Combinatorics, Probability and Computing (2010) 19, 343–369. © Cambridge University Press 2009
doi:10.1017/S0963548309990484

Relative Tutte Polynomials for Coloured Graphs and Virtual Knot Theory

Y. DIAO[†] and G. HETYEI[‡]

Department of Mathematics and Statistics, UNC Charlotte, Charlotte, NC 28223, USA
(e-mail: ydiao@uncc.edu, ghetyei@uncc.edu)

Received 19 May 2008; accepted 8 September 2009; first published online 18 November 2009

We introduce the concept of a relative Tutte polynomial of coloured graphs. We show that this relative Tutte polynomial can be computed in a way similar to the classical spanning tree expansion used by Tutte in his original paper on this subject. We then apply the relative Tutte polynomial to virtual knot theory. More specifically, we show that the Kauffman bracket polynomial (and hence the Jones polynomial) of a virtual knot can be computed from the relative Tutte polynomial of its face (Tait) graph with some suitable variable substitutions. Our method offers an alternative to the ribbon graph approach, using the face graph obtained from the virtual link diagram directly.

1. Introduction

In this paper, we introduce and study the *relative Tutte polynomial* for a coloured graph, a generalization of the (ordinary) Tutte polynomial for a coloured graph. Let G be a connected graph. An important property that a classical Tutte polynomial possesses is that it can be computed by contracting and deleting the edges of G in an arbitrary order. Assume now that $\mathcal{H}$ is a subset of the edges of G. We would like to define the relative Tutte polynomial $T_{\mathcal{H}}$ of G with respect to $\mathcal{H}$ so that it would have a similar property, but only for the edges of G not in $\mathcal{H}$. More specifically, we would like to be able to compute the relative Tutte polynomial $T_{\mathcal{H}}(G)$ via two stages: first by the contracting/deleting process on edges of G that are not in $\mathcal{H}$, then by assigning variables to the remaining graphs (whose edges are all from $\mathcal{H}$) with a totally different rule. The variable assigning rule in the second stage can be quite arbitrary and that is what makes the relative Tutte polynomial more general and different from the (ordinary) Tutte polynomial.

Tutte defined his polynomial of an un-coloured graph [29] in terms of counting *activities* with respect to a specific labelling of the edges of the graph, and his main result in [29]

[†] Partially supported by NSF grant DMS-0712958.
[‡] Partially supported by NSA grant H98230-07-10073.

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

shows that the polynomial he introduced can be computed via a spanning tree expansion by counting activities of the edges with respect to the spanning trees and a given labelling of the edges, and that the polynomial is actually independent of the labelling, thus truly an invariant of the graph. This is equivalent to saying that the Tutte polynomial can be computed through the contracting/deleting process and the order of edges appearing in this process does not matter. The greatest challenge in generalizing Tutte's polynomial to coloured graphs is to preserve the independence of the labelling. This challenge is typically met by considering the Tutte polynomial of a coloured graph as an element of a polynomial ring, modulo certain relations between the variables. The most general result here is due to Bollobás and Riordan [2], who give a necessary and sufficient set of relations modulo which a Tutte polynomial of a coloured graph is labelling-independent. It turns out that the relative Tutte polynomial defined in this paper also possesses this property. That is, it can also be computed by a process similar to the spanning tree expansion by counting the activities of the edges of G (that are not in $\mathcal{H}$) with respect to a given order of the edges of G (that are not in $\mathcal{H}$). Furthermore, the polynomial so defined is also independent of the order on the edges. It is worthwhile pointing out that since our polynomial is defined on coloured graphs, it also generalizes the *set-pointed* Tutte polynomial introduced and discussed by Las Vergnas [26] in the graph-theoretical setting. Our approach closely follows the one used in Bollobás and Riordan [2] and our result is analogous to that of [2]. In other words, our result is also the most general, in the sense that we have given a necessary and sufficient set of relations modulo which a relative Tutte polynomial of a coloured graph is labelling-independent.

Our main motivation to introduce and study the relative Tutte polynomial comes from knot theory. It is well known that the Jones polynomial of a link can be computed from the Kauffman bracket polynomial. On the other hand, the Kauffman bracket polynomial of a link can be computed from the (signed) Tutte polynomial of the face graph of a regular projection of the link. This was first shown for alternating links and the ordinary Tutte polynomial by Thistlethwaite [27], then generalized to arbitrary links and a signed Tutte polynomial by Kauffman [20]. This enables applications of the ordinary Tutte polynomials and their signed generalizations to classical knot theory such as those in [12, 13, 17]. For virtual knots the situation is a little more complicated. An appropriate generalization of the Kauffman bracket polynomial was developed by Kauffman himself [21]. However, until very recently, no appropriate generalization of the Tutte polynomial to face graphs of virtual links was known. In a series of papers, Chmutov, Pak and Voltz [9, 10, 11] developed a generalization of Thistlethwaite's theorem first to checkerboard-colourable [10] and then to arbitrary [9, 11] virtual link diagrams. These express the Jones polynomial of a virtual link in terms of a signed generalization of the Bollobás–Riordan polynomial [3, 4] of a ribbon graph, obtained from the virtual link diagram. In this paper we will show that a relative variant of the *other* generalization of the Tutte polynomial, also due to Bollobás and Riordan [2], may also be used to compute the Jones polynomial of a virtual link, this time directly from the face graph of the virtual link diagram. The application of the relative Tutte polynomials is not just limited to virtual knot theory. We remind the reader that the Bollobás–Riordan polynomial defined in [2] is in a sense the most general Tutte polynomial that may be defined for coloured

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

graphs [2, Theorem 2]. Several examples of less general Tutte polynomials arising as a homomorphic image of this coloured Tutte polynomial are given in [2], and the relative variant of the coloured Bollobás–Riordan Tutte polynomial may also be used in the study of all models where the original, non-relative variant has proved itself useful. For example, it can be applied to networks with different layers of structures as well, generalizing some results of Fortuin and Kasteleyn [16].

This paper is organized in the following way. In the Preliminaries (Section 2) we review the main result of Bollobás and Riordan [2], providing in a sense the most general notion of a coloured Tutte polynomial that is labelling-independent if we generalize Tutte's original approach [29] of counting activities. We then turn to the introduction and discussions of the relative Tutte polynomial of a coloured graph in Section 3. There we state and prove our main theorem about the relative Tutte polynomial for connected graphs and matroids. In Section 4, we extend the relative Tutte polynomial to disconnected graphs. We also give some examples of the relative Tutte polynomials in this section. In one example, we show how an ordinary coloured Tutte polynomial can be recovered from a relative Tutte polynomial. In Section 5, we apply the relative Tutte polynomial to virtual knot theory. There we will state and prove our main theorem in the application of the relative Tutte polynomial.

2. Preliminaries

2.1. Matroids associated to graphs

The Tutte polynomial and its coloured generalizations are *matroid invariants*. There are many equivalent definitions of a matroid; a good basic reference is [30]. A matroid is *graphic* if its elements may be represented by the edges of a graph G such that the minimal dependent sets are the cycles in the graph. Independent sets then correspond to forests, maximal independent sets or bases to spanning forests. If the graph is connected then the bases are the spanning trees. Since the matroid associated to a graph depends only on the cycle structure, two graphs have the same underlying matroid structure if they have the same 2-edge connected components. The Tutte polynomial and the generalizations we consider are matroid invariants in the sense that they depend only on the matroid associated to the graph. Most statements and proofs we make or were made about (generalized) Tutte polynomials of graphs may be easily generalized to matroids.

In this paper we will often rely on the notion of *matroid duality*. Perhaps the easiest way to define the dual $\mathcal{M}^*$ of a matroid $\mathcal{M}$ is by giving the maximal independent sets of $\mathcal{M}^*$: the set B^* is a basis in $\mathcal{M}^*$ if and only if its complement is a basis in $\mathcal{M}$. It is well known that the deletion and contraction operations are duals of each other. The dual of a graphic matroid is not necessarily graphic, in fact, only planar graphs have a dual graph. However, the dual notion of a cycle in a graph is well known. Assume that a matroid is represented as the cycle matroid of a connected graph. Then a set is a cocycle (= cycle in the dual) if and only if it is a minimal cut, *i.e.*, a minimal disconnecting set. If we make a pair of dual statements about a graph, it is sufficient to prove only one of the two statements if that proof generalizes immediately to matroids. The proof of the

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

Table 1. The variable assignment of an edge with respect to a spanning tree T.

internally active	X_λ	externally active	Y_λ
internally inactive	x_λ	externally inactive	y_λ

dual statement may be obtained by replacing each notion with its dual in the proof of the original statement.

2.2. Tutte polynomials

In this section we review the results of Bollobás and Riordan [2] concerning the Tutte polynomial for a coloured connected graph, as well as some results we had obtained in our earlier work [14]. A graph G with vertex set V and edge set E is a coloured graph if every edge of G is assigned a value from a colour set Λ. The following notion of 'activities' was first introduced by Tutte [29] for non-coloured graphs to express the ordinary Tutte polynomial as a sum of contributions over all spanning trees of a connected graph.

