltimes T$ be the action groupoid with unit space identified with T, so that

$$r(m,t) = m \cdot t, \quad s(y,t) = t \quad \text{and} \quad (m, k \cdot t)(k,t) = (mk,t).$$

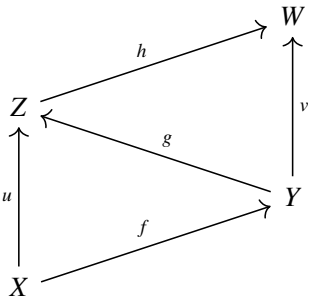
The *action Fell bundle* is

$$\mathscr{B} \ltimes T = \{(b,t) : s(\overline{p}(b)) = \rho(t)\},$$

with source and range $s(b,t) = t$, $r(b,t) = \overline{p}(b) \cdot t$, and operations

$$(b, \overline{p}(b') \cdot t)(b',t) = (bb',t) \quad \text{and} \quad (b,t)^* = (b^*, \overline{p}(b) \cdot t).$$

The crux of the *ladder technique* mentioned in the introduction is a straightforward observation about maps between sets: if



is a commutative diagram of sets and maps such that u and v are bijections, then f, g and h are also bijections. We refer to such a diagram as a *ladder*, with *vertical struts* u and v, and *rungs* f, g and h. We refer to the preceding observation as the *Ladder Lemma*.

3. Fell bundle ideals

The notion of a Fell bundle ideal and the Rieffel correspondence for equivalent Fell bundles is central here. For details, we refer to [5, Sections 2.7 and 3]. We recall some of the basics for convenience. A Fell subbundle $\mathscr{I}$ of a Fell bundle $\mathscr{B} \to G$ is an *ideal* of $\mathscr{B}$ if $bc \in \mathscr{I}$ whenever $(b,c) \in \mathscr{B}^{(2)}$ and either $b \in \mathscr{I}$ or $c \in \mathscr{I}$. In particular, an ideal of a Fell bundle is itself a Fell bundle and is therefore closed under the adjoint operation.

Suppose that $q : \mathscr{E} \to T$ is a $\mathscr{B}$–$\mathscr{C}$ Fell bundle equivalence. The *Rieffel correspondence* for Fell bundle equivalences [5, Theorem 3.10] says that there are lattice isomorphisms among the ideals of $\mathscr{B}$, the full Banach $\mathscr{B}$–$\mathscr{C}$ submodules of $\mathscr{E}$ and

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the ideals of $\mathscr{C}$. Specifically, if $\mathscr{I}$ is an ideal in $\mathscr{B}$, then the corresponding full Banach $\mathscr{B}-\mathscr{C}$ submodule, $\mathscr{M}_{\mathscr{L}}$, is

$$\mathscr{I}\cdot\mathscr{E} := \bigcup_{s(h)=\rho(t)} J_h\cdot E_t.$$

Describing the ideal $\mathscr{I}_{\mathscr{M}}$ in $\mathscr{B}$ corresponding to a full Banach $\mathscr{B}-\mathscr{C}$ submodule, $\mathscr{M}$, takes a bit more fussing. Extending the usual conventions for imprimitivity bimodules, we let ${}_L\langle\mathscr{M}\ ,\ \mathscr{E}\rangle$ denote the span of all possible left-inner products ${}_L\langle m\ ,\ e\rangle$ with $m\in\mathscr{M}$ and $e\in\mathscr{E}$ as in [5, (3.2)]. Then,

$$\mathscr{I}_{\mathscr{M}} = \coprod_{h\in H} \overline{{}_L\langle\mathscr{M}\ ,\ \mathscr{E}\rangle\cap B_h}, \tag{3.1}$$

where the closure is taken in the norm topology on B_h.

We can describe (3.1) a bit more elegantly using a simple observation (which probably should have been noted in [5]).

