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HETEROGENEITY IN INFLATION EXPECTATIONS AND MACROECONOMIC DYNAMICS UNDER EVOLUTIONARILY SATISFICING LEARNING

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Drawing on the empirical evidence that heterogeneity in inflation expectations is persistent and endogenously time-varying, we embed two inflation forecasting strategies—one based on costly *ex ante* perfect foresight, and the second based on costless *ex ante* extrapolative trend-following—in a macrodynamic model. Drawing also on the empirical evidence that inflation forecast errors may have to exceed some threshold before agents abandon their previously selected forecasting strategy, we describe agents as switching between forecasting strategies according to evolutionarily satisficing learning. Convergence to a long-run equilibrium consistent with output growth, unemployment and inflation at their natural levels may be achieved even if heterogeneity in inflation forecasting strategies (with predominance of the extrapolative foresight strategy) is an attractor of an evolutionarily satisficing dynamic perturbed by mutant agents. Thus, in keeping with the empirical evidence, heterogeneity in strategies to form inflation expectations (with prevalence of bounded rationality) can be a stable long-run equilibrium.

Keywords: Heterogeneous Inflation Expectations, Perfect Foresight, Extrapolative Foresight, Satisficing Learning Dynamics

1. INTRODUCTION

There is compelling empirical evidence from survey data and laboratory experiments that inflation expectations are persistently heterogeneous and formed

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mostly through boundedly rational, rule-of-thumb mechanisms or heuristics (see, e.g., Hommes (2013), for a detailed overview). Meanwhile, there is also robust evidence from survey data that sometimes inflation forecast errors must pass some threshold level before decision-makers choose to abandon their previously selected inflation predictor (Branch 2004, 2007).

Motivated by these several pieces of empirical evidence (which we report and discuss more thoroughly in the next section), we explore implications of heterogeneous inflation expectations that vary over time as driven by *evolutionarily satisficing learning dynamics* (with and without mutation). From a methodological perspective, the contribution of this paper lies in particular in an alternative way of modeling of the dynamics of inflation predictor switching when inflation forecast errors have to cross some threshold in order to induce a switch. Our proposed modeling strategy is intended to incorporate this documented threshold or inertia effect in a more representative way than has been carried out in the existing literature (Branch 2004, 2007; Lines and Westerhoff 2010), in which such threshold is modeled as a component of the cost related to the use of a predictor (and a cost which is usually taken as exogenously fixed and deterministic). It is true, as argued in Branch (2004, 2007), first, that if forecast errors have to pass some threshold in order to induce a switch, there is inertia in predictor switching; and second, that costs act as thresholds through which forecast errors must cross to induce switching between prediction methods. In our model, however, although the existence of a cost incurred by agents to use a predictor also imposes inertia, satisficing predictor choice in and of itself is already an inherent source of inertia in predictor switching. It is therefore analytically convenient that these two distinct inherent sources of inertia in predictor switching are not bunched together in a single channel. For instance, in our model, as in Lines and Westerhoff (2010), there is an exogenously fixed cost associated with perfect inflation prediction. Yet, unlike in Lines and Westerhoff (2010), we make it explicit that this cost is associated with the fact that perfect inflation prediction requires perfectly knowing the frequency distribution of inflation predictors in the population of agents. As it was early pointed out by Brock and de Fontnouvelle (2000), the perfect foresight predictor involves very complex calculations given that it has to take into account the equilibrium effects of (all) the other predictor(s). Thus, it is analytically convenient that those two sources of inertia in predictor switching are incorporated to the model separately. Meanwhile, if the cost attached to the use of an inflation predictor, when broadly conceived, ensures that the empirical estimate of the proportion of agents using it most closely fit the data, as documented in Branch (2004, 2007), our suggested satisficing choice protocol can be seen as similarly empirically accommodating. As acknowledged in the closing of Branch (2004), the data are supportive of the model of the paper, but not completely so, given that such model could not account for inertia-generating bias toward a predictor. The author concludes that the empirical evidence found in the paper indicate that a theoretical account of these biases or predispositions is called for. In this context, our model suggests an explanation of inflation predictor choice based on evolutionarily satisficing learning.

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As further described in Section 3, our formal methodological approach follows the spirit of the suggestive contributions of Herbert Simon and conceives of choice as intending to meet an acceptability threshold rather than to select the best of all possible alternatives, hence the notion of *satisficing behavior*. Consequently, while in both Branch (2004, 2007) and Lines and Westerhoff (2010) the inflation predictor selection follows the discrete choice model in Brock and Hommes (1997), in our paper such predictor choice follows an evolutionarily satisficing learning protocol subject to a perturbation analogous to mutation in natural environments. In effect, by replacing the demand for choosing the best inflation predictor possible for the agent with a conceptually and formally precise notion of being acceptably or tolerably inaccurate an inflation prediction, our evolutionarily satisficing learning protocol offers a fruitful alternative way of rationalizing the inertia finding in Branch (2004, 2007). In the latter, the finding that agents abandon a previously selected predictor only after forecast errors cross some threshold is seen as reflecting that agents have inherent predispositions to use one predictor over another. In our model, the same finding is seen as reflecting that agents have an inherent predisposition not to one predictor over another, but to satisficing choice behavior. As our proposed satisficing behavior is subject to mutation, we are ascribing to agents another inherent predisposition, which is a predisposition to experiment once and a while (as explained shortly). In Herbert Simon's original usage of the term "satisficing", choices are identified that are suitable enough by comparing their attributes to some standard. While in Simon (1955, 1956) choices are compared to the "aspiration level" of how good a choice might be made, in the model herein inflation predictor choices are compared analogously to the aspiration level of how acceptably or tolerably inaccurate an inflation predictor choice might be made in light of the uncertainties surrounding the future.[1]

As a result, while in Branch (2004) agents are admittedly perceived as inherently predisposed to using an inflation predictor in a way that does not depend on accuracy, our conception that agents are instead inherently predisposed to satisficing choice does not mean that accuracy is overlooked. What satisficing predictor choice means is that agents, instead of looking for the highest of all possible accuracies, in fact look for an acceptable level of accuracy or, to put it another way, are satisfied with (or tolerate) a positive level of inaccuracy in inflation prediction. Therefore, our notion of satisficing predictor choice is consistent with the viewpoint stated in Branch (2004) that "the agent will not switch his prediction method because the forecast error is 'reasonable'. Only forecast errors above a threshold will induce changes in predictor choice". (p. 613) Besides, we plausibly assume that the level of inaccuracy in inflation prediction that is deemed reasonable or acceptable or tolerable is agent-specific and randomly distributed across agents. This heterogeneity across agents in acceptable prediction errors is in accordance with the empirical evidence found in Branch (2004).

The evolutionarily satisficing learning protocol suggested here, besides incorporating the threshold effect documented in Branch (2004, 2007) in a representative, conceptually sound and intuitive way, carries relevant implications for

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the long-run equilibrium configuration of a macroeconomic model as the one set forth in the related contribution by Lines and Westerhoff (2010). The evolutionarily satisficing dynamics across inflation predictor choice suggested in this paper is coupled to the equilibrium dynamics of the endogenous macroeconomic variables. Though our suggested evolutionarily satisficing choice protocol is embedded in a specific macro-model featuring inflation prediction, readers are nonetheless invited to contemplate the possibility that other instances of predictor (or, more broadly still, behavior) switching are also subject to satisficing evolutionary learning.

In keeping with the empirical evidence on inflation expectations and following Lines and Westerhoff (2010), the aggregate expected inflation is a weighted average of the inflation forecast of agents who play the costly *ex ante full rationality* or *perfect foresight* strategy and the inflation forecast of agents who play a costless *ex ante bounded rationality* or *extrapolative trend-following* foresight strategy. Differently from the discrete choice framework adopted in Lines and Westerhoff (2010), however, our evolutionarily satisficing learning protocol, if not perturbed by a noise analogous to mutation in natural environments, necessarily generates pervasive extrapolative trend-following foresight (all agents eventually decide to use the bounded rationality inflation forecasting strategy) as a long-run equilibrium. Meanwhile, in the presence of mutation, the evolutionarily satisficing learning protocol necessarily culminates with heterogeneity in strategies to establish inflation expectations as a long-run equilibrium, which is in keeping with the empirical evidence. In either situation (with and without mutation), the long-run equilibrium is nonetheless characterized by the natural values of the output growth, unemployment, and inflation being achieved. Yet, a long-run equilibrium configuration consistent with the achievement of the natural values of the endogenous macroeconomic variables does not emerge because all agents adopt the perfect foresight inflation forecasting strategy. Instead, it is either all agents (without mutation) or the greater proportion of them (with mutation) that eventually decide to follow the bounded rationality inflation forecasting strategy because the natural values of the output growth, unemployment, and inflation are being achieved.

As the general case, our model features "mutation" as a perturbation in the satisficing evolutionary protocol, leading some decision-makers to select an inflation forecasting strategy at random. We should emphasize that "mutation" is a metaphor for all sorts of payoff-unrelated reasons, outside the more explicit part of the model, for why agents might switch inflation predictor, and we take the rate of mutation as being independent across agents and over time. Two rationales for the presence of mutation are that an agent exits the economy with some (fixed) probability and is replaced with a new agent who still does not know anything about the decision-making environment, or that each agent simply experiments occasionally with exogenously fixed probability. Thus, our model features three layers of randomness: first, the level of acceptable inflation forecast inaccuracy is randomly and independently determined across agents and over time; second, the

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specific agents who behave as mutants in each period, whose measure is given by a mutation rate which is fixed and independent across agents and over time, are randomly drawn from the population; and third, agents who behave as mutants choose an inflation foresight strategy at random.

A key question that arises is whether the existence of mutation prevents convergence toward a long-run equilibrium configuration consistent with the achievement of the natural values of the endogenous macroeconomic variables, and the answer is not necessarily. The long-run equilibrium distribution of inflation forecasting strategies across agents and its local stability properties depend on whether the satisficing learning dynamics are perturbed by mutation: in the absence (presence) of mutation, the long-run equilibrium, which may be a local attractor, is characterized by the natural values of the macroeconomic variables and the extinction of the perfect foresight strategy (survival of both inflation forecasting strategies). It follows that universal perfect foresight is neither a necessary condition for the achievement of the natural values of the rates of output growth, unemployment, and inflation, nor an inevitable consequence of the achievement of such values. In fact, heterogeneity in inflation expectations on the part of agents need not necessarily thwart the successful handling of stabilization policymaking. This is the case even when inflation expectations are not only formed heterogeneously but also, as a direct outcome of the propensity of decisionmakers to switch between inflation forecasting strategies using evolutionarily satisficing criteria, endogenously time-varying.

The remainder of the paper is organized as follows. Section 2 presents the motivating empirical evidence and briefly discusses (and broadly compares the present contribution to) a representative sample of the related literature, while Section 3 lays out the structure of the basic macroeconomic model on which our analysis is based. Section 4 explores the implications for macroeconomic stability in the long-run equilibrium of heterogeneous inflation expectations that market participants switch between as described by evolutionarily satisficing learning dynamics with and without mutation. Section 5 explores global dynamics of the model through numerical simulations. Section 6 concludes.

