Probability in the Engineering and Informational Sciences, **13**, 1999, 333–358. Printed in the U.S.A.

A DYNAMIC COMPETING RISKS MODEL

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We consider a series system with p components where the failure rate of each component depends on the residual number of defects present. Successive tasks are given to the system with each task completion time independent of each component failure time. Based on the outcomes over a fixed testing period, the asymptotic properties of the estimators of the component parameters, task completion parameters, and eventual system reliability are obtained.

1. INTRODUCTION AND DESCRIPTION OF MODEL

Consider a system consisting of p components (or subsystems) structured in series so the system fails when a component fails. Let $\boldsymbol{T} = (T_1,\ldots,T_p)^{\mathrm{t}}$, where '$^{\mathrm{t}}$' is vector/matrix transpose, denote the random vector of the component lifetimes, so the system lifetime is $\min\{T_1,\ldots,T_p\}$. In many situations, such as electronic, computer software, biological, and mechanical systems, it is desired that the system be able to perform a mission, task, or job with high reliability, that is, a high probability of success. Such missions will usually have their own completion times. (For instance, when sorting an input file consisting of thousands of records using

*Part of this work was done while M. Agustin was visiting the Department of Statistics, University of Pittsburgh, Pittsburgh, PA 15260. He wishes to acknowledge the support of this department, particularly Professors Henry Block, Allan Sampson, and Thomas Savits.

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https://doi.org/10.1017/S0269964899133060 Published online by Cambridge University Press

a bug-free sorting program, a random amount of CPU time, denoted by C, will be consumed.) For a given mission with completion time C, the probability that it will be successfully completed by the system, also referred to as the *system reliability*, is $\mathbf{P}\{C < \min\{T_1,\ldots,T_p\}\}$. In the development of such systems, or while they are deployed, the components are constantly being improved in the following iterative way. When the system fails before mission or task completion, the component that caused the failure is improved through redesign and/or elimination of defects (such as "bugs" in software), and a similarly structured system but with the improved component is then further tested or redeployed. As this process of testing and component improvement proceeds, the system lifetime will increase stochastically, and as a consequence, the system reliability $\mathbf{P}\{C < \min\{T_1,\ldots,T_p\}\}$ will also increase, so after a certain period of testing and component improvement, this reliability will exceed a certain desired reliability. In this paper we consider one particular model and tackle the problems of estimating the system reliability after a certain period of testing and component improvement, as well as the parameters of the distributions of the completion times and the component lifetimes. The parameters of a component lifetime will be classified into two types: the first will represent the number of defects in the component, and the other will represent the contribution of each defect to the failure intensity of the component. The resulting model will extend the one software reliability model of Jelinski and Moranda [15] to a system of components, with the added feature that missions or tasks have their own random completion times. The resulting model can be described as a *dynamic competing risks model*.

We describe the specific model of interest. We suppose that the system will be tested for a period $\mathcal{T} = [0,\tau]$, where $0 < \tau < \infty$ is some fixed time. The case where the testing period is sequentially determined will be considered in another paper, and makes substantial use of the results presented in this paper. We will postulate the following:

i. The random completion time of a mission has hazard function $\Lambda_0^0(\cdot)$ where $\Lambda_0^0(\cdot)$ belongs to some class $\mathcal{C}$ of hazard functions. In the area of software reliability, the specification of these tasks is called a program's operational profile (Musa and Ackerman [20]).

ii. At time $\tau_0 = 0$, for the jth component, $(j = 1,\ldots,p)$, there exists a parameter vector (B_j^0,γ_j^0) where $B_j^0 \in \mathbb{N} \equiv \{0,1,2,\ldots\}$ and $\gamma_j^0 \geq 0$ and such that the hazard function of the failure time of the jth component is given by $\Lambda_j(t;B_j^0,\gamma_j^0) = B_j^0\gamma_j^0 t$. This is the assumption in the Jelinski and Moranda [15] model for the failure time of the software, where B_j^0 represents the initial number of defects in the software, and γ_j^0 is the failure rate contribution of each of the defects in the software.

iii. When a mission or task is assigned to the system, one of $p+2$ events could occur: (a) the mission is completed successfully, indicated by setting $\delta = 0$; (b) the jth component, $(j = 1,\ldots,p)$, fails before mission completion, indi-

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cated by $\delta = j$; or (c) both completion time and system lifetime are right-censored, such as when a power failure occurs, and this will be indicated by setting $\delta = -1$. We shall denote by $\tau_1 < \tau_2 < \ldots$ the successive (operational) times of event occurrences, and by $\delta_1, \delta_2, \ldots$ the successive event indicators, and for notational convenience, we set $\tau_n = \tau_{n+1} = \ldots = \infty$ for $\tau_n > \tau$ and with $\delta_n = \delta_{n+1} = \ldots = \Delta^*$ where $\Delta^* \notin \{-1, 0, 1, \ldots, p\}$ and is some arbitrary value.

iv. At each of the event occurrences, the B_j^0's get updated according to the rule

$$(\text{new } B_j^0) \leftarrow (\text{old } B_j^0) - I\{\delta = j\},$$

where $I\{\cdot\}$ is the indicator function. In other words, the number of defects of the jth component decreases by 1 if it caused the system failure; otherwise, it remains the same.

v. When the system is restarted or redeployed for further testing, the mission or task completion hazard rate and component lifetimes hazard rates are still identical to those specified in (i) and (ii) except for the change in the B_j^0's.

vi. The completion time and component lifetime variables are mutually stochastically independent.

Thus, the parameter vector of the model is $\boldsymbol{\theta}^0 = (\Lambda_0^0(\cdot), \gamma_1^0, \ldots, \gamma_p^0, B_1^0, \ldots, B_p^0)^{\mathrm{t}}$ aside from the nuisance parameters involved in the right-censoring mechanism. The reliability of the system just after time τ, the time of test termination, becomes

$$\rho_\tau(\boldsymbol{\theta}^0) = \int_0^\infty \exp\left\{-t \sum_{j=1}^{p} \gamma_j^0 B_j(\tau)\right\} \lambda_0^0(t) \exp\{-\Lambda_0^0(t)\}\, \mathrm{d}t = \varphi^0\left(\sum_{j=1}^{p} \gamma_j^0 B_j(\tau)\right), \tag{1.1}$$

where $\varphi^0(\cdot)$ is the Laplace transform of the completion time, $\lambda_0^0(\cdot)$ is the hazard rate function of the completion time, and $B_j(\tau)$ is the remaining number of defects in component j at the end of the testing period. We therefore seek to obtain estimators of $\boldsymbol{\theta}^0$ and, more importantly, of the system reliability $\rho_\tau(\boldsymbol{\theta}^0)$ just after test termination.

Several papers have dealt with the problem of estimating the parameters governing the lifetimes of components, as well as the system reliability. Most of the papers, however, were limited to the case of one component (or subsystem). Moreover, the situation where some task or mission is to be completed seems not to have been considered in the literature. A partial, but not exhaustive, list of papers include Jelinski and Moranda [15], Littlewood and Verrall [18], Musa [19], and Goel and Okumoto [13] in the software reliability area; and Blumenthal and Marcus [6] and Joe and Reid [16] in system fault estimation. The case where the termination of the testing process is sequentially determined has been considered, among others, by Forman and Singpurwalla [12], Ross [22], Dalal and Mallows [9], and Singpurwalla

https://doi.org/10.1017/S0269964899133060 Published online by Cambridge University Press

[24] in the software reliability area; and by Chow and Schechner [8] and Ferguson and Hardwick [10] in proofreading.

We now outline the contents of this paper. In Section 2, we present the mathematical setting and derive the estimators. The asymptotic properties of the estimators are examined in Section 3, while Section 4 deals with the estimator of system reliability at the end of the testing period. A concrete example is provided in Section 5, where the case of exponentially distributed task completion times is considered. A simulation is presented in Section 6 in order to investigate the finite-sample properties of the system reliability estimator. We have deferred the formal proofs of lemmas and theorems in Section 7 in order to focus on the main results.

2. RELEVANT PROCESSES AND ESTIMATORS

2.1. Formalization of the Model

We approach these problems by utilizing the framework of counting processes and martingales, so we define in this section the relevant stochastic processes and present the partial likelihood process which will be utilized in making inference about $\boldsymbol{\theta}$. Let $\boldsymbol{F} = \{\mathcal{F}_t : t \geq 0\}$ be the natural filtration on $(\Omega, \mathcal{F})$ with $\mathcal{F}_t = \sigma\{(\tau_n, \delta_n) : \tau_n \leq t\}$. Then $\{(\tau_n, \delta_n) : n = 1, 2, \ldots\}$ is a marked point process with respect to the filtration $\boldsymbol{F}$, where the δ_n's represent the marks, with the marks belonging to the set $\boldsymbol{D} = \{-1, 0, 1, \ldots, p\}$ (Arjas [4]). Associated with this marked point process is the $(p+1)$-dimensional process $\boldsymbol{N}(t) = \{(N_0(t), N_1(t), \ldots, N_p(t))^{\mathrm{t}} : t \geq 0\}$, where

$$N_d(t) = \sum_{i=1}^{\infty} \mathrm{I}\{\tau_i \leq t; \delta_i = d\}, \qquad d \in \boldsymbol{D}. \tag{2.1}$$

With respect to the filtration $\boldsymbol{F}$, $\{\boldsymbol{N}(t) : t \geq 0\}$ is a multivariate counting process. Let $\boldsymbol{A}(t; \boldsymbol{\theta}^0) = \{(A_0(t; \boldsymbol{\theta}^0), A_1(t; \boldsymbol{\theta}^0), \ldots, A_p(t; \boldsymbol{\theta}^0))^{\mathrm{t}} : t \geq 0\}$ denote the $\boldsymbol{F}$-compensator vector of $\boldsymbol{N}$ such that $\boldsymbol{A}(0; \boldsymbol{\theta}^0) = \mathbf{0}$, and

$$\boldsymbol{M}(t; \boldsymbol{\theta}^0) = \boldsymbol{N}(t) - \boldsymbol{A}(t; \boldsymbol{\theta}^0), \qquad t \geq 0, \tag{2.2}$$

is a vector of local square-integrable $\boldsymbol{F}$-martingales. With these processes, we formalize the notion of a modified Jelinski–Moranda model for a series system.

DEFINITION 2.1: *Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a probability space and let $\boldsymbol{F} = \{\mathcal{F}_t : t \geq 0\}$ be the natural filtration generated by the marked point process $\{(\tau_n, \delta_n) : n = 1, 2, \ldots\}$, where $0 \equiv \tau_0 < \tau_1 < \tau_2 < \ldots$ and $\delta_n \in \boldsymbol{D}$. This marked point process, or equivalently its associated $(p+1)$-dimensional multivariate counting process $\{\boldsymbol{N}(t) : t \geq 0\}$, satisfies a* modified Jelinski–Moranda model for a series system *with true parameter vector $\boldsymbol{\theta}^0 \equiv (\Lambda_0^0(\cdot), \gamma_1^0, \ldots, \gamma_p^0, B_1^0, \ldots, B_p^0)^{\mathrm{t}} \in \mathcal{C} \times \Re_+^p \times \mathbb{N}^p$, where $\mathcal{C}$ is some class of cumulative hazard functions, if its $\boldsymbol{F}$-compensator $\boldsymbol{A}(t; \boldsymbol{\theta}^0) = (A_0(t; \boldsymbol{\theta}^0), A_1(t; \boldsymbol{\theta}^0), \ldots, A_p(t; \boldsymbol{\theta}^0))^{\mathrm{t}}$ is*

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$$A_d(\mathrm{d}t;\boldsymbol{\theta}) = \begin{cases} \sum_{n=1}^{\infty} I\{\tau_{n-1} < t \le \tau_n\}\Lambda_0^0(\mathrm{d}t - \tau_{n-1}) & \text{if } d = 0 \\ \gamma_d(B_d - N_d(t-))\,\mathrm{d}t & \text{if } d = 1,\ldots,p \end{cases}. \tag{2.3}$$

This definition of a modified Jelinski–Moranda model for a series system formalizes the model described in Section 1. The assumed independent right-censoring property allows us to express the vector of compensators $\boldsymbol{A}$ as in (2.3). Following Definition 2.1, the $\boldsymbol{F}$-adapted process $\boldsymbol{M}(t;\boldsymbol{\theta}^0) = \{(M_0(t;\boldsymbol{\theta}^0), M_1(t;\boldsymbol{\theta}^0),\ldots, M_p(t;\boldsymbol{\theta}^0))^{\mathrm{t}} : t \ge 0\}$ with

$$M_d(t;\boldsymbol{\theta}^0) = \begin{cases} N_0(t) - \sum_{n=1}^{\infty} \int_0^t I\{\tau_{n-1} < u \le \tau_n\}\Lambda_0^0(\mathrm{d}u - \tau_{n-1}) & \text{for } d = 0 \\ N_d(t) - \int_0^t \gamma_d^0(B_d^0 - N_d(u-))\,\mathrm{d}u & \text{for } d = 1,\ldots,p \end{cases} \tag{2.4}$$

is a vector of square-integrable local $\boldsymbol{F}$-martingales with predictable quadratic variation processes (Fleming and Harrington [11]) given by

$$\langle M_d(\cdot;\boldsymbol{\theta}^0), M_{d'}(\cdot;\boldsymbol{\theta}^0)\rangle(t) = \begin{cases} A_d(t;\boldsymbol{\theta}^0) & \text{if } d = d' = 0,1,\ldots,p \\ 0 & \text{if } d \ne d' \end{cases}.$$