LEMMA 3.1. *Suppose that $p:\mathscr{B}\to G$ is a Fell bundle and that $\mathscr{S}$ is any subset of $\mathscr{B}$. Then, there is a smallest ideal, $\mathrm{Id}_{\mathscr{B}}(\mathscr{S})$, of $\mathscr{B}$ such that $\mathscr{S}\subset\mathrm{Id}_{\mathscr{B}}(\mathscr{S})$. We call $\mathrm{Id}_{\mathscr{B}}(\mathscr{S})$ the ideal of $\mathscr{B}$ generated by $\mathscr{S}$.*

PROOF. It will suffice to see that the arbitrary intersections of ideals is an ideal. To this end, let $\mathscr{I}^a$ be an ideal in $\mathscr{B}$ for each $a\in A$. By [5, Proposition 2.30], $I^a=\Gamma_0(G^{(0)};\mathscr{I}_a)$ is a G-invariant ideal in $\Gamma_0(G^{(0)};\mathscr{B})$, and $\mathscr{I}^a=\{b\in\mathscr{B}: b^*b\in I^a_{s(b)}\}$. Then, $I=\bigcap I^a$ is a G-invariant ideal in $\Gamma_0(G^{(0)};\mathscr{B})$, and $\mathscr{I}=\{b\in\mathscr{B}: b^*b\in I_{s(b)}\}$ is an ideal in $\mathscr{B}$ by [5, Proposition 2.30]. Now, it is clear that $\mathscr{I}=\bigcap\mathscr{I}^a$. □

It is not hard to see that

$$\mathscr{I}_{\mathscr{M}} = \mathrm{Id}_{\mathscr{B}}({}_L\langle\mathscr{B}\ ,\ \mathscr{M}\rangle).$$

REMARK 3.2. Of course, it is possible in the proof of Lemma 3.1 that $I=\bigcap I^a$ is the zero ideal. However, then, $\mathscr{L}=\{0_x: x\in G\}$.

4. The vertical struts

Let H be a wide closed subgroupoid of M. (Here, 'wide' simply means that $H^{(0)}=M^{(0)}$.) Since our standing assumptions are that both H and M have open range (and source) maps, it follows from [6, Corollary 2.50] that M is an $(M\rtimes M/H, H)$ equivalence. Specifically, the moment map for the left-action is given by $\rho(m)=mH$, where we have identified the unit space of $M\rtimes M/H$ with the orbit space M/H. Then, $(m,nH)\cdot n=mn$. The moment map for the right action is $\sigma(m)=s(m)$ and then $m\cdot h=mh$. Note that if $\rho(m)=\rho(n)$, then $nH=mH$ and there is a unique $h\in H$ such that $mh=n$ and $[m,n]_H=h$. However, if $\sigma(m)=\sigma(n)$, then $(mn^{-1},nH)\cdot n=m$ and ${}_{M\rtimes M/H}[m,n]=(mn^{-1},nH)$.

THEOREM 4.1. *Let $p:\mathscr{B}\to M$ be a Fell bundle and let H be a closed subgroupoid of M that is wide in the sense that $H^{(0)}=M^{(0)}$. Since M acts on M/H, we can form*

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the action Fell bundle $\mathscr{B} \rtimes M/H$ as defined following [3, Remark 4.12]. *Then, $\mathscr{B}$ is a $\mathscr{B} \rtimes M/H - \mathscr{B}|_H$ equivalence, with operations:*

(a) *left action,* $(b, mH) \cdot c = bc$;
(b) *right action,* $b \cdot c = bc$;
(c) *left inner product,* ${}_L\langle b\ ,\ c\rangle = (bc^*, p(c)H)$;
(d) *right inner product,* $\langle b\ ,\ c\rangle_R = b^*c$.

PROOF. The left-action defined in (a) clearly makes sense on $(\mathscr{B} \rtimes M/H) * \mathscr{B}$ and covers the $M \rtimes M/H$ action on M. Also, on the one hand,

$$((b, mH)(c, nH)) \cdot d = (bc, nH) \cdot d = bcd,$$

while on the other hand,

$$(b, mH) \cdot ((c, nH) \cdot d) = (b, mH) \cdot cd = bcd.$$

Therefore, the pairing is associative. Since

$$\|(b, mH) \cdot c\| = \|bc\| \le \|b\|\,\|c\| = \|(b, mH)\|\,\|c\|,$$

the pairing is submultiplicative. Therefore, we have an action of the semidirect product Fell bundle $\mathscr{B} \rtimes M/H$ on the Banach bundle $\mathscr{B}$ as required.