Definition 2.1. Let G be a connected graph with edges labelled $1, 2, \ldots, n$, and let T be a spanning tree of G. An edge e of T is said to be *internally active* if for any edge $f \neq e$ in G such that $(T \setminus e) \cup f$ is a spanning tree of G, the label of e is less than the label of f. Otherwise, e is said to be *internally inactive*. On the other hand, an edge f of $G \setminus T$ is said to be *externally active* if f has the smallest label among the edges in the unique cycle contained in $T \cup f$. Otherwise, f is said to be *externally inactive.*

Bollobás and Riordan [2] use Tutte's notion of activities but generalize Tutte's variable assignments as follows. Let G be a coloured and connected graph and let T be a spanning tree of G. For each edge e in G with colour λ, we assign one of the variables X_λ, Y_λ, x_λ and y_λ to it according to the activities of e as shown in Table 1 (with respect to the tree T).

Definition 2.2. Let G be a connected coloured graph. For a spanning tree T of G, let $C(T)$ be the product of the variable contributions from each edge of G according to the variable assignment above; then the Tutte polynomial $T(G)$ is defined as the sum of all the $C(T)$s over all possible spanning trees of G.

Tutte's original variable assignment may be recovered by setting all $X_\lambda = x$, $Y_\lambda = y$, $x_\lambda = 1$ and $y_\lambda = 1$ for all $\lambda \in \Lambda$. It is Tutte's main result that the total contribution of all spanning trees is labelling-independent in the non-coloured case. This property does not generalize to the coloured case. To remedy the situation, in the definition of most coloured Tutte polynomials in the literature one needs to factor the polynomial ring $\mathbb{Z}[\Lambda] := \mathbb{Z}[X_\lambda, Y_\lambda, x_\lambda, y_\lambda : \lambda \in \Lambda]$ with an appropriate ideal I, such that the formula for $T(G)$ in $\mathbb{Z}[\Lambda]/I$ becomes labelling-independent. An exact description of all such ideals was given by Bollobás and Riordan [2, Theorem 2].

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

Proposition 2.3 (Bollobás and Riordan). *Assume I is an ideal of $\mathbb{Z}[\Lambda]$. Then the homomorphic image of $T(G)$ in $\mathbb{Z}[\Lambda]/I$ is independent of the labelling of the edges of G if and only if*

$$\det\begin{pmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} \in I, \tag{2.1}$$

$$Y_\nu \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} - Y_\nu \det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix} \in I \tag{2.2}$$

and

$$X_\nu \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} - X_\nu \det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix} \in I \tag{2.3}$$

hold for all $\lambda, \mu, \nu \in \Lambda$.

Bollobás and Riordan use I_0 to denote the ideal generated by the differences listed in Proposition 2.3. The homomorphic image of $T(G)$ in $\mathbb{Z}[\Lambda]/I_0$ is the most general coloured Tutte polynomial whose definition is independent of the labelling. Many important polynomials may be obtained from this most general coloured Tutte polynomial by substitution, and most such substitutions map $\mathbb{Z}[\Lambda]/I_0$ into an integral domain in such a way that the image of the variables $x_\lambda, X_\lambda, y_\lambda$ and Y_λ is non-zero. As is implicitly noted in [2, Corollary 3], all such substitutions factor through the canonical map $\mathbb{Z}[\Lambda]/I_0 \to \mathbb{Z}[\Lambda]/I_1$, where I_1 is the ideal generated by all polynomials of the form

$$\det\begin{pmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix} \tag{2.4}$$

and

$$\det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix}. \tag{2.5}$$

As I_1 properly contains I_0, the canonical image of the Tutte polynomial in $\mathbb{Z}[\Lambda]/I_1$ is labelling-independent. Moreover, we highlight the following algebraic observation, making [2, Corollary 3] truly useful.

Lemma 2.4 ([14]). *The ideal I_1 is a prime ideal. More generally, given any integral domain $\mathcal{R}$, the ideal I_1 generated by all elements of the form* (2.4) *and* (2.5) *in $\mathcal{R}[\Lambda] := \mathcal{R}[X_\lambda, Y_\lambda, x_\lambda, y_\lambda : \lambda \in \Lambda]$ is prime.*

As noted in [2, Remark 3] and in [14], the above definitions and statements may be generalized to *matroids* without essential adjustment. In fact, the definitions of activities may be restated by replacing the word 'spanning tree' with 'matroid basis' and interpreting the word 'cycle' as a 'minimal dependent set'. In particular, there is a trivial generalization of our Tutte polynomial to *disconnected graphs*, by replacing the word 'spanning tree' with *spanning forest*. Given a disconnected graph G with connected components $G_1, \ldots, G_k$, the Tutte polynomial $T(G)$ obtained via this generalization is the product of the Tutte

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

polynomials of its components. There are also other generalizations to disconnected graphs which keep track of the number of connected components. Such generalizations are discussed in [2, Section 3.4]. In order to be able to represent a larger class of graph-theoretic polynomials by substitution into the variables of their coloured Tutte polynomial, Bollobás and Riordan [2] introduced a multiplicative constant $\alpha_{k(G)}$ (depending only on the number $k(G)$ of connected components in G) to $T(G)$. Consequently they generalized their result [2, Theorem 2], providing a necessary and sufficient condition to have a labelling-independent Tutte polynomial $\alpha_{k(G)}T(G)$. We wish to stress that these disconnected generalizations depend on the number of connected components which *cannot be recovered from the cycle matroid of the graph.*

Finally, since the Tutte polynomials considered above are labelling-independent, we have the following recursive formula:

$$T(G) = \begin{cases} y_\lambda T(G \setminus e) + x_\lambda T(G/e) & \text{if } e \text{ is neither a loop nor a bridge,} \\ Y_\lambda T(G \setminus e) & \text{if } e \text{ is a loop,} \\ X_\lambda T(G/e) & \text{if } e \text{ is a bridge,} \end{cases} \tag{2.6}$$

where λ is the colour of e, $G \setminus e$ is the graph obtained from G by deleting e and G/e is the graph obtained from G by contracting e. (See [2, (3.14)] and the Preliminaries in [14].)

3. Relative Tutte polynomials

In this section we introduce the relative Tutte polynomial of a connected coloured graph G, with respect to a set of edges $\mathcal{H} \subset E(G)$ (where $E(G)$ is the edge set of G) and prove our generalization of Theorem 2.3 of Bollobás and Riordan [2] to our relative Tutte polynomials. The definitions and the main result may be easily generalized to coloured matroids, by keeping the relative Tutte polynomial $\psi(G)$ of a graph satisfying $\mathcal{H} = E(G)$ a matroid invariant. However, as we will see below, a larger class of mappings ψ fits our theory. In Section 4 these other generalizations will allow us to consider the Tutte polynomials of disconnected graphs introduced by Bollobás and Riordan [2, Section 3.4] as a special instance of our relative Tutte polynomials, even without extending our definitions to disconnected graphs. In our presentation we will focus on graphs and indicate along the way in remarks how the immediate generalization to matroids may be made, when applicable. However, each time we omit half of the proof of a lemma, it is implied that the statement should be about matroids, and what is left to prove is the dual of what we have already shown, using only the matroid structure.

Definition 3.1. Let G be a connected graph and let $\mathcal{H}$ be a subset of its edge set $E(G)$. A subset $\mathcal{C}$ of the edge set $E(G) \setminus \mathcal{H}$ is called a *contracting set* of G with respect to $\mathcal{H}$ if $\mathcal{C}$ contains no cycles and $E(G) \setminus (\mathcal{C} \cup \mathcal{H})$ contains no cocycles. For a contracting set $\mathcal{C}$ of G with respect to $\mathcal{H}$, the corresponding edge set $E(G) \setminus (\mathcal{C} \cup \mathcal{H})$ is called a *deleting set* and is denoted by $\mathcal{D}$.

Note that $\mathcal{C}$ and $\mathcal{D}$ mutually determine each other, by their disjoint union being $E(G) \setminus \mathcal{H}$.

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

Remark 3.2. Definition 3.1 may be used without any change to define a contracting set $\mathcal{C}$ and a deleting set $\mathcal{D}$ for any matroid with respect to a set of elements $\mathcal{H}$. For matroids we obtain a 'self-dual' dual notion: $(\mathcal{C},\mathcal{D})$ is a pair of contracting and deleting sets for a matroid $\mathcal{M}$ if and only if $(\mathcal{D},\mathcal{C})$ is a pair of contracting and deleting sets for the dual matroid $\mathcal{M}^*$.

Remark 3.3. Recalling that the cocycles in a connected graph G are exactly the minimal cuts, $\mathcal{D}$ contains no cocycles if and only if the deletion of $\mathcal{D}$ does not disconnect the graph G.