2. MOTIVATING EVIDENCE AND RELATED LITERATURE

The extensive empirical literature on inflation expectations formation can be divided into two groups depending on the type of evidence used: studies based on survey data and laboratory experiments with human subjects. In either case, a common finding is the empirical support for time-varying heterogeneity in inflation expectations formation with predominance of some form of bounded rationality (see, e.g., Duffy (2008), and Hommes (2011, 2013), for thorough overviews of experiments dealing with formation of expectations about macroeconomic variables). The main reasons for this heterogeneity that have been suggested in the literature are that market participants rely on different models,

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have access to different information sets, or have different cognitive capacities for processing information.

In the experimental literature, Adam (2007) documents that subjects' inflation expectations are not captured by the rational expectation predictor, whereas predictors based on lagged inflation capture inflation expectations fairly well. Roos and Luhan (2013) find that adaptive and static (or naïve) inflation expectations are often observed across subjects. Pfajfar and Zakelj (2018) find evidence of persistent heterogeneity in the inflation forecasting process both across subjects and time, while Pfajfar and Zakelj (2014) find that switching between different mechanisms to form inflation expectations describes subjects' behavior better than a single rule. Heterogeneous and time-varying inflation expectations have been found in studies based on survey data as well. Mankiw et al. (2004) and Pfajfar and Santoro (2010) find significant heterogeneity in inflation expectations using US survey data, and Blanchflower and MacCoille (2009) find robust evidence of heterogeneous and backward-looking inflation expectations using UK survey data.

The literature also features macro-models with heterogeneous inflation expectations from which predictions are sometimes tested empirically. Branch (2004) develops a model where agents form their forecasts of inflation by selecting a predictor from a set of costly alternatives and may rationally choose a method other than the most accurate due to an inherent inclination to use one predictor over another. Agents are nonetheless seen as *rationally heterogeneous* in that each predictor choice is optimal for them; strictly speaking, agents' expectations are seen as boundedly rational and consistent with optimizing behavior. In our framework, agents' expectations are seen as boundedly rational precisely because they are instead consistent with satisficing behavior; in sum, agents are seen instead as *satisficingly heterogeneous*. In Brazier et al. (2008) and De Grauwe (2010), agents employ two heuristics to forecast inflation: one is based on one-period lagged inflation and the other on the official inflation target. Meanwhile, Lustenhouwer (in press) develops a model featuring heterogeneous inflation expectations which intuitively captures the notion of anchoring of expectations. Agents form inflation expectations randomly choosing a value from a discrete distribution around the inflation target of the monetary authority. Inflation expectations are anchored to (unanchored from) the official target when such distribution has a small (large) range. While in these models heuristic switching is described by a discrete choice model à la Brock and Hommes (1997), in our model agents switch between foresight strategies as described by evolutionarily satisficing learning dynamics with and without mutation. Branch and McGough (2010) embed predictor selection in a model with heterogeneity in inflation forecasts. They follow Brock and Hommes (1997) in setting the degree of heterogeneity to endogenously vary over time depending on past forecast errors (net of a fixed cost). Agents choose between a costly rational predictor and a costless adaptive predictor. In our model, the choice of foresight strategy involves costly rational and costless trend-following strategies, with agents switching between strategies as described by evolutionarily

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satisficing learning dynamics instead of the discrete choice model in Brock and Hommes (1997).

Our contribution is mostly related to the model set forth in Lines and Westerhoff (2010), where the interplay between heterogeneity in inflation expectations and the coupled dynamics of inflation, output growth, and unemployment is explored. Agents switch between trend-following (which is costless *ex ante*) and perfect foresight (which is costly *ex ante*) strategies to predict the future inflation, and the proportions of agents using one or the other inflation predictor is updated over time as described by the discrete choice model in Brock and Hommes (1997). The authors show analytically that the unique equilibrium of inflation, output growth, and unemployment features the natural values of these variables, but this equilibrium loses stability through a bifurcation at a critical value that depends on the cost associated with perfect foresight and the parameter measuring how sensitive agents are to selecting the most attractive predictor. The authors also do numerical simulations of the model to explore its global dynamic behavior and investigate conditions under which it leads to endogenous, complex fluctuations in macroeconomic variables and the proportions of agents using one or the other predictor.

Our contribution differs from the interesting one offered in Lines and Westerhoff (2010) in several broad respects (a comparison of specific results is carried out in Section 4). First, although both contributions employ the same dynamic macro-model (featuring Okun's law, an expectations-augmented Phillips curve and an aggregate demand relation), the two models differ considerably in the way they incorporate the empirical evidence documented in Branch (2004) that inflation forecast errors must exceed a threshold in order to induce a switch of inflation predictors. Lines and Westerhoff (2010) follow the suggestion made in Branch (2004, 2007) to treat the cost associated with the use of a predictor in the Brock and Hommes (1997) framework as also reflecting (along with information collection and processing costs) such inertial behavior (or predisposition effect) based on acceptable ranges of forecast errors. In our model, though we maintain the assumption made in Lines and Westerhoff (2010) that there is an *ex ante* cost associated with perfect inflation foresight, the inertial behavior found in Branch (2004) is modeled more characteristically by appealing to Herbert Simon's notion of satisficing choice behavior. This cogent notion conceives of choice as intending to meet an acceptability threshold rather than to select the best of all possible alternatives, which makes it an effectively representative way to model inertial behavior arising from acceptable ranges of forecast errors. Besides, we treat the threshold effect found empirically in Branch (2004) as randomly agent-specific. In Brock and Hommes (1997), agents use a discrete choice model to select a predictor where the deterministic part of the utility of the predictor is the performance measure, and the intensity of choice parameter specifies how good a measure the deterministic component is of the predictor utility. The standard discrete choice model features deterministic and random individual-specific characteristics, while in our model the payoff associated with an inflation

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foresight strategy features a deterministic component and a random agent-specific error-tolerance threshold. Yet in our model such random component is not random shocks faced by agents, but instead random agent-specific error-acceptance (or, equivalently, error-tolerance) thresholds; although in both models the random component affects the payoff of each of the possible choices. Incidentally, as it turns out, our satisficing evolutionary choice dynamics with agent-specific acceptable prediction errors that are randomly and independently distributed across agents and over time can be seen as in keeping with the following suggestion in Branch (2004): "It would be fruitful for future research to examine a 'richer' dynamical structure then the model at hand. One possible accounting for the inertia finding is that a predictor choice in period t is not independent from the choice in $t-1$. The serial correlation could be the result of time-varying random costs or random utility shocks". (p. 616)

Second, in Lines and Westerhoff (2010), the distribution of inflation predictors in the population of agents evolves over time as described by the predictor switching mechanism in Brock and Hommes (1997), while in the model herein this distribution follows evolutionarily satisficing learning dynamics with and without mutation. As a result, in Lines and Westerhoff (2010), as in Brock and Hommes (1997), for a strictly positive and finite intensity of choice (or intensity with which firms react to increases in relative net benefit) and information cost, there is a strictly positive share of firms using the strictly dominated perfect foresight strategy even in the long-run equilibrium. In our model, in turn, with a strictly positive and finite information cost and the absence of mutation, a strictly dominated inflation foresight strategy inevitably vanishes asymptotically. Thus, our long-run evolutionarily satisficing equilibrium is characterized either by all firms playing the trend-following extrapolative foresight strategy (in the absence of mutation) or by most firms playing such strategy (in the presence of mutation). In either situation, though, as in Lines and Westerhoff (2010), the long-run equilibrium features the natural values of the output growth, unemployment, and inflation being achieved.

Third, the role played by mutation in our model is analytically analogous to the one played by a finite intensity of choice in the discrete choice framework in Lines and Westerhoff (2010). The analogy arises because in both cases the implication is that not all agents will select the best performing inflation predictor as reflected in the respective relative net attractiveness, but the underlying rationale is different in each model. Lines and Westerhoff (2010) define the intensity of choice as a measure of how sensitive agents are to selecting the most attractive predictor. In our satisficing framework, a mutant is either an agent who exits the economy with fixed probability and is replaced with a new agent who is unfamiliar with predictor choice, or an agent who experiments occasionally with fixed probability. In either case, a mutant agent chooses each available predictor with equal probability. Also, recall that Brock and Hommes (1997) and Branch (2004) define the intensity of choice as a measure of how *fast* agents switch predictors for a given change in relative net benefit. Thus, in the context of a discrete choice framework,

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one might suggest that it would be more representative to treat the evidence of inertia (or predisposition effect) found in Branch (2004) not as a component of the cost of the use of a predictor, but instead as a reflection of a finite intensity of choice. In fact, Branch (2004) admits that the assumption of a finite intensity of choice is a simple way to incorporate inertia. Besides, although Branch (2004) declares to have found separate evidence of a finite intensity of choice and inertia stemming from the costs, he acknowledges that this inertia may be a reflection of agents' "inattentiveness", as emphasized in Brock and Hommes (1997), and that his findings support a rich variety of behavior. In this context, our model suggests an alternative explanation of predictor choice behavior based on satisficing evolutionary learning which is consistent with these findings.[2]

Fourth, as the model in Lines and Westerhoff (2010), our model (with and without mutation) can exhibit local instability of the (unique) long-run equilibrium and complicated global equilibrium dynamics, which is also explored through numerical simulations. In Lines and Westerhoff (2010), a rational choice between a cheap *ex ante* destabilizing (boundedly rational) predictor and a costly *ex ante* (perfect foresight) stabilizing predictor leads to the existence of complicated dynamics when the intensity of choice to switch predictors is high. In our model, the evolutionary competition between costless *ex ante* bounded rationality and costly *ex ante* sophisticated inflation prediction may be a source of local stability even if there is exogenous mutation. Therefore, although instability is inherent in such circumstances when more sophisticated prediction methods are more expensive (Brock and Hommes 1997), the evolutionarily satisficing learning suggested herein may take the strategy-revision process to a locally stable long-run equilibrium with either the majority or all agents playing a costless *ex ante* trend-following extrapolative foresight strategy.[3]

3. THE MODEL

3.1. The Macroeconomic Setting

As in Lines and Westerhoff (2010), for comparability of results, the macro-model on which the analysis in this paper is conducted can be stated as follows:

$$u_t - u_{t-1} = -\beta(g_t - g_n), \tag{1}$$

$$\pi_t = \pi_t^e - \alpha(u_t - u_n), \tag{2}$$

$$g_t = m - \pi_t, \tag{3}$$

where u_t denotes the unemployment rate in period t, u_n represents the natural (or normal) unemployment rate, g_t is the output growth rate in period t, g_n is the natural (or normal) output growth rate, m stands for the nominal money growth rate, which is exogenously fixed, π_t is the actual rate of inflation in period t, π_t^e is the aggregate expected rate of inflation for period t, which is formed by agents in period $t-1$, and lower case Greek letters denote strictly positive parameters.

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Equation (1) is simply Okun's law, according to which changes in the rate of unemployment are negatively related to the rate of output growth, equation (2) is an expectations-augmented Phillips curve, and equation (3) is a simple linear aggregate demand relation.

Inserting (2) into (3) and the resulting expressions into (1), we get:

$$u_t - u_n = \frac{1}{1+\alpha\beta}(u_{t-1} - u_n) + \frac{\beta}{1+\alpha\beta}(\pi_t^e - \pi_n), \tag{4}$$

where $\pi_n = m - g_n$ is the natural (or normal) inflation rate. Substituting (4) in (2), we obtain:

$$\pi_t - \pi_n = -\frac{\alpha}{1+\alpha\beta}(u_{t-1} - u_n) + \frac{1}{1+\alpha\beta}(\pi_t^e - \pi_n). \tag{5}$$

Consequently, for a given vector of structural and policy parameters represented by $(\alpha, \beta, u_n, \pi_n = m - g_n)$, the state transition of the unemployment rate and the actual inflation rate depends not only on the current unemployment rate, but also on the aggregate expected inflation.

