Proc. Camb. Phil. Soc. (1970), 68, 345
PCPS 68–40
Printed in Great Britain

Dimensionally dependent identities

By DAVID LOVELOCK
Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada

(*Received* 25 *August* 1969)

Abstract. In the general theory of relativity a number of apparently unrelated identities peculiar to a 4-dimensional space are frequently used. However, the proofs usually presented appear to have no common ideas and, furthermore, it is not clear at what stage the dimensionality restriction plays a significant role. In this note a technique is presented which explicitly exhibits the dimensionality dependence and thereby enables the above identities to be generalized for n-dimensional spaces. It is also shown that the above identities are all special cases of a more general identity valid for $n = 4$.

1. *Introduction.* In an n-dimensional Riemannian space with line element†

$$ds^2 = g_{ij}\,dx^i\,dx^j,$$

the curvature tensor $R_i{}^j{}_{kh}$ is introduced by means of the commutation relations

$$X^j{}_{;kh} - X^j{}_{;hk} = R_i{}^j{}_{kh}\,X^i,$$

where a semi-colon denotes partial covariant differentiation and X^i is any contravariant vector field. The Ricci tensor, the curvature scalar and the Weyl conformal curvature tensor are defined respectively by

$$R_{ij} = R_i{}^h{}_{jh},$$

$$R = g^{ij}R_{ij},$$

and

$$C_j{}^h{}_{im} = R_j{}^h{}_{im} + \frac{1}{n-2}(R_i^h g_{jm} + R_{jm}\,\delta_i^h - R_{ij}\,\delta_m^h - R_m^h g_{ij}) + \frac{R}{(n-1)(n-2)}(g_{ij}\,\delta_m^h - g_{jm}\,\delta_i^h)$$

(for $n > 2$). $C_j{}^h{}_{im}$ has all the symmetry properties of the curvature tensor $R_j{}^h{}_{im}$ together with

$$C_j{}^h{}_{ih} = 0.$$

We shall always raise and lower indices by means of g^{ij} and g_{ij} respectively.

In the literature there exist what may be regarded as *dimensionally dependent identities*. By a dimensionally dependent identity we mean an identity which is a direct consequence of the dimension of the space taking on a particular value and which therefore is not valid for arbitrary n in general.

† Unless otherwise stated the summation convention (according to which repeated indices are summed from 1 to n) is used throughout this note.

https://doi.org/10.1017/S0305004100046144 Published online by Cambridge University Press

To clarify ideas we cite five well-known examples:

Ex. 1. If $n = 3$ then $C_i^j{}_{kl} = 0$ (Weyl (1), p. 404).

Ex. 2. If $n = 4$ then $R_{ij} = \lambda g_{ij}$

implies† $3g_{[ij}R_{kh]rs} = 2\lambda g_{[rj}g_{sk}g_{jh]}$

and conversely (Petrov (2), p. 77).

Ex. 3. If $n = 4$ and ϕ^i is any vector for which

$$C_i^j{}_{kl}\phi^i = 0,$$

then either $C_i^j{}_{kl} = 0$

or $\phi_i\phi^i = 0$

(Brinkmann (3), p. 277).

Ex. 4. If $n = 4$ then

$$2R_{jh}R^{ih} + 2R^{hk}R_h{}^i{}_{kj} - RR_j^i - R_{hjlk}R^{hilk} = \tfrac{1}{4}\delta_j^i(4R_{hk}R^{hk} - R^2 - R_{rshk}R^{rshk}),$$

(Lanczos (4), pp. 847–849). This latter identity may be rewritten in the simpler form

$$C^{ihkm}C_{jhkm} = \tfrac{1}{4}\delta_j^i(C^{rshk}C_{rshk}),$$

(Lovelock (5), p. 188).

Ex. 5. If F_{ij} is an antisymmetric tensor and T_{ij} is defined by

$$T_{ij} = F_{ih}F_j^h - \tfrac{1}{4}g_{ij}(F_{rs}F^{rs})$$

then for $n = 4$ $T_{ij}T^{jk} = \tfrac{1}{4}\delta_i^k(T_{jh}T^{jh})$,

(Rainich (6), p. 118; Misner and Wheeler (7), p. 242).

Exs. 1–5 are dimensionally dependent identities—without further restrictions none of these results is valid for $n \geqslant 5$.