Sometimes we will refer to $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{H}$ as *graphs*, and in each such instance we consider them as the subgraphs of G induced by the respective set of edges. Since the spanning trees of G are the cycle-free connected subgraphs of G, in the light of Remark 3.3 the following lemma is obvious.

Lemma 3.4. *In the above definition, if $\mathcal{H} = \emptyset$, then $\mathcal{C} \subseteq E(G)$ is a contracting set if and only if the subgraph $\mathcal{C}$ is a spanning tree of $E(G)$.*

To prove the generalizations of our statements for matroids, the following observation will be useful.

Lemma 3.5. *$\mathcal{C}$ is a contracting set with respect to $\mathcal{H}$ if and only if there is a basis $B \subset \mathcal{C} \cup \mathcal{H}$ that contains $\mathcal{C}$.*

Proof. Assume first that $\mathcal{C}$ is a contracting set. Then $\mathcal{D} := E(G) \setminus (\mathcal{C} \cup \mathcal{H})$ contains no cocycle, so it is co-independent, *i.e.*, contained in a dual basis. The complement of this dual basis is a basis contained in $\mathcal{C} \cup \mathcal{H}$. Thus $\mathcal{C} \cup \mathcal{H}$ has full rank. Since $\mathcal{C}$ contains no cycle, it is independent. Extending the independent set $\mathcal{C}$ to a maximal independent subset of $\mathcal{C} \cup \mathcal{H}$ yields a basis containing $\mathcal{C}$ and contained in $\mathcal{C} \cup \mathcal{H}$.

Assume now that there is a basis $B \subset \mathcal{C} \cup \mathcal{H}$ that contains $\mathcal{C}$. Then $\mathcal{C} \subset B$ is independent and contains no cycle. On the other hand, $E(G) \setminus B$ is a dual basis containing $\mathcal{D}$. Hence $\mathcal{D}$ is co-independent and contains no cocycle. □

Note that even if $E(G) \setminus \mathcal{H}$ contains edges that are not loops, it is possible that $\mathcal{C} = \emptyset$ (as the following example shows). This is not the case when $\mathcal{H} = \emptyset$. On the other hand, it is important to note that for any edge $e \in \mathcal{D}$, the graph $\{e\} \cup \mathcal{C} \cup \mathcal{H}$ must contain a cycle with e in it (although such a cycle may not be unique), for otherwise e would be a cut edge of $\{e\} \cup \mathcal{C} \cup \mathcal{H}$, contradicting the fact that deleting $\mathcal{D}$ cannot disconnect the graph.

Example 3.6. It is possible that $\mathcal{C} = \emptyset$ is a contracting set, as the following example shows. In Figure 1, $\mathcal{H} = \{e_2\}$. So by definition, $\{e_1\}$ can serve either as $\mathcal{C}$ or $\mathcal{D}$.

Lemma 3.7. *Let G be a connected graph and let $\mathcal{H}$ be a subset of $E(G)$. Let $\mathcal{C}$ be a contracting set of G with respect to $\mathcal{H}$, let $\mathcal{D}$ be the corresponding deleting set and let $e \in \mathcal{C}$*

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

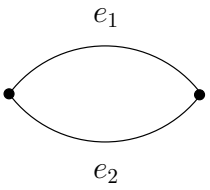


Figure 1. A graph G of two edges with $\mathcal{H}$ containing one edge.

be any edge in $\mathcal{C}$. *Then, for any* $f \in \mathcal{D}$, $\mathcal{C}' = \{f\} \cup (\mathcal{C} \setminus \{e\})$ *is also a contracting set with respect to* $\mathcal{H}$ *if the triplet* $(\mathcal{C}, e, f)$ *has either of the following properties:*

(i) $\mathcal{C} \cup \{f\}$ *contains a cycle containing* $\{e\}$,

(ii) $\mathcal{D} \cup \{e\}$ *contains a cocycle containing* $\{f\}$.

Moreover, if the triplet $(\mathcal{C}, e, f)$ *satisfies* (i) *or* (ii) *then the triplet* $(\mathcal{C}', f, e)$ *has the same property.*

Proof. Assume first that (i) holds. Since $\mathcal{C}$ contains no cycle, adding f creates at most one cycle C_0. By our assumption, $e \in C_0$. By Lemma 3.5 there is a basis B containing $\mathcal{C}$, contained in $\mathcal{C} \cup \mathcal{H}$. Adding f to B creates only one cycle, so this cycle is still C_0. Thus $B' := B \cup \{f\} \setminus \{e\}$ contains no cycle, and it is a basis. Clearly B' is a basis containing $\mathcal{C}'$ and contained in $\mathcal{C}' \cup \mathcal{H}$. Thus $\mathcal{C}'$ is a contracting set by Lemma 3.5. Furthermore $\mathcal{C}' \cup \{e\}$ contains the cycle C_0 which contains f.

The statements about property (ii) follow by 'dualizing' the above argument. □

Remark 3.8. The statement and proof of Lemma 3.7 may be generalized to matroids immediately. For graphs, property (ii) is equivalent to the following:

(ii′) e is a bridge in $\mathcal{C} \cup \mathcal{H}$ and $(\mathcal{C} \cup \mathcal{H} \cup \{f\}) \setminus \{e\}$ is connected.

It should be noted that properties (i) and (ii) are not necessarily mutually exclusive for a triplet $(\mathcal{C}, e, f)$. As seen in the proof of Lemma 3.7, if a triplet $(\mathcal{C}, e, f)$ has property (i) then the cycle C_0 contained in $\mathcal{C} \cup \{f\}$ is unique, and equal to the unique cycle contained in $\mathcal{C}' \cup \{e\}$. If a triplet $(\mathcal{C}, e, f)$ has property (ii) but does not have property (i), then every cycle contained in $\mathcal{C} \cup \mathcal{H} \cup \{f\}$ and containing $\{e, f\}$ contains at least one element of $\mathcal{H}$, and there is at least one such cycle.

In the next definition, we define the relative activities of edges of $E(G) \setminus \mathcal{H}$ with respect to a contracting set $\mathcal{C}$ of G based on a particular labelling of the edges of G.

Definition 3.9. Let G be a connected graph and let $\mathcal{H}$ be a subset of $E(G)$. Let us assume that a labelling of G is given in such a way that all edges in $\mathcal{H}$ are labelled with number 0 and all other edges are labelled with distinct positive integers. Such a labelling is called a *proper labelling* or a *relative labelling* (with respect to $\mathcal{H}$). In other words, a proper labelling of the edges of G with respect to $\mathcal{H}$ is a map $\phi : E(G) \longrightarrow \mathbb{Z}$ such that $\phi(e) = 0$ for any $e \in \mathcal{H}$ and ϕ is an injective map from $E(G) \setminus \mathcal{H}$ to $\mathbb{Z}^+$. We say that e_1 is larger than e_2 if $\phi(e_1) > \phi(e_2)$. Let $\mathcal{C}$ be a contracting set of G with respect to $\mathcal{H}$. Then:

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

(a) an edge $e \in \mathcal{C}$ is called *internally active* if $\mathcal{D} \cup \{e\}$ contains a cocycle D_0 in which e is the smallest edge, and otherwise it is *internally inactive*,
(b) an edge $f \in \mathcal{D}$ is called *externally active* if $\mathcal{C} \cup \{f\}$ contains a cycle C_0 in which f is the smallest edge, and otherwise it is *externally inactive*.

Remark 3.10. For any $f \in \mathcal{D}$, if $\mathcal{C} \cup \{f\}$ contains a cycle then this cycle is unique. For a fixed $\mathcal{C}$ we may identify the internally or externally active edges by comparing the label of each $e \in \mathcal{C}$ with the label of each $f \in \mathcal{D}$ such that the triplet $(\mathcal{C}, e, f)$ has at least one of the properties considered in Lemma 3.7.

Remark 3.11. An internal edge $e \in \mathcal{C}$ is active only if it closes a cocycle in $\mathcal{D}$. For a connected graph, $\mathcal{D} \cup \{e\}$ contains a cocycle if and only if removing $\mathcal{D} \cup \{e\}$ disconnects the graph. This is equivalent to stating that e is not contained in any cycle of $\mathcal{C} \cup \mathcal{H}$, or that e is a bridge in $\mathcal{C} \cup \mathcal{H}$. This rephrasing remains true for matroids in general if we define 'bridge' as 'coloop'. (The proof is left to the reader.) A bridge e in $\mathcal{C} \cup \mathcal{H}$ is active exactly when it is smaller than any $f \in \mathcal{D}$ connecting the two components of $\mathcal{C} \cup \mathcal{H} \setminus \{e\}$. Hence we may restate the condition for internal activity as follows: an edge $e \in \mathcal{C}$ is internally active if, whenever an edge $f \notin \mathcal{C}$ closes a cycle in $\{f\} \cup \mathcal{C} \cup \mathcal{H}$ containing e, e is always smaller than f; otherwise e is said to be *internally inactive*. This last rephrasing holds again for matroids in general. In particular, if e is on a cycle containing only edges from $\mathcal{C} \cup \mathcal{H}$, then e is internally inactive.