2.2. Likelihood Process

In Definition 2.1 the true parameter vector $\boldsymbol{\theta}^0$ is typically unknown, and the statistical problem is to make inferences about this parameter based on the realization of the marked point process $\{(\tau_n, \delta_n) : n = 1,2,\ldots\}$ or the multivariate counting process $\{\boldsymbol{N}(t) : t \ge 0\}$. For our statistical model we suppose that there is a family of probability measures $\mathcal{P} = \{\boldsymbol{P}_{(\boldsymbol{\theta},\boldsymbol{v})} : (\boldsymbol{\theta},\boldsymbol{v}) \in \Theta \times \Upsilon\}$ on $(\Omega, \mathcal{F})$, where $\boldsymbol{\theta} = (\Lambda_0(\cdot), \boldsymbol{\gamma}^{\mathrm{t}}, \boldsymbol{B}^{\mathrm{t}})^{\mathrm{t}}$ is the parameter vector of interest and $\boldsymbol{v}$ is a nuisance parameter vector associated with the right-censoring mechanism. To incorporate our statistical model in the filtration, we redefine $\boldsymbol{F} = \{\mathcal{F}_t : t \ge 0\}$ according to $\mathcal{F}_t = \sigma\{((\tau_n, \delta_n) : \tau_n \le t), (\mathcal{L}(u), u \le t)\}$. Here $\mathcal{L}(\cdot)$ is defined via $\mathcal{L}(t) = (\boldsymbol{A}(u;\boldsymbol{\theta}); \boldsymbol{\theta} \in \Theta, u \le t)$, and it represents information concerning the possible compensators for $\boldsymbol{N}$. Following Jacod [14] (see also Andersen, Borgen, Gill, and Keiding [2]), the relevant portion of the likelihood function, given $\mathcal{F}_t$, for making inferences about $\boldsymbol{\theta}$ is

$$L_t(\boldsymbol{\theta}) = \boldsymbol{\pi}_0^t \left\{ \left[\prod_{d=0}^{p} [A_d(\mathrm{d}u;\boldsymbol{\theta})]^{\Delta N_d(u)} \right] [1 - A(\mathrm{d}u;\boldsymbol{\theta})]^{1-\Delta N(u)} \right\},$$

where $N(\cdot) = \sum_{d=0}^{p} N_d(\cdot)$, $A(\cdot;\boldsymbol{\theta}) = \sum_{d=0}^{p} A_d(\cdot;\boldsymbol{\theta})$, and $\boldsymbol{\pi}_0^t$ denotes the product integral on $[0,t]$ (see Andersen et al. [2]). Let us consider the case where $\Lambda_0(\cdot)$ is absolutely continuous with hazard rate function $\lambda_0(\cdot)$ parametrized by a $q \times 1$

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parameter vector $\boldsymbol{\beta}$. Thus, the parameter vector of interest can be represented by $\boldsymbol{\theta} = (\beta^{\mathrm{t}}, \boldsymbol{\gamma}^{\mathrm{t}}, \boldsymbol{B}^{\mathrm{t}})^{\mathrm{t}}$. By using (2.3), the relevant portion of the log-likelihood function can be written as

$$\begin{aligned} l_t(\boldsymbol{\theta}) = & \int_0^t \log\left\{ \sum_{n=1}^{\infty} I\{\tau_{n-1} < u \le \tau_n\} \lambda_0(u - \tau_{n-1}; \boldsymbol{\beta}) \right\} \mathrm{d}N_0(u) \\ & - \sum_{n=1}^{\infty} \int_0^t I\{\tau_{n-1} < u \le \tau_n\} \lambda_0(u - \tau_{n-1}; \boldsymbol{\beta}) \, \mathrm{d}u \\ & + \sum_{d=1}^{p} \left\{ \int_0^t \log\{\gamma_d(B_d - N_d(u-))\} \, \mathrm{d}N_d(u) - \int_0^t \gamma_d(B_d - N_d(u-)) \, \mathrm{d}u \right\}. \end{aligned}$$

Since $\Delta N_0(u) = 0$ whenever $u \neq \tau_n$ for some n, then

$$\begin{aligned} & \int_0^t \log\left\{ \sum_{n=1}^{\infty} I\{\tau_{n-1} < u \le \tau_n\} \lambda_0(u - \tau_{n-1}; \boldsymbol{\beta}) \right\} \mathrm{d}N_0(u) \\ & \quad = \sum_{\{n:\tau_n \le t\}} [\log\{\lambda_0(\tau_n - \tau_{n-1}; \boldsymbol{\beta})\}] \Delta N_0(\tau_n); \\ & \sum_{n=1}^{\infty} \int_0^t I\{\tau_{n-1} < u \le \tau_n\} \lambda_0(u - \tau_{n-1}; \boldsymbol{\beta}) \, \mathrm{d}u \\ & \quad = \sum_{k=1}^{K(t)} \Lambda_0(\tau_k - \tau_{k-1}; \boldsymbol{\beta}) + \Lambda_0(t - \tau_{K(t)}; \boldsymbol{\beta}), \end{aligned} \tag{2.5}$$

where $K(t) = \max\{n : \tau_n \le t\}$. Note that $K(t)$ is $\mathcal{F}_t$-measurable. Thus,

$$\begin{aligned} l_t(\boldsymbol{\theta}) = & \sum_{k=1}^{K(t)} \log\{\lambda_0(\tau_k - \tau_{k-1}; \boldsymbol{\beta})\} \Delta N_0(\tau_k) \\ & - \sum_{k=1}^{K(t)} \Lambda_0(\tau_k - \tau_{k-1}; \boldsymbol{\beta}) - \Lambda_0(t - \tau_{K(t)}; \boldsymbol{\beta}) \\ & + \sum_{d=1}^{p} \left\{ \int_0^t \log\{\gamma_d(B_d - N_d(u-))\} \, \mathrm{d}N_d(u) - \int_0^t \gamma_d(B_d - N_d(u-)) \, \mathrm{d}u \right\}. \end{aligned}$$

2.3. Score Process and the Maximum Likelihood Estimator

The score process $\{\boldsymbol{U}_t(\boldsymbol{\theta}) : t \in [0, \tau]\}$ has the form

$$\underbrace{\boldsymbol{U}_t(\boldsymbol{\theta})}_{(q+2p)\times 1} = \frac{\partial l_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \left(\underbrace{\boldsymbol{U}_t^{\beta}(\boldsymbol{\theta})^{\mathrm{t}}}_{q\times 1}, \underbrace{\boldsymbol{U}_t^{\gamma}(\boldsymbol{\theta})^{\mathrm{t}}}_{p\times 1}, \underbrace{\boldsymbol{U}_t^{\boldsymbol{B}}(\boldsymbol{\theta})^{\mathrm{t}}}_{p\times 1} \right)^{\mathrm{t}},$$

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where, with

$$\bar{\omega}(u;\boldsymbol{\beta}) = \sum_{n=1}^{\infty} I\{\tau_{n-1} < u \le \tau_n\}\lambda_0(u - \tau_{n-1};\boldsymbol{\beta}), \tag{2.6}$$

we have

$$\begin{aligned}
\boldsymbol{U}_t^{\boldsymbol{\beta}}(\boldsymbol{\theta}) &= \int_0^t \frac{\partial}{\partial \boldsymbol{\beta}} \log \bar{\omega}(u;\boldsymbol{\beta})\, \mathrm{d}M_0(u;\boldsymbol{\theta});\\
\boldsymbol{U}_t^{\boldsymbol{\gamma}}(\boldsymbol{\theta}) &= \left(\int_0^t \frac{\mathrm{d}M_d(u;\boldsymbol{\theta})}{\gamma_d}\right)_{d=1,\ldots,p} = \left(\frac{M_d(t;\boldsymbol{\theta})}{\gamma_d}\right)_{d=1,\ldots,p};\\
\boldsymbol{U}_t^{\boldsymbol{B}}(\boldsymbol{\theta}) &= \left(\int_0^t \frac{I\{B_d - N_d(u-) > 0\}}{(B_d - n_d(u-))}\, \mathrm{d}M_d(u;\boldsymbol{\theta})\right)_{d=1,\ldots,p}.
\end{aligned}$$

Let $\Xi(u;\boldsymbol{\gamma}) = (\xi_1(u;\boldsymbol{\gamma}),\ldots,\xi_p(u;\boldsymbol{\gamma}))^{\mathrm{t}}$ and $\boldsymbol{\Psi}(u;\boldsymbol{B}) = (\psi_1(u;\boldsymbol{B}),\ldots,\psi_p(u;\boldsymbol{B}))^{\mathrm{t}}$, where

$$\xi_d(u;\boldsymbol{\gamma}) = \frac{1}{\gamma_d} \quad \text{and} \quad \psi_d(u;\boldsymbol{B}) = \frac{I\{B_d - N_d(u-) > 0\}}{B_d - N_d(u-)}, \quad \text{for } d = 1,\ldots,p.$$

Then a more compact representation of the score process is

$$\boldsymbol{U}_t(\boldsymbol{\theta}) = \int_0^t \boldsymbol{H}(u;\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{M}(u;\boldsymbol{\theta}), \tag{2.7}$$

where $\boldsymbol{H}(u;\boldsymbol{\theta})$ is the $(q + 2p) \times (1 + p)$ matrix given by

$$\boldsymbol{H}(u;\boldsymbol{\theta}) = \begin{bmatrix} \dfrac{\partial}{\partial \boldsymbol{\beta}} \log \bar{\omega}(u;\boldsymbol{\beta}) & \mathbf{0} \\ \mathbf{0} & \mathrm{Diag}(\Xi(u;\boldsymbol{\gamma})) \\ \mathbf{0} & \mathrm{Diag}(\boldsymbol{\Psi}(u;\boldsymbol{B})) \end{bmatrix}, \tag{2.8}$$

with the convention that, if $\boldsymbol{x}$ is a $p \times 1$ vector, then $\mathrm{Diag}(\boldsymbol{x})$ is the $p \times p$ diagonal matrix with diagonal elements being the elements of $\boldsymbol{x}$. The elements of $\boldsymbol{H}(\cdot;\boldsymbol{\theta})$ are locally bounded, since they are left-continuous adapted processes. Moreover, the elements of $\boldsymbol{H}(\cdot;\boldsymbol{\theta})$ are predictable processes. Under $\boldsymbol{\theta} = \boldsymbol{\theta}^0 \equiv ((\boldsymbol{\beta}^0)^{\mathrm{t}},(\boldsymbol{\gamma}^0)^{\mathrm{t}},(\boldsymbol{B}^0)^{\mathrm{t}})^{\mathrm{t}}$, $\boldsymbol{U}_t(\boldsymbol{\theta}^0)$ is a $(q + 2p) \times 1$ vector of square-integrable local martingales whose matrix of quadratic covariation processes is the $(q + 2p) \times (q + 2p)$ matrix

$$\begin{aligned}
\langle \boldsymbol{U}_\cdot(\boldsymbol{\theta}^0), \boldsymbol{U}_\cdot(\boldsymbol{\theta}^0)\rangle(t) &= \int_0^t \boldsymbol{H}(u;\boldsymbol{\theta}^0)\langle \boldsymbol{M}(\cdot;\boldsymbol{\theta}^0), \boldsymbol{M}(\cdot;\boldsymbol{\theta}^0)\rangle(\mathrm{d}u)\boldsymbol{H}(u;\boldsymbol{\theta}^0)^{\mathrm{t}}\\
&= \int_0^t \boldsymbol{H}(u;\boldsymbol{\theta}^0)\mathrm{Diag}(A_0(\mathrm{d}u;\boldsymbol{\theta}^0),\ldots,A_p(\mathrm{d}u;\boldsymbol{\theta}^0))\boldsymbol{H}(u;\boldsymbol{\theta}^0)^{\mathrm{t}}.
\end{aligned}$$

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As a possible estimator of $\boldsymbol{\theta}^0$, given $\mathcal{F}_\tau$, one could use the maximum likelihood estimator (MLE), denoted by $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\beta}}^{\mathrm{t}}, \hat{\boldsymbol{\gamma}}^{\mathrm{t}}, \hat{\boldsymbol{B}}^{\mathrm{t}})^{\mathrm{t}}$, which is the $\boldsymbol{\theta}$ that maximizes $l_\tau(\boldsymbol{\theta})$. One may obtain this as a solution of the system of $(q + 2p)$ equations:

$$\sum_{k=1}^{K(\tau)} \frac{\partial}{\partial \boldsymbol{\beta}} \log \lambda_0(\tau_k - \tau_{k-1}; \hat{\boldsymbol{\beta}}) \Delta N_0(\tau_k) = \sum_{k=1}^{K(\tau)} \frac{\partial}{\partial \boldsymbol{\beta}} \Lambda_0(\tau_k - \tau_{k-1}; \hat{\boldsymbol{\beta}}) + \frac{\partial}{\partial \boldsymbol{\beta}} \Lambda_0(\tau - \tau_{K(\tau)}; \hat{\boldsymbol{\beta}}); \tag{2.9}$$

$$\boldsymbol{N}_*(\tau) \otimes (1/\hat{\boldsymbol{\gamma}}) = \int_0^\tau (\hat{\boldsymbol{B}} - \boldsymbol{N}_*(u-))\, \mathrm{d}u; \tag{2.10}$$

$$\int_0^\tau \boldsymbol{\Psi}(u; \hat{\boldsymbol{B}}) \otimes (\mathrm{d}\boldsymbol{N}_*(u)) = \int_0^\tau \hat{\boldsymbol{\gamma}} \otimes \boldsymbol{Q}(u; \hat{\boldsymbol{B}})\, \mathrm{d}u; \tag{2.11}$$

where $\otimes$ denotes componentwise product, and

$$\boldsymbol{N}_*(t) = (N_1(t), \ldots, N_p(t))^{\mathrm{t}};$$

$$\boldsymbol{Q}(t; \boldsymbol{B}) = (q_1(t; \boldsymbol{B}), \ldots, q_p(t; \boldsymbol{B}))^t,$$

with $q_d(t; \boldsymbol{B}) = \mathrm{I}\{B_d - N_d(t-) > 0\}$, for $d = 1, \ldots, p$. Notice that the first equation is functionally independent from the second and third equations. On the other hand, (2.10) yields the $p \times 1$ vector $\hat{\boldsymbol{\gamma}}$ which has components

$$\hat{\gamma}_d = \frac{N_d(\tau)}{\displaystyle\int_0^\tau (\hat{B}_d - N_d(u-))\, \mathrm{d}u}, \qquad d = 1, \ldots, p. \tag{2.12}$$

Notice that $\hat{\gamma}_d$ can be interpreted as the ratio of the total number of failures caused by component d on $[0, \tau]$ and the total time that the defects are present in the time period $[0, \tau]$ for component d. Substituting (2.12) into (2.11), one obtains the set of equations

$$\int_0^\tau \frac{\left\{ \dfrac{I\{\hat{B}_d - N_d(t-) > 0\}}{\displaystyle\int_0^\tau I\{\hat{B}_d - N_d(u-) > 0\}\, \mathrm{d}u} \right\}}{\left\{ \dfrac{\hat{B}_d - N_d(t-)}{\displaystyle\int_0^\tau (\hat{B}_d - N_d(u-))\, \mathrm{d}u} \right\}} \mathrm{d}\left\{ \frac{N_d(t)}{N_d(\tau)} \right\} = 1, \qquad d = 1, \ldots, p, \tag{2.13}$$

from which $\hat{\boldsymbol{B}}$ can be obtained. The terms in braces can be interpreted as relative rates.