3.2. Inflation Expectations Formation

At a given period t, each agent plays either the costly *ex ante* perfect foresight strategy, so that her expected rate of inflation is given by:

$$\pi_t^R = \pi_t, \tag{6}$$

or a costless *ex ante* extrapolative trend-following foresight strategy and form inflation expectations given by:

$$\pi_t^E = \pi_{t-1} + \gamma(\pi_{t-1} - \pi_{t-2}), \tag{7}$$

where $\gamma \in \mathbb{R}_{++}$ is a parameter. Therefore, again for comparability of results, these two inflation foresight strategies are the same as those available to agents in Lines and Westerhoff (2010).

At a given period t there is a fraction $x_t \in [0, 1] \subset \mathbb{R}$ of the population of agents, which may vary from one period to the next one, forming extrapolative trend-following inflation expectations, that is, they use the bounded rationality inflation predictor in (7). We refer to this type of agent as an *extrapolative forecaster*. The remaining fraction, $1 - x_t$, is made up of fully informed agents that pay an exogenously fixed and strictly positive *ex ante* cost to play the perfect foresight strategy in (6). We refer to this type of agent as a *perfect forecaster*. Given this predictor distribution across the population of agents, the average aggregate expected rate of inflation rate can then be expressed as a weighted average of these two predictors as follows:

$$\pi_t^e = x_{t-1}\pi_t^E + (1 - x_{t-1})\pi_t^R. \tag{8}$$

Therefore, the cost associated with perfect foresight is actually the cost associated with a potential heterogeneity in the choice of inflation forecasting strategy

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across agents. In fact, as shown in (8), playing the perfect inflation foresight strategy inevitably requires perfectly knowing the frequency distribution of inflation forecasting strategies across agents.

Using (6), (7), and (8), the gap between the average aggregate expected rate of inflation and the natural (or normal) rate of inflation can be expressed as follows:

$$\pi_t^e - \pi_n = x_{t-1}[(1+\gamma)(\pi_{t-1} - \pi_n) - \gamma(\pi_{t-2} - \pi_n)] + (1 - x_{t-1})(\pi_t - \pi_n). \quad (9)$$

3.3. Average Inflation Expectations Dynamics as Evolutionarily Satisficing Learning

Although the frequency distribution of predictors is predetermined at a given period t, it evolves over time according to evolutionarily satisficing learning dynamics. Yet, as in Lines and Westerhoff (2010), we assume that agents strictly prefer inflation predictors with a high forecasting accuracy and, therefore, rely on squared prediction error as a (publicly observable) fitness measure. Following the nomenclature in Lines and Westerhoff (2010), the attractiveness of the extrapolative trend-following forecasting strategy can be defined as follows:

$$a_t^E = -(\pi_t^E - \pi_t)^2. \quad (10)$$

Given (6), it follows that $\pi_t^R - \pi_t = 0$. Thus, the attractiveness of the perfect foresight strategy, as in Lines and Westerhoff (2010), is given by:

$$a_t^R = -(\pi_t^R - \pi_t)^2 - \kappa = -\kappa, \quad (11)$$

where $\kappa \in \mathbb{R}_+$ stands for the exogenously fixed cost associated with playing the perfect foresight strategy, as this playing behavior does necessitate knowing the frequency distribution of inflation forecasting strategies across agents, which is not explicitly acknowledged in Lines and Westerhoff (2010). Besides, in our model, more closely in accordance with the evidence in Branch (2004) that inflation prediction errors have to exceed some threshold to induce inflation predictor switching, the attractiveness of a prediction strategy is compared with the attractiveness the agent considers acceptable.

Let us then describe the evolutionarily satisficing learning dynamics embedded in the law of motion of the proportion of extrapolative forecasters, x_t, which is in the spirit of the contributions of Herbert Simon.[4] As classically elaborated by Simon (1955, 1956), satisficing is a theory of choice centered on the process through which available alternatives are examined and evaluated. By conceiving of choice as intending to meet an *acceptability threshold* rather than to select the best of all possible alternatives, satisficing theory evidently contrasts with optimization theory. As Simon interestingly suggests, this contrast is similar to "looking for the sharpest needle in the haystack" (i.e., optimizing) versus "looking for a needle sharp enough to sew with" (i.e., satisficing) (Simon (1987), p. 244). Thus, as in Lines and Westerhoff (2010), in our model agents choose an inflation predictor from a set of alternative predictors whose *ex ante* cost ordering

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increases in a predictor's *ex ante* precision, whereas the proportion of agents in the population using a predictor varies positively with a formally precise measure of relative attractiveness. In our model, however, as it is plausibly assumed that there is heterogeneity in the choice idiosyncrasies of satisficing agents, an acceptable range of prediction error is explicitly and separately modeled as randomly varying across agents.[5]

In our specification, an extrapolative forecaster i takes the current attractiveness given by (10) and compares it with the attractiveness she considers *acceptable*, which is denoted by a^i. Taking $t-1$ as the *current* period, if $\left|a^i\right| < \left|a^E_{t-1}\right|$, the extrapolative forecaster i considers changing her strategy for forming inflation expectations in t, that is, the extrapolative forecaster i becomes a strategy reviser. The attractiveness that is deemed acceptable by an agent depends, *inter alia*, on her idiosyncrasies. Reasonably, we assume that the attribute of acceptable attractiveness is randomly and independently determined across agents and over time. We assume that $\left|a^i\right| = -a^i$ is a random variable with cumulative distribution function $F : \mathbb{R}_+ \to [0, 1] \subset \mathbb{R}$ which is continuously differentiable. Consequently, the probability that a randomly drawn agent i in the subpopulation of extrapolative forecasters will consider that the currently observed attractiveness of her foresight strategy is unacceptable is simply:

$$\Pr\left(\left|a^i\right| < \left|a^E_{t-1}\right|\right) = \Pr(-a^i < -a^E_{t-1}) = F(-a^E_{t-1}). \tag{12}$$

The measure of extrapolative forecasters who become perfect forecasters is then given by:

$$x_{t-1}F(-a^E_{t-1}). \tag{13}$$

Analogously, the measure of perfect forecasters who become extrapolative forecasters is given by:

$$(1-x_{t-1})\Pr\left(\left|a^i\right| < \left|a^R_{t-1}\right|\right) = (1-x_{t-1})\Pr(-a^i < -a^R_{t-1}) = (1-x_{t-1})F(-a^R_{t-1}). \tag{14}$$

Therefore, the subtraction of (13) from (14) and use of (7), (10), and (11) yields the following evolutionarily satisficing learning dynamic:

$$x_t - x_{t-1} = (1-x_{t-1})F(\kappa) - x_{t-1}F\left(g(\pi_{t-1}, y_{t-1}, z_{t-1})\right), \tag{15}$$

where:

$$g(\pi_{t-1}, y_{t-1}, z_{t-1}) \equiv -a^E_{t-1} = \left[(1+\gamma)(y_{t-1}-\pi_n) - \gamma(z_{t-1}-\pi_n) - (\pi_{t-1}-\pi_n)\right]^2, \tag{16}$$

which is specified using (7), (10) and the following auxiliary variables:

$$y_t = \pi_{t-1} \text{ and } z_t = y_{t-1}. \tag{17}$$

As it turns out, an increase in the relative attractiveness of the extrapolative trend-following foresight strategy in the current period (a decrease in $g(\cdot)$) leads to an

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increase in the proportion of agents playing this inflation foresight strategy in the next period, whereas the opposite occurs when the relative attractiveness of the perfect inflation foresight strategy rises (a decrease in κ). Thus, the evolutionarily satisficing learning dynamics in (15) reflects the operation of a selection protocol according to which the proportion of agents playing a given inflation forecasting strategy at a given period in time varies positively with the relative fitness or attractiveness of such strategy.

In order to further enrich the predictor switching process and gain in analytical generality, we also consider the possibility that the evolutionarily satisficing learning dynamics in (15) operate in the presence of a perturbation mechanism, analogous to mutation in natural environments. In a biological setting, mutation is interpreted literally as consisting of random changes in genetic codes. In economic settings, as pointed out by Samuelson (1997, Ch. 7), mutation refers to a situation in which a player refrains from comparing payoffs and switches strategy at random. Thus, the present extension features mutation as an exogenous perturbation in the evolutionarily satisficing learning protocol leading some decision-makers to choose an inflation foresight strategy at random. Following Kandori et al. (1993), two rationales can be provided to this random choice. First, at every period, a given measure of agents exit the economy with some (exogenously fixed) probability and are replaced one-to-one with new agents who happen to know nothing about (or are relatively inexperienced in) the decision-making process in question. Second, each agent simply "experiments" occasionally with some exogenously fixed probability. More generally, this noise component could also capture the effect of sporadic exogenous institutional factors such as changes of administration in the monetary authority or other similar changes in the policy-making framework. However, we would have to further assume that these changes do not affect the structure of the basic macroeconomic model in (1)–(3) (i.e., these institutional changes do not affect the structural stability of the model in (1)–(3)).

Drawing on Gale et al. (1995), mutation can be incorporated into the evolutionarily satisficing learning protocol in (15) as follows. Let $\theta \in (0, 1) \subset \mathbb{R}$ be the measure of mutant agents that choose an inflation foresight strategy in a given revision period independently both of their acceptable forecast error and of the attractiveness of the inflation foresight strategy just played. Thus, there are θx_{t-1} extrapolative forecasters and $\theta(1 - x_{t-1})$ perfect forecasters behaving as mutants. As we assume that mutant agents choose either one or the other of the available inflation foresight strategies with equal probability, there are then $\theta x_{t-1}/2$ extrapolative forecaster mutant agents and $\theta(1 - x_{t-1})/2$ perfect forecaster mutant agents changing foresight strategy. The net flow of mutant agents becoming extrapolative forecaster agents in a given revision period, which can be either positive or negative, is then the following:

$$\theta\,(1 - x_{t-1})\,\frac{1}{2} - \theta x_{t-1}\frac{1}{2} = \theta\left(\frac{1}{2} - x_{t-1}\right). \tag{18}$$

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Following Gale et al. (1995), this exogenous perturbation can be linearly added to the evolutionary protocol in (15) to yield the following *noisy evolutionarily satisficing learning dynamic* of the frequency distribution of inflation foresight strategies across agents:

$$x_t - x_{t-1} = (1-\theta)\left[(1-x_{t-1})F(\kappa) - x_{t-1}F\left(g(y_{t-1}, z_{t-1}, \pi_{t-1})\right)\right] + \theta\left(\frac{1}{2} - x_{t-1}\right). \tag{19}$$

4. BEHAVIOR OF THE MODEL IN THE LONG-RUN EVOLUTIONARY EQUILIBRIUM

Note that the macrodynamics of the rates of output growth, unemployment, and inflation is coupled with the microdynamics of the frequency distribution of inflation foresight strategies across firms. In fact, plugging (9) into (5) and considering the auxiliary variables defined in (17), we can establish that:

$$\pi_t - \pi_n = \frac{1}{\alpha\beta + x_{t-1}}\{-\alpha(u_{t-1} - u_n) + x_{t-1}\left[(1+\gamma)(\pi_{t-1} - \pi_n) - \gamma(y_{t-1} - \pi_n)\right]\}. \tag{20}$$

Meanwhile, inserting (20) into (9) and the resulting expressions into (4), we get:

$$u_t - u_n = \frac{x_{t-1}}{\alpha\beta + x_{t-1}}\left\{u_{t-1} - u_n + \beta\left[(1+\gamma)(\pi_{t-1} - \pi_n) - \gamma(y_{t-1} - \pi_n)\right]\right\}. \tag{21}$$

As a result, the state transition of the economy is determined by the first-order difference equation system given by (19)–(21) and the auxiliary variables defined in (17), whose state space is given by $\Theta = \{(x_t, \pi_t, u_t, y_{t-1}, z_{t-1}) \in \mathbb{R}^5 : 0 \leq x_t \leq 1, 0 \leq u_t \leq 1\}$.