Remark 3.12. The following statements offer an equivalent definition of the activity of a regular edge of G. An edge $e \in \mathcal{C}$ is internally active if and only if it becomes a bridge once all edges in $\mathcal{D}$ larger than e are deleted. On the other hand, an edge $f \in \mathcal{D}$ is externally active if and only if it becomes a loop after all edges in $\mathcal{C}$ larger than f are contracted.

Before we give a formal definition of our relative Tutte polynomial of a coloured graph G with respect to a subset $\mathcal{H}$ of edges of G, let us explain the role we intend the set $\mathcal{H}$ to play. In the classical case where $\mathcal{H} = \emptyset$, when we apply the recursive formula (2.6) to compute $T(G)$, every edge of G is either deleted or contracted in the process. Finally, only one vertex is left. Naturally, the Tutte polynomial of a vertex is defined to be 1. In our case, we would like to be able to compute our relative Tutte polynomial by the same recursive rule (2.6); however, we want to preserve the special zero edges (edges in $\mathcal{H}$) in this process, as these can be special edges that may not allow the use of a contraction/deletion formula similar to (2.6) (which is the case in our application to virtual knot theory). By doing so, at the end of the process, we will end up with graphs $\mathcal{H}_{\mathcal{C}}$ obtained from G by contracting the edges in $\mathcal{C}$ and deleting the edges in $\mathcal{D}$. We have the option of defining a weight $\psi(\mathcal{H}_{\mathcal{C}})$ for these graphs in a different manner. In the proof of our main result we will only need to be able to guarantee that, for any triplet $(\mathcal{C}, e, f)$ satisfying property (i) or (ii) in Lemma 3.7, we associate the same value to the graphs $\mathcal{H}_{\mathcal{C}}$ and $\mathcal{H}_{\mathcal{C}'}$. To guarantee this, we require the mapping ψ to have the following property.

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

Definition 3.13. Let ψ be a mapping defined on the isomorphism classes of finite connected graphs with values in a ring $\mathcal{R}$. We say that ψ is a *block invariant* if, for any positive integer n, there is a function $f_n : \mathcal{R}^n \to \mathcal{R}$ that is symmetric under permuting its input variables such that, for any connected graph G having n blocks $G_1, \ldots, G_n$, we have

$$\psi(G) = f_n(\psi(G_1), \ldots, \psi(G_n)).$$

In other words, we require the ability to compute $\psi(G)$ from the value of ψ on the blocks of G, and this computation should not depend on the order in which the blocks are listed. The function $\psi(G) = z^{r(G)}$ is such an example (where $r(G)$ is the rank of the graph G).

Lemma 3.14. *Let G be a connected graph and let $\mathcal{H}$ be a subset of $E(G)$. Assume that $\mathcal{C}$ is a contracting set with respect to $\mathcal{H}$ and that the triplet $(\mathcal{C}, e, f)$ has at least one of the properties listed in Lemma 3.7. Let $\mathcal{C}' := (\mathcal{C} \cup \{f\}) \setminus \{e\}$. Then the multiset of blocks of $\mathcal{H}_{\mathcal{C}}$ is the same as the multiset of blocks of $\mathcal{H}_{\mathcal{C}'}$.*

Proof. The statement is obvious when $(\mathcal{C}, e, f)$ has property (i), since then $\mathcal{H}_{\mathcal{C}}$ and $\mathcal{H}_{\mathcal{C}'}$ are isomorphic as graphs. If $(\mathcal{C}, e, f)$ has property (ii), then $\mathcal{H}_{\mathcal{C}'}$ is obtained from $\mathcal{H}_{\mathcal{C}}$ by expanding the edge e (which becomes a bridge), removing e, adding a new bridge f, and contracting f. Clearly $\mathcal{H}_{\mathcal{C}}$ and $\mathcal{H}_{\mathcal{C}'}$ have the same blocks. □

Recall that the cycle matroid of a graph depends only on the 2-edge connected components that are subsets of its blocks. Thus a matroid invariant of connected graphs is also a block invariant. When we generalize our notion of the relative Tutte polynomial to matroids we want to require ψ to be a matroid invariant with values in a fixed integral domain $\mathcal{R}$. Then we need the following variant of Lemma 3.14.

Lemma 3.15. *Let M be a matroid and let $\mathcal{H}$ be a subset of its elements. Assume that $\mathcal{C}$ is a contracting set with respect to $\mathcal{H}$ and that the triplet $(\mathcal{C}, e, f)$ has at least one of the properties listed in Lemma 3.7. Let $\mathcal{C}' := (\mathcal{C} \cup \{f\}) \setminus \{e\}$. Then the cycle matroid of the graph $\mathcal{H}_{\mathcal{C}}$ is the same as the cycle matroid of the graph $\mathcal{H}_{\mathcal{C}'}$.*

Proof. Assume that $(\mathcal{C}, e, f)$ has property (i). A set $X \subseteq \mathcal{H}$ is independent in $\mathcal{H}_{\mathcal{C}}$ if and only if there is a basis B of the original graph containing $X \cup \mathcal{C}$. As seen in the proof of Lemma 3.7, the set $B' := (B \cup \{f\}) \setminus \{e\}$ is also a basis of the original graph, and this basis contains $X \cup \mathcal{C}'$. Thus, if $X \subseteq \mathcal{H}$ is independent in $\mathcal{H}_{\mathcal{C}}$, it is also independent in $\mathcal{H}_{\mathcal{C}'}$. The converse is also true since, by Lemma 3.7, the triplet $(\mathcal{C}', f, e)$ also has property (i).

The proof for the case when $(\mathcal{C}, e, f)$ has property (ii) may be obtained by 'dualizing' the above argument. □

Let G be a connected graph and $\mathcal{H} \subseteq E(G)$. Assume we are given a mapping c from $E(G) \setminus \mathcal{H}$ to a colour set Λ. Assume further that ψ is a block invariant associating an

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

element of a fixed integral domain $\mathcal{R}$ to each connected graph. For any contracting set $\mathcal{C}$ of G with respect to $\mathcal{H}$, let $\mathcal{H}_\mathcal{C}$ be the graph obtained by deleting all edges in $\mathcal{D}$ and contracting all edges in $\mathcal{C}$ (so that the only edges left in $\mathcal{H}_\mathcal{C}$ are the zero edges). Finally, we will assign a proper labelling ϕ to the edges of G. We now define the relative Tutte polynomial of G with respect to $\mathcal{H}$ and ψ as

$$T^\psi_\mathcal{H}(G) = \sum_\mathcal{C} \left(\prod_{e \in G \setminus \mathcal{H}} w(G, c, \phi, \mathcal{C}, e) \right) \psi(\mathcal{H}_\mathcal{C}) \in \mathcal{R}[\Lambda], \tag{3.1}$$

where the summation is taken over all contracting sets $\mathcal{C}$ and $w(G, c, \phi, \mathcal{C}, e)$ is the *weight* of the edge e with respect to the contracting set $\mathcal{C}$, which is defined as (assume that e has colour λ):

$$w(G, c, \phi, \mathcal{C}, e) = \begin{cases} X_\lambda & \text{if } e \text{ is internally active,} \\ Y_\lambda & \text{if } e \text{ is externally active,} \\ x_\lambda & \text{if } e \text{ is internally inactive,} \\ y_\lambda & \text{if } e \text{ is externally inactive.} \end{cases} \tag{3.2}$$

To simplify the notation somewhat, we will be using $T_\mathcal{H}(G)$ for $T^\psi_\mathcal{H}(G)$, with the understanding that some ψ has been chosen, unless there is a need to stress what ψ really is. Following [2], we then write

$$W(G, c, \phi, \mathcal{C}) = \prod_{e \in G \setminus \mathcal{H}} w(G, c, \phi, \mathcal{C}, e)$$

so that

$$T_\mathcal{H}(G, \phi) = \sum_\mathcal{C} W(G, c, \phi, \mathcal{C}) \psi(\mathcal{H}_\mathcal{C}). \tag{3.3}$$

We are now able to extend Theorem 2.3 of Bollobás and Riordan [2] to $T_\mathcal{H}$.

Theorem 3.16. *Assume I is an ideal of $\mathcal{R}[\Lambda]$. Then the homomorphic image of $T_\mathcal{H}(G, \phi)$ in $\mathcal{R}[\Lambda]/I$ is independent of ϕ (for any G and ψ) if and only if*

$$\det \begin{pmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{pmatrix} - \det \begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} \in I \tag{3.4}$$

and

$$\det \begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} - \det \begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix} \in I \tag{3.5}$$

hold for all $\lambda, \mu \in \Lambda$.