Recall that the major aim of this paper is to provide an estimator of the system reliability at the end of the testing process. The system reliability, which is defined

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in general in (1.1), is denoted by $\rho_\tau(\boldsymbol{\theta}^0)$. Since at $\tau+$ the remaining number of defects for component d is $B_d(\tau) = B_d^0 - N_d(\tau)$, then

$$\rho_\tau(\boldsymbol{\theta}^0) = \varphi^0[\mathbf{1}^{\mathrm{t}}(\boldsymbol{\gamma}^0 \otimes (\boldsymbol{B}^0 - \boldsymbol{N}_*(\tau))), \boldsymbol{\beta}^0], \tag{2.14}$$

where

$$\varphi^0(u,\boldsymbol{\beta}) = \mathbf{E}[\exp(-uC)] = \int_0^\infty \exp\{-uc\}\lambda_0(c;\boldsymbol{\beta})\exp\{-\Lambda_0(c;\boldsymbol{\beta})\}\,\mathrm{d}c$$

is the Laplace transform of the completion time at the parameter value $\boldsymbol{\beta}$. If $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\beta}}^{\mathrm{t}},\hat{\boldsymbol{\gamma}}^{\mathrm{t}},\hat{\boldsymbol{B}}^{\mathrm{t}})^{\mathrm{t}}$ is the MLE of $\boldsymbol{\theta}^0 = ((\boldsymbol{\beta}^0)^{\mathrm{t}},(\boldsymbol{\gamma}^0)^{\mathrm{t}},(\boldsymbol{B}^0)^{\mathrm{t}})^{\mathrm{t}}$ based on $\mathcal{F}_\tau$, then the MLE of $\rho_\tau(\boldsymbol{\theta}^0)$ is

$$\hat{\rho}_\tau = \varphi^0[\mathbf{1}^{\mathrm{t}}(\hat{\boldsymbol{\gamma}} \otimes (\hat{\boldsymbol{B}} - \boldsymbol{N}_*(\tau))), \hat{\boldsymbol{\beta}}]. \tag{2.15}$$

From (2.15), it is clear that, in order to obtain the properties of $\hat{\rho}_\tau$, it would be necessary to derive the properties of $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\beta}}^{\mathrm{t}},\hat{\boldsymbol{\gamma}}^{\mathrm{t}},\hat{\boldsymbol{B}}^{\mathrm{t}})^{\mathrm{t}}$. The next section therefore presents results on the properties of $\hat{\boldsymbol{\theta}}$.

3. ASYMPTOTIC PROPERTIES OF ESTIMATORS

In this section we will derive large-sample properties of the MLE $\hat{\boldsymbol{\theta}}$. We will consider a sequence of models which follow the model described in Section 2, indexed by n, and let $s^{(n)}$ denote the size of a component relative to some unit of measure (in the software reliability setting, size can be measured in terms of thousands of lines of code). We assume that $s^{(n)} \to \infty$ as $n \to \infty$, and that the initial number of defects in component d per unit size is equal to some unknown quantity α_d^0 and is the same for all models, $n = 1,2,\ldots$. Thus, the initial number of defects in the nth model for component d is $B_d^{(n)} = s^{(n)}\alpha_d^0$ (van Pul [25]). In this setting, we redefine the parameter vector to be $\boldsymbol{\theta} = (\boldsymbol{\beta}^{\mathrm{t}},\boldsymbol{\gamma}^{\mathrm{t}},\boldsymbol{\alpha}^{\mathrm{t}})^{\mathrm{t}}$.

Let us examine the compensator of $N_0(t)$. From (2.4) and (2.5), this can be expressed via $A_0(t;\boldsymbol{\beta}) = \sum_{k=1}^{K(t)} \Lambda_0(\tau_k - \tau_{k-1};\boldsymbol{\beta}) + \Lambda_0(t - \tau_{K(t)};\boldsymbol{\beta})$. Let $N_\bullet(t) = \sum_{d=-1}^{p} N_d(t)$, so $K(t) = N_\bullet(t) = N_\bullet(t-) + \Delta N_\bullet(t)$. Then

$$A_0(t;\boldsymbol{\beta}) = \sum_{k=1}^{N_\bullet(t)} \Lambda_0(\tau_k - \tau_{k-1};\boldsymbol{\beta}) + \Lambda_0(t - \tau_{N_\bullet(t)};\boldsymbol{\beta}).$$

The following lemma easily follows by noting that either $\Delta N_\bullet(t) = 1$ or $\Delta N_\bullet(t) = 0$. The main essence of the lemma is that $A_0(t;\boldsymbol{\beta})$ depends only on $\mathcal{F}_{t-}$.

LEMMA 3.1: *For each* $t \geq 0$, $A_0(t;\boldsymbol{\beta}) = \sum_{k=1}^{N_\bullet(t-)} \Lambda_0(\tau_k - \tau_{k-1};\boldsymbol{\beta}) + \Lambda_0(t - \tau_{N_\bullet(t-)};\boldsymbol{\beta})$.

The following corollary immediately follows from Lemma 3.1 and (2.6).

COROLLARY 3.1: *The intensity process of* $N_0(t)$ is $\bar{\omega}(t;\boldsymbol{\beta}) = \lambda_0(t - \tau_{N_\bullet(t-)};\boldsymbol{\beta})$.

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Consider now the sequence of counting processes $\boldsymbol{N}^{(n)} = (N_0^{(n)}, N_1^{(n)}, \ldots, N_p^{(n)})^{\mathrm{t}}$ associated with the model described in Section 2 with intensity processes $\boldsymbol{\lambda}^{(n)}$ of form

$$\boldsymbol{\lambda}^{(n)}(t;\boldsymbol{\theta}|\mathcal{F}_{t-}^{(n)}) = s^{(n)} \begin{bmatrix} \dfrac{1}{s^{(n)}}\, \lambda_0^{(n)}(t - \tau_{N_\cdot^{(n)}(t-)};\boldsymbol{\beta}) \\ \gamma \otimes \left(\boldsymbol{\alpha} - \dfrac{\mathbf{N}_*^{(n)}(t-)}{s^{(n)}} \right) \end{bmatrix}. \tag{3.1}$$

Let $(N_\cdot^{(n)})^{-1}(k) = \inf\{t : N_\cdot^{(n)} \geq k\}$. It follows that $\tau_k = (N_\cdot^{(n)})^{-1}(k)$ and, in particular,

$$\tau_{N_\cdot^{(n)}(t-)} = (N_\cdot^{(n)})^{-1}(N_\cdot^{(n)}(t-)) = \left(\frac{N_\cdot^{(n)}}{s^{(n)}} \right)^{-1} \left(\frac{N_\cdot^{(n)}(t-)}{s^{(n)}} \right), \tag{3.2}$$

so that

$$\lambda_0^{(n)}(t - \tau_{N_\cdot^{(n)}(t-)};\boldsymbol{\beta}) = \lambda_0^{(n)} \left(t - \left(\frac{N_\cdot^{(n)}}{s^{(n)}} \right)^{-1} \left(\frac{N_\cdot^{(n)}(t-)}{s^{(n)}} \right); \boldsymbol{\beta} \right).$$

Let $x_{-1}(\cdot), \ldots, x_p(\cdot)$ be nondecreasing elements of $\mathcal{D}[0,\infty)$, the space of cádlág (right-continuous with left-hand limits) functions on $[0,\infty)$ endowed with the Skorohod topology, and define $\boldsymbol{x}_*(t) = (x_1(t), \ldots, x_p(t))^{\mathrm{t}}$, $\boldsymbol{x}(t) = (x_0(t), \boldsymbol{x}_*(t)^{\mathrm{t}})^{\mathrm{t}}$, and $\boldsymbol{x}_\star(t) = (x_{-1}(t), \boldsymbol{x}(t)^{\mathrm{t}})^{\mathrm{t}}$. Also, define

$$\phi(t;\boldsymbol{x}^{t-},\boldsymbol{\theta}) \equiv \begin{bmatrix} \lambda_0^{(0)}(t - x_\cdot^{-1}(x_\cdot(t-));\boldsymbol{\beta}) \\ \boldsymbol{\gamma} \otimes (\boldsymbol{\alpha} - \boldsymbol{x}_*(t-)) \end{bmatrix}, \tag{3.3}$$

where $x_\cdot(t) = \sum_{d=-1}^{p} x_d(t)$, and $\lambda_0^{(0)}(\cdot;\boldsymbol{\beta})$ is some hazard rate function. Consider the case where $\Lambda_0^{(n)}(\cdot;\boldsymbol{\beta})$ is of form

$$\Lambda_0^{(n)}(\cdot;\boldsymbol{\beta}) = s^{(n)} \Lambda_0^{(0)}(\cdot;\boldsymbol{\beta}) + o(1). \tag{3.4}$$

If we let $\boldsymbol{r}^{(n)}(t) = (o(1), 0, \ldots, 0)^{\mathrm{t}}$, then

$$\boldsymbol{\lambda}^{(n)}(t;\boldsymbol{\theta}|\mathcal{F}_{t-}^{(n)}) = s^{(n)} \boldsymbol{\phi} \left(t; \frac{\boldsymbol{N}^{(n)t-}}{s^{(n)}}, \boldsymbol{\theta} \right) + \boldsymbol{r}^{(n)}(t). \tag{3.5}$$

Furthermore, consider the following conditions on $\lambda_0^{(0)}(\cdot;\boldsymbol{\beta})$ which are similar to the conditions given in Kurtz [17] (see also van Pul [25] and Andersen et al. [2]). For every $\boldsymbol{x}_\star, \boldsymbol{y}_\star \in \mathcal{D}[0,\infty)^{p+2}$, and for each $\boldsymbol{\beta}$,

(K1) $\sup_{t \leq \tau} \kappa(t, x_\cdot^{t-}, \boldsymbol{\beta}) < \infty$; and
(K2) there exists a constant C not depending on t such that $\forall\, t \leq \tau$,

$$|\kappa(t, x_\cdot^{t-}, \boldsymbol{\beta}) - \kappa(t, y_\cdot^{t-}, \boldsymbol{\beta})| \leq C |x_\cdot(t) - y_\cdot(t)|,$$

where $\kappa(t, x_\cdot^{t-}, \boldsymbol{\beta}) = \lambda_0^{(0)}(t - x_\cdot^{-1}(x_\cdot(t-));\boldsymbol{\beta})$. In the following theorem we describe the limiting vector function to which $\boldsymbol{N}^{(n)}/s^{(n)}$ converges. For brevity of notation, we suppress the superscript n. Moreover, let $(\mathbf{1} - \exp(-\boldsymbol{\gamma}^0 t)) = (1 -$

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$\exp(-\gamma_1^0 t),\ldots,1-\exp(-\gamma_p^0 t))^{\mathrm{t}}$. Recall that $\boldsymbol{\theta}^0 = ((\boldsymbol{\beta}^0)^{\mathrm{t}},(\boldsymbol{\gamma}^0)^t,(\boldsymbol{\alpha}^0)^{\mathrm{t}})^{\mathrm{t}}$ is the true value of the parameter vector.

