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A classical ergodic property for IFS: a simple proof

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Abstract. Let $\{w_i, p_i\}$ be a contractive iterated function system (IFS) [**1**, pp. 79–80] with probabilities, i.e. a set of contraction maps $w_i : X \to X$ with associated probabilities p_i, $i = 1, 2, \ldots, N$. We provide a simple proof that for almost every address sequence σ and for all x the limit $\lim_{n\to\infty} 1/n \sum_{i\le n} f(w_{\sigma_n} \circ w_{\sigma_{n-1}} \circ \cdots \circ w_{\sigma_1}(x))$ exists and is equal to $\int_X f(z)\, d\mu(z)$, where μ is the invariant measure of the IFS. This is the so called 'ergodic property' for the IFS and was proved by Elton in [**3**]. However, the uniqueness of the invariant measure was not previously exploited. This provides considerable simplification to the proof.

Let X be a compact metric space and let $\{w_i\}_{i=1}^{L}$ be a collection of L contraction maps on X. Let $\{p_i\}$ be a collection of L probabilities (i.e. $\sum_i p_i = 1$).

In [**4**] Hutchinson proves that there exists a unique measure μ invariant under the Markov operator M defined by

$$M(\nu)B = \sum_i p_i \nu(w_i^{-1}(B)),$$

where ν is a probability measure on X and B is a Borel subset of X. In fact $M^n(\nu) \to \mu$ for any probability measure ν since the w_i's are contractive (see [**4**, p. 733]).

The operator U defined as

$$U(f)(x) = \sum_i p_i f(w_i(x))$$

(where x is a point in X and f is a continuous function on X) is the adjoint to M and will play an important role in what follows.

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Let

$$\Sigma = \prod_{\mathbf{N}} \{1, 2, \ldots, L\}$$

be the code space (see [**1**, p. 364]) with P the product measure induced by the measure $p(\{i\}) = p_i$ on each factor.

We will need the projections $\pi^n : \Sigma \to \Sigma^n$ defined by $\pi^n(\sigma) = (\sigma_n, \sigma_{n-1}, \ldots, \sigma_1)$. We will denote $\pi^n(\sigma)$ by σ^n.

For $\alpha^n \in \Sigma^n$ we denote by p_{α^n} the product $p_{\alpha_n} p_{\alpha_{n-1}} \ldots p_{\alpha_1}$. Furthermore, we denote by w_{α^n} the composition

$$w_{\alpha_n} \circ w_{\alpha_{n-1}} \circ \cdots \circ w_{\alpha_1}.$$

The following theorem was proved by Elton in [**3**]. We provide a simplified proof of this result.

THEOREM 1. *For any continuous function f on X and any $x \in X$ we have*

$$\lim_{n\to\infty} 1/n \sum_{i\le n} f(w_{\sigma_i} \circ w_{\sigma_{i-1}} \circ \cdots \circ w_{\sigma_1}(x)) = \int_X f(z)\, d\mu(z)$$

for P almost all address sequences $\sigma \in \Sigma$.

Proof. Let f be a continuous function on X and let x be a fixed element of X.

We wish to show that

$$1/n \sum_{i\le n} f(w_{\sigma^i}(x))$$

converges. Let ν-lim be a Banach Limit on $l^\infty(\mathbb{N})$ (see [**2**, p. 82] for a nice discussion of Banach Limits). Recall that a Banach Limit is a 'generalized limit' in the sense that it is a bounded linear functional on l^∞ which does not depend on the first terms of the sequence in l^∞. We will show that any two Banach limits will give the same value for P almost every $\sigma \in \Sigma$ so that the limit exists almost everywhere.

Now $f \mapsto \nu\text{-lim}_{n\to\infty} 1/n \sum_{i\le n} f(w_{\sigma^i}(x))$ is a bounded linear functional on $C(X)$. Thus, by the Riesz representation theorem it corresponds to a measure μ_ν on X. Since if $f = 1$, we get the limit equals 1, we know that this measure is a probability measure.

We show that $\mu_\nu = \mu$ for P almost all σ. This will show that for almost all σ the limit exists and is what we wish it to be.

Let $S : \Sigma \to \Sigma$ denote the shift map on Σ. Since each w_i is contractive and X is compact, we know that

$$\left| 1/n \sum_{i\le n} f(w_{\sigma^i}(x)) - 1/n \sum_{i\le n} f(w_{S(\sigma)^i}(x)) \right| \to 0$$

as $n \to \infty$. Thus,

$$\nu\text{-lim}\, 1/n \sum_{i\le n} f(w_{\sigma^i}(x)) = \nu\text{-lim}\, 1/n \sum_{i\le n} f(w_{S(\sigma)^i}(x)).$$

Since the shift map on Σ is ergodic, we know that $\nu\text{-lim}_{n\to\infty} 1/n \sum_{i\le n} f(w_{\sigma^i}(x))$ is constant for P almost all σ.

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To show that $\mu = \mu_\nu$ it suffices to show that

$$\int_X f(z)\,d\mu_\nu(z) = \int_X Uf(z)\,d\mu_\nu(z)$$

which is the same as showing that

$$\nu\text{-}\lim_{n\to\infty} 1/n \sum_{i\le n} f(w_{\sigma^i}(x)) = \nu\text{-}\lim_{n\to\infty} 1/n \sum_{i\le n}\sum_j p_j f(w_j \circ w_{\sigma^i}(x))$$

Computing, we get

$$\begin{aligned}
&\nu\text{-}\lim_{n\to\infty} 1/n \sum_{i\le n}\sum_j p_j f(w_j \circ w_{\sigma^i}(x)) \\
&\quad = \int_{\sigma\in\Sigma} \nu\text{-}\lim_{n\to\infty} 1/n \sum_{i\le n}\sum_j p_j f(w_j \circ w_{\sigma^i}(x))\,dP(\sigma) \\
&\quad = \sum_j p_j \int_{\sigma\in\Sigma} \nu\text{-}\lim_{n\to\infty} 1/n \sum_{i\le n} f(w_j \circ w_{\sigma^i}(x))\,dP(\sigma).
\end{aligned}$$

Performing the change of variable $\sigma \equiv (\sigma_1, \sigma_2, \ldots) \to (j, \sigma_1, \sigma_2, \ldots)$ we get $dP \to dP/p_j$ so this integral becomes

$$\begin{aligned}
&\sum_j p_j/p_j \int_{\sigma_1=j} \nu\text{-}\lim_{n\to\infty} 1/n \sum_{i\le n} f(w_{\sigma^{i+1}}(x))\,dP(\sigma) \\
&\quad = \int_{\sigma\in\Sigma} \nu\text{-}\lim_{n\to\infty} 1/n \sum_{i\le n} f(w_{\sigma^i}(x))\,dP(\sigma) \\
&\quad = \nu\text{-}\lim_{n\to\infty} 1/n \sum_{i\le n} f(w_{\sigma^i}(x))
\end{aligned}$$

for P almost all σ.

Therefore, for P almost all σ we know that μ_ν is invariant under M so $\mu_\nu = \mu$. However, since ν was arbitrary this shows that for P almost all σ

$$\lim_{n\to\infty} 1/n \sum_{i\le n} f(w_{\sigma^i}(x)) = \int_X f(z)\,d\mu(z)$$

for all f and all $x \in X$. □

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