Proof. We will follow the approach and notation of [2] as closely as possible, so the reader familiar with that paper can follow easily. In this approach, for the sufficient condition, it suffices to show that the relative Tutte polynomial defined in equation (3.1) is the same for any two proper labellings of G that differ only by a transposition under conditions (3.4) and (3.5). That is, if two proper labellings ϕ and ϕ' differ only on two

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

regular edges e and f such that $\phi(f) - \phi(e) = 1$ and $\phi'(e) = \phi(f)$, $\phi'(f) = \phi(e)$, then the relative Tutte polynomial defined in equation (3.1) is the same for ϕ and ϕ' under conditions (3.4) and (3.5). Without loss of generality let us assume that $\phi(e) = \phi'(f) = i$ and $\phi(f) = \phi'(e) = i + 1$.

We now proceed to prove this fact. Let $\mathcal{C}$ be any contracting set of G with respect to $\mathcal{H}$ and $\mathcal{D} = E(G) \setminus (\mathcal{C} \cup \mathcal{H})$. Clearly, by the definition of activities, if both edges e and f are in $\mathcal{C}$ or in $\mathcal{D}$, then they will never be compared to each other with respect to their activities, therefore their label comparison to any other regular edge stays the same under the two labellings. This is even more obvious if one uses the equivalent definition of activities in Remark 3.12. Thus, in that case, all regular edges will make the same contributions under the two labellings. Furthermore, as noted at the end of Remark 3.10, the labels of e and f are still not compared to each other unless the triplet $(\mathcal{C}, e, f)$ or the triplet $(\mathcal{C}, f, e)$ has at least one of the properties listed in Lemma 3.7. Without loss of generality we may assume that $e \in \mathcal{C}$, $f \in \mathcal{D}$, and the triplet satisfies at least one of the properties listed in Lemma 3.7. By Lemma 3.7, the set $\mathcal{C}' := (\mathcal{C} \setminus \{e\}) \cup \{f\}$ is also a contracting set, and the triplet $(\mathcal{C}', f, e)$ has the same properties as $(\mathcal{C}, e, f)$. As noted in the proof of Theorem 2.3 in [2] in a similar situation, it suffices to show that for such pairs $(\mathcal{C}, \mathcal{C}')$ the total contribution of the two contracting sets is the same with respect to ϕ and ϕ':

$$W(G, c, \phi, \mathcal{C}) + W(G, c, \phi, \mathcal{C}') - W(G, c, \phi', \mathcal{C}) - W(G, c, \phi', \mathcal{C}') \in I. \tag{3.6}$$

There are three cases, depending on whether the triplet $(\mathcal{C}, e, f)$ has only one or both properties listed in Lemma 3.7. Let $\lambda = c(e)$, $\mu = c(f)$.

Case 1: $(\mathcal{C}, e, f)$ has property (ii) but not (i). In this case $\mathcal{C} \cup \{f\}$ contains no cycle. Indeed, since $\mathcal{C}$ is cycle free, f is a bridge in $(\mathcal{C} \cup \mathcal{H} \cup \{f\}) \setminus \{e\}$, so a cycle contained in $\mathcal{C} \cup \mathcal{H} \cup \{f\}$ contains f, and a cycle containing f in $\mathcal{C} \cup \mathcal{H} \cup \{f\}$ also contains e. We obtained that a cycle contained in $\mathcal{C} \cup \{f\}$ must contain e, but we assumed property (i) to be false. Therefore, $\mathcal{C} \cup \{f\}$ contains no cycle. It follows that f is externally inactive with respect to $\mathcal{C}$, under both labellings. Similarly, $\mathcal{C}' \cup \{e\}$ does not contain any cycle, and so e is externally inactive with respect to $\mathcal{C}'$, under both labellings. Since we assume $\phi(e) < \phi(f)$, f is internally inactive with respect to $\mathcal{C}'$, under ϕ. Similarly $\phi'(f) < \phi'(e)$ implies that e is internally inactive with respect to $\mathcal{C}$, under ϕ'. So far we have obtained that e and f contribute the following factors to the polynomials $W(G, c, \phi, \mathcal{C})$, $W(G, c, \phi, \mathcal{C}')$, $W(G, c, \phi', \mathcal{C})$, $W(G, c, \phi', \mathcal{C}')$:

	$\mathcal{C}$	$\mathcal{C}'$
ϕ	$(?)y_\mu$	$x_\mu y_\lambda$
ϕ'	$x_\lambda y_\mu$	$(?)y_\lambda$

The question marks indicate that we are left to determine whether e is internally active with respect to $\mathcal{C}$ under ϕ and whether f is internally active with respect to $\mathcal{C}'$ under ϕ'. The information at hand does not allow us to determine whether or not these edges are active in the given context. However, we can show that the answer is either simultaneously 'yes' or simultaneously 'no' to both questions. In fact, e is internally active with respect to $\mathcal{C}$ under ϕ if and only if all edges of $\mathcal{D}$ that close a cycle containing e with

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

$\mathcal{C} \cup \mathcal{H} \setminus \{e\} = \mathcal{C}' \cup \mathcal{H} \setminus \{f\}$ have a larger label than the label of e. (This question has the same answer under ϕ and ϕ'.) Similarly f is internally active with respect to $\mathcal{C}'$ under ϕ' if and only if all edges of $\mathcal{D}$ that close a cycle containing f with $\mathcal{C}' \cup \mathcal{H} \setminus \{f\} = \mathcal{C} \cup \mathcal{H} \setminus \{e\}$ have a larger label than the label of e. We are comparing the labels of the same set of deleting edges with the adjacent labels of e and f.

If the answer to both remaining questions is 'no', then the left-hand side of (3.6) becomes zero; otherwise (3.6) becomes

$$W_0(X_\lambda y_\mu + x_\mu y_\lambda - x_\lambda y_\mu - X_\mu y_\lambda) \in I. \tag{3.7}$$

Here W_0 is the product of $\psi(\mathcal{H}_\mathcal{C}) = \psi(\mathcal{H}_{\mathcal{C}'})$ and of the variables associated to the regular edges that are different from e and f. In this case (3.6) follows from conditions (3.4) and (3.5).

Case 2: $(\mathcal{C}, e, f)$ has property (i) but not (ii). This case is the 'dual' of the previous one and may be handled in a similar manner. The edge e cannot be a bridge in $\mathcal{C} \cup \mathcal{H}$, otherwise we may show that $\mathcal{C} \cup \mathcal{H} \cup \{f\}$ contains a cycle containing e and $(\mathcal{C}, e, f)$ has property (ii). Therefore there is at least one cycle in $\mathcal{C} \cup \mathcal{H}$ containing e, and e is internally inactive with respect to $\mathcal{C}$ under both ϕ and ϕ'. Similarly f is internally inactive with respect to $\mathcal{C}'$ under both ϕ and ϕ'. Using $\phi(e) < \phi(f)$ and $\phi'(f) < \phi'(e)$, we obtain that e and f contribute the following factors to the polynomials $W(G, c, \phi, \mathcal{C})$, $W(G, c, \phi, \mathcal{C}')$, $W(G, c, \phi', \mathcal{C})$, $W(G, c, \phi', \mathcal{C}')$:

	$\mathcal{C}$	$\mathcal{C}'$
ϕ	$x_\lambda y_\mu$	$x_\mu(??)$
ϕ'	$x_\lambda(??)$	$x_\mu y_\lambda$

The double question marks indicate that we are left to determine whether e is externally active with respect to $\mathcal{C}'$ under ϕ and whether f is externally active with respect to $\mathcal{C}$ under ϕ'. Again the answer to both questions the answer is simultaneously 'yes' or 'no': to decide we must compare the labels of the remaining edges to the label of e resp. f, on the the unique cycle contained in $\mathcal{C}' \cup \{e\}$ which is the same as the unique cycle contained in $\mathcal{C} \cup \{f\}$. If the answer to both remaining questions is 'no', then the left-hand side of (3.6) becomes zero, otherwise (3.6) becomes

$$W_0(x_\lambda y_\mu + x_\mu Y_\lambda - x_\lambda Y_\mu - x_\mu y_\lambda) \in I, \tag{3.8}$$

which follows from condition (3.4).

Case 3: $(\mathcal{C}, e, f)$ has both properties, (i) and (ii). Now $\phi(e) < \phi(f)$ implies that f is externally inactive with respect to $\mathcal{C}$ and internally inactive with respect to $\mathcal{C}'$ under ϕ. Similarly, $\phi'(f) < \phi'(e)$ implies that e is internally inactive with respect to $\mathcal{C}$ and externally inactive with respect to $\mathcal{C}'$ under ϕ'. We obtain that e and f contribute the following factors to the polynomials $W(G, c, \phi, \mathcal{C})$, $W(G, c, \phi, \mathcal{C}')$, $W(G, c, \phi', \mathcal{C})$, $W(G, c, \phi', \mathcal{C}')$:

	$\mathcal{C}$	$\mathcal{C}'$
ϕ	$(?)y_\mu$	$x_\mu(??)$
ϕ'	$x_\lambda(??)$	$(?)y_\lambda$

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

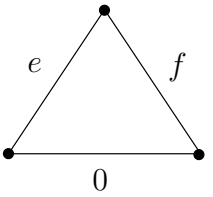


Figure 2. A triangle graph G with two regular edges and one zero edge.