For the associativity of the bimodule actions,

$$((b, mH) \cdot c) \cdot d = (bc) \cdot d = bcd = (b, mH) \cdot (c \cdot d).$$

The left pairing $\mathscr{B} *_\sigma \mathscr{B} \to \mathscr{B} \rtimes M/H$ covers the pairing $M *_\sigma M \to M \rtimes M/H$ because

$$p({}_L\langle b\ ,\ c\rangle) = p(bc^*, p(c)H) = (p(b)p(c)^{-1}, p(c)H) = {}_{M\rtimes M/H}[p(b), p(c)].$$

For the adjoint property of the left pairing,

$$\begin{aligned} {}_L\langle b\ ,\ c\rangle^* = (bc^*, p(c)H)^* &= (cb^*, p(bc^*)p(c)H) \\ &= (cb^*, p(b)p(c)^{-1}p(c)H) = (cb^*, p(b)H) = {}_L\langle c\ ,\ b\rangle. \end{aligned}$$

Next,

$$\begin{aligned} {}_L\langle (b\ ,\ mH) \cdot c, d\rangle = {}_L\langle bc\ ,\ d\rangle &= (bcd^*, p(d)H) \\ &= (b, mH) \cdot (cd^*, p(d)H) = (b, mH) \cdot {}_L\langle c\ ,\ d\rangle, \end{aligned}$$

where the operations are valid because

$$mH = p(c)H = p(cd^*)p(d)H.$$

The verifications of the remaining axioms involving the right pairing are similar and easier.

Finally, for $m \in M$, we verify that B_m is a $(\mathscr{B} \rtimes M/H)_{\rho(m)} - (\mathscr{B}|_H)_{\sigma(m)}$ imprimitivity bimodule. Unravelling the definitions, we must show that B_m is a $(B_{r(m)} \times \{mH\}) - B_{s(m)}$ imprimitivity bimodule. Since the left-module action of $B_{r(m)} \times \{mH\}$ on B_m is

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essentially just the left action of $B_{r(m)}$, the axiom follows immediately from the fact that B_m is a $B_{r(m)}-B_{s(m)}$ imprimitivity bimodule. □

Recall that if G acts on H by isomorphisms, then we can form the semidirect product $H \propto G$ as in [2, Definition 5.1]. Then, if $p : \mathscr{A} \to H$ is a Fell bundle and G acts on $\mathscr{A}$ by isomorphisms, we can form the semidirect product Fell bundle $\mathscr{A} \propto G$ as defined prior to [2, Lemma 5.5].

COROLLARY 4.2. *Let G act by isomorphisms on a Fell bundle $p\colon \mathscr{A} \to H$. Then, $\mathscr{A} \propto G$ is an $\mathscr{A} \propto G \rtimes G - \mathscr{A}$-equivalence, with operations:*

(a) *left action,* $(a,\gamma,\eta)\cdot(b,\eta) = (a(\gamma\cdot b),\gamma\eta)$;
(b) *right action,* $(b,\eta)\cdot c = (b(\eta\cdot c),\eta)$;
(c) *left inner product,* ${}_{L}\langle (a,\gamma)\ ,\ (b,\eta)\rangle = (a(\gamma\eta^{-1}\cdot b^*),\gamma\eta^{-1},\eta)$;
(d) *right inner product,* $\langle (a,\gamma)\ ,\ (b,\gamma)\rangle_{R} = \gamma^{-1}\cdot(a^*b)$.

PROOF. We appeal to Theorem 4.1, with $M = H \propto G$ and $\mathscr{B} = \mathscr{A} \propto G$. Then, we can identify H with the wide subgroupoid $H * G^{(0)} = \{(h,u) \in H \times G^{(0)} : \rho(h) = u\}$ of the semidirect product $H \propto G$. Then, H acts on the right of $M = H \propto G$ and $[h,x] \mapsto x$ is a homeomorphism of $(H \propto G)/H$ onto G, where as usual, we have written $[h,x]$ for the H-orbit of $(h,x) \in H \propto G$. Furthermore, this map intertwines the natural $H \propto G$-action on the quotient $(H \propto G)/H$ with the composition of the natural map on $H \propto G$ onto G with left translation. Therefore, the horizontal maps induced by $[h,x] \mapsto x$ induce a diagram

$$\begin{array}{ccc} (\mathscr{A} \propto G) \rtimes (H \propto G)/H & \longrightarrow & (\mathscr{A} \propto G) \rtimes G \\ \downarrow & & \downarrow \\ (H \propto G) \rtimes (H \propto G)/H & \longrightarrow & (H \propto G) \rtimes G \end{array}$$

which gives a Fell bundle isomorphism of $\mathscr{B} \rtimes M/H = (\mathscr{A} \propto G) \rtimes (H \propto G)/H$ with $(\mathscr{A} \propto G) \rtimes G$.