THEOREM 3.1: *Assume that* $\lambda_0^{(0)}(\cdot;\boldsymbol{\beta})$ *satisfies conditions* (K1) *and* (K2). *Then, as* $n \to \infty$,

i.
$$\sup_{t\le\tau}\left|\frac{\boldsymbol{N}(t)}{s} - \hat{\boldsymbol{X}}(t;\boldsymbol{\theta}^0)\right| \xrightarrow{\mathrm{pr}} 0,$$

where $\hat{\boldsymbol{X}}(\cdot;\boldsymbol{\theta}^0)$ *is the unique solution of*

$$\hat{\boldsymbol{X}}(t;\boldsymbol{\theta}^0) = \int_0^t \boldsymbol{\phi}(u;\hat{\boldsymbol{X}}^{u-},\boldsymbol{\theta}^0)\,\mathrm{d}u; \tag{3.6}$$

ii. *if* $\lambda_0^{(0)}(\cdot;\boldsymbol{\beta}^0)$ *is right-continuous at* 0 *with* $\zeta_0(\boldsymbol{\beta}^0) \equiv \lambda_0^{(0)}(0;\boldsymbol{\beta}^0) > 0$, *then*

$$\hat{\boldsymbol{X}}(t;\boldsymbol{\theta}^0) = \begin{bmatrix} \zeta_0(\boldsymbol{\beta}^0)t \\ \boldsymbol{\alpha}^0 \otimes (\mathbf{1}-\exp(-\boldsymbol{\gamma}^0 t)) \end{bmatrix}. \tag{3.7}$$

PROOF: See Section 7. ■

We now prove that the MLE $\hat{\boldsymbol{\theta}}$ is a consistent estimator of $\boldsymbol{\theta}^0$. We assume the following conditions for $\lambda_0^{(0)}(\cdot;\boldsymbol{\beta})$.

(a) The partial derivatives of $\lambda_0^{(0)}(t;\boldsymbol{\beta})$ and $\log \lambda_0^{(0)}(t;\boldsymbol{\beta})$ of the first, second, and third orders with respect to $\boldsymbol{\beta}$ exist and are continuous in $\boldsymbol{\beta}$.

(b) For $k,l,m = 1,\ldots,q$, the following functions are continuous functions of $t \in [0,\tau]$:
$$\frac{\partial}{\partial\beta_k}\lambda_0^{(0)}(t;\boldsymbol{\beta}), \quad \frac{\partial^2}{\partial\beta_k\partial\beta_l}\lambda_0^{(0)}(t;\boldsymbol{\beta}), \quad \text{and} \quad \frac{\partial^3}{\partial\beta_k\partial\beta_l\partial\beta_m}\lambda_0^{(0)}(t;\boldsymbol{\beta}).$$

(c) The matrix $\boldsymbol{\Sigma}_1^0 = (\sigma_{kl}(\boldsymbol{\beta}^0))_{k,l=1,\ldots,q}$ is positive semidefinite, where
$$\sigma_{kl}(\boldsymbol{\beta}^0) = \frac{\left[\frac{\partial}{\partial\beta_k}\lambda_0^{(0)}(0;\boldsymbol{\beta}^0)\right]\left[\frac{\partial}{\partial\beta_l}\lambda_0^{(0)}(0;\boldsymbol{\beta}^0)\right]}{\lambda_0^{(0)}(0;\boldsymbol{\beta}^0)}\tau.$$

(d) There exist predictable processes $H_0(\cdot)$ and $G_0(\cdot)$ such that, for some neighborhood $\mathcal{B}^0$ of $\boldsymbol{\beta}^0$,
$$\sup_{\boldsymbol{\beta}\in\mathcal{B}^0}\left|\frac{\partial^3}{\partial\beta_k\partial\beta_l\partial\beta_m}\lambda_0^{(0)}(t;\boldsymbol{\beta})\right| \le H_0(t) \quad \text{and}$$
$$\sup_{\boldsymbol{\beta}\in\mathcal{B}^0}\left|\frac{\partial^3}{\partial\beta_k\partial\beta_l\partial\beta_m}\log\lambda_0^{(0)}(t;\boldsymbol{\beta})\right| \le G_0(t),$$

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for all $k, l, m = 1, \ldots, q$. Moreover,

$$\int_0^t H_0(u - \tau_{N_\cdot(u-)})\, \mathrm{d}u, \qquad \frac{1}{s}\int_0^t G_0(u - \tau_{N_\cdot(u-)})\lambda_0^{(0)}(u - \tau_{N_\cdot(u-)}; \boldsymbol{\beta}^0)\, \mathrm{d}u,$$

and

$$\int_0^t \left[\frac{\partial^2}{\partial\beta_k \partial\beta_l} \log \lambda_0^{(0)}(u - \tau_{N_\cdot(u-)}; \boldsymbol{\beta}^0)\right]^2 \lambda_0^{(0)}(u - \tau_{N_\cdot(u-)}; \boldsymbol{\beta}^0)\, \mathrm{d}u$$

all converge in probability to finite quantities as $n \to \infty$, and for $\epsilon > 0$,

$$\frac{1}{s}\int_0^t G_0(u - \tau_{N_\cdot(u-)}) \mathrm{I}\{|s^{-1} G_0(u - \tau_{N_\cdot(u-)})| > \epsilon\}$$

$$\times \lambda_0^{(0)}(u - \tau_{N_\cdot(u-)}; \boldsymbol{\beta}^0)\, \mathrm{d}u \xrightarrow{\text{pr}} 0.$$

Consider the $(q + 2p) \times (q + 2p)$ matrix $\boldsymbol{\Sigma}^0$ given by

$$\boldsymbol{\Sigma}^0(\tau) \equiv \boldsymbol{\Sigma}^0 = \begin{bmatrix} \boldsymbol{\Sigma}_1^0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2^0 & \boldsymbol{\Sigma}_4^0 \\ \mathbf{0} & \boldsymbol{\Sigma}_4^0 & \boldsymbol{\Sigma}_3^0 \end{bmatrix}, \tag{3.8}$$

where, with $\boldsymbol{\eta}(\tau) = (\eta_1(\tau), \ldots, \eta_p(\tau))^{\mathrm{t}} = (\exp(-\gamma_1\tau), \ldots, \exp(-\gamma_p\tau))^{\mathrm{t}}$, we have

$$\boldsymbol{\Sigma}_2^0 = \mathrm{Diag}(\boldsymbol{\alpha}^0 \otimes (\mathbf{1} - \boldsymbol{\eta}^0(\tau)) \otimes (\boldsymbol{\gamma}^0)^{-2}) = \mathrm{Diag}\left(\frac{\alpha_i^0(1 - \eta_i^0(\tau))}{(\gamma_i^0)^2}, i = 1, \ldots, p\right);$$

$$\boldsymbol{\Sigma}_3^0 = \mathrm{Diag}((\mathbf{1} - \boldsymbol{\eta}^0(\tau)) \otimes (\boldsymbol{\alpha}^0 \otimes \boldsymbol{\eta}^0(\tau))^{-1}) = \mathrm{Diag}\left(\frac{1 - \eta_i^0(\tau)}{\alpha_i^0 \eta_i^0(\tau)}, i = 1, \ldots, p\right);$$

$$\boldsymbol{\Sigma}_4^0 = \mathrm{Diag}(\tau, \ldots, \tau).$$

In the next theorem, we show that with probability going to one, there exists a solution to the likelihood equation which is consistent for $\boldsymbol{\theta}^0$.

THEOREM 3.2: *Suppose conditions* (a)–(d) *are satisfied. Furthermore, assume* $\lambda_0^{(0)}(\cdot; \boldsymbol{\beta})$ *satisfies all the conditions given in Theorem* 3.1. *Let* $N_{-1}(\cdot)$ *be the counting process associated with the right-censored values with intensity process* $\lambda_{-1}(\cdot)$ *that satisfies conditions similar to* (K1) *and* (K2). *Then there is a solution* $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\beta}}^{\mathrm{t}}, \hat{\boldsymbol{\gamma}}^{\mathrm{t}}, \hat{\boldsymbol{\alpha}}^{\mathrm{t}})^{\mathrm{t}}$ *of the equation* $\mathbf{U}_\tau(\boldsymbol{\theta}) = \mathbf{0}$ *such that* $\hat{\boldsymbol{\theta}} \xrightarrow{\text{pr}} \boldsymbol{\theta}^0 = ((\boldsymbol{\beta}^0)^{\mathrm{t}}, (\boldsymbol{\gamma}^0)^{\mathrm{t}}, (\boldsymbol{\alpha}^0)^{\mathrm{t}})^{\mathrm{t}}$.

PROOF: See Section 7. ■

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Consider the $(q + 2p) \times (q + 2p)$ matrix $(\boldsymbol{\Sigma}^0)^-$ given by

$$(\boldsymbol{\Sigma}^0(\tau))^- \equiv (\boldsymbol{\Sigma}^0)^- = \begin{bmatrix} (\boldsymbol{\Sigma}_1^0)^- & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\boldsymbol{\Sigma}_2^0)^{-1} + \boldsymbol{F}\boldsymbol{E}^{-1}\boldsymbol{F}^t & -\boldsymbol{F}\boldsymbol{E}^{-1} \\ \mathbf{0} & -\boldsymbol{E}^{-1}\boldsymbol{F}^t & \boldsymbol{E}^{-1} \end{bmatrix}, \tag{3.9}$$

where $(\boldsymbol{\Sigma}_1^0)^-$ is the generalized inverse of $\boldsymbol{\Sigma}_1^0$ given by

$$(\boldsymbol{\Sigma}_1^0)^- = \begin{bmatrix} \sigma_{11}^{-1}(\boldsymbol{\beta}^0) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \boldsymbol{F} = (\boldsymbol{\Sigma}_2^0)^{-1}\boldsymbol{\Sigma}_4^0 \quad \text{and} \quad \boldsymbol{E} = \boldsymbol{\Sigma}_3^0 - \boldsymbol{\Sigma}_4^0\boldsymbol{F}$$

(Rao [21]). In the next theorem, we present a result on the asymptotic normality of $\hat{\boldsymbol{\theta}}$. This result will be used in establishing the asymptotic distribution of the estimator of system reliability at test termination. Another application of this result is in interval estimation of the completion and/or component lifetime parameters. However, since the system reliability is of primary importance in this paper, this latter aspect will not be discussed further.

THEOREM 3.3: *If the conditions of Theorem* 3.2 *hold, and* $\hat{\boldsymbol{\theta}}$ *is a consistent solution of the equation* $\boldsymbol{U}_\tau(\boldsymbol{\theta}) = \mathbf{0}$, *then* $\sqrt{s}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0) \xrightarrow{d} \boldsymbol{N}_{q+2p}(\mathbf{0}, (\boldsymbol{\Sigma}^0)^-)$.

PROOF: See Section 7. ■

Notice that the asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}$ need not be of full rank. This is a consequence of the fact that the limiting covariance matrix $\boldsymbol{\Sigma}_1^0$ is not of full rank when $\boldsymbol{\beta}$ is not a scalar parameter as seen in condition (c). In such a case, the completion time parameters are linearly dependent.

4. SYSTEM RELIABILITY AT TEST TERMINATION

In this section we present the main result of this paper. We consider the sequence of models in Section 3, which specifies that $\boldsymbol{B}^{(n)} = s^{(n)}\boldsymbol{\alpha}^0$ and $\Lambda_0^{(n)}(\cdot;\boldsymbol{\beta}) = s^{(n)}\Lambda_0^{(0)}(\cdot;\boldsymbol{\beta}) + o(1)$ (Eq. (3.4)). For the nth model the true system reliability and its corresponding estimator are given, respectively, by

$$\rho_\tau^{(n)}(\boldsymbol{\theta}^0) = \varphi^{(n)}\left[\mathbf{1}^t\left(\boldsymbol{\gamma}^0 \otimes \left(\boldsymbol{\alpha}^0 - \frac{\boldsymbol{N}_*^{(n)}(\tau)}{s^{(n)}}\right)\right)s^{(n)}, \boldsymbol{\beta}^0\right];$$

and

$$\hat{\rho}_\tau^{(n)} = \varphi^{(n)}\left[\mathbf{1}^t\left(\hat{\boldsymbol{\gamma}}^{(n)} \otimes \left(\hat{\boldsymbol{\alpha}}^{(n)} - \frac{\boldsymbol{N}_*^{(n)}(\tau)}{s^{(n)}}\right)\right)s^{(n)}, \hat{\boldsymbol{\beta}}^{(n)}\right],$$

where $\varphi^{(n)}(\cdot, \boldsymbol{\beta})$ is the Laplace transform associated with the hazard function $\Lambda_0^{(n)}(\cdot;\boldsymbol{\beta})$. Let us assume that there exists a function $\varphi^{(*)}(\cdot,\cdot)$ such that in a neigh-

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borhood $\mathcal{B}^0$ of $\boldsymbol{\beta}^0$, $\varphi^{(*)}(\cdot,\boldsymbol{\beta})$ for $\boldsymbol{\beta} \in \mathcal{B}^0$ is a Laplace transform associated with a hazard function $\Lambda_0^{(*)}(\cdot;\boldsymbol{\beta})$, and such that

$$\varphi^{(n)}(us^{(n)},\boldsymbol{\beta}) = \varphi^{(*)}(u,\boldsymbol{\beta}) + o(1) \quad \text{as } n \to \infty. \tag{4.1}$$

This is the case, for instance, if $s^{(n)}\Lambda_0^{(0)}(c/s^{(n)};\boldsymbol{\beta}) = \Lambda_0^{(*)}(c;\boldsymbol{\beta}) + o(1)$ as $n \to \infty$. Furthermore, if $\varphi^{(*)}(u,\boldsymbol{\beta})$ is continuous at $(u,\boldsymbol{\beta}) = (\mathbf{1}^{\mathrm{t}}(\boldsymbol{\gamma}^0 \otimes \boldsymbol{\alpha}^0 \otimes \boldsymbol{\eta}^0(\tau)),\boldsymbol{\beta}^0)$, then it follows from Theorem 3.1 and Theorem 3.2 that

$$\hat{\rho}_\tau^{(n)} \xrightarrow{\text{pr}} \rho_\tau^{(*)} \equiv \varphi^{(*)}[\mathbf{1}^{\mathrm{t}}(\boldsymbol{\gamma}^0 \otimes \boldsymbol{\alpha}^0 \otimes \boldsymbol{\eta}^0(\tau)),\boldsymbol{\beta}^0] = \lim_{n\to\infty} \rho_\tau^{(n)}(\boldsymbol{\theta}^0),$$

so that $\hat{\rho}_\tau^{(n)}$ is a consistent sequence of estimators.