In the following proposition, we establish the existence and uniqueness of the evolutionary long-run equilibrium of the dynamic system given by (19)–(21) and the auxiliary variables defined in (17).

PROPOSITION 1. *For a given vector of parameters* $(\alpha, \beta, \gamma, u_n, \pi_n = m - g_n, \theta)$*, the dynamic system given by (19)–(21) and the auxiliary variables in (17) has a unique long-run evolutionary equilibrium which is given by* $(x^*, \pi_n, u_n, \pi_n, \pi_n) \in \Theta$*, with* $x^* = \frac{(1-\theta)F(\kappa)+\theta/2}{(1-\theta)F(\kappa)+\theta} \in (1/2, 1] \subset \mathbb{R}$.

Proof: See Appendix 1

In the long-run equilibrium, therefore, the extrapolative forecasting strategy is either the only strategy to survive ($x^* = 1$), in the absence of mutant behavior ($\theta = 0$), or the most played foresight strategy ($1/2 < x^* < 1$), in the presence of mutation ($0 < \theta < 1$). The long-run equilibrium values of the endogenous

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macroeconomic variables (π_t, u_t, and g_t) are the same in either case. Although pervasive extrapolative inflation foresight (all agents eventually decide to follow the boundedly rational inflation forecasting strategy) is an evolutionarily satisficing long-run equilibrium outcome when mutation is absent, this equilibrium configuration is nonetheless characterized by the natural values of the rates of inflation, output growth, and unemployment being achieved. The intuition is straightforward: when mutation is absent, once the economy converges to the long-run equilibrium, the extrapolative foresight strategy fares as well as the perfect foresight strategy, but it is less costly. Given the nature of the information imperfection imposing a cost on perfect foresight, all agents following the perfect foresight strategy ($x = 0$) is not a long-run equilibrium. The reason is that the substance of this result ($x = 0$) is that all agents are paying the cost to learn the measure of a behavioral heterogeneity that does not exist. Meanwhile, when mutation is absent, $x^* = 1$ is a long-run equilibrium, for the substance of this result is that no agent is paying the cost to find out the measure of a non-existing behavioral heterogeneity. Therefore, by the same logic, $1/2 < x^* < 1$ is a long-run equilibrium in the presence of mutant agents because in this situation there are agents paying a behavioral heterogeneity cost that actually has to be paid.

As in Lines and Westerhoff (2010), in our model, the unique long-run equilibrium features the natural (or normal) values of the macroeconomic variables (rates of inflation, unemployment, and output growth). Moreover, in each model, the long-run equilibrium solution for the frequency distribution of inflation foresight strategies across agents depends on the respective key parameters: information cost associated with perfect foresight and intensity of choice (in Lines and Westerhoff (2010)) and information cost associated with perfect foresight and mutation rate (in our model). As elaborated in Section 2, a strictly positive mutation rate is nonetheless analytically analogous to a finite strictly positive intensity of choice, as they both imply that, due to some source of randomness, not all agents will select the best performing predictor as reflected in the respective relative net attractiveness. In fact, we would suggest that the rationales we have put forward for the presence of mutation may apply to (and add to the existing justifications for) a finite strictly positive intensity of choice.

While in our model the proportion of extrapolative trend-follower agents in the long-run equilibrium is given by the expression for x^* in Proposition 1, the respective value in Lines and Westerhoff (2010, p. 250) is given by:

$$\mu = \frac{1}{1 + e^{-\lambda\kappa}} \in (1/2, 1] \subset \mathbb{R}, \tag{22}$$

where $\lambda \in \mathbb{R}_+$ is the intensity of choice and $\kappa \in \mathbb{R}_+$ is also the exogenously fixed cost associated with playing the perfect foresight strategy. In the limit for $\lambda \to +\infty$, it follows that all agents choose the extrapolative trend-following strategy in the long-run equilibrium ($\mu \to 1^-$). Therefore, as when mutation is absent in our model ($\theta = 0$), which is equivalent to the intensity of choice tending to positive infinity in the Lines and Westerhoff (2010) model, the perfect foresight

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strategy also ceases to be played in the long-run equilibrium ($x^* = 1$); and this is so no matter how low happens to be any strictly positive *ex ante* cost associated with playing the perfect foresight strategy. The intuition is that in the limit for either $\lambda \to +\infty$ or $\theta \to 0^+$, the long-run equilibrium is characterized by both foresight strategies being equally successful in predicting inflation, but whereas the extrapolative trend-following strategy does so without any *ex ante* cost, the perfect foresight strategy does so at a strictly positive *ex ante* cost. Consequently, except in the limit for $\lambda \to +\infty$ (in the model in Lines and Westerhoff (2010)) or in the limit for $\theta \to 0^+$ (in our model), in the unique long-run equilibrium, which is common to both models, although both strategies are effective in predicting inflation, there is nonetheless a tail of the population of agents that selects the least attractive inflation predictor due to randomness. And, per (10) and (11), this least attractive inflation predictor is the perfect foresight one. Meanwhile, as in the limit for $\theta \to 1^-$ in our model, in the limit for $\lambda \to 0^+$ in the model in Lines and Westerhoff (2010), it follows that exactly half of the agents play each inflation foresight strategy in the long-run equilibrium. The intuition is straightforward: in either case, agents' choice of inflation predictor becomes completely random.

In the limit for $\kappa \to 0^+$, the result obtains that $\mu = x^* = 1/2$, whose substance is that the two available foresight strategies become equivalent in the long-run equilibrium due to their having the same level of attractiveness. Meanwhile, in the limit for $\kappa \to +\infty$, it follows that $\mu \to 1^-$, whereas $x^* \to 1 - \theta/2$. Thus, in the limit for the information cost associated with the perfect foresight strategy tending to positive infinity, while such strategy fails to survive in the long-run equilibrium in the Lines and Westerhoff (2010) model, in our model the existence of mutation prevents the extinction of such strategy in the long-run equilibrium, albeit it survives in a minority position for any $\theta \in (0, 1) \subset \mathbb{R}$. The randomness introduced by mutation comes to rescue the perfect foresight strategy in such equilibrium where the extrapolative trend-following strategy is permanently more attractive. In this situation, intuitively then, the higher the rate of mutation, the lower the predominance of the extrapolative strategy in the long-run equilibrium.

A natural question that arises regards whether the dynamic described in (19)–(21) and (17) can take the economy to the long-run equilibrium configuration established in Proposition 1. For instance, in an economy with no mutant agents and where initially there is heterogeneity in inflation expectations, can all agents eventually learn to perfectly predict the rate of inflation, but without having to play the costly *ex ante* perfect foresight strategy? The answer is yes, as formally established in the following proposition.

PROPOSITION 2. *For a given vector of parameters* $(\alpha, \beta, \gamma, u_n, \pi_n = m - g_n, \theta)$*, the unique long-run evolutionary equilibrium given by* $(x^*, \pi_n, u_n, \pi_n, \pi_n) \in \Theta$ *of the dynamic system given by (19)–(21) and the auxiliary variables in (17) is locally asymptotically stable if* $\gamma < \frac{\alpha\beta + x^*}{2x^*} \equiv \gamma_c$*. This unique long-run evolutionary equilibrium loses stability through a Neimark–Sacker (NS) bifurcation at critical value* γ_c*.*

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Proof: See Appendix 2.

Consequently, the predictor switching game explored in this paper is subject to evolutionary learning dynamics that can take it to a long-run equilibrium configuration in which, albeit either the majority or even all agents choose to play the extrapolative foresight strategy to form inflation expectations, the rates of inflation, output growth, and unemployment all achieve their natural values. Thus, inflation expectations that are persistently heterogeneous need not prevent the successful conduct of macroeconomic stabilization policy through the setting of the growth rate of nominal money by the monetary authority. Besides, when there is mutation, so that the two available inflation forecasting strategies survive in the long-run equilibrium solution (yet the extrapolative forecasting strategy predominates), such long-run equilibrium level of heterogeneity varies expectedly, considering (A5) in Appendix A, with the *ex ante* cost associated with perfect foresight and the measure of mutant agents:

$$\frac{\partial x^*}{\partial \kappa} = \frac{\theta(1-\theta)F'(\kappa)}{2[(1-\theta)F(\kappa)+\theta]^2} > 0, \tag{23}$$

$$\frac{\partial x^*}{\partial \theta} = \frac{-F(\kappa)}{2[(1-\theta)F(\kappa)]+\theta]^2} < 0. \tag{24}$$

Intuitively, in the long-run equilibrium configuration with coexistence of the two inflation forecasting strategies, the higher the *ex ante* cost associated with perfect foresight, the greater the predominance of agents playing the extrapolative foresight strategy (recall from Proposition 1 that, in the presence of mutation, the extrapolative forecasting strategy is the most followed strategy in the long-run evolutionary equilibrium ($1/2 < x^* < 1$)). Yet the predominance of agents who select the extrapolative foresight strategy varies negatively with the rate of mutation. The intuition is that, the greater the random noise in the selection process of inflation forecasting strategy, the greater has to be the proportion of perfect forecasters in the long-run equilibrium to ensure that the aggregate expected inflation is consistent with the natural rate of inflation.[6]

While in our model the unique long-run equilibrium is locally asymptotically stable if the extrapolation parameter γ is lower than the critical value given by the expression for γ_c in Proposition 2, the respective critical value in the Lines and Westerhoff (2010, p. 250) model is given by $\gamma_c^{LW} \equiv \frac{\alpha\beta+\mu}{2\mu}$. Intuitively, in both models, the corresponding critical value for the extrapolation parameter varies positively with the parameters α and β, and negatively with the long-run equilibrium value of the proportion of extrapolative trend-following agents, which in turn varies positively with the information cost associated with perfect foresight. Thus, in the limit for $\lambda \to +\infty$ (in the Lines and Westerhoff (2010) model) or in the limit for $\theta \to 0^+$ (in our model), as all agents choose the extrapolative trend-following strategy in the common long-run equilibrium, the likewise common basin of attraction (or corridor of stability) given by $\gamma_c^{LW} = \gamma_c = 0.5(1+\alpha\beta)$ attains its smallest possible size. Meanwhile, as in the limit for $\theta \to 1^-$ in our

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model, in the limit for $\lambda \to 0^+$ in the model in Lines and Westerhoff (2010) (or in the limit for $\kappa \to 0^+$ in both models), since exactly half of the agents choose each inflation foresight strategy in the long-run equilibrium, the resulting common basin of attraction given by $\gamma_c = \gamma_c^{LW} = 0.5 + \alpha\beta$ attains its largest possible size. Interestingly, therefore, in both models a larger extent of a specific kind of randomness makes for a more locally stability-prone long-run equilibrium. In effect, in both models, the critical value for the extrapolation parameter varies negatively with the long-run equilibrium value of the proportion of extrapolative trend-following agents, which in turn varies negatively with the extent of randomness in inflation predictor switching as measured respectively by the mutation rate and intensity of choice (recall that our model has three layers of randomness, of which the distribution of tolerable prediction inaccuracy across agents is the only one unrelated to mutation).