Since $(\mathscr{A} \propto G)|_H$ is isomorphic (as a Fell bundle) to $\mathscr{A}$, we can realise $\mathscr{A} \propto G$ as an $(\mathscr{A} \propto G) \rtimes G - \mathscr{A}$-equivalence. Then, the formulae for the actions and inner products follow from routine computations. □

For convenience and easy reference, we record some results from [3]. Recall from [3, Proposition 6.3] that if G acts freely and properly on H by isomorphisms, then H becomes an $H \propto G - G\backslash H$ equivalence, where the left-action of $H \propto G$ on H is given by $(h,x)\cdot k = h(x\cdot k)$ and the right action of $G\backslash H$ on H is given by $h\cdot(G\cdot k) = hk$ provided $s(h) = r(k)$. Then, we have the following result.

THEOREM 4.3 [3, Theorem 7.2]. *Let $\mathscr{B} \to H$ be a Fell bundle and let G act on $\mathscr{B}$ principally by isomorphisms. Then, $\mathscr{B}$ is a $\mathscr{B} \propto G - G\backslash\mathscr{B}$ equivalence, with operations:*

(a) *left action,* $(b,\gamma)\cdot c = b(\gamma\cdot c)$;
(b) *right action,* $b\cdot(G\cdot c) = bc$;

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(c) *left inner product,* ${}_L\langle b\ ,\ c\rangle = (b(x\cdot c^*), x)$ *where* $x = {}_G[s(b), s(c)]$;
(d) *right inner product,* $\langle b\ ,\ c\rangle_R = G\cdot b^*c$.

Concretely, we can suppose that K and G act on (the left) of a space T such that the actions commute and the moment map $\rho : T \to K^{(0)}$ is a free and proper G-bundle. Then, G acts freely and properly by isomorphisms on the action groupoid $K \rtimes T$ via $x\cdot(k,t) = (k, x\cdot t)$. We can identify $G\backslash(H\rtimes T)$ with K and $K\rtimes T$ is a $(K\rtimes T)\propto G - K$ equivalence. This gives the following result.

COROLLARY 4.4 [3, Theorem 7.1]. *Let K, G and T be as above. Let $p : \mathscr{A} \to K$ be a Fell bundle. Then, $\mathscr{A}\rtimes T$ is an $\mathscr{A}\rtimes T\propto G - \mathscr{A}$ equivalence, with operations:*

(a) *left action,* $(a,t,x)\cdot(b,t') = (ab, x\cdot t')$;
(b) *right action,* $(a,t)\cdot b = (ab, p(b)^{-1}\cdot t)$;
(c) *left inner product,* ${}_L\langle(a,t)\ ,\ (b,t')\rangle = (ab^*, p(b)\cdot t, x)$ *where x is the unique element of G such that $t = x\cdot t'$;*
(d) *right inner product,* $\langle(a,t)\ ,\ (b,t')\rangle_R = a^*b$.

Note that we have commuting left actions of G on itself—the first just left translation and the second given by $x\cdot y = yx^{-1}$. Furthermore, the moment map $r : G\to G^{(0)}$ for left translation is a free and proper G-bundle for the action by right translation. Thus, if $p : \mathscr{A}\to G$ is a Fell bundle, then the corresponding G-action on the action bundle $\mathscr{A}\rtimes G$ is given by $x\cdot(b,y) = (b, yx^{-1})$. Consequently, we have the following result.