The proofs usually supplied for Exs. 1–5 appear to have no common ideas. Furthermore, it is not clear at what stage the dimensionality restriction plays a significant role. For example, Exs. 2 and 4 are usually proved by firstly transforming to a system of coordinates in which $g_{ii} = g^{ii} = 1$ (no summation) and $g_{ij} = 0$ $(i \neq j)$, and secondly considering certain appropriate expressions in this coordinate system. Ex. 1 may be proved in a similar fashion (Eisenhart (8), pp. 32, 33). The proof of Ex. 3 involves a geometrical argument where use is made of the 3-dimensional space, orthogonal to ϕ^i, in which $C_i^j{}_{kh}$ lies. Finally the proof of Ex. 5 involves the concept of duality rotations or the properties of certain 4×4 matrices (Adler, Bazin and Schiffer (9), pp. 417 ff.).

Without a technique which explicitly displays the dimensionality dependence it is difficult to formulate theorems which, in some sense, may be regarded as generalizations of Exs. 1–5 for arbitrary n. The purpose of this note is to supply such a technique, thereby leading to dimensionally dependent identities for arbitrary n. In the process we show that, in fact, Exs. 2–5 are all special cases of a more general result valid for $n = 4$.

† The square brackets denote antisymmetrization, e.g.

$$B_{[i_1\ldots i_N]} = \frac{1}{N!}(B_{i_1 i_2\ldots i_N} - B_{i_2 i_1\ldots i_N} + \ldots).$$

In Ex. 2 double antisymmetrization is implied—the left-hand side is antisymmetrized over *jkh* and *irs* independently while the right-hand side is antisymmetrized over *iskj* and *rh* independently.

https://doi.org/10.1017/S0305004100046144 Published online by Cambridge University Press

2. *General identities in an n-dimensional space.* For some positive integer m (> 1) we shall consider the quantities $K_{i_1 i_2 \dots i_m}{}^{j_1 j_2 \dots j_m}$, where the indices

$$i_1, i_2, \dots, i_m, j_1, j_2, \dots, j_m$$

may take on any of the values $1, 2, \dots, n$. We assume that $K_{i_1 \dots i_m}{}^{j_1 \dots j_m}$ has the following properties:

P 1. $K_{i_1 \dots i_m}{}^{j_1 \dots j_m}$ is antisymmetric under interchange of j_p and j_q for any distinct p and q, $1 \leqslant p \leqslant m$, $1 \leqslant q \leqslant m$, e.g.

$$K_{i_1 \dots i_m}{}^{j_1 j_2 j_3 \dots j_m} = -K_{i_1 \dots i_m}{}^{j_2 j_1 j_3 \dots j_m}.$$

P 2. $K_{i_1 \dots i_m}{}^{j_1 \dots j_m}$ is antisymmetric under interchange of i_p and i_q for any distinct p and q, $1 \leqslant p \leqslant m$, $1 \leqslant q \leqslant m$.

P 3. $$K_{i_1 \dots i_{m-1} i}{}^{j_1 \dots j_{m-1} i} = 0.$$

We observe that conditions P1 to P 3 imply that $K_{i_1 \dots i_m}{}^{j_1 \dots j_m}$ will vanish whenever any one of $i_1 \dots i_m$ coincides with any one of $j_1 \dots j_m$ (and are then summed from 1 to n, being repeated indices). We do not make any assumptions as to the tensorial character (or otherwise) of $K_{i_1 \dots i_m}{}^{j_1 \dots j_m}$. An example of a tensor which satisfies P 1–P 3 for $m = 2$ is the Weyl conformal curvature tensor $C_{ij}{}^{kh}$. Other examples will be given in the sequel.

It is easily seen that the total number of independent quantities $K_{i_1 \dots i_m}{}^{j_1 \dots j_m}$ is

$$\left[\frac{n(n-1)\dots(n-m+1)}{m!}\right]^2 - \left[\frac{n(n-1)\dots(n-m+2)}{(m-1)!}\right]^2$$
$$= \left[\frac{n(n-1)\dots(n-m+2)}{m!}\right]^2 (n+1)(n-2m+1),$$

which will be negative or zero if $\quad n \leqslant 2m-1$.

Thus we have

THEOREM 1. *If* $n \leqslant 2m-1$ *then*

$$K_{i_1 \dots i_m}{}^{j_1 \dots j_m} = 0.$$

This theorem may also be established in a different fashion which is more in the spirit of the rest of this note. To this end we consider the quantity

$$T_{i_1 \dots i_m}{}^{j_1 \dots j_m} = \delta^{j_1 \dots j_m k_1 \dots k_m}_{i_1 \dots i_m h_1 \dots h_m} K_{k_1 \dots k_m}{}^{h_1 \dots h_m},$$

where $\delta^{i_1 i_2 \dots i_N}_{j_1 j_2 \dots j_N}$ is the generalized Kronecker delta (Synge and Schild (10), pp. 242, 278) which may be written in the form

$$\delta^{i_1 \dots i_N}_{j_1 \dots j_N} = \det \begin{vmatrix} \delta^{i_1}_{j_1} & \dots & \delta^{i_1}_{j_N} \\ \vdots & & \vdots \\ \delta^{i_N}_{j_1} & \dots & \delta^{i_N}_{j_N} \end{vmatrix}.$$