Moreover, in our model, when mutant behavior is absent ($\theta = 0$), the resulting critical value for the extrapolation parameter is given by $\gamma_c = 0.5(1 + \alpha\beta)$, which is strictly lower than $\gamma_c^{LW} = 0.5\left[1 + (1 + e^{-\lambda\kappa})\alpha\beta\right]$ for any $0 \leq \lambda < +\infty$ and $0 \leq \kappa < +\infty$ in the Lines and Westerhoff (2010) model. In words, in the absence of mutation, the long-run equilibrium in our model is more locally instability-prone than the long-run equilibrium in the Lines and Westerhoff (2010) model with any strictly finite intensity of choice and information cost. In the latter model, however, either in the limit for $\lambda \to +\infty$ or in the limit for $\kappa \to +\infty$, the critical value for the extrapolation parameter is $\gamma_c^{LW} = 0.5(1 + \alpha\beta)$, which is strictly lower than $\gamma_c = (\alpha\beta + x^*)/2x^*$ for any $0 < \theta \leq 1$ and $0 \leq \kappa \leq +\infty$ in our model, since in that case $0 < x^* < 1 - \theta/2 < 1$. In words, when the model in Lines and Westerhoff (2010) features either the intensity of choice or the information cost tending to positive infinity, its long-run equilibrium becomes more locally instability-prone than the long-run equilibrium in our model with any strictly positive mutation rate and any (even if extreme value) information cost.[7]

Meanwhile, when the information cost attached to perfect foresight, the intensity of choice and the mutation rate all assume non-extreme values (i.e., $0 < \kappa < +\infty$, $0 < \lambda < +\infty$, and $0 < \theta < 1$), so that $1/2 < \mu < 1$ and $1/2 < x^* < 1$, respectively, in both models the extent of the long-run predominance of extrapolative trend-followers varies positively (and hence the size of the basin of attraction of the respective unique long-run equilibrium varies negatively) with the information cost associated with perfect foresight. In fact, as discussed earlier, in the limit for $\kappa \to 0^+$, it follows that $\mu \to 1/2$ and $x^* \to 1/2$ in that limit. Meanwhile, in the limit for $\kappa \to +\infty$, it follows that $\mu \to 1$ and $x^* \to 1 - \theta/2$ in that limit. Furthermore, in the limit for $\kappa \to +\infty$, it follows that $(\mu - x^*) \to \theta/2 > 0$ (whose strictly positive value, in the presence of mutation, intuitively varies positively with the mutation rate) in that limit. Consequently, it follows that $1/2 < x^* < \mu < 1$ for a sufficiently high value of the information cost attached to perfect foresight, whose implication that $\gamma_c > \gamma_c^{LW}$ means that the unique long-run equilibrium in our model is more (locally) stability-prone than the (also unique) long-run equilibrium in the Lines and Westerhoff (2010) model

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when the information cost associated with perfect foresight becomes relatively high.

Note, however, that it is not possible to unambiguously evaluate how the difference in terms of local stability properties as expressed by $\gamma_c - \gamma_c^{LW}$ varies with the information cost associated with perfect foresight. In fact, this difference is given by:

$$\gamma_c - \gamma_c^{LW} = \frac{1}{2}\alpha\beta\left[\frac{\theta}{\theta - 2(\theta - 1)F(\kappa)} - e^{-\lambda\kappa}\right], \tag{25}$$

which varies with the information cost associated with perfect foresight according to:

$$\frac{\partial(\gamma_c - \gamma_c^{LW})}{\partial\kappa} = \alpha\beta\left\{\frac{1}{2}\lambda e^{-\lambda\kappa} - \frac{\theta(1-\theta)F'(\kappa)}{[\theta + 2(1-\theta)F(\kappa)]^2}\right\}, \tag{26}$$

whose sign is ambiguous. Considering (A5), (22), and (25), in the limit for $\kappa \to 0^+$, it follows that $\mu \to 1/2$ and $x^* \to 1/2$ in that limit, so that $(\gamma_c - \gamma_c^{LW}) \to 0$ in that limit, which implies that in both models the common basin of attraction of the common unique long-run equilibrium attains its largest possible size in that limit. Meanwhile, in the limit for $\kappa \to +\infty$, it follows that $\mu \to 1$ and $x^* \to 1 - \theta/2$ in that limit, so that $(\mu - x^*) \to \theta/2 > 0$ (which intuitively varies positively with the rate of mutation) in that limit. As a result, it follows that $x^* < \mu$ (and hence $\gamma_c > \gamma_c^{LW}$) for a sufficiently high value of the information cost associated with perfect foresight.

5. NUMERICAL SIMULATION RESULTS

In order to appropriately complement the analytical results regarding local dynamics derived in the preceding section, we now perform numerical simulation analyses of global dynamics of the model. Given that the steady state of a further simplified version of the model in (1)–(3) without expected inflation is known to be (globally) stable, it follows that any instability in our heterogeneous expectations-augmented model can be attributed to the specified evolutionary dynamics of expectation formation. Nonetheless, our numerical simulations suggest that the global dynamic behavior of the extended model over much of the studied parameter space is motion or, or close to, stable cycles of different periods (including, of course, the unique steady state, which is a cycle of period 1). We have experimented extensively with the parameters related to expectation formation before setting representative and reasonable values for them. As in Lines and Westerhoff (2010), again to ensure comparability of results, the common exogenous variables and parameters of the macroeconomic model in (1)–(3) are set according to the following empirically plausible values: natural output growth rate, $g_n = 0.02$; natural unemployment rate, $u_n = 0.05$; nominal money growth rate, $m = 0.05$; $\alpha = 1$; and $\beta = 0.35$. Furthermore, we typically set the cost associated with following the perfect foresight strategy at $\kappa = 0.0001$, which

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means that such cost has the same weight as a 1% predictor error in the respective attractiveness function in (11).

Recall that our evolutionarily satisficing learning protocol does not feature the intensity of choice parameter that plays a significant part in the predictor switching protocol (based on a discrete-choice model) adopted in Lines and Westerhoff (2010). Our alternative predictor switching protocol in (19), meanwhile, operates in the presence of a perturbation analogous to mutation in natural environments, and we typically set the respective exogenous mutation rate at 1%, so that $\theta = 0.01$. As for the cumulative distribution function of the acceptable attractiveness as a random variable in (19), we assume that it is exponential as represented by $F(x_t) = 1 - e^{-\delta x_t}$, where $x_t = -a_t^E, -a_t^R$ for any t and we set $\delta = 1\,200$ as a reasonable value for this additional parameter.[8] The implication of this value as regards predictor switching dynamics can be tested in a similar way as Lines and Westerhoff (2010, p. 251) do the same testing with their assumed value of the intensity of choice parameter. Suppose that the prediction error of a given strategy A is 1% less than the prediction error of a given alternative strategy B, which in turn involves an error of 0.01. In order to isolate the influence of a differing prediction capacity across strategies, let us assume that $\kappa = 0$. For the basic parameter values assumed above, it follows that the 1% difference in prediction error implies that the proportion of agents playing strategy A would be approximately 50.1% as compared to a proportion of approximately 50% if there were no difference in prediction errors. Therefore, a 1% prediction improvement results in a 0.1% switch of agents from using predictor B to using predictor A, which is a reasonable switching elasticity in light of the empirical evidence documented in Branch (2004) that there is inertia in predictor switching.

Figure 1 top pictures' inflation rate limit sets by means of parameter basins with respect to the pair of parameters given by (κ, γ), with $\kappa \in (0,\ 0.0002) \subset \mathbb{R}$ and $\gamma \in (0.65,\ 1.25) \subset \mathbb{R}$, by assigning different colors to the stable steady state and stable cycles of different periods. Therefore, as in Lines and Westerhoff (2010), extrapolative forecasters predict that from 65% to 125% of last period's inflation trend will continue, while costs are allowed to vary from nonexistent to twice the typically assumed value. It should be noted that such so-called double parameter bifurcation diagram only provides an approximate categorization of the dynamics of the inflation rate limit sets. Also, we eschew detecting only local dynamics of the variables of interest by specifying initial conditions at some reasonable distance from the unique long-run equilibrium inflation rate and proportion of extrapolative forecasters, which (per Proposition 1) are given by $\pi_n = m - g_n = 0.03$ and $x^* = 0.959$, respectively. Note that the resulting critical value of the extrapolation parameter (per Proposition 2) is given by $\gamma_c = 0.682$. We set the initial values of the rates of unemployment and inflation at $u = 0.04$ and $\pi = 0.02$, respectively, and of the proportion of extrapolative forecasters at $x = 0.9$. Concerning the lagged values of the inflation rate given by the auxiliary variables in (17), we set initial values given by $y = 0.02$ and $z = 0.02$, respectively. In Figure 1, the algorithm nicely sought stable long-run cycles of periods

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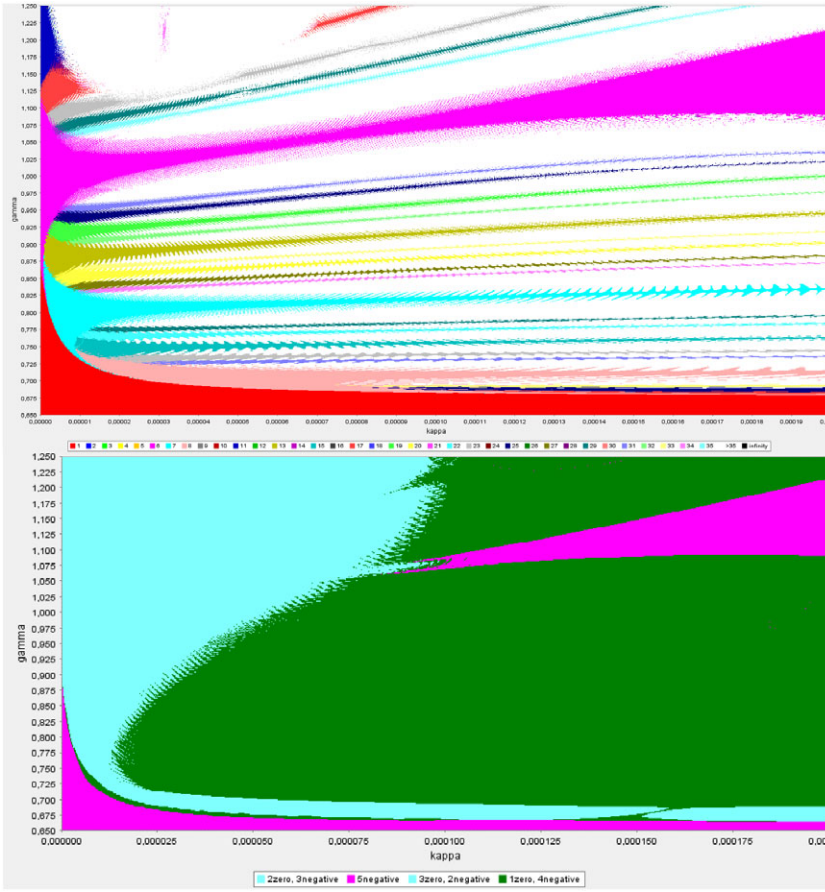


FIGURE 1. Inflation rate limit sets in parameter space (κ, γ) (top) and respective Lyapunov characteristic exponents (bottom).

from 1 (steady state) to 35 (the maximum allowed by the software package used in the simulations), while in Lines and Westerhoff (2010) the maximum cycle sought was of period 12. In the search reported here, we set 5000 transient iterations (i.e., the number of initial iterations that have been ignored by the algorithm before searching for cycles) for every coordinate couple of parameter values. The algorithm for this simulation routine also requires the specification of two approximation parameters. The precision parameter indicates how fine-grained the plot is wanted to be, so that the smaller this parameter, the closer a value must be to a given point to be taken (and hence plotted) as that point. We set this precision parameter at 0.001. The infinity parameter specifies at what value of the respective variable its trajectory is taken to be leaning to infinity. Cautiously, we set such parameter defining a cutoff value for unstable orbits at 5000.