COROLLARY 4.5 [3, Theorem 8.1]. *Let $p : \mathscr{A}\to G$ be a Fell bundle. Then, $\mathscr{A}\rtimes G$ is an $\mathscr{A}\rtimes G\propto G - \mathscr{A}$-equivalence, with operations:*

(a) *left action,* $(a,x,y)\cdot(b,z) = (ab, zy^{-1})$;
(b) *right action,* $(a,z)\cdot b = (ab, p(b)^{-1}z)$;
(c) *left inner product,* ${}_L\langle(a,z)\ ,\ (b,y)\rangle = (ab^*, p(b)z, z^{-1}y)$;
(d) *right inner product,* $\langle(a\ ,\ \zeta),(b,\gamma)\rangle_R = a^*b$.

Now that we have equivalent Fell bundles, by [5, Theorem 3.10], we have an associated *Rieffel correspondence*, giving a lattice isomorphism between the sets of Fell-bundle ideals. In the case in which a groupoid G acts consistently on the Fell bundles, the Rieffel correspondence restricts to the invariant ideals. More precisely, we have the following result.

THEOREM 4.6. *In Corollary 4.2, if $\mathscr{I}$ is a G-invariant ideal of $\mathscr{A}$, then the ideal of $\mathscr{A}\propto G\rtimes G$ associated to $\mathscr{I}$ via the Rieffel correspondence is $\mathscr{I}\propto G\rtimes G$. In fact, the Rieffel correspondence restricts to a bijection between the G-invariant ideals of $\mathscr{A}$ and the G-invariant ideals of $\mathscr{A}\propto G\rtimes G$ for the usual G-action on an action bundle as above.*

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PROOF. First, we show that

$$(\mathscr{A} \propto G) \cdot \mathscr{I} = \mathscr{I} \propto G. \tag{4.1}$$

If $(a, x) \in \mathscr{A} \propto G$ and $i \in \mathscr{I}$, then

$$(a, x) \cdot i = (a(x \cdot i), x).$$

Putting $j = a(x \cdot i)$, we have $j \in \mathscr{I}$, because $\mathscr{I}$ is a G-invariant ideal, and so $(j, x) \in \mathscr{I} \propto G$.

For the opposite containment, consider $(i, x) \in \mathscr{I} \propto G$. Let $h = p(i)$. Then, $i \in J_h = A_h \cdot J_{s(h)}$ by [5, Lemma 2.26]. Then, $i = aj$ with $a \in A_h$ and $j \in J_{s(h)}$ by [5, Remark 2.2]. Note that left multiplication by x is a bijection of $A|_{H_{s(x)}}$ onto $A|_{H_{r(x)}}$. (Recall that if $u \in G^{(0)}$, then we let $H_u = \rho^{-1}(u)$, where $\rho : H \to G^{(0)}$ is the moment map; see [5, Definition 4.9].) Since $\mathscr{I}$ is G-invariant, left-multiplication by x also induces a bijection of $\mathscr{I}|_{H_{s(x)}}$ onto $\mathscr{I}|_{H_{r(x)}}$. In particular, $j = x \cdot j'$ for some $j' \in J_{s(h)}$. Then,

$$(i, x) = (aj, x) = (a, x) \cdot j' \in (\mathscr{A} \propto G) \cdot \mathscr{I}.$$

Hence, (4.1) holds as claimed.

Now, let $\mathscr{L}$ be the ideal of $\mathscr{A} \propto G \rtimes G$ associated to $\mathscr{I}$ via the Rieffel correspondence. By the argument above, to show that

$$\mathscr{L} \subset (\mathscr{I} \propto G) \rtimes G,$$

it suffices to consider an element of the form ${}_{L}\langle (a, x)\ ,\ (i, y)\rangle$ with $(a, x) \in \mathscr{A} \propto G$ and $(i, y) \in \mathscr{I} \propto G$. (Here, we have written ${}_{L}\langle \cdot\ ,\ \cdot \rangle$ in place of the more cumbersome ${}_{\mathscr{A} \propto G \rtimes G}\langle \cdot\ ,\ \cdot \rangle$.) Since the inner product is

$$(a(xy^{-1} \cdot i^*), xy^{-1}, y),$$

and since

$$a(xy^{-1} \cdot i^*) \in \mathscr{I}$$

because $\mathscr{I}$ is a G-invariant ideal and ideals are self-adjoint, we see that