As in the previous two cases, the question marks indicate missing activity determination. The argument used in Case 1 is applicable here to show that e is internally active with respect to $\mathcal{C}$ under ϕ if and only if f is internally active with respect to $\mathcal{C}'$ under ϕ'. Hence either both single question marks need to be replaced by upper-case letters, or both need to be replaced by lower-case letters. Similarly, we may re-use the argument from Case 2 to show that e is externally active with respect to $\mathcal{C}'$ under ϕ if and only if f is externally active with respect to $\mathcal{C}$ under ϕ'. Hence either both double question marks need to be replaced by upper-case letters, or both need to be replaced by lower-case letters. Depending on the choice of letter case for the missing variables we have $2 \times 2 = 4$ possibilities: either the left-hand side of (3.6) becomes zero, or (3.6) becomes (3.7), or (3.6) becomes (3.8), or (3.6) becomes

$$W_0(X_\lambda y_\mu + x_\mu Y_\lambda - x_\lambda Y_\mu - X_\mu y_\lambda) \in I. \tag{3.9}$$

We only need to observe that (3.9) is the same as (3.4).

We now turn to the necessity of the conditions of the theorem. The necessity of (3.4) follows from the double-edge graph example given in [2] (as shown in Figure 1 earlier but with both edges as regular edges). For the necessity of (3.5), let us look at the graph given in Figure 2.

In the figure, let $c(e) = \lambda$, $c(f) = \mu$, $\phi(e) = 1 = \phi'(f)$, and $\phi(f) = 2 = \phi'(e)$. There are only three choices for the contracting set $\mathcal{C}$: $\mathcal{C}_1 = \{e\}$, $\mathcal{C}_2 = \{f\}$, or $\mathcal{C}_3 = \{e, f\}$. With respect to $\mathcal{C}_1$, e is internally active under ϕ because deleting f (since it has a larger label) will make e a bridge, and f is externally inactive because it does not close a cycle with $\mathcal{C}_1$. Thus their combined contribution under ϕ with respect to $\mathcal{C}_1$ is $X_\lambda y_\mu$. With respect to $\mathcal{C}_2$, e is externally inactive under ϕ because it does not become a loop after f is contracted first, and f is internally inactive because it is not a bridge, as no edge can be deleted first. Thus their combined contribution under ϕ with respect to $\mathcal{C}_2$ is $y_\lambda x_\mu$. Finally, with respect to $\mathcal{C}_3$, e and f are both internally inactive as they are both non-bridge since there is no any edge to be deleted first. Thus their combined contribution under ϕ with respect to $\mathcal{C}_3$ is $x_\lambda x_\mu$. By definition, we have

$$T_{\mathcal{H}}(G, \phi) = \sum_{1 \leqslant j \leqslant 3} W(G, c, \phi, \mathcal{C}_j)\psi(\mathcal{H}_{\mathcal{C}_j}) = x_\lambda x_\mu \psi(H_0) + (X_\lambda y_\mu + x_\mu y_\lambda)\psi(H_1), \tag{3.10}$$

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

where H_0 is the graph that contains only a zero loop edge and H_1 is the graph that contains only a simple zero edge. Similarly, we have

$$T_{\mathcal{H}}(G,\phi') = \sum_{1\leqslant j\leqslant 3} W(G,c,\phi',\mathcal{C}_j)\psi(\mathcal{H}_{\mathcal{C}_j}) = x_\lambda x_\mu \psi(H_0) + (x_\lambda y_\mu + X_\mu y_\lambda)\psi(H_1). \tag{3.11}$$

So, in order to have $T_{\mathcal{H}}(G,\phi) = T_{\mathcal{H}}(G,\phi')$ (with $\psi = 1$), we must have $X_\lambda y_\mu + x_\mu y_\lambda - x_\lambda y_\mu - X_\mu y_\lambda \in I$. But this, together with (3.4), implies (3.5). This finishes our proof of the theorem. □

From now on, we will assume that $T_{\mathcal{H}}(G,\phi)$ is defined in $\mathcal{R}[\Lambda]/I_1$ so that it is independent of the proper labellings of G, and we will simply write $T_H(G)$ for $T_H(G,\phi)$. An immediate consequence of this fact is the following corollary.

Corollary 3.17. *$T_{\mathcal{H}}(G)$ can be computed via the following recursive formula, valid for any regular edge e, i.e., any $e \notin \mathcal{H}$:*

$$T_{\mathcal{H}}(G) = \begin{cases} y_\lambda T_{\mathcal{H}}(G\setminus e) + x_\lambda T_{\mathcal{H}}(G/e) & \text{if } e \text{ is neither a bridge nor a loop,} \\ X_\lambda T_{\mathcal{H}}(G/e) & \text{if } e \text{ is a bridge,} \\ Y_\lambda T_{\mathcal{H}}(G\setminus e) & \text{if } e \text{ is a loop.} \end{cases} \tag{3.12}$$

In the above, $e \notin \mathcal{H}$ is a regular edge, $\lambda = c(e)$, $G\setminus e$ is the graph obtained from G by deleting e and G/e is the graph obtained from G by contracting e.

Proof. We may assign to e the largest label. So, in the case that e is not a bridge nor a loop, it is always inactive. Thus we have

$$\begin{aligned} T_{\mathcal{H}}(G) &= \sum_{\mathcal{C}} W(G,c,\phi,\mathcal{C})\psi(\mathcal{H}_{\mathcal{C}}) \\ &= \sum_{e\in\mathcal{C}} W(G,c,\phi,\mathcal{C})\psi(\mathcal{H}_{\mathcal{C}}) + \sum_{e\notin\mathcal{C}} W(G,c,\phi,\mathcal{C})\psi(\mathcal{H}_{\mathcal{C}}) \\ &= x_\lambda \sum_{\mathcal{C}'} W(G/e,c,\phi,\mathcal{C}')\psi(H_{\mathcal{C}/e}) + y_\lambda \sum_{\mathcal{C}} W(G\setminus e,c,\phi,\mathcal{C}\setminus e)\psi(\mathcal{H}_{\mathcal{C}}) \\ &= x_\lambda T_{\mathcal{H}}(G/e) + y_\lambda T_{\mathcal{H}}(G\setminus e). \end{aligned}$$

If e is a bridge, then it has to be in $\mathcal{C}$ (the cocycle $\{e\}$ cannot by contained in $\mathcal{D}$ by Definition 3.1) and it must be internally active by Remark 3.12. So its contribution for any chosen $\mathcal{C}$ is always X_λ. Finally, if e is a loop, then it has to be in $\mathcal{D}$ (the cycle $\{e\}$ cannot be contained in $\mathcal{C}$ by Definition 3.1) and is always externally active by Remark 3.12, so its contribution for any $\mathcal{C}$ will be Y_λ. □

Remark 3.18. A careful reader may have realized that the ideal I_1 in Theorem 3.16 has to be replaced by the ideal I_0 generated by the differences listed in Proposition 2.3 if we restrict ourselves to the class of edge sets $\mathcal{H}$ containing only bridges and loops. This is because in this case the activities of the regular edges will not be affected by the zero edges at all (and the graphs $\mathcal{H}_{\mathcal{C}}$ are all isomorphic).

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

We conclude this section with an interesting observation that was communicated to us by Sergei Chmutov [8]. In the ring $\mathcal{R}[\Lambda]/I_1$, the relations (3.4) and (3.5) become equations and may be used to eliminate all variables X_λ and Y_λ. Indeed, we should first note that all relations (3.4) and (3.5) generate the same ideal as all relations (3.5) and all relations of the form

$$\det\begin{pmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix} \in I. \tag{3.13}$$

Since I_1 is a prime ideal (by Lemma 2.4), the ring $\mathcal{R}[\Lambda]/I_1$ is a domain which we may localize by the semigroup generated by all variables x_λ, y_λ ($\lambda \in \Lambda$). In this localized ring, the relations (3.13) may be rewritten as

$$\frac{X_\lambda - x_\lambda}{y_\lambda} = \frac{X_\mu - x_\mu}{y_\mu} \quad \text{for all } \lambda, \mu, \tag{3.14}$$

whereas the relations (3.5) may be rewritten as

$$\frac{Y_\lambda - y_\lambda}{x_\lambda} = \frac{Y_\mu - y_\mu}{x_\mu} \quad \text{for all } \lambda, \mu. \tag{3.15}$$

Introducing X for the common value of all $(X_\lambda - x_\lambda)/y_\lambda$ and Y for the common value of all $(Y_\lambda - y_\lambda)/x_\lambda$, we obtain the following.