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Suppose that the cost associated with playing the perfect foresight strategy is fixed at its typically assumed value of $\kappa = 0.0001$. A continuing increase in the extrapolation parameter from its minimum considered value of $\gamma = 0.65$ eventually leads to a Neimark–Sacker bifurcation at the corresponding critical value when the dynamic changes from a stable steady state, 1-cycle (the red region) to a stable 8-cycle (the pink region). As the extrapolation parameter keeps increasing, the system undergoes a sequence of bifurcations that take its dynamics to cycles of different periods, eventually crossing white regions of unclassifiable dynamics whose meaning will be further explored shortly. For instance, as the extrapolation parameter approaches the value of $\gamma = 1.025$ and keeps increasing toward its maximum considered value of $\gamma = 1.25$, while the cost associated with perfect foresight remains fixed at $\kappa = 0.0001$, a sequence of bifurcations take the system first to stable 6-cycles (the purple regions), then to stable 35-cycles (the light blue regions), then to stable 29-cycles (the dark green regions), and finally to stable 23-cycles (the gray regions). Meanwhile, there is a widening to the right of the region of stable 1-cycle (steady state) as the cost continually falls from its maximum considered value of $\kappa = 0.0002$. In effect, Proposition 2 implies that when there is mutation (so that both inflation forecasting strategies survive in the long-run equilibrium), it follows that the critical value of the extrapolation parameter γ varies positively with the long-run equilibrium proportion of perfect forecasters, which in turn (per (23)) is positively affected by a fall in the cost associated with following the perfect foresight strategy.

The white regions in Figure 1 top contain unclassifiable couples of parameter values for which the respective trajectory points do not converge within the specified precision distance, which excludes the possibility that the type of attractor is a fixed point. In the case of these unclassifiable limit sets, although the trajectory does not explode, it does not settle on to a cycle of periodicity within the specified maximum cycle. These white regions include parameter couples for which the dynamics could be either periodic (but of periodicity higher than the maximum specified in these simulations, which is 35), quasiperiodic (dynamics that share with periodic cycles the feature that the sequences lie on an invariant curve), or chaotic (where trajectory points lie on a strange attractor). Since we could extensively search for stable long-run cycles of periods from 1 to 35 with convenient precision and approximation parameters, the aggregate of these white regions representing parameter coordinates with unclassifiable dynamics is not very large in Figure 1 top. Nevertheless, in Figure 1 bottom, we use plots indicating the corresponding Lyapunov characteristic exponents (LCEs) over the same parameter space as in the top panel to reduce the ambiguity regarding those parameter coordinates with unclassifiable dynamics. For higher dimensional systems such as the five-dimensional one explored in this paper, the LCEs measure the asymptotic, average, exponential rate of stretching (or contraction) of a vector in the tangent space at the given initial conditions (Medio and Lines (2001), Ch. 7.1). Therefore, our model features five LCEs measuring the divergence of nearby orbits, one for each dimension. In general, periodic (including period 1 or fixed point) behavior is

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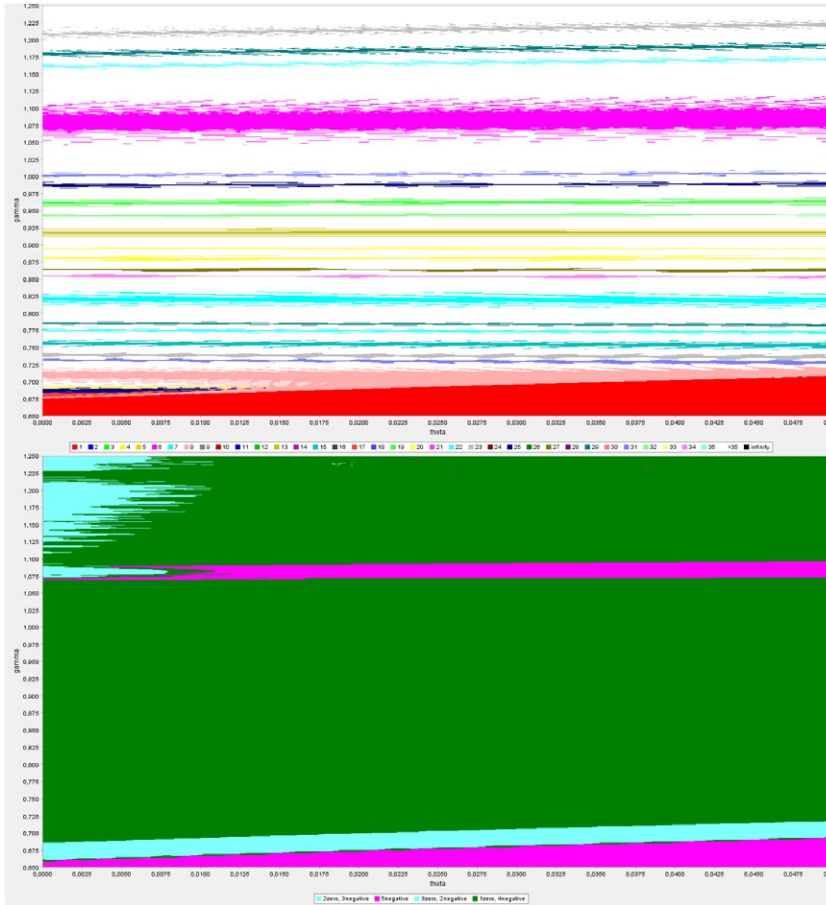


FIGURE 2. Inflation rate limit sets in parameter space (θ, γ) (top) and respective Lyapunov characteristic exponents (bottom).

characterized by five negative values, quasiperiodic behavior by the largest (and possibly all the others but one) LCE being equal to zero, and chaotic (but with bounded trajectories) behavior by the largest LCE being positive, with nearby trajectories then diverging exponentially in the respective direction. As suggested in the bottom panel in Figure 1, parameter coordinates with unclassifiable dynamics in the top panel do not seem to generate chaotic dynamics.[9]

As the predictor switching dynamics in (19) are governed by a noisy satisficing protocol, Figure 2 top represents inflation rate limit sets over parameter space (θ, γ), with $\theta \in (0, 0.05) \subset \mathbb{R}$ as noise parameter and $\gamma \in (0.65, 1.25) \subset \mathbb{R}$. Except for the setting of the cost related to perfect foresight at its typical value of $\kappa = 0.0001$ and the varying of the mutation rate over the interval given by $\theta \in (0, 0.05) \subset \mathbb{R}$, all the other numerical specifications are as in Figure 1 top.

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Suppose that the mutation rate is fixed at a level twice as higher than its typical value, so that $\theta = 0.02$. A continuing increase in the extrapolation parameter γ from a level below its respective critical value eventually causes a Neimark–Sacker bifurcation when the dynamic changes from a stable steady state, 1-cycle (the red region) to a stable 8-cycle (the pink region). As the extrapolation parameter keeps increasing toward its maximum assumed value of $\gamma = 1.25$, the system undergoes a sequence of bifurcations that take its dynamics to several regions of stable periodic cycles and unclassifiable dynamics. For instance, as the extrapolation parameter approaches the value of $\gamma = 1.025$ and keeps increasing, while the mutation rate remains fixed at $\theta = 0.02$, a sequence of bifurcations expectedly similar to those observed in Figure 1 top take the system first to stable 6-cycles (the purple regions), then to stable 35-cycles (the light blue regions), then to stable 29-cycles (the dark green regions), and finally to stable 23-cycles (the gray regions). Meanwhile, there is a widening to the right of the region of stable 1-cycles as the mutation rate steadily rises from levels close to zero to its maximum considered level of $\theta = 0.05$. Recall that Proposition 2 implies that when the strategy switching protocol features mutation (so that both inflation forecasting strategies survive in the long-run equilibrium), the critical value of the extrapolation parameter varies positively with the long-run equilibrium proportion of perfect forecasters, which in turn (per (24)) is positively affected by an increase in the mutation rate. Also, as suggested in the bottom panel in Figure 2, in which the represented LCEs were calculated using the same precision parameter used in Figure 1 bottom, parameter coordinates with unclassifiable dynamics in the respective top panel do not seem to generate chaotic dynamics.

Figure 3 represents the dynamics of the system from a different perspective. The top and middle panels portray one parameter bifurcation diagrams describing the long-run behavior of the inflation rate and the proportion of extrapolative inflation forecasters, respectively, both as a function of the extrapolation parameter, while the bottom panel is a plot of the largest LCE (and ranges of the second and third largest ones) as a function of the same parameter. All required numerical specifications are as in Figures 1 and 2, so the cost associated with perfect foresight and the mutation rate are set at their typical values of $\kappa = 0.0001$ and $\theta = 0.01$, respectively. In the top panel, for instance, only the fixed point and some low-period cycles shown in Figure 1 top can be clearly identified (a prominent example being the stable 6-cycles at $\gamma = 1.075$ in the purple region), as the high-period cycles involve a large number of variable values. In the bottom panel, the possibility of a small range of values around $\gamma = 1$ featuring a positive value for the largest Lyapunov exponent is not confirmed by the more reliable spectrum of LCEs in the bottom panels in Figures 1 and 2, which were computed using a more appropriate value for the precision parameter when the purpose is to look for evidence of a positive LCE (as explained in the preceding footnote 8). Nonetheless, over most of the state space trajectories are attracted in the long run to a limit set different from the unique fixed point with the rate of inflation being at its natural value π_n, which is locally stable only up to $\gamma_c \approx 0.682$.