To obtain the asymptotic distribution of $\hat{\rho}_\tau^{(n)}$, we note that it is a function of

$$\hat{\boldsymbol{R}}_\tau^{(n)} \equiv \left((\hat{\boldsymbol{\beta}}^{(n)})^{\mathrm{t}}, (\hat{\boldsymbol{\gamma}}^{(n)})^{\mathrm{t}}, (\hat{\boldsymbol{\alpha}}^{(n)})^{\mathrm{t}}, \frac{(\boldsymbol{N}_*^{(n)}(\tau))^{\mathrm{t}}}{s^{(n)}} \right)^{\mathrm{t}}.$$

Thus, upon obtaining the asymptotic distribution of $\hat{\boldsymbol{R}}_\tau^{(n)}$, and together with some conditions on the transform $\varphi^{(*)}(\cdot,\cdot)$, the δ-method can be invoked to find the asymptotic distribution of $\hat{\rho}_\tau^{(n)}$. Let $\boldsymbol{R}^0(\tau) = ((\boldsymbol{\beta}^0)^{\mathrm{t}}, (\boldsymbol{\gamma}^0)^{\mathrm{t}}, (\boldsymbol{\alpha}^0)^{\mathrm{t}}, (\hat{\boldsymbol{X}}_*(\tau;\boldsymbol{\theta}^0))^{\mathrm{t}})^{\mathrm{t}}$ where $\hat{\boldsymbol{X}}_*(\tau;\boldsymbol{\theta}^0) = \boldsymbol{\alpha}^0 \otimes (\mathbf{1} - \boldsymbol{\eta}^0(\tau))$. In the following lemma, we obtain a stochastic integral representation of

$$\sqrt{s^{(n)}}[\hat{\boldsymbol{R}}_\tau^{(n)} - \boldsymbol{R}^0(\tau)] = \sqrt{s^{(n)}} \begin{bmatrix} \hat{\boldsymbol{\theta}}^{(n)} - \boldsymbol{\theta}^0 \\ \dfrac{\boldsymbol{N}_*^{(n)}(\tau)}{s^{(n)}} - \hat{\boldsymbol{X}}_*(\tau;\boldsymbol{\theta}^0) \end{bmatrix}.$$

For brevity of notation, we suppress the superscript n. Recall that $\boldsymbol{\theta}^0 = ((\boldsymbol{\beta}^0)^{\mathrm{t}},(\boldsymbol{\gamma}^0)^{\mathrm{t}},(\boldsymbol{\alpha}^0)^{\mathrm{t}})^{\mathrm{t}}$.

LEMMA 4.1: *Suppose the conditions of Theorem* 3.3 *are satisfied and suppose* $\hat{\boldsymbol{\theta}}$ *is a consistent solution of the equation* $\boldsymbol{U}_\tau(\boldsymbol{\theta}) = \mathbf{0}$. *Then*,

$$\sqrt{s}[\hat{\boldsymbol{R}}_\tau - \boldsymbol{R}^0(\tau)] = \begin{bmatrix} (\boldsymbol{\Sigma}^0(\tau))^- & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q}_*^0(\tau) \end{bmatrix} \frac{1}{\sqrt{s}} \int_0^\tau \begin{bmatrix} \boldsymbol{H}(t;\boldsymbol{\theta}^0) \\ \boldsymbol{J}(t;\boldsymbol{\theta}^0) \end{bmatrix} \mathrm{d}\boldsymbol{M}(t;\boldsymbol{\theta}^0) + o_p(1)$$

where $\boldsymbol{Q}_*^0(\tau) = \mathrm{Diag}(\boldsymbol{\eta}^0(\tau))$, $\boldsymbol{J}(t;\boldsymbol{\theta}^0) = [\mathbf{0}, \quad \mathrm{Diag}((\boldsymbol{\eta}^0(t))^{-1})]$, $\boldsymbol{H}(t;\boldsymbol{\theta}^0)$ *is defined in* (2.8), *and* $\boldsymbol{\Sigma}^0(\tau)$ *is defined in* (3.8).

PROOF: See Section 7. ■

Using the representation in Lemma 4.1, it follows from the Martingale Central Limit Theorem (see, e.g., [2, p. 83]) that, as $n \to \infty$,

$$\sqrt{s}[\hat{\boldsymbol{R}}_\tau - \boldsymbol{R}^0(\tau)] \xrightarrow{d} \boldsymbol{N}_{q+3p}(\mathbf{0},\boldsymbol{\Gamma}^0(\tau)),$$

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where

$$\begin{aligned}\boldsymbol{\Gamma}^0(\tau) = \lim_{n\to\infty} \frac{1}{s}\begin{bmatrix} (\boldsymbol{\Sigma}^0(\tau))^- & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q}_*^0(\tau) \end{bmatrix} \\ \times \left\{ \int_0^\tau \begin{bmatrix} \boldsymbol{H}(t;\boldsymbol{\theta}^0) \\ \boldsymbol{J}(t;\boldsymbol{\theta}^0) \end{bmatrix} \mathrm{Diag}(\boldsymbol{A}(\mathrm{d}t;\boldsymbol{\theta}^0)) \begin{bmatrix} \boldsymbol{H}(t;\boldsymbol{\theta}^0) \\ \boldsymbol{J}(t;\boldsymbol{\theta}^0) \end{bmatrix}^t \right\} \\ \times \begin{bmatrix} (\boldsymbol{\Sigma}^0(\tau))^- & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q}_*^0(\tau) \end{bmatrix}^t.\end{aligned}$$

Upon simplifying, we obtain

$$\boldsymbol{\Gamma}^0(\tau) = \begin{bmatrix} (\boldsymbol{\Sigma}^0(\tau))^- & (\boldsymbol{\Gamma}_1^0(\tau))^t \\ \boldsymbol{\Gamma}_1^0(\tau) & \boldsymbol{\Gamma}_2^0(\tau) \end{bmatrix},$$

where

$$\begin{aligned}\underbrace{\boldsymbol{\Gamma}_1^0(\tau)}_{p\times(2p+q)} &= [\underbrace{\mathbf{0}}_{p\times q}, \underbrace{\mathbf{0}}_{p\times p}, \underbrace{\mathrm{Diag}(\boldsymbol{\alpha}^0 \otimes \boldsymbol{\eta}^0(\tau))}_{p\times p}]; \\ \underbrace{\boldsymbol{\Gamma}_2^0(\tau)}_{p\times p} &= \mathrm{Diag}(\boldsymbol{\alpha}^0 \otimes \boldsymbol{\eta}^0(\tau) \otimes (\mathbf{1} - \boldsymbol{\eta}^0(\tau))).\end{aligned}$$

Finally, invoking the multivariate δ-method (Serfling [23]), and after lengthy but straightforward manipulations, the following theorem is obtained (for details, see Agustin [1]).

THEOREM 4.1: *Suppose the conditions of Theorem* 3.3 *are satisfied and the partial derivatives* $\varphi_{10}^{(*)}(u,\boldsymbol{\beta}) = (\partial/\partial u)\varphi^{(*)}(u,\boldsymbol{\beta})$ *and* $\varphi_{01}^{(*)}(u,\boldsymbol{\beta}) = (\partial/\partial\boldsymbol{\beta})\varphi^{(*)}(u,\boldsymbol{\beta})$ *exist at* $(u,\boldsymbol{\beta}) = (\nu^0(\tau),\boldsymbol{\beta}^0)$, *where* $\nu^0(\tau) \equiv \mathbf{1}^t(\boldsymbol{\gamma}^0 \otimes \boldsymbol{\alpha}^0 \otimes \boldsymbol{\eta}^0(\tau))$. *Furthermore, suppose the* $(q + 3p) \times 1$ *matrix*

$$\mathrm{D}^0(\tau) = \begin{bmatrix} \varphi_{01}^{(*)}(\nu^0(\tau),\boldsymbol{\beta}^0) \\ -\varphi_{10}^{(*)}(\nu^0(\tau),\boldsymbol{\beta}^0)(\boldsymbol{\alpha}^0 \otimes \boldsymbol{\eta}^0(\tau)) \\ -\varphi_{10}^{(*)}(\nu^0(\tau),\boldsymbol{\beta}^0)\boldsymbol{\gamma}^0 \\ \varphi_{10}^{(*)}(\nu^0(\tau),\boldsymbol{\beta}^0)\boldsymbol{\gamma}^0 \end{bmatrix}$$

is nonzero. Then $\sqrt{s}[\hat{\rho}_\tau - \rho_\tau^{(*)}] \xrightarrow{d} N(0,\vartheta^2(\tau;\boldsymbol{\theta}^0))$, *where*

$$\begin{aligned}\vartheta^2(\tau;\boldsymbol{\theta}^0) = [\varphi_{01}^{(*)}(\nu^0(\tau),\boldsymbol{\beta}^0)]^t[\boldsymbol{\Sigma}_1^0(\tau)]^-[\varphi_{01}^{(*)}(\nu^0(\tau),\boldsymbol{\beta}^0)] \\ + \varphi_{10}^{(*)}(\nu^0(\tau),\boldsymbol{\beta}^0)^2 \sum_{d=1}^{p} \alpha_d^0(\gamma_d^0\eta_d^0(\tau))^2 \\ \times \left\{ \frac{2\tau\gamma_d^0 + 1 - (\eta_d^0(\tau))^2 + (\tau\gamma_d^0)^2(1 + \eta_d^0(\tau))}{(1 - \eta_d^0(\tau))^2 - (\tau\gamma_d^0)^2\eta_d^0(\tau)} \right\}. \qquad \textbf{(4.2)}\end{aligned}$$

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As a consequence of Theorem 4.1, an asymptotic lower confidence bound for the system reliability can be obtained from the log-odds transformation of $\hat{\rho}_\tau$. By the δ-method, we immediately have

$$\log\left(\frac{\hat{\rho}_\tau}{1-\hat{\rho}_\tau}\right) \sim \text{AN}\left(\log\left(\frac{\rho_\tau^{(*)}}{1-\rho_\tau^{(*)}}\right), \frac{\vartheta^2(\tau;\boldsymbol{\theta}^0)}{s[\,\rho_\tau^{(*)}(1-\rho_\tau^{(*)})]^2}\right). \tag{4.3}$$

Let z_ε denote the $100(1-\varepsilon)$th percentile of the standard normal distribution. Using (4.3) and after straightforward algebra, an asymptotic $100(1-\varepsilon)\%$ lower confidence bound for the system reliability is

$$\hat{\rho}_\tau \times \left[\hat{\rho}_\tau + (1-\hat{\rho}_\tau)\exp\left(\frac{z_\varepsilon \vartheta(\tau;\hat{\boldsymbol{\theta}})}{\sqrt{s}[\,\hat{\rho}_\tau(1-\hat{\rho}_\tau)]}\right)\right]^{-1}, \tag{4.4}$$

where $\vartheta(\tau;\hat{\boldsymbol{\theta}})$ is as given in (4.2) with $\boldsymbol{\theta}^0$ replaced by $\hat{\boldsymbol{\theta}}$.

It is also of interest to determine the behavior of the limiting variance $\vartheta^2(\tau;\boldsymbol{\theta}^0)$ for relatively large τ's, which will coincide with the situation where the testing is performed for a relatively long period of time. To provide us with an idea of this behavior we allow τ to approach ∞. First, note that from their respective definitions, as $\tau \to \infty, \boldsymbol{\eta}^0(\tau) = o(1)$ and $(\boldsymbol{\Sigma}_1^0(\tau))^- \to \mathbf{0}$. Consequently, $\lim_{\tau\to\infty} \nu^0(\tau) = 0$; hence,

$$\lim_{\tau\to\infty} \varphi_{10}(\nu^0(\tau), \boldsymbol{\beta}^0) = \varphi_{10}(0, \boldsymbol{\beta}^0) = -\mathbf{E}(C^*) \equiv \mu^*,$$

where C^* is a random variable with hazard function $\Lambda_0^{(*)}(\cdot;\boldsymbol{\beta}^0)$. On the other hand,

$$\lim_{\tau\to\infty} \sum_{d=1}^{p} \alpha_d^0(\gamma_d^0\eta_d^0(\tau))^2 \left\{\frac{2\tau\gamma_d^0 + 1 - (\eta_d^0(\tau))^2 + (\tau\gamma_d^0)^2(1+\eta_d^0(\tau))}{(1-\eta_d^0(\tau))^2 - (\tau\gamma_d^0)^2\eta_d^0(\tau)}\right\} = 0.$$

Thus, for a relatively long testing period, the limiting variance of the standardized system reliability estimator tends to 0. This result is intuitively expected since $\hat{\rho}_\tau$ converges to 1 as $n \to \infty$ and as $\tau \to \infty$.

5. CONCRETE EXAMPLE—EXPONENTIAL COMPLETION TIMES

Let us consider the situation where the successive completion times $C_1, C_2, \ldots$ are i.i.d. with common constant hazard rate, that is, $\Lambda_0(t;\beta) = \beta t$. Hence, (2.9) simplifies to $N_0(\tau) = \hat{\beta}\tau$. The MLE of β is therefore

$$\hat{\beta} = \frac{N_0(\tau)}{\tau} = \frac{\text{Number of completed jobs on } [0,\tau]}{\tau}.$$

The estimators of $\boldsymbol{B}$ and $\boldsymbol{\gamma}$, denoted by $\hat{\boldsymbol{B}}$ and $\hat{\gamma}$, respectively, are not in closed forms but can be obtained numerically from (2.12) and (2.13).