Any easy calculation making use of P1–P 3 shows that

$$T_{i_1 \dots i_m}{}^{j_1 \dots j_m} = \delta^{j_1 \dots j_m}_{h_1 \dots h_m} \delta^{k_1 \dots k_m}_{i_1 \dots i_m} K_{k_1 \dots k_m}{}^{h_1 \dots h_m}$$
$$= (m!)^2 K_{i_1 \dots i_m}{}^{j_1 \dots j_m}.$$

https://doi.org/10.1017/S0305004100046144 Published online by Cambridge University Press

However, *if* $n < N$ *then* $\quad \delta^{i_1 \dots i_N}_{j_1 \dots j_N} = 0$ *identically,*

by virtue of the fact that at least two of $i_1 \dots i_N$ must be the same. Theorem 1 follows by taking $N = 2m$ in the above analysis, since in this case

$$T_{i_1 \dots i_m}{}^{j_1 \dots j_m} = 0.$$

We now turn to the case $n = 2m$, i.e. n even. In this case we consider the quantity

$$P_{i_1 \dots i_{m+1}}{}^{j_1 \dots j_{m+1}} = \delta^{j_1 \dots j_{m+1} k_1 \dots k_m}_{i_1 \dots i_{m+1} h_1 \dots h_m} K_{k_1 \dots k_m}{}^{h_1 \dots h_m}.$$

By using arguments analogous to those in the preceding paragraph, it is not difficult to establish

THEOREM 2. *If* $n = 2m$ *then*

$$K_{[i_1 \dots i_m}{}^{[j_1 \dots j_m} \delta^{j_{m+1}]}_{i_{m+1}]} = 0.$$

By virtue of P 1 and P 2 the left-hand side of the latter consists of $(m+1)^2$ terms.

We remark that Theorem 2 can also be proved directly from Theorem 1 as follows. Consider the quantity $S_{i_1 \dots i_{m+1}}{}^{j_1 \dots j_{m+1}}$ defined by

$$S_{i_1 \dots i_{m+1}}{}^{j_1 \dots j_{m+1}} = K_{[i_1 \dots i_m}{}^{[j_1 \dots j_m} \delta^{j_{m+1}]}_{i_{m+1}]}.$$

If $n = 2m$ then $S_{i_1 \dots i_{m+1}}{}^{j_1 \dots j_{m+1}}$ has all the properties P 1–P 3 and hence by Theorem 1 (with m replaced by $m+1$)

$$S_{i_1 \dots i_{m+1}}{}^{j_1 \dots j_{m+1}} = 0,$$

which is Theorem 2.

3. *Special cases.* In this section we shall first consider the simplest case, i.e. $m = 2$. For this case conditions P 1–P 3 read

$$K_{ij}{}^{kh} = -K_{ij}{}^{hk} = -K_{ji}{}^{kh}$$

together with

$$K_{ij}{}^{kj} = 0,$$

while Theorems 1 and 2 reduce to

THEOREM 3. (*a*) *If* $n = 3$ *then* $\quad K_{ij}{}^{kh} = 0.$

(*b*) *If* $n = 4$ *then* $\quad K_{[ij}{}^{[kh} \delta^{a]}_{b]} = 0.$

We assert that the results cited in section 1, namely *Examples* 1–5, *are all special cases of Theorem* 3. We treat each of these examples in turn.

Ex. 1. Theorem 3 (*a*) is clearly a generalization of Weyl's result. The latter follows when $K_{ij}{}^{kh}$ is chosen as $C_{ij}{}^{kh}$.

Ex. 2. If in Theorem 3 (*b*) we choose $K_{ij}{}^{kh}$ as $C_{ij}{}^{kh}$ we find

COROLLARY 1. *If* $n = 4$ *then* $\quad C_{[ij}{}^{[kh} \delta^{a]}_{b]} = 0.$

It is easily established that Ex. 2 follows from this result.