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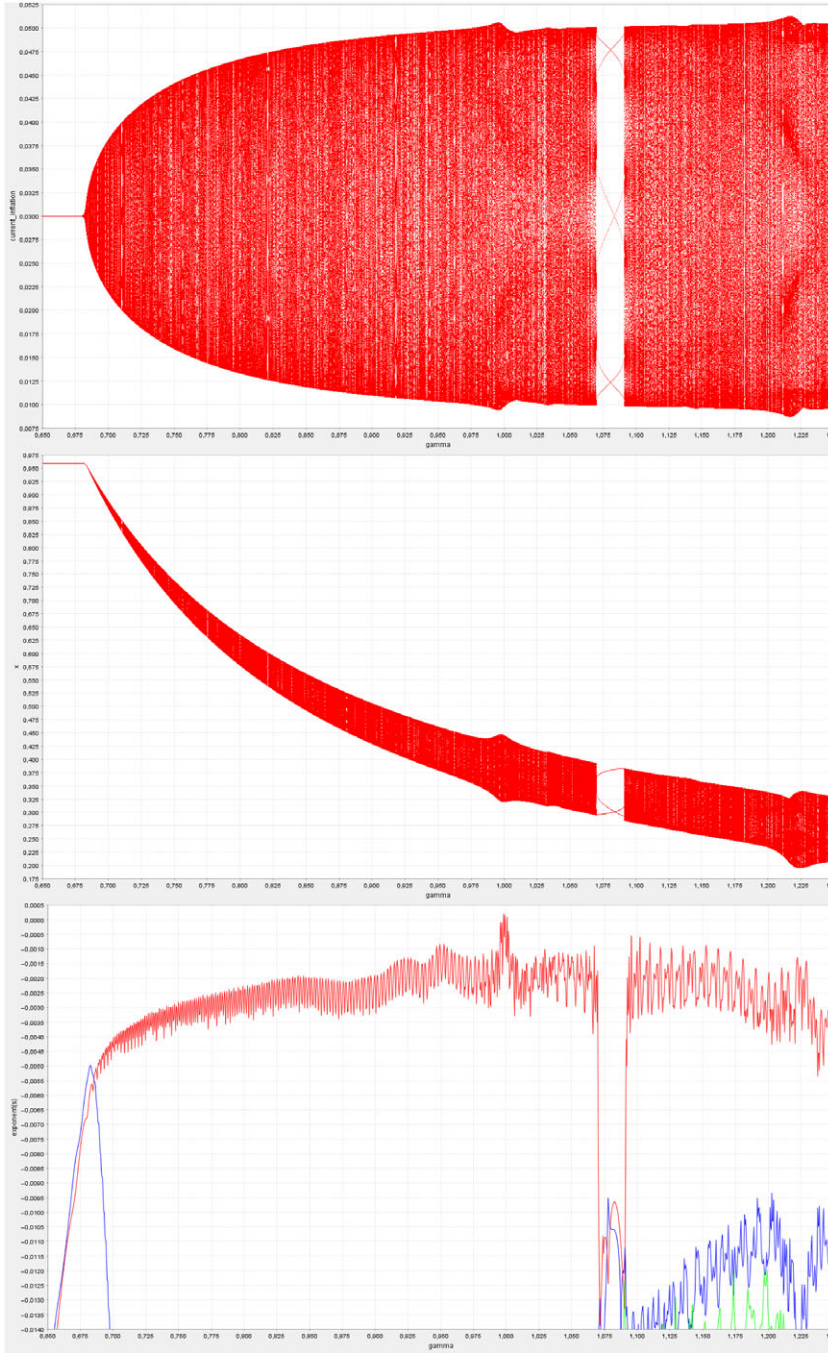


FIGURE 3. One parameter bifurcation diagrams (top: inflation rate; middle: proportion of extrapolative forecasters) and respective largest Lyapunov characteristic exponents (bottom).

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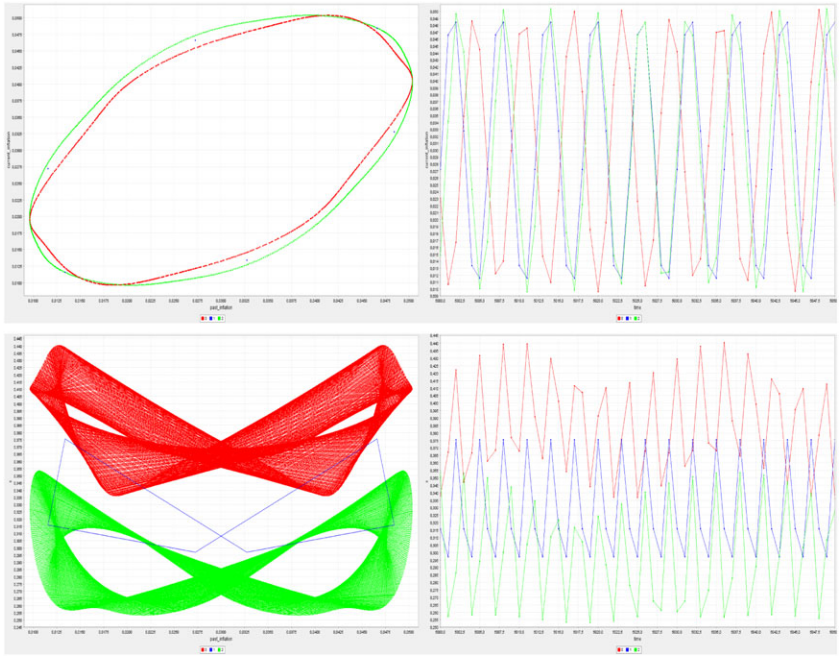


FIGURE 4. Left panels: state space for diverse values of γ (red: 0.99; blue: 1.075; light green: 1.16), with $(\pi_{t-1},\ \pi_t)$ (top) and $(\pi_{t-1},\ x_t)$ (bottom). Right panels: time slices, with π (top) and x (bottom).

In order to gain further qualitative understanding of global dynamics of the model, in the left panels in Figure 4, we plot cycles of different periodicities corresponding to different values of the extrapolation parameter γ in the state space with past inflation and current inflation $(\pi_{t-1},\ \pi_t)$ (top) and in the plane with past inflation and the proportion of extrapolative forecasters $(\pi_{t-1},\ x_t)$ (bottom). In the right panels, meanwhile, selected time series of π (top) and x (bottom), along with Figures 1 and 2, then suggest that the different values of γ under consideration (red: 0.99, blue: 1.075, and light green: 1.16) generate a 25-cycle, a 6-cycle, and a 35-cycle, respectively.

6. CONCLUSION

The empirical motivation for this paper comes from two series of evidence on the formation of inflation expectations. First, inflation expectations are endogenously time-varying, persistently heterogeneous across decision-makers and formed mostly through boundedly rational methods. Second, inflation forecast errors may have to pass some threshold before decision-makers choose to abandon their previously selected inflation predictor. Against this backdrop, the theoretical contribution of this paper lies in the exploration of macroeconomic implications

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of heterogeneous inflation expectations which are endogenously time-varying according to evolutionarily satisficing learning dynamics subject to a random noise analogous to mutation in natural environments. In the alternative modeling approach to inertia in inflation predictor switching devised in this paper, a fundamental role is played by Herbert Simon's understanding of choice as intending to meet some acceptability threshold rather than to select the best of all possible alternatives, hence the core idea of satisficing instead of optimal choice.

Using a specification of inflation predictor revisable choice featuring perfect foresight in competition with extrapolative trend-following, we find that the frequency distribution of inflation prediction strategies across agents in the long-run equilibrium depends on the relative importance of an objective measure of accuracy in inflation prediction as compared to random components operating at different layers of the predictor choice process. Moreover, there is a definite relationship between the frequency distribution of inflation prediction strategies and the local stability properties as two distinct features of the unique long-run equilibrium, which is characterized by the endogenous macroeconomic variables at their natural values and at least one half of the population of agents following the extrapolative strategy. The higher the proportion of extrapolative trend-following agents in the long-run equilibrium is, the less (locally) stability-prone will be such evolutionary equilibrium. Meanwhile, the more the inflation predictor choice is subject to random disturbance, the lower will be the proportion of extrapolative trend-following agents in the long-run equilibrium. Thus, randomness in inflation predictor choice plays a locally stabilizing role in our suggested evolutionarily satisficing learning dynamics. These results are both derived analytically and explored through numerical simulations.

Lastly, we have suggested the notion of evolutionarily satisficing learning in the context of a given dynamic macroeconomic model and applied it to the exploration of the specific but relevant issue of whether endogenously time-varying (and possibly persistent) heterogeneity in strategies to predict inflation is necessarily detrimental to successful stabilization policy, and the answer was found to be no. Yet the likely prospect that other instances of choice behavior are also subject to satisficing learning deserves future investigation.

NOTES

1. Experimental evidence on satisficing choice behavior as defined by Simon (1955) is offered in Caplin et al. (2011), who find that subjects stop searching when a satisficing level of reservation utility is achieved. Hey et al. (2017) find experimental evidence for Manski's (2017) theoretical contribution on when and how an agent adopts satisficing behavior as defined by Simon (1955). Manski (2017) theorizes satisficing as a class of decision strategies available to an agent who is seeking to optimize in a setting where deliberation is costly.

2. Speaking of "inattentiveness", Lima and Silveira (2015) offer evolutionary micro-foundations to the nominal adjustment of the price level in response to monetary shocks by introducing the notion of "boundedly rational inattentiveness". A firm can pay an information cost (with a random part) to set the optimal price or use costless non-updated information to set the lagged optimal price. The distribution of information-updating strategies across firms and the extent of the nominal adjustment of the

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price level to monetary shocks co-evolve as driven by evolutionary dynamics. While Lima and Silveira (2015) specify an asymmetric evolutionary protocol (only firms that do not pay the information cost rely on random pairwise comparisons of losses), the model herein specifies a satisficing evolutionary protocol based on random agent-specific acceptable prediction errors. Rational inattentiveness in updating information sets in the presence of heterogeneous expectations is also addressed in Dräger (2016).

3. In most of the above respects, our model likewise differs from the one in Lines and Westerhoff (2012), which is closely related (and obtains qualitative results similar) to Lines and Westerhoff (2010). These two contributions employ the same macroeconomic model and specify inflation predictor switching following Brock and Hommes (1997). Yet in Lines and Westerhoff (2012) extrapolative expectations are paired with regressive expectations (which predict the return of inflation toward its normal value) as costly predictor instead of rational expectations. Even so, the associated cost is again supposed to reflect a predisposition effect in addition to information collection and processing.

4. The formal derivation of this evolutionarily satisficing learning protocol (without mutation) is based on Vega-Redondo (1996, p. 91).

5. We should note that the model in Lines and Westerhoff (2010) also features "a permanent evolutionary competition between the prediction rules" (p. 247). The authors explicitly allude to Simon's suggestion that agents should be seen as boundedly rational when they recognize that because agents "lack the cognitive capabilities to derive fully optimal actions. . . they strive to do the right thing" (p. 249). However, although the gist of Simon's approach of choice behavior is that decision-makers satisfice rather than optimize, the model in Lines and Westerhoff (2010) does not incorporate the evidence in Branch (2004) that inflation forecast errors have to cross some threshold in order to induce a switch by appealing to Simon's fitting notion of satisficing behavior.

6. Besides, since the stable eigenvalue corresponding to the evolutionarily satisficing learning dynamics is given by $\lambda_1 = (1-\theta)[1-F(\kappa)]$ (see AppendixB), the lower is the measure of mutant agents or the higher is the *ex ante* cost carried by perfect inflation foresight, the lower is the speed of convergence of the frequency distribution of inflation forecasting strategies to its long-run equilibrium configuration. The intuition is that perfect forecasters play a crucial role in guiding the evolutionarily satisficing learning dynamics.

7. Admittedly, mutant agents, who are presently assumed to play either one or the other of the inflation foresight strategies with equal probability, might eventually switch to playing the extrapolative strategy with higher probability as the information cost associated with perfect foresight would become to be considered too high. In fact, an agent might even become reluctant to intentionally behave as mutant for experimentation reasons in such a situation. However interesting, we abstract from these analytical possibilities in our paper and leave them for future research. Incidentally, the robust empirical evidence provided in Branch (2004, 2007) is that the intensity of choice and information costs are both finite.