For the asymptotic properties of the estimators, we consider the sequence of models specified in Section 3 such that $\Lambda_0^{(n)}(t;\beta) = s^{(n)}\Lambda_0^{(0)}(t;\beta) = s^{(n)}\beta t$. Note that the hazard rate associated with job completion increases as n increases. This increase is in order to compensate for the fact that, as n increases, the initial number of defects in each component also increases, thereby shortening the time to encounter-

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ing a defect. In the sequel, we let $\boldsymbol{\theta}^0 = (\beta^0,(\boldsymbol{\gamma}^0)^{\mathrm{t}},(\boldsymbol{\alpha}^0)^{\mathrm{t}})^{\mathrm{t}}$ denote the true, but unknown, parameter vector.

Under this exponential completion times model, conditions (K1) and (K2) are clearly satisfied. Since $\lambda_0^{(0)}(\cdot;\beta)$ is right-continuous at 0 with $\lambda_0^{(0)}(0;\beta^0) = \beta^0$, then, if $\beta^0 \in (0,\infty)$, Theorem 3.1 holds with $\hat{X}_0(t;\boldsymbol{\theta}^0) = \beta^0 t$. Conditions (a)–(d) are also satisfied with $\Sigma_1^0(\tau) = \sigma_{11}(\beta^0) = \tau/\beta^0$, $H_0(t) = 0$, and $G_0(t) = 2/(\beta^0 - \epsilon)^3$ for some $\epsilon > 0$. Therefore, by Theorem 3.2, there exists a $\hat{\boldsymbol{\theta}} = (\hat{\beta},\hat{\gamma}_1,\ldots,\hat{\gamma}_p,\hat{\alpha}_1,\ldots,\hat{\alpha}_p)^{\mathrm{t}}$ with $\hat{\beta} = N_0(\tau)/\tau$ and $(\hat{\boldsymbol{\gamma}},\hat{\boldsymbol{B}}) \equiv (\hat{\boldsymbol{\gamma}},s^{(n)}\hat{\boldsymbol{\alpha}})$ solving (2.12) and (2.13) such that $\hat{\boldsymbol{\theta}} \xrightarrow{\mathrm{pr}} \boldsymbol{\theta}^0$. In addition, by Theorem 3.3, as $n \to \infty$, $\hat{\boldsymbol{\theta}} \sim \mathrm{AN}(\boldsymbol{\theta}^0,(s^{(n)}\boldsymbol{\Sigma}^0(\tau))^{-1})$, where $\boldsymbol{\Sigma}^0(\tau)$ is as defined in (3.8) with $\Sigma_1^0(\tau) = \tau/\beta^0$. In this situation, $(\boldsymbol{\Sigma}^0(\tau))^- = (\boldsymbol{\Sigma}^0(\tau))^{-1}$.

The Laplace transform of $\Lambda_0(t;\beta) = \beta t$ is $\varphi^0(u,\beta) = \beta/(u+\beta)$ for $u > -\beta$. From (2.15), the reliability function estimator at the end of the testing process is therefore

$$\hat{\rho}_\tau = \frac{\hat{\beta}}{\mathbf{1}^{\mathrm{t}}(\hat{\boldsymbol{\gamma}} \otimes (\hat{\boldsymbol{B}} - \boldsymbol{N}_*(\tau))) + \hat{\beta}} = \frac{N_0(\tau)}{\tau \sum_{j=1}^{p} \hat{\gamma}_d(\hat{B}_d - N_d(\tau)) + N_0(\tau)}.$$

To obtain its asymptotic properties, we note that the Laplace transform associated with $\Lambda_0(\cdot;\beta)$ satisfies $\varphi(us,\beta) = \beta/(u+\beta) \equiv \varphi^{(*)}(u,\beta)$, so $\varphi_{10}^{(*)}(u,\beta) = -\beta/(u+\beta)^2$ and $\varphi_{01}^{(*)}(u,\beta) = u/(u+\beta)^2$, which are both nonzero at $(u,\beta) = (\nu^0(\tau),\beta^0)$ with $\nu^0(\tau) = \mathbf{1}^{\mathrm{t}}(\boldsymbol{\gamma}^0 \otimes \boldsymbol{\alpha}^0 \otimes \boldsymbol{\eta}^0(\tau)) = \sum_{d=1}^{p} \alpha_d^0 \gamma_d^0 \exp(-\gamma_d^0 \tau)$. From Theorem 4.1, as $n \to \infty$, $\hat{\rho}_\tau^{(n)}$ is therefore asymptotically normal with mean $\rho_\tau^{(*)}$ and asymptotic variance $\vartheta^2(\tau;\boldsymbol{\theta}^0)/s^{(n)}$, where

$$\hat{\rho}_\tau^{(n)} = \frac{N_0(\tau)}{\tau \sum_{d=1}^{p} \hat{\gamma}_d(s^{(n)}\hat{\alpha}_d - N_d(\tau)) + N_0(\tau)};$$

$$\rho_\tau^{(*)} = \frac{\beta^0}{\nu^0(\tau) + \beta^0}; \tag{5.1}$$

$$\vartheta^2(\tau;\boldsymbol{\theta}^0) = \frac{\beta^0}{(\nu^0(\tau)+\beta^0)^4} \times \left\{ \frac{\nu^0(\tau)^2}{\tau} + \beta^0 \sum_{d=1}^{p} \alpha_d^0(\gamma_d^0 \eta_d^0(\tau))^2 \times \left[\frac{2\tau\gamma_d^0 + 1 - (\eta_d^0(\tau))^2 + (\tau\gamma_d^0)^2(1+\eta_d^0(\tau))}{(1-\eta_d^0(\tau))^2 - (\tau\gamma_d^0)^2\eta_d^0(\tau)} \right] \right\}.$$

Finally, a $100(1-\varepsilon)\%$ lower confidence bound for the system reliability can be obtained using (4.4) and (5.1), with $\boldsymbol{\theta}^0$ replaced by $\hat{\boldsymbol{\theta}}$.

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6. SOME SIMULATION RESULTS

To serve as numerical illustrations and to gain some insights into the finite-sample properties of the system reliability estimator, modest simulations were performed. In these simulations, the task completion times are exponentially distributed, and the series system is assumed to have two components, so that $p = 2$. An arbitrarily chosen value of $\boldsymbol{\theta}^0 = (\beta^0, \gamma_1^0, \alpha_1^0, \gamma_2^0, \alpha_2^0) = (8, 0.20, 50, 0.25, 55)$ was fixed, and the values of τ, given by $\tau = 10$ and $\tau = 15$, were considered. For this value of $\boldsymbol{\theta}^0$, the system reliability at $\tau = 0$ is 0.2519. Furthermore, we took $s^{(n)} = 10$ and $s^{(n)} = 50$, where we recall that $s^{(n)}$ represents the "size" of the system. For each pair of $(\tau, s^{(n)})$, 500 realizations of $\hat{\rho}_\tau$ were generated. The simulation program was coded in Fortran and subroutines from the IMSL Math/Stat Library were used to generate the failure times. The histograms of the realizations of $\hat{\rho}_\tau$ for the pairs of values of τ and $s^{(n)}$ are presented in Figure 1, with these plots generated using S-Plus.

Observe that for the larger value of τ, the simulated sampling distribution of $\hat{\rho}_\tau$ is more skewed to the left. This left-skewness could be attributed to the fact that the values of $\hat{\rho}_\tau$ are getting closer to unity as τ increases, which is expected since a longer testing period will result in a more reliable system. Notice, however, that as $s^{(n)}$ increases, the normal approximation becomes better. Note that the higher degree of skewness at larger values of τ for fixed $s^{(n)}$ does not contradict the asymptotic results presented in Section 4 since the asymptotic set-up there fixes τ while letting $s^{(n)} \to \infty$.

For the pairs of values of τ and $s^{(n)}$, we also compared the standard deviations of $\hat{\rho}_\tau$ obtained from the simulated data with the asymptotic variance given by (5.1). Recall that $\hat{\rho}_\tau \sim \text{AN}(\rho_\tau^{(*)}, \vartheta^2(\tau; \boldsymbol{\theta}^0)/s^{(n)})$. Moreover, the achieved (simulated) coverage probability of the 90% lower confidence bound presented in (4.4) is examined. These results are summarized in Table 1. We see that the achieved coverage probability of the lower confidence bound becomes more consistent with the desired asymptotic level as $s^{(n)}$ gets larger. Note that for $\tau = 15$, the coverage probability is greater than the desired asymptotic level which results in a more conservative bound. This could be attributed to the left-skewness exhibited by $\hat{\rho}_\tau$. Furthermore, note that the mean of the simulated values of $\hat{\rho}_\tau$ is close, but tends to be larger, to the expected system reliability at τ. Finally, the difference between the standard deviations of $\hat{\rho}_\tau$ and the asymptotic standard deviation $\vartheta(\tau; \boldsymbol{\theta}^0)/\sqrt{s^{(n)}}$ diminishes as we increase $s^{(n)}$ from 10 to 15 with τ fixed. These observations are to be expected, since the asymptotic set-up involves increasing $s^{(n)}$ while holding the value of τ fixed.

7. PROOFS OF THEOREMS AND LEMMA 4.1

Proof of Theorem 3.1: Part (i) follows from Kurtz's Lemma [2, pp. 87–88] because of the relationship in (3.5). Since $N_d(t) \leq s\alpha_d^0$ for every $d = 1, \ldots, p$, then $\hat{X}_d(t; \boldsymbol{\theta}^0) \leq \alpha_d^0$, so $\phi_d(t; \hat{\boldsymbol{X}}^{t-}, \boldsymbol{\theta}^0) = \gamma_d^0(\alpha_d^0 - \hat{X}_d(t-; \boldsymbol{\theta}^0))$, for $d = 1, \ldots, p$. Thus, it follows from Andersen et al. [2, pp. 430–432] that $\hat{X}_d(t; \boldsymbol{\theta}^0) = \alpha_d^0(1 - \exp(-\gamma_d^0 t))$, $d = 1, \ldots, p$ [25]. It remains to find $\hat{X}_0(t; \boldsymbol{\theta}^0)$. Clearly, since $N_\bullet(t)/s$ is nondecreasing

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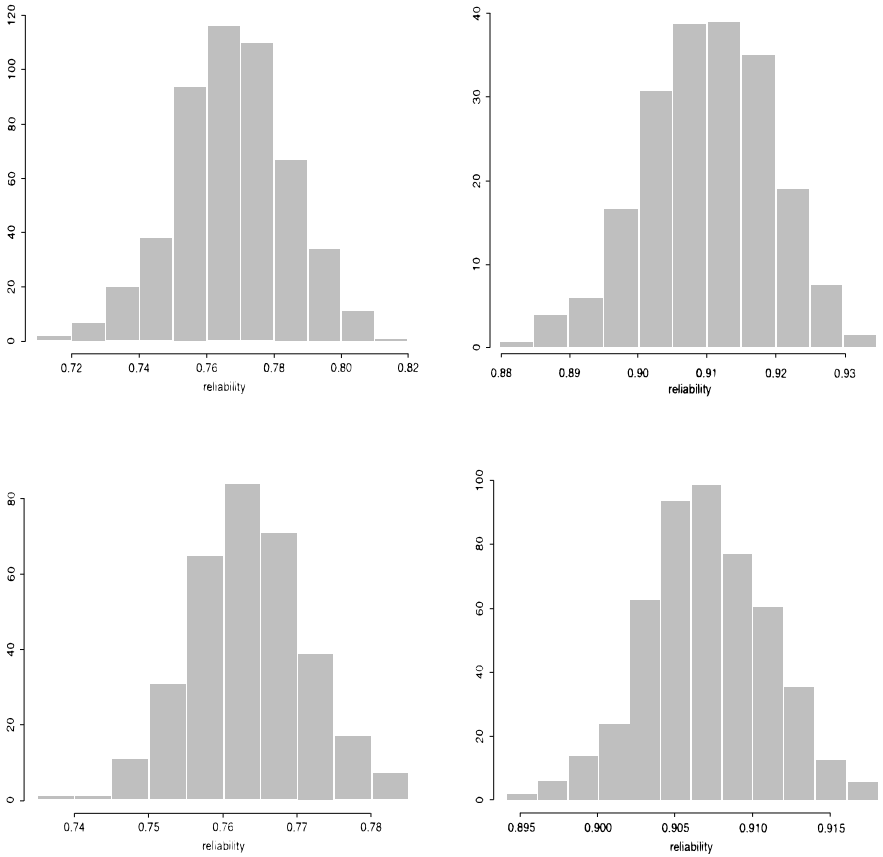


FIGURE 1. Histograms of 500 simulated values of $\hat{\rho}_\tau$ for $(\tau, s^{(n)}) \in \{(10,10), (15,10), (10,50), (15,50)\}$ when $\boldsymbol{\theta}^0 = (\beta^0, \gamma_1^0, \alpha_1^0, \gamma_2^0, \alpha_1^0) = (8, 0.20, 50, 0.25, 55)$. The histograms in clockwise order, starting from the top left, correspond to (10,10), (15,10), (15,50), and (10,50), respectively.

in t, then $\hat{X}_\cdot(t;\boldsymbol{\theta}^0)$ is also nondecreasing in t. If $\hat{X}_\cdot(t;\boldsymbol{\theta}^0)$ is strictly increasing, then $\hat{X}_\cdot^{-1}(\hat{X}_\cdot(t-;\boldsymbol{\theta}^0);\boldsymbol{\theta}^0) = t$ for every $t \geq 0$, and by (i),

$$\hat{X}_0(t;\boldsymbol{\theta}^0) = \int_0^t \lambda_0^{(0)}(0;\boldsymbol{\beta}^0)\, du = \zeta_0(\boldsymbol{\beta}^0)t.$$