Theorem 3.19. *The ring $\mathcal{R}[\Lambda]/I_1$, localized by all variables x_λ, y_λ ($\lambda \in \Lambda$), is isomorphic to the polynomial ring $\mathcal{R}[x_\lambda, y_\lambda : \lambda \in \Lambda][X, Y]$, localized by all variables x_λ, y_λ ($\lambda \in \Lambda$). Under this isomorphism, each X_λ corresponds to $x_\lambda + Xy_\lambda$ and each Y_λ corresponds to $y_\lambda + Yx_\lambda$.*

Therefore the coloured relative Tutte may be considered as an element of a localized polynomial ring that does not need to be factorized by any algebraic relation.

Remark 3.20. For some other (non-relative) generalizations of the Bollobás–Riordan coloured Tutte polynomial of signed graphs, the idea of eliminating all variables X_λ and Y_λ appears in the work of Ellis-Monaghan and Traldi; see [15, Corollary 5.2] and [28, Corollary 1.3]. In the situations considered by these sources the Tutte polynomial originally considered is not an element of (a variant of) $\mathcal{R}[\Lambda]/I_1$, but of (a variant of) $\mathcal{R}[\Lambda]/I_0$. Thus, assuming that one may take the inverses of the variables x_λ and y_λ in these situations involves 'giving up a certain degree of generality'. The same applies to the relative Tutte polynomial in the situation mentioned in Remark 3.18.

4. Disconnected graphs and some applications

4.1. Disconnected graphs

Our first example below shows that the coloured Tutte polynomial of a disconnected graph introduced by Bollobás and Riordan [2, Section 3.4] is equivalent to the relative Tutte polynomial of a related *connected* graph with a suitably chosen function ψ.

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

Example 4.1. Let $\alpha_1, \alpha_2, \ldots$ be an infinite list of variables, let $\mathcal{R} = \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$ and let ψ be the mapping that associates α_{k+1} to each graph that has k edges. Consider a disconnected graph G whose edges are coloured by a colour set Λ. Select a vertex in each component and add a tree on the selected vertices and colour each edge on this added tree with a colour $0 \notin \Lambda$. Call the resulting connected graph $\widetilde{G}$. Let $\mathcal{H}$ be the set of the zero coloured edges. By Remark 3.18, the relative Tutte polynomial $T_{\mathcal{H}}(\widetilde{G})$ is exactly the disconnected Tutte polynomial introduced by Bollobás and Riordan [2, Section 3.4] (as an element of $\mathbb{Z}[\alpha_1, \alpha_2, \ldots][\Lambda]/I_0$).

Although the above example indicates a subtle trick that allows us to reduce the study of disconnected graphs to connected ones, it is more straightforward to generalize our notion of a relative Tutte polynomial to disconnected graphs as follows. Let G be any graph, and let $\mathcal{H}$ be a subset of $E(G)$. We define the deleting and contracting sets as in Definition 3.1, keeping in mind that a cocycle in a disconnected graph is a minimal set of edges whose removal increases the number of connected components. We introduce colours and a labelling on the regular edges as before. Regarding the map ψ, we amend Definition 3.13 to require that the value of ψ should remain the same if we rearrange the blocks of a graph *within the same connected component*. To make this notion precise, recall that *vertex splicing* is an operation that merges two disjoint graphs by picking a vertex from each and identifying these selected vertices, thus creating a cut point. The opposite operation is *vertex splitting*, which creates two disjoint graphs by replacing a cut point v with two copies v_1 and v_2, and makes each block containing v contain exactly one of v_1 and v_2.

Definition 4.2. Let G be a graph that has a cut point u. A *vertex pivot* is a sequence of vertex splitting and vertex splicing as follows. First we split G by creating two copies of u and two disjoint graphs G_1 and G_2. Then we take a vertex $v_1 \in V(G_1)$ from the connected component of u_1 and a vertex $v_2 \in V(G_2)$ in the connected component of u_2 and we merge G_1 and G_2 by identifying u_1 with u_2.

We require ψ to be a mapping that assigns to each graph an element in an integral domain $\mathcal{R}$ and that the value of ψ remains unchanged if we perform a vertex pivot on its input. For example, assigning to each graph the number of its vertices is such an operation. We may now adapt Lemma 3.14 by observing that after dropping the requirement of G being connected, the graph $\mathcal{H}_{\mathcal{C}'}$ is either equal to $\mathcal{H}_{\mathcal{C}}$ or may be obtained from it by a single vertex pivot. The definition of the relative Tutte polynomial is then given by (3.1), and Theorem 3.16 holds for this Tutte polynomial as before.

Example 4.3. Let $\alpha_1, \alpha_2, \ldots$ be an infinite list of variables, let $\mathcal{R} = \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$ and let ψ be the mapping that associates α_k to each graph that has k connected components (so ψ is invariant under the vertex pivot operation). Consider a disconnected graph G with coloured edges. Let $G_1, G_2, \ldots, G_m$ be the connected components of G and assume that $\mathcal{H}_j$ is a subgraph of G_j, and let $\mathcal{H} = \cup_{1 \leqslant j \leqslant m} \mathcal{H}_j$. The relative Tutte polynomial $T_{\mathcal{H}}(G)$ is then $\alpha_m \prod_{1 \leqslant j \leqslant m} T_{\mathcal{H}_j}(G_j)$, since the deletion/contraction process does not change the number

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

of components in G. If $\mathcal{H}$ contains only bridges and loops, then I_1 is replaced with I_0. In particular, if $\mathcal{H} = \emptyset$, then we obtain exactly $\alpha_{k(G)} \cdot T(G)$.

Example 4.4. Let G be a graph whose edges are all regular of the same colour λ. Add a loop of colour zero to each vertex of G to get the graph $\widetilde{G}$. Let ψ be the mapping that associates $(-1)^e \cdot (-x)^k$ to each graph having e (zero-coloured) edges and k connected components. Thus ψ maps into the polynomial ring $\mathbb{Z}[x]$. Substituting $1-x$ into X_λ and x_λ and 0 into Y_λ and y_λ yields

$$\begin{aligned} T_{\mathcal{H}}(\widetilde{G})\Big|_{\substack{X_\lambda = x_\lambda = 1-x \\ Y_\lambda = y_\lambda = 0}} &= (-1)^{v(G)} \cdot (-x)^{k(G)} T(G)(1-x,0) \\ &= (-1)^{v(G)-k(G)} \cdot x^{k(G)} T(G)(1-x,0). \end{aligned}$$

Here $T(G)(x,y)$ is the ordinary Tutte polynomial of G, and so $T_{\mathcal{H}}(\widetilde{G})$ generalizes the chromatic polynomial of G (see [1, Chapter X, Section 4, Theorem 6]).

Example 4.5. Let G be a graph with a special edge set $\mathcal{H}$. Let Λ_1 and Λ_2 be two colour sets. Assume that the edges of $E(G)\backslash\mathcal{H}$ are coloured with colours from the set Λ_1 and that the edges of $\mathcal{H}$ are coloured with colours from the set Λ_2. Define $\psi(\mathcal{H}_\mathcal{C})$ to be the ordinary (coloured) Tutte polynomial of the graph $\mathcal{H}_\mathcal{C}$ (as an element of $\mathbb{Z}[\Lambda_2]/I_2$, where I_2 is the ideal generated by polynomials of the form (2.4) and (2.5) using colours from Λ_2), then the Tutte polynomial $T(G)$ (as an element of $\mathbb{Z}[\Lambda_1, \Lambda_2]/I$, where I is the ideal generated by polynomials of the form (2.4) and (2.5) using colours from $\Lambda_1 \cup \Lambda_2$) equals the relative Tutte polynomial $T_{\mathcal{H}}(G)$.

We leave the verification of the above example to our reader.

4.2. The set-pointed Tutte polynomial of Las Vergnas

Given a matroid M on the set E, *pointed* by a subset $A \subseteq E$, Las Vergnas [26, (3.1)] defines the *Tutte polynomial of M pointed by A* as the 3-variable polynomial

$$t(M; A; x, y, z) = \sum_{X \subseteq E \setminus A} (x-1)^{r(M)-r_M(X \cup A)} (y-1)^{|X|-r_M(X)} z^{r_M(X \cup A)-r_M(X)}. \tag{4.1}$$

Here $r(M)$ is the rank of M and r_M is the rank function. As noted in [26, p. 978], the choice $A = \emptyset$ yields the the ordinary Tutte polynomial, whereas the choice $|A| = 1$ yields the Tutte polynomial of M pointed by a single edge e, introduced by Brylawski [5, 6].