8. As in Lines and Westerhoff (2010), all numerical simulation results presented in this section (along with their graphical representation) were obtained with the open-source software iDMC, version 2.0.10, which is available at https://code.google.com/archive/p/idmc/downloads. Several routines of the iDMC software package are nicely illustrated in Lines (2007), and we are quite grateful to Marji Lines for kindly clarifying several further queries on these routines over several e-mail correspondences. The source code used to generate the numerical results presented in this section is obtainable upon request to the authors.

9. The algorithm for calculating the LCEs in Figure 1 bottom also requires the specification of a precision parameter. As recommended in Lines (2007), for distinguishing a zero or negative LCE, it may be appropriate to set such parameter at 0.001, while for looking for evidence of a positive LCE a less precise value might be appropriate, say 0.01. The reason is that in this case the LCE is equal to 0 if its absolute value is lower than 0.01 (the range is symmetrical), so that to be positive the LCE must be more than just a tiny bit positive. We follow Lines' (2007) suggestion in setting the precision parameter to calculate the LCEs in Figure 1 bottom to 0.01 (with 1000 iterations).

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APPENDIX A: PROOF OF PROPOSITION 1

A long-run equilibrium of the dynamic system given by (19)–(21) and (17) has to satisfy the conditions represented by $x_t = x_{t-1} \equiv x^*$, $u_t = u_{t-1} \equiv u^*$, and $\pi_t = \pi_{t-1} = \pi_{t-2} = \pi_{t-3} \equiv \pi^*$ for any $t \in \{1, 2, 3, ...\} \subset \mathbb{N}$. Thus, we can use (20) and (21) to establish that:

$$\alpha\beta(\pi^* - \pi_n) + \alpha(u^* - u_n) = 0, \tag{A1}$$

$$\beta x^*(\pi^* - \pi_n) - \alpha\beta(u^* - u_n) = 0. \tag{A2}$$

It is straightforward that for any given $x^* \in [0, 1] \subset \mathbb{R}$, there is a unique solution for the homogeneous linear system (A1)–(A2), given by $\pi^* - \pi_n = u^* - u_n = 0$, such that:

$$\pi^* = \pi_n \text{ and } u^* = u_n. \tag{A3}$$

Considering (A3), it follows that $\pi_t = \pi_{t-1} = \pi_{t-2} = \pi_{t-3} = \pi_n$. Thus, based on (17), we have that $y_{t-1} = \pi_{t-2} = \pi_n$ and $z_{t-1} = y_{t-2} = \pi_{t-3} = \pi_n$ in the long-run evolutionary equilibrium. Considering (16), it follows that $F\left(-a^E_{t-1}\right) = F\left(g(\pi_{t-1}, y_{t-1}, z_{t-1})\right) = F(0) = 0$ in the long-run equilibrium solution. From the noisy evolutionarily satisficing learning dynamic in (19), the following condition has then to be satisfied:

$$(1 - \theta)(1 - x^*)F(\kappa) + \theta\left(\frac{1}{2} - x^*\right) = 0. \tag{A4}$$

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The solution for this condition is given by:

$$x^* = \frac{(1-\theta)F(\kappa)+\theta/2}{(1-\theta)F(\kappa)+\theta}. \tag{A5}$$

Let us prove that $x^* \in (1/2, 1] \subset \mathbb{R}$ for all $\theta \in [0, 1) \subset \mathbb{R}$. Given that $\theta \in [0, 1) \subset \mathbb{R}$ and $F(\kappa) \subset (0, 1) \subset \mathbb{R}$, it follows that $(1-\theta)F(\kappa) > 0$. Therefore, we can use the latter inequality to establish that:

$$2(1-\theta)F(\kappa)+\theta > (1-\theta)F(\kappa)+\theta \Rightarrow x^* = \frac{(1-\theta)F(\kappa)+\theta/2}{(1-\theta)F(\kappa)+\theta} > \frac{1}{2}, \tag{A6}$$

for all $\theta \in [0, 1) \subset \mathbb{R}$, and:

$$(1-\theta)F(\kappa)+\theta/2 < (1-\theta)F(\kappa)+\theta \Rightarrow x^* = \frac{(1-\theta)F(\kappa)]+\theta/2}{(1-\theta)F(\kappa)]+\theta} < 1, \tag{A7}$$

for all $\theta \in (0, 1) \subset \mathbb{R}$. Finally, it is easy to check that $x^* = 1$ for $\theta = 0$. This completes the proof that there is one, and only one, equilibrium solution $(x^*, \pi_n, u_n, \pi_n, \pi_n) \in \Theta$ of the system represented by (19)–(21) and (17).

APPENDIX B: PROOF OF PROPOSITION 2

Consider the Jacobian matrix of the linearization around the evolutionary long-run equilibrium of the system given by (19)–(21) and (17):

$$J(x^*, \pi_n, u_n, \pi_n, \pi_n) = \left[\begin{array}{c|c|c|c|c} (1-\theta)\,(1-F(\kappa)) & 0 & 0 & 0 & 0 \\ \hline 0 & \dfrac{(1+\gamma)x^*}{\alpha\beta+x^*} & \dfrac{-\alpha}{\alpha\beta+x^*} & \dfrac{-\gamma x^*}{\alpha\beta+x^*} & 0 \\ \hline 0 & \dfrac{\beta(1+\gamma)x^*}{\alpha\beta+x^*} & \dfrac{x^*}{\alpha\beta+x^*} & \dfrac{-\beta\gamma x^*}{\alpha\beta+x^*} & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \end{array}\right]. \tag{B1}$$

Let λ be an eigenvalue of the Jacobian matrix (B1) and I the 5×5 identity matrix. We can then set the following characteristic equation of the linearization around the equilibrium:

$$|J-\lambda I| = \left[\begin{array}{c|c|c|c|c} (1-\theta)\,(1-F(\kappa))-\lambda & 0 & 0 & 0 & 0 \\ \hline 0 & \dfrac{(1+\gamma)x^*}{\alpha\beta+x^*}-\lambda & \dfrac{-\alpha}{\alpha\beta+x^*} & \dfrac{-\gamma x^*}{\alpha\beta+x^*} & 0 \\ \hline 0 & \dfrac{\beta(1+\gamma)x^*}{\alpha\beta+x^*} & \dfrac{x^*}{\alpha\beta+x^*}-\lambda & \dfrac{-\beta\gamma x^*}{\alpha\beta+x^*} & 0 \\ \hline 0 & 1 & 0 & -\lambda & 0 \\ \hline 0 & 0 & 0 & 1 & -\lambda \end{array}\right]. \tag{B2}$$

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We can use the Laplace expansion to express (B2) as follows:

$$|J-\lambda I| = [(1-\theta)(1-F(\kappa))-\lambda]\,\lambda \begin{vmatrix} \dfrac{(1+\gamma)x^*}{\alpha\beta+x^*}-\lambda & \dfrac{-\alpha}{\alpha\beta+x^*} & \dfrac{-\gamma x^*}{\alpha\beta+x^*} \\ \dfrac{\beta(1+\gamma)x^*}{\alpha\beta+x^*} & \dfrac{x^*}{\alpha\beta+x^*}-\lambda & \dfrac{-\beta\gamma x^*}{\alpha\beta+x^*} \\ 1 & 0 & -\lambda \end{vmatrix} = 0. \tag{B3}$$

Consequently, two of the eigenvalues can already be explicitly determined, which are given by $\lambda_1 = (1-\theta)[1-F(\kappa)]$ and $\lambda_2 = 0$. Given that $\theta \in [0,1) \subset \mathbb{R}$ and $F(\kappa) \subset (0,1) \subset \mathbb{R}$, it follows that $0 < \lambda_1 < 1$.

Considering (B3), the remaining eigenvalues have to satisfy:

$$\begin{vmatrix} \dfrac{(1+\gamma)x^*}{\alpha\beta+x^*}-\lambda & \dfrac{-\alpha}{\alpha\beta+x^*} & \dfrac{-\gamma x^*}{\alpha\beta+x^*} \\ \dfrac{\beta(1+\gamma)x^*}{\alpha\beta+x^*} & \dfrac{x^*}{\alpha\beta+x^*}-\lambda & \dfrac{-\beta\gamma x^*}{\alpha\beta+x^*} \\ 1 & 0 & -\lambda \end{vmatrix} = 0, \tag{B4}$$

which can be re-written as:

$$\lambda^3 - a\lambda^2 - b\lambda - c = 0, \tag{B4a}$$

where:

$$a = \frac{(2+\gamma)x^*}{\alpha\beta+x^*},\ b = \frac{(1+2\gamma)x^*}{\alpha\beta+x^*},\ \text{and}\ c = \frac{\gamma x^*}{\alpha\beta+x^*}. \tag{B5}$$

Given (B4a), we can then make use of a set of stability conditions for a third-order characteristic equation to determine if the remaining three eigenvalues are inside the unit circle. In fact, based on Farebrother (1973, p. 397, inequalities 3), we can then establish, as done in Lines and Westerhoff (2010, p. 256), the following set of necessary and sufficient for the cubic polynomial (B4a) to have roots that are strictly smaller than one in absolute value:

$$1-a-b-c>0,\ 1+a-b+c>0,\ 1+b+ac-c^2>0,\ \text{and}\ 3-a+b+3c>0. \tag{B6}$$

Inserting (B5) into (B6), we obtain:

$$1-a-b-c = \frac{\alpha\beta}{\alpha\beta+x^*} > 0, \tag{B7}$$

$$1+a-b+c = \frac{\alpha\beta+4(1+\gamma)x^*}{\alpha\beta+x^*} > 0, \tag{B8}$$

$$1+b+ac-c^2 = \frac{\alpha\beta[\alpha\beta+x^*-2\gamma x^*]}{(\alpha\beta+x^*)^2} > 0,\ \text{if}\ \gamma < \frac{\alpha\beta+x^*}{2x^*} \equiv \gamma_c, \tag{B9}$$

$$3-a+b+3c = \frac{3\alpha\beta}{\alpha\beta+x^*} > 0. \tag{B10}$$

Therefore, if $\gamma < \frac{\alpha\beta+x^*}{2x^*} \equiv \gamma_c$, all the eigenvalues of (B3) are strictly less than one in absolute value, so the equilibrium configuration given by $(x^*, \pi_n, u_n, \pi_n, \pi_n) \in \Theta$ of the system given by (19)–(21) and (17) is locally asymptotically stable.

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At $\gamma = \gamma_c$, the inequality (B9) becomes an equality. Besides, it can be easily verified that the inequalities represented by $-1 < -c = -\frac{\gamma_c x^*}{\alpha\beta + x^*} = -\frac{1}{2} < 2 - a = 2 - \frac{(2+\gamma_c)x^*}{\alpha\beta + x^*} = \frac{3}{2} - \frac{2x^*}{\alpha\beta + x^*}$ and $-2 - a = -2 - \frac{(2+\gamma)x^*}{\alpha\beta + x^*} < -c = -\frac{x^*}{\alpha\beta + x^*} < 1$ hold. Thus, based on Lines et al. (2019), we can infer that the Jacobian matrix (B4) has two complex conjugate eigenvalues whose absolute value is equal to 1 and one real eigenvalue whose absolute value is less than 1. As the Jacobian matrix (B2) has three real eigenvalues whose absolute value is less than 1 and a complex conjugate pair that simultaneously crosses the unit circle, we can then conclude that a Neimark–Sacker bifurcation occurs at $\gamma = \gamma_c$. □

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