Ex. 3. If we multiply the equation in Theorem 3 (*b*) by $\phi^i \psi_k$ we find

COROLLARY 2. *If* $n = 4$ *then for any quantities* ϕ^i, ψ_k *such that*

$$K_{ij}{}^{kh} \phi^i = 0$$

and

$$K_{ij}{}^{kh} \psi_k = 0,$$

https://doi.org/10.1017/S0305004100046144 Published online by Cambridge University Press

either $$\phi^i\psi_i = 0$$

or $$K_{ij}{}^{kh} = 0.$$

Ex. 3 is a special case of this corollary where

$$K_{ij}{}^{kh} = C_{ij}{}^{kh}$$

and $$\psi_k = g_{ki}\phi^i.$$

Exs. 4 and 5. In order to derive these examples we multiply the equation in Theorem 3 (*b*) by $K_{kh}{}^{ij}$ to find

COROLLARY 3. *If* $n = 4$ *then*

$$K_{ij}{}^{kh}K_{kh}{}^{ir} = \tfrac{1}{4}\delta^r_j(K_{is}{}^{kh}K_{kh}{}^{is}).$$

As has been pointed out elsewhere (Lovelock (5), pp. 193, 194), Exs. 4 and 5 follow by choosing $$K_{ij}{}^{kh} = C_{ij}{}^{kh}$$

and

$$K_{ij}{}^{kh} = -12F_{ij}F^{hk} + 6(\delta^h_i T^k_j + \delta^k_j T^h_i - \delta^h_j T^k_i - \delta^k_i T^h_j) + F_{rs}F^{rs}(\delta^h_i\delta^k_j - \delta^h_j\delta^k_i),$$

respectively.

Obviously Theorems 1 and 2 are the direct generalizations of Exs. 1 and 2 for arbitrary n, while it is possible to generalize Exs. 3–5 by way of Corollaries 2 and 3, namely

COROLLARY 4. *If* $n = 2m$ *then for any quantities* ϕ^i, ψ_k *such that*

$$K_{i_1\dots i_{m-1}i}{}^{j_1\dots j_m}\phi^i = 0$$

and $$K_{i_1\dots i_m}{}^{j_1\dots j_{m-1}j}\psi_j = 0,$$

either $$\phi^i\psi_i = 0$$

or $$K_{i_1\dots i_m}{}^{j_1\dots j_m} = 0.$$

COROLLARY 5. *If* $n = 2m$ *then*

$$K_{i_1\dots i_{m-1}i}{}^{j_1\dots j_m}K_{j_1\dots j_m}{}^{i_1\dots i_{m-1}j} = \frac{1}{n}\delta^j_i(K_{i_1\dots i_m}{}^{j_1\dots j_m}K_{j_1\dots j_m}{}^{i_1\dots i_m}).$$

By way of example we briefly discuss the case $m = 3$. The quantity $H_{ijk}{}^{rst}$ defined, for $n > 4$, by

$$H_{ijk}{}^{rst} = A_{ijk}{}^{rst} + \frac{3^2}{4-n}A_{[ijk}{}^{r[bc}\delta^{a]}_{i]} + \frac{2}{(4-n)(3-n)}A_{[rs[i}{}^{s[c}\delta^a_{j}\delta^{b]}_{k]}$$

$$+ \frac{6}{(4-n)(3-n)(2-n)}A_{[rst]}{}^{rst}\delta^{[a}_{[i}\delta^b_j\delta^{c]}_{k]},$$

where $$A_{ijk}{}^{abc} = 4C_{[ij}{}^{h[a}C_{k]h}{}^{bc]}$$

has all the properties P 1–P 3. Consequently we find the following:

If $n = 5$ *then* $$H_{ijk}{}^{rst} = 0,$$

while if $n = 6$ $$H_{[ijk}{}^{[rst}\delta^{b]}_{a]} = 0.$$

Thus in 5- and 6-dimensional spaces there exist identities which are quadratic in the curvature tensor.

https://doi.org/10.1017/S0305004100046144 Published online by Cambridge University Press

REFERENCES

(1) WEYL, H. *Math. Z.* **2** (1918), 384–411.
(2) PETROV, A. Z. *Einstein spaces*, English edition (Pergamon Press, Oxford, 1969).
(3) BRINKMANN, H. W. *Math. Ann.* **91** (1924), 269–278.
(4) LANCZOS, C. *Ann. of Math.* (2), **39** (1938), 842–850.
(5) LOVELOCK, D. *Atti. Accad. Naz. Lincei* (VIII), **42** (1967), 187–194.
(6) RAINICH, G. Y. *Trans. Amer. Math. Soc.* **27** (1925), 106–130.
(7) WHEELER, J. A. *Geometrodynamics. Topics of modern physics*, vol. 1 (Academic Press, New York, 1962).
(8) EISENHART, L. P. *Riemannian geometry*, 6th printing (Princeton University Press, 1966).
(9) ADLER, R., BAZIN, M. and SCHIFFER, M. *Introduction to general relativity* (McGraw-Hill, London, 1965).
(10) SYNGE, J. L. and SCHILD, A. *Tensor calculus* (University of Toronto Press, Toronto, 1949).

https://doi.org/10.1017/S0305004100046144 Published online by Cambridge University Press