It remains to show, therefore, that $\hat{X}_\cdot(t;\boldsymbol{\theta}^0)$ is strictly increasing. Since $\hat{X}_d(t;\boldsymbol{\theta}^0) \geq 0$ for $d = -1, 0, \ldots, p$, it suffices to show that $\hat{X}_0(t;\boldsymbol{\theta}^0)$ is strictly increasing. Since $\lambda_0^{(0)}(0;\boldsymbol{\beta}^0) > 0$, then there exists an $\epsilon > 0$ such that $\lambda_0^{(0)}(t;\boldsymbol{\beta}^0) > 0$ for every $t \in (0, \epsilon)$. Suppose there exists an interval $[a, b)$ such that $\hat{X}_0(t;\boldsymbol{\theta}^0) = \hat{X}_0(a;\boldsymbol{\theta}^0)$ for every

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TABLE 1. Summary of Results with $\boldsymbol{\theta}^0 = (\beta^0, \gamma_1^0, \alpha_1^0, \gamma_2^0, \alpha_1^0) = (8, 0.20, 50, 0.25, 55)$

	$s^{(n)} = 10$		$s^{(n)} = 50$	
τ	10	15	10	15
Mean of $\hat{\rho}_\tau$	.7680	.9102	.7634	.9071
Value of $\rho_\tau^{(*)}$	.7632	.9069	.7632	.9069
Coverage probability of 90% lower confidence bound	.524	.968	.887	.988
Standard deviation of $\hat{\rho}_\tau$	.0166	.0093	.0078	.0040
Value of $\vartheta(\tau;\boldsymbol{\theta}^0)/\sqrt{s^{(n)}}$	.0376	.0164	.0168	.0073

$t \in [a, b)$, that is, $\hat{X}_0(\cdot;\boldsymbol{\theta}^0)$ is not strictly increasing. Choose a $w \in (a, b)$ such that $w - a < \epsilon$. Then,

$$\hat{X}_0(w;\boldsymbol{\theta}^0) = \hat{X}_0(a;\boldsymbol{\theta}^0) + \int_a^w \lambda_0^{(0)}(u - \hat{X}_\cdot^{-1}(\hat{X}_\cdot(u-;\boldsymbol{\theta}^0);\boldsymbol{\theta}^0);\boldsymbol{\beta}^0)\, du.$$

Since $0 < u - \hat{X}_\cdot^{-1}(\hat{X}_\cdot(u-;\boldsymbol{\theta}^0);\boldsymbol{\theta}^0) < \epsilon$ for $u \in (a, w]$, and $\lambda_0^{(0)}(t;\boldsymbol{\beta}^0) > 0$ for $t \in (0, \epsilon)$, then

$$\int_a^w \lambda_0^{(0)}(u - \hat{X}_\cdot^{-1}(\hat{X}_\cdot(u-;\boldsymbol{\theta}^0);\boldsymbol{\theta}^0);\boldsymbol{\beta}^0)\, du > 0.$$

Consequently, $\hat{X}_0(w;\boldsymbol{\theta}^0) > \hat{X}_0(a;\boldsymbol{\theta}^0)$ which contradicts the supposition. It must be the case, therefore, that $\hat{X}_\cdot(\cdot;\boldsymbol{\theta}^0)$ is strictly increasing. ■

PROOF OF THEOREM 3.2: We need to show the following:

$$s^{-1} U_\tau^k(\boldsymbol{\theta}^0) \xrightarrow{\text{pr}} 0; \tag{7.1}$$

$$s^{-1} \mathcal{I}_\tau^{kl}(\boldsymbol{\theta}^0) \xrightarrow{\text{pr}} -\sigma_{kl}(\boldsymbol{\theta}^0); \tag{7.2}$$

and there exists a constant $G > 0$, not depending on $\boldsymbol{\theta}$ such that

$$\lim_{n\to\infty} \mathbf{P}\{|s^{-1} R_\tau^{klm}(\boldsymbol{\theta})| < G\} = 1, \tag{7.3}$$

for all $\boldsymbol{\theta}$ in a neighborhood around $\boldsymbol{\theta}^0$. Recall that for $k, l, m = 1, \ldots, q + 2p$,

$$U_\tau^k(\boldsymbol{\theta}) = \int_0^\tau \sum_{d=0}^{p} \left[\frac{\partial}{\partial \theta_k} \log \lambda_d(t;\boldsymbol{\theta} | \mathcal{F}_{t-}) \right] dM_d(t;\boldsymbol{\theta}),$$

$$\mathcal{I}_\tau^{kl}(\boldsymbol{\theta}) = \frac{\partial}{\partial \theta_l} U_\tau^k(\boldsymbol{\theta}) \quad \text{and} \quad R_\tau^{klm}(\boldsymbol{\theta}) = \frac{\partial}{\partial \theta_m} \mathcal{I}_\tau^{kl}(\boldsymbol{\theta}).$$

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The proof will then follow along the lines of Borgan [7] who uses the standard arguments of Cramér given in Billingsley [5]. To establish (7.1), we show that

$$s^{-1}\langle U_\cdot^k(\boldsymbol{\theta}^0), U_\cdot^k(\boldsymbol{\theta}^0)\rangle(\tau) \equiv s^{-1}\langle U_\cdot^k(\boldsymbol{\theta}^0)\rangle(\tau) \xrightarrow{\text{pr}} \sigma_{kk}(\boldsymbol{\theta}^0), \tag{7.4}$$

and then apply Lenglart's Inequality (see [2, p. 86]) to obtain (7.1). For $k = 1, \ldots, q$,

$$\frac{1}{s}\langle U_\cdot^k(\boldsymbol{\theta}^0)\rangle(\tau) = \int_0^\tau \frac{\left[\dfrac{\partial}{\partial\beta_k}\lambda_0^{(0)}(t - \tau_{N_\cdot(t-)}; \boldsymbol{\beta}^0)\right]^2}{\lambda_0^{(0)}(t - \tau_{N_\cdot(t-)}; \boldsymbol{\beta}^0)}\,\mathrm{d}t.$$

Equation (7.4) will then follow if we can show that

$$\sup_{t\le\tau}\left|\frac{\left[\dfrac{\partial}{\partial\beta_k}\lambda_0^{(0)}(t - \tau_{N_\cdot(t-)}; \boldsymbol{\beta}^0)\right]^2}{\lambda_0^{(0)}(t - \tau_{N_\cdot(t-)}; \boldsymbol{\beta}^0)} - \frac{\left[\dfrac{\partial}{\partial\beta_k}\lambda_0^{(0)}(0; \boldsymbol{\beta}^0)\right]^2}{\lambda_0^{(0)}(0; \boldsymbol{\beta}^0)}\right| \xrightarrow{\text{pr}} 0. \tag{7.5}$$

This result allows the interchange of limiting and integration operations. From Kurtz's Lemma, $N_{-1}(t)/s$ converges in probability to some cádlág function $\hat{X}_{-1}(t)$. Hence,

$$\sup_{t\le\tau}\left|\frac{N_\cdot(t)}{s} - \hat{X}_\cdot(t; \boldsymbol{\theta}^0)\right| \xrightarrow{\text{pr}} 0,$$

where $\hat{X}_\cdot(\cdot; \boldsymbol{\theta}^0)$ is strictly increasing and cádlág. Using the representation in (3.2) and the relationship $\tau_{N_\cdot(t-)} \le t \le \tau_{N_\cdot(t)}$, (7.5) follows, since

$$\sup_{t\le\tau}\left|\tau_{N_\cdot(t)} - \tau_{N_\cdot(t-)}\right| \xrightarrow{\text{pr}} 0.$$

For each of the p components, we apply the results in [2, pp. 430–432], which were obtained for the one component case. For $k = q+1, \ldots, q+p$,

$$\frac{1}{s}\langle U_\cdot^k(\boldsymbol{\theta}^0), U_\cdot^k(\boldsymbol{\theta}^0)\rangle(\tau) \xrightarrow{\text{pr}} \frac{\alpha_{k-q}^0(1 - \eta_{k-q}^0(\tau))}{(\gamma_{k-q}^0)^2} = \sigma_{kk}(\boldsymbol{\theta}^0),$$

and for $k = q+p+1, \ldots, q+2p$,

$$\frac{1}{s}\langle U_\cdot^k(\boldsymbol{\theta}^0), U_\cdot^k(\boldsymbol{\theta}^0)\rangle(\tau) \xrightarrow{\text{pr}} \frac{1 - \eta_{k-q-p}^0(\tau)}{\alpha_{k-q-p}^0\eta_{k-q-p}^0(\tau)} = \sigma_{kk}(\boldsymbol{\theta}^0).$$

Hence, (7.4) is satisfied. Next, we show that (7.2) is satisfied. Note that we can write

$$\begin{aligned} s^{-1}\mathcal{I}_\tau^{kl}(\boldsymbol{\theta}^0) = {} & \frac{1}{s}\sum_{d=0}^{p}\int_0^\tau\left[\frac{\partial^2}{\partial\theta_k\partial\theta_l}\log\lambda_d(t; \boldsymbol{\theta}^0 \mid \mathcal{F}_{t-})\right]\mathrm{d}M_d(t; \boldsymbol{\theta}^0) \\ & - \frac{1}{s}\sum_{d=0}^{p}\int_0^\tau\left[\frac{\dfrac{\partial}{\partial\theta_k}\lambda_d(t; \boldsymbol{\theta}^0 \mid \mathcal{F}_{t-})\dfrac{\partial}{\partial\theta_l}\lambda_d(t; \boldsymbol{\theta}^0 \mid \mathcal{F}_{t-})}{\lambda_d(t; \boldsymbol{\theta}^0 \mid \mathcal{F}_{t-})}\right]\mathrm{d}t. \end{aligned} \tag{7.6}$$

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The first term converges to zero in probability by using condition (d), Theorem 3.1, and Lenglart's Inequality. By condition (c) and Theorem 3.1, we have

$$\frac{1}{s}\sum_{d=0}^{p}\int_0^{\tau}\left[\frac{\frac{\partial}{\partial\theta_k}\lambda_d(t;\boldsymbol{\theta}^0\,|\,\mathcal{F}_{t-})\,\frac{\partial}{\partial\theta_l}\lambda_d(t;\boldsymbol{\theta}^0\,|\,\mathcal{F}_{t-})}{\lambda_d(t;\boldsymbol{\theta}^0\,|\,\mathcal{F}_{t-})}\right]\mathrm{d}t \xrightarrow{\text{pr}} \sigma_{kl}(\boldsymbol{\theta}^0).$$

Hence, (7.2) is satisfied. Finally, to show (7.3), note that for $d = 1,\ldots,p$,

$$\frac{\partial^3}{\partial\gamma_d^3}\log\lambda_d(t;\boldsymbol{\theta}|\,\mathcal{F}_{t-}) = \frac{2}{\gamma_d^3} \quad\text{and}\quad \frac{\partial^3}{\partial\alpha_d^3}\log\lambda_d(t;\boldsymbol{\theta}|\,\mathcal{F}_{t-}) = \frac{2}{\left[\alpha_d - \dfrac{N_d(t-)}{s}\right]^3},$$

while the remaining third order partial derivatives of $\lambda_d(t;\boldsymbol{\theta}|\,\mathcal{F}_{t-})$ and $\log\lambda_d(t;\boldsymbol{\theta}|\,\mathcal{F}_{t-})$ with respect to γ_i and $\alpha_i, (i = 1,\ldots,p,\ i \neq d)$, are equal to 0. For $\epsilon > 0$, consider the set $\boldsymbol{O}_\epsilon = \{\boldsymbol{\theta} : \|\boldsymbol{\theta} - \boldsymbol{\theta}^0\| \leq \epsilon\}$. Hence, for $\boldsymbol{\theta} \in \boldsymbol{O}_\epsilon$,

$$\left|\frac{R_\tau^{klm}(\boldsymbol{\theta})}{s}\right| \leq \sum_{d=0}^{p}\int_0^{\tau}\frac{G_d(t)}{s}\,\mathrm{d}N_d(t) + \int_0^{\tau}H_0(t-\tau_{N_\cdot(t-)})\,\mathrm{d}t,$$

where $G_d(t) = \max\{2/[\gamma_d^0 - \epsilon]^3, 2/[\alpha_d^0\exp(-\gamma_d^0 t) - \epsilon]^3\}$ for $d = 1,\ldots,p$. By condition (d),

$$\frac{1}{s}\int_0^{\tau}G_d(t)I\{|s^{-1}G_d(t)| > \epsilon\}\lambda_d(t;\boldsymbol{\theta}|\,\mathcal{F}_{t-})\,\mathrm{d}t \xrightarrow{\text{pr}} 0.$$

Using the Martingale Central Limit Theorem,

$$\frac{1}{s}\int_0^{\tau}G_d(t)\lambda_d(t;\boldsymbol{\theta}^0\,|\,\mathcal{F}_{t-})\,\mathrm{d}t \quad\text{and}\quad \frac{1}{s}\int_0^{\tau}G_d(t)\,\mathrm{d}N_d(t)$$

converge to the same finite quantity. By condition (d), $\int_0^{\tau}H_0(t-\tau_{N_\cdot(t-)})\,\mathrm{d}t$ converges to some finite quantity. Thus, (7.3) follows. ■