$${}_{L}\langle (a, x)\ ,\ (i, y)\rangle \in \mathscr{I} \propto G \rtimes G.$$

For the opposite containment, consider $(i, z, y) \in \mathscr{I} \propto G \rtimes G$. Let $h = p(i)$. Then, $i \in J_h = A_h \cdot J_{s(h)}$. Let $i = a \cdot j$ with $a \in A_h$ and $j \in J_{s(h)}$ as above. We can assume $j = z \cdot j_1$. Note that $(a, zy) \in \mathscr{A} \propto G$, while $(j_1^*, y) \in \mathscr{I} \propto G$. Hence,

$$(i, z, y) = (aj, z, y) = (a(zyy^{-1} \cdot j_1), zyy^{-1}, y) = {}_{L}\langle (a, zy)\ ,\ (j_1^*, y)\rangle \in \mathscr{L}.$$

Hence, $\mathscr{L} = \mathscr{I} \propto G \rtimes G$ as required.

Now, suppose that $\mathscr{L}$ is a G-invariant ideal in $\mathscr{A} \propto G \rtimes G$. Then, the corresponding ideal, $\mathscr{I}$, in $\mathscr{A}$ is generated by pairings of the form $\langle (a, x, y) \cdot (b, y)\ ,\ (c, xy)\rangle_{R}$ with $(a, x, y) \in \mathscr{L}$ and $(b, y), (c, xy) \in \mathscr{A} \propto G$. That is, $\mathscr{I}$ is generated by elements of the form

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$$(xy)^{-1} \cdot [(a(x \cdot b))^* c].$$

However,

$$\begin{aligned} z \cdot ((xy)^{-1} \cdot [(a(x \cdot b))^* c]) &= (xyz^{-1})^{-1} \cdot [(a(x \cdot b))^* c] \\ &= \langle (a, x, yz^{-1}) \cdot (b, yz^{-1}) \, , \, (c, xyz^{-1}) \rangle_R . \end{aligned}$$

Since the G-action is isometric and bijective on the fibres, it now follows from (3.1) that $\mathscr{I}$ is G-invariant in $\mathscr{A}$.

Since $\mathscr{I} \propto G \rtimes G$ is clearly G-invariant in $\mathscr{A} \propto G \rtimes G$ and since the Rieffel correspondence is a bijection, the result follows. □

Similarly, starting with action-product Fell bundles gives the next result.

THEOREM 4.7. *In Corollary 4.5, if $\mathscr{I}$ is an ideal of $\mathscr{A}$, then the ideal $\mathscr{L}$ of $\mathscr{A} \rtimes G \propto G$ associated to $\mathscr{I}$ via the Rieffel correspondence is $\mathscr{I} \rtimes G \propto G$.*

PROOF. First, we show that in analogy with (4.1),

$$(\mathscr{A} \rtimes G) \cdot \mathscr{I} = \mathscr{I} \rtimes G.$$

If $(a, z) \in \mathscr{A} \rtimes G$ and $i \in \mathscr{I}$, then

$$(a, z) \cdot i = (ai, p(i)^{-1} z),$$

which is in $\mathscr{I} \rtimes G$ because $\mathscr{I}$ is an ideal.

For the opposite containment, let $(i, z) \in \mathscr{I} \rtimes G$ with $p(i) = x$. Note that, $s(x) = r(z)$. As in the proof of Theorem 4.6, we can factor $i = aj$ with $a \in A_x$ and $j \in J_{s(x)}$. Then,

$$(i, z) = (aj, z) = (a, z) \cdot j \in (\mathscr{A} \rtimes G) \cdot \mathscr{I}.$$

Now, let $\mathscr{L}$ be the ideal of $\mathscr{A} \rtimes G \propto G$ associated to $\mathscr{I}$ via the Rieffel correspondence. For the containment

$$\mathscr{L} \subset \mathscr{I} \rtimes G \propto G,$$

let $(a, z) \in \mathscr{A} \rtimes G$ and $(i, y) \in \mathscr{I} \rtimes G$. Then,

$${}_L\langle (a, z) \, , \, (i, y) \rangle = (ai^*, p(i) z, z^{-1} y) \in \mathscr{I} \rtimes G \propto G$$

because $\mathscr{I}$ is an ideal and, hence, self-adjoint.

For the opposite containment, given $(i, z, y) \in \mathscr{I} \rtimes G \propto G$, we can factor $i = aj$ with $a \in A_x$ and $j \in J_{s(x)}$, where $x = p(i)$. Since $s(x) = r(z)$,

$$(i, z, y) = {}_L\langle (a, z) \, , \, (j^*, zy) \rangle \in \mathscr{L}$$

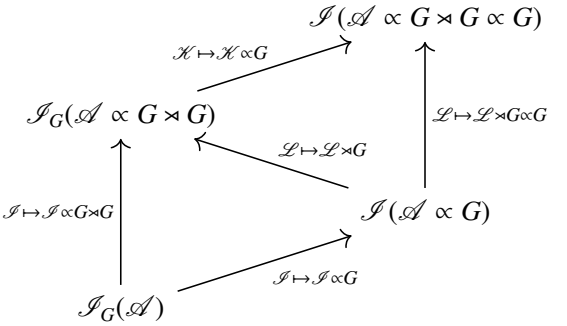
because $(j^*, zy) \in \mathscr{I} \rtimes G$. □

5. Ladders

If $p : \mathscr{A} \to H$ is a Fell bundle on which G acts by isomorphisms, then we let $\mathscr{I}_G(\mathscr{A})$ be the set of G-invariant ideals of the Fell bundle $\mathscr{A}$ and $\mathscr{I}_G(\mathscr{A} \propto G \rtimes G)$

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be the G-invariant ideals of the action Fell bundle $\mathscr{A} \propto G \rtimes G$ as above. Consider the diagram



(5.1)

This diagram commutes and constitutes our basic ladder diagram. Moreover, the vertical arrows coincide with the Rieffel correspondences, by Theorems 4.6 and 4.7. Therefore, by the Ladder Lemma, all the arrows are bijections.

We record part of this formally in the following result.

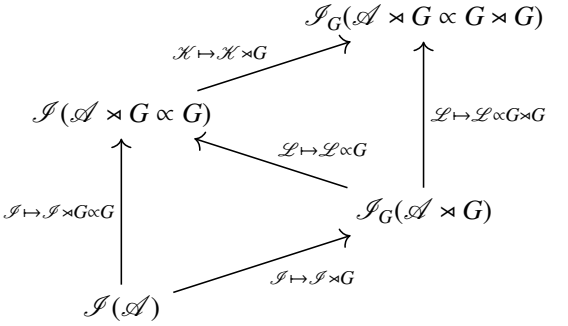
THEOREM 5.1. *If a groupoid G acts on a Fell bundle $\mathscr{A} \to H$, then every ideal of $\mathscr{A} \propto G$ is of the form $\mathscr{I} \propto G$ for a G-invariant ideal of $\mathscr{A}$ and the map*

$$\mathscr{I} \mapsto \mathscr{I} \propto G$$

gives a lattice isomorphism between the G-invariant ideals of $\mathscr{A}$ and all ideals in $\mathscr{A} \propto G$.

Note that the ladder (5.1) could be continued indefinitely upward. Additionally, it could be started with the structure on the right-hand side, so that the first rung would be an action product (rather than a semidirect product).

To see how that would look, we consider a Fell bundle $p : \mathscr{A} \to G$ over a groupoid G (to continue the ladder (5.1), just replace G with $G \propto G$.) We can form the following ladder:



The argument that the right strut is a bijection between G-invariant ideals is a special case of Theorem 4.6. Hence, we get a companion result to Theorem 5.1.

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THEOREM 5.2. *If $\mathscr{A} \to G$ is a Fell bundle, then the map*

$$\mathscr{I} \mapsto \mathscr{I} \rtimes G$$

gives a lattice isomorphism between the ideals of $\mathscr{A}$ and the G-invariant ideals of $\mathscr{A} \rtimes G$.

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S. KALISZEWSKI, School of Mathematical and Statistical Sciences,
Arizona State University, Tempe, Arizona 85287, USA
e-mail: kaliszewski@asu.edu

JOHN QUIGG, School of Mathematical and Statistical Sciences,
Arizona State University, Tempe, Arizona 85287, USA
e-mail: quigg@asu.edu

DANA P. WILLIAMS, Department of Mathematics,
Dartmouth College, Hanover, New Hampshire 03755-3551, USA
e-mail: dana.williams@dartmouth.edu

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