Proof of Theorem 3.3: As in the previous theorem, our proof will follow the same arguments as Borgan [7]. We need to show that

$$\frac{1}{\sqrt{s}}\boldsymbol{U}_\tau(\boldsymbol{\theta}^0) \xrightarrow{d} \boldsymbol{N}_{q+2p}(\mathbf{0},\boldsymbol{\Sigma}^0); \tag{7.7}$$

and

$$s^{-1}\mathcal{I}_\tau^{kl}(\boldsymbol{\theta}^*) \xrightarrow{\text{pr}} -\sigma_{kl}(\boldsymbol{\theta}^0), \qquad k,l = 1,\ldots,q+2p, \tag{7.8}$$

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whenever $\boldsymbol{\theta}^* \xrightarrow{\text{pr}} \boldsymbol{\theta}^0$. To show (7.7), for $\epsilon > 0$, let

$$\boldsymbol{W}_t(\boldsymbol{\theta}^0) = \left(\frac{U_t^k(\boldsymbol{\theta}^0)}{\sqrt{s}} : k = 1, \ldots, q + 2p \right),$$

$$\boldsymbol{W}_t^{\epsilon}(\boldsymbol{\theta}^0) = \left(\frac{U_t^{k\epsilon}(\boldsymbol{\theta}^0)}{\sqrt{s}} : k = 1, \ldots, q + 2p \right),$$

where

$$U_t^{k\epsilon}(\boldsymbol{\theta}^0) = \sum_{d=0}^{p} \int_0^t \left[\frac{\partial}{\partial \theta_k} \log \lambda_d(u; \boldsymbol{\theta}^0 \,|\, \mathcal{F}_{u-}) \right] \times I\left\{ \left| \frac{1}{\sqrt{s}} \frac{\partial}{\partial \theta_k} \log \lambda_d(u; \boldsymbol{\theta}^0 \,|\, \mathcal{F}_{u-}) \right| > \epsilon \right\} \mathrm{d}M_d(u; \boldsymbol{\theta}^0).$$

Recall from (7.4) that $s^{-1}\langle U_\cdot^k(\boldsymbol{\theta}^0)\rangle(\tau) \xrightarrow{\text{pr}} \sigma_{kk}(\boldsymbol{\theta}^0)$. When $k, l = 1, \ldots, q$, $(k \neq l)$,

$$\begin{aligned} & s^{-1}\langle U_\cdot^k(\boldsymbol{\theta}^0), U_\cdot^l(\boldsymbol{\theta}^0)\rangle(\tau) \\ &= \int_0^\tau \left[\frac{\partial}{\partial \beta_k} \log \lambda_0^{(0)}(t - \tau_{N_\cdot(t-)}; \boldsymbol{\beta}^0) \right] \left[\frac{\partial}{\partial \beta_l} \log \lambda_0^{(0)}(t - \tau_{N_\cdot(t-)}; \boldsymbol{\beta}^0) \right] \\ &\quad \times \lambda_0^{(0)}(t - \tau_{N_\cdot(t-)}; \boldsymbol{\beta}^0)\, \mathrm{d}t \\ &\xrightarrow{\text{pr}} \left[\frac{\partial}{\partial \beta_k} \log \lambda_0^{(0)}(0; \boldsymbol{\beta}^0) \right] \left[\frac{\partial}{\partial \beta_l} \log \lambda_0^{(0)}(0; \boldsymbol{\beta}^0) \right] \lambda_0^{(0)}(0; \boldsymbol{\beta}^0)\tau \equiv \sigma_{kl}(\boldsymbol{\theta}^0), \end{aligned}$$

where the integration and limiting operations can be interchanged because of (7.5). When $k = q + 1, \ldots, q + p$ and $l = q + p + 1, \ldots, q + 2p$,

$$s^{-1}\langle U_\cdot^k(\boldsymbol{\theta}^0), U_\cdot^l(\boldsymbol{\theta}^0)\rangle(\tau) = s^{-1}\langle U_\cdot^l(\boldsymbol{\theta}^0), U_\cdot^k(\boldsymbol{\theta}^0)\rangle(\tau) = \tau \equiv \sigma_{kl}(\boldsymbol{\theta}^0) = \sigma_{lk}(\boldsymbol{\theta}^0).$$

Thus,

$$\langle \boldsymbol{W}(\boldsymbol{\theta}^0), \boldsymbol{W}(\boldsymbol{\theta}^0)\rangle(\tau) \xrightarrow{\text{pr}} \boldsymbol{\Sigma}^0. \tag{7.9}$$

Next, we show that

$$\langle W_\cdot^{k\epsilon}(\boldsymbol{\theta}^0), W_\cdot^{k\epsilon}(\boldsymbol{\theta}^0)\rangle(\tau) \xrightarrow{\text{pr}} 0. \tag{7.10}$$

First, consider the case where $k = 1, \ldots, q$, so that

$$\begin{aligned} & \frac{1}{s}\langle W_\cdot^{k\epsilon}(\boldsymbol{\theta}^0), W_\cdot^{k\epsilon}(\boldsymbol{\theta}^0)\rangle(\tau) \\ &= \int_0^\tau \frac{\left[\frac{\partial}{\partial \beta_k} \lambda_0^{(0)}(t - \tau_{N_\cdot(t-)}; \boldsymbol{\beta}^0) \right]^2}{\lambda_0^{(0)}(t - \tau_{N_\cdot(t-)}; \boldsymbol{\beta}^0)} \\ &\quad \times \mathrm{I}\left\{ \left| \frac{1}{\sqrt{s}} \frac{\partial}{\partial \beta_k} \log \lambda_0^{(0)}(t - \tau_{N_\cdot(t-)}; \boldsymbol{\beta}^0) \right| > \epsilon \right\} \mathrm{d}t. \end{aligned}$$

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By condition (b), the first term converges in probability to

$$\frac{\left[\dfrac{\partial}{\partial\beta_k}\lambda_0^{(0)}(0;\boldsymbol{\beta}^0)\right]^2}{\lambda_0^{(0)}(0;\boldsymbol{\beta}^0)}$$

which is finite, while the second term converges in probability to 0. Hence, by the same arguments used in showing (7.4), (7.10) is satisfied for $k = 1,\ldots,q$. Next, consider the case where $k = q+1,\ldots,q+p$, so that

$$\frac{1}{s}\langle W_\cdot^{k\epsilon}(\boldsymbol{\theta}^0), W_\cdot^{k\epsilon}(\boldsymbol{\theta}^0)\rangle(\tau) = \int_0^\tau \left[\frac{\left(\alpha_{k-1}^0 - \dfrac{N_{k-q}(t-)}{s}\right)}{\gamma_{k-q}^0}\right] \mathrm{I}\left\{\left|\frac{1}{\sqrt{s}\gamma_{k-q}^0}\right| > \epsilon\right\} \mathrm{d}t.$$

Applying (3.7), the first factor converges in probability to a finite quantity. Thus, (7.10) is satisfied. Finally, consider the case where $k = q+p+1,\ldots,q+2p$, so that

$$s^{-1}\langle W_\cdot^{k\epsilon}(\boldsymbol{\theta}^0), W_\cdot^{k\epsilon}(\boldsymbol{\theta}^0)\rangle(\tau) = \int_0^\tau \left[\frac{\gamma_{k-q-p}^0}{\alpha_{k-q-p}^0 - \dfrac{N_{k-q-p}(t-)}{s}}\right] \times \mathrm{I}\left\{\left|\frac{1}{\sqrt{s}\left[\alpha_{k-q-p}^0 - \dfrac{N_{k-q-p}(t-)}{s}\right]}\right| > \epsilon\right\} \mathrm{d}t.$$

Applying (3.7) results in (7.10). This completes the proof of (7.10) for $k = 1,\ldots,q+2p$. Combining this result with (7.9), (7.7) follows from the Martingale Central Limit Theorem. That (7.8) holds is evident by taking a Taylor expansion of $\mathcal{I}_\tau^{kl}(\boldsymbol{\theta}^*)$ around $\boldsymbol{\theta}^0$ and using the results obtained in Theorem 3.2. Using the definition of the generalized inverse [21] and applying Theorem 2.4.5 of Anderson [3] guarantees that the limiting distribution is valid. This completes the proof of the theorem. ■

Proof of Lemma 4.1: Using (3.6), we obtain

$$\begin{aligned}
&\sqrt{s}\left[\frac{\boldsymbol{N}_*(\tau)}{s} - \hat{\boldsymbol{X}}_*(\tau;\boldsymbol{\theta}^0)\right] \\
&\quad = \sqrt{s}\left[\frac{\boldsymbol{N}_*(\tau)}{s} - \int_0^\tau \boldsymbol{\phi}_*\left(t;\frac{\boldsymbol{N}^{t-}}{s},\boldsymbol{\theta}^0\right)\mathrm{d}t\right] \\
&\qquad + \sqrt{s}\left[\int_0^\tau \boldsymbol{\phi}_*\left(t;\frac{\boldsymbol{N}^{t-}}{s},\boldsymbol{\theta}^0\right)\mathrm{d}t - \hat{\boldsymbol{X}}_*(\tau;\boldsymbol{\theta}^0)\right] \\
&\quad = \frac{1}{\sqrt{s}}\int_0^\tau \mathrm{d}\boldsymbol{M}_*(t;\boldsymbol{\theta}^0) + \sqrt{s}\left[\int_0^\tau \left\{\boldsymbol{\phi}_*\left(t;\frac{\boldsymbol{N}^{t-}}{s},\boldsymbol{\theta}^0\right) - \boldsymbol{\phi}_*(t;\hat{\boldsymbol{X}}^{t-},\boldsymbol{\theta}^0)\,\mathrm{d}t\right\}\right],
\end{aligned}$$

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where $\boldsymbol{M}_*(t;\boldsymbol{\theta}) = (M_1(t;\boldsymbol{\theta}),\ldots,M_p(t;\boldsymbol{\theta}))^{\mathrm{t}}$ and $\boldsymbol{\phi}_*(t;\boldsymbol{x}^{t-},\boldsymbol{\theta}) = \boldsymbol{\gamma} \otimes (\boldsymbol{\alpha} - \boldsymbol{x}_*(t-))$. Let $\boldsymbol{\phi}'_*(t;\boldsymbol{y},\boldsymbol{\theta}) = (\partial/\partial \boldsymbol{y})\boldsymbol{\phi}_*(t;\boldsymbol{y},\boldsymbol{\theta})$ and $\boldsymbol{\phi}''_*(t;\boldsymbol{y},\boldsymbol{\theta}) = (\partial/\partial \boldsymbol{y}^t)\boldsymbol{\phi}'_*(t;\boldsymbol{y},\boldsymbol{\theta})$. Using a Taylor series expansion, we have

$$\begin{aligned}
&\boldsymbol{\phi}_*\left(t;\frac{\boldsymbol{N}^{t-}}{s},\boldsymbol{\theta}^0\right) - \boldsymbol{\phi}_*(t;\hat{\boldsymbol{X}}^{t-},\boldsymbol{\theta}^0) \\
&\quad = \boldsymbol{\phi}'_*(t;\hat{\boldsymbol{X}}^{t-},\boldsymbol{\theta}^0)\left(\frac{\boldsymbol{N}_*(t-)}{s} - \hat{\boldsymbol{X}}_*(t-;\boldsymbol{\theta}^0)\right) + \frac{1}{2}\left(\frac{\boldsymbol{N}_*(t-)}{s} - \hat{\boldsymbol{X}}_*(t-;\boldsymbol{\theta}^0)\right)^{\mathrm{t}} \\
&\qquad \times \boldsymbol{\phi}''_*(t;\boldsymbol{Y}^{t-},\boldsymbol{\theta}^0)\left(\frac{\boldsymbol{N}_*(t-)}{s} - \hat{\boldsymbol{X}}_*(t-;\boldsymbol{\theta}^0)\right), \qquad \textbf{(7.11)}
\end{aligned}$$

for

$$\boldsymbol{Y} \in \left[\hat{\boldsymbol{X}}_*, \frac{\boldsymbol{N}_*}{s}\right].$$

Since $(\partial/\partial \boldsymbol{y})\boldsymbol{\phi}_*(t;\boldsymbol{y},\boldsymbol{\theta}^0) = -\boldsymbol{\gamma}^0$, then $\boldsymbol{\phi}''_*(t;\boldsymbol{y},\boldsymbol{\theta}^0) = \boldsymbol{0}$. Thus, the second term in (7.11) is equal to $\boldsymbol{0}$. Hence,

$$\begin{aligned}
&\sqrt{s}\left[\frac{\boldsymbol{N}_*(\tau)}{s} - \hat{\boldsymbol{X}}_*(\tau;\boldsymbol{\theta}^0)\right] \\
&\quad = \frac{1}{\sqrt{s}}\int_0^\tau \mathrm{d}\boldsymbol{M}_*(t;\boldsymbol{\theta}^0) + \frac{1}{\sqrt{s}}\int_0^\tau (-\gamma^0)^{\mathrm{t}}(\boldsymbol{N}_*(t-) - s\hat{\boldsymbol{X}}_*(t-;\boldsymbol{\theta}^0))\,\mathrm{d}t. \qquad \textbf{(7.12)}
\end{aligned}$$

Using Theorem II.6.3 of [2], (7.12) simplifies to

$$\sqrt{s}\left[\frac{\boldsymbol{N}_*(\tau)}{s} - \hat{\boldsymbol{X}}_*(\tau;\boldsymbol{\theta}^0)\right] = \frac{1}{\sqrt{s}}\,\boldsymbol{\eta}^0(\tau)\int_0^\tau (\boldsymbol{\eta}^0(t)^{-1})^{\mathrm{t}}\,\mathrm{d}\boldsymbol{M}_*(t;\boldsymbol{\theta}^0),$$

which proves the lemma. ■

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