Lemma 4.6. *Let G be a graph with edge set E, pointed by the subset A of E. Colour all elements of $E \setminus A$ with the same colour λ and all elements of A with the colour zero, i.e., let $\mathcal{H}$ be the subgraph whose set of edges is A. Let us also define $\psi(G) = z^{r(G)}$, where $r(G)$ is the rank of G. Then the Tutte polynomial $t(G; A; x, y, z)$ of G pointed by A may be obtained from the relative Tutte polynomial $T_{\mathcal{H}}(G)$ by substituting $x_\lambda \mapsto 1$, $y_\lambda \mapsto 1$, $X_\lambda \mapsto x$ and $Y_\lambda \mapsto y$.*

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

Indeed, both $T_{\mathcal{H}}(G)$ and $t(G;A;x,y,z)$ satisfy analogous deletion/contraction rules for edges $e \in E \setminus A$, and the statement is trivially true when $A = E$. Lemma 4.6 may be generalized from graphs to all (set-pointed) matroids without any substantial change.

As noted in [26, p. 987], an 'almost equivalent' problem to computing the Tutte polynomial of a set-pointed matroid is the problem of finding the Tutte polynomial [26, (5.1)] of a *matroid perspective* (for a definition, see [26, p. 977]). Applications of the matroid perspective approach may be found in [22, 23, 25]. As a consequence of Lemma 4.6, a coloured generalization of these applications could be considered, using the relative Tutte polynomial.

Another generalization of the Tutte polynomial of a set-pointed matroid is the Tutte polynomial of a matroid perspective sequence, studied by Chaiken [7]; the term 'matroid perspective sequence' is used by Las Vergnas [26, p. 975]. Finding a common generalization of the relative Tutte polynomial of a coloured matroid and of the Tutte polynomial of a matroid perspective sequence seems an interesting question for future research.

4.3. The random-cluster model

We conclude this section with an application of the relative Tutte polynomial to the random-cluster model introduced by Fortuin and Kasteleyn [16]. A random-cluster model can be thought of as a graph $G(V,E)$ that is associated with a function $p : E \longrightarrow [0,1]$. We may think of $p(e)$ as the probability that the edge $e \in E$ 'survives an accident', and $q(e) = 1 - p(e)$ for the probability that the edge e 'breaks' in an accident. Fortuin and Kasteleyn [16] introduced the following polynomial of the variable κ as a cluster generating function $Z(G;p,\kappa)$:

$$Z(G;p,\kappa) = \sum_{C \subseteq E} p^C q^{E \setminus C} \kappa^{k(C)}. \tag{4.2}$$

Here p^C is shorthand for the product $\prod_{e \in C} p_e$, $q^{E \setminus C}$ is shorthand for the product $\prod_{e \in E \setminus C} q_e$, and $k(C)$ is the number of connected components in the subset of edges C. The quantity $p^C q^{E \setminus C}$ represents the probability that exactly the subset of connections C survives. Thus the function $Z(G;p,\kappa)$ gives the expectation $\mathbb{E}(\kappa^{k(G)})$, which is an indirect measure of the mean average number of connected components of the network after an accident happens. The following example concerns only a very special network setting since we do not intend to explore the general cases in this paper.

Example 4.7. Let us consider the following problem. Suppose that a network of communications is to be built among a number of 'stations' and two types of communication methods are available. Type A is cheap and fast (such as the internet) so communication between two stations does not need to be direct so long as it can be routed through a sequence of stations connected by type A communication. However, type A communication may break in the event of some 'accident'. On the other hand, type B communication is dependable and will not break in the event of an accident. However, it is costly, slow and hence communication between two stations using type B cannot be routed through a sequence of stations where at least two pairs of stations are communicated by the type B method. If station 1 and station 2 cannot be communicated through a sequence of stations

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

connected by type A communication, then either a direct type B communication has to be established between them, or they have to be routed to two stations 3 and 4 through type A communications first and then through a direct type B communication between station 3 and station 4, if the latter is cheaper. It thus makes sense to build a network with a two-layer structure: a primary structure that uses type A communication to connect all stations and a secondary structure that would enable a type B communication between any two stations when needed after an accident strikes.

This two-layer structure network can then be represented by a graph G, the stations are the vertices of G, and a type A communication between two stations is a regular edge and a type B communication between two stations is a 'zero edge'. Each zero edge f comes with a positive weight c_f, namely the cost to operate that communication line (and there is a zero edge between any two vertices so the zero edges alone give us a complete graph), and each regular edge e comes with a positive weight $p_e < 1$, namely the probability that edge e survives in the event of an accident (so $q_e = 1 - p_e$ is the probability that the edge breaks in the event of an accident). We will assume further that the events that the regular edges break are all independent of each other. We wish to compute the mean cost of maintaining a functional network in the event of an accident. And this turns out to be a relative Tutte polynomial with a suitable choice of ψ and the assignment of the variables to the regular edges.

Since the zero edges alone give us a complete graph, no regular edges will be active. For a regular edge e, we will then assign $x_e = p_e$ and $y_e = q_e$ (we are using e as the colour of e since every regular edge is treated as being coloured differently). Notice that for a given contracting set $\mathcal{C}$, there is at least one edge between any two vertices in the graph $\mathcal{H}_\mathcal{C}$ (though $\mathcal{H}_\mathcal{C}$ may now have loop edges and multiple edges). If we remove all loop edges from $\mathcal{H}_\mathcal{C}$, and for any two vertices of $\mathcal{H}_\mathcal{C}$, keep only the edge incident to them that has the smallest weight among all edges incident to these two vertices, then we obtain a complete graph with the minimum total weight (the summation of all weights in the graph), which we will denote as $\mathcal{H}'_\mathcal{C}$. We then define $\psi(\mathcal{H}_\mathcal{C})$ to be the total weight of $\mathcal{H}'_\mathcal{C}$. Of course ψ so defined is invariant under the vertex pivot operation. $T_\mathcal{H}(G)$ then represents precisely the mean cost of operating a functional network in the event of an accident.

5. Applications to virtual knot theory

In this section we apply the relative Tutte polynomial to virtual knot theory. More specifically, we relate the relative Tutte polynomial to the Kauffman bracket polynomial of a virtual knot. It is well known that a classical link diagram can be converted to a plane graph (called the *face graph* or the *Tait graph* of the diagram) where the edges of the face graph are coloured with the colour set $\{+, -\}$. Furthermore, the Tutte polynomial of this coloured plane graph can be converted to the Kauffman bracket polynomial via suitable variable substitutions. However, for a virtual link diagram, the corresponding face graph created using the old approach creates some special 'zero' edges which cannot be handled via the traditional deletion/contraction approach when one tries to define or compute the Tutte polynomial of such a graph. One way to overcome this difficulty is to change the virtual link diagram into a ribbon graph where there are no more zero edges,

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

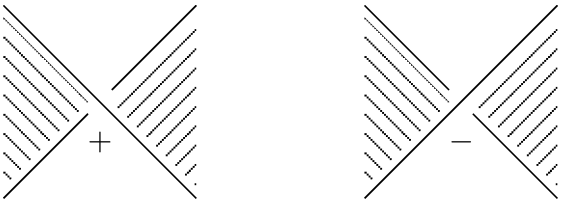


Figure 3. The sign of a classic crossing with respect to a checkerboard colouring in a link diagram.

and the Bollobás–Riordan polynomial [3, 4] of the ribbon graph may be used to express the Jones polynomial [9, 10, 11, 18, 19]. The earlier results along these lines [10, 18, 19] are only applicable to 'checkerboard-colourable' virtual link diagrams. Figure 4 shows the virtual trefoil, which is not checkerboard-colourable. The best current generalization in this direction is due to Chmutov and Voltz [9, 11], providing a formula for *all* virtual links. Our approach is closer to Kauffman's [20], and uses only the underlying graph structure of the face graph.

First, we need some preparation on how we will handle a graph with only the zero edges. In other words, we would like to choose the function ψ in such a way that we may apply the relative Tutte polynomial to a virtual link diagram.

5.1. The face graph of a link diagram

A regular link diagram K can be viewed as a plane 4-valent graph from which one can obtain a so-called 'face graph' G. Here is a brief description of this process. One starts from the regular projection K and shades the regions in its projection either 'white' or 'dark' in a checkerboard fashion, so that no two dark regions are adjacent, and no two white regions are adjacent (this can always be done for a 4-valent plane graph). We usually consider the infinite region surrounding the knot projection to be white. Note that as we move diagonally over a knot crossing, we go from a white region to a white region, or from a dark region to a dark region. Next we construct a dual graph of K by converting the dark regions in K into vertices in a graph G and converting the crossings in K between two dark regions into edges incident to the corresponding vertices in G. So if we can move diagonally over a knot crossing from one dark region to another, then these two dark regions and the crossing will be represented in G as two vertices connected by an edge. Note that we may obtain parallel edges from some knot projections. Now we have our unsigned graph. To obtain the signed version, we look at each crossing in the knot projection. If, after the upper strand passes over the lower, the dark region is to the left of the upper strand, then we denote this as a positive crossing. If the dark region is to the right of the upper strand, we denote it as a negative crossing. See Figure 3. Then our signed graph is obtained by marking each edge of G with the same sign as the crossing of K to which it corresponds. In the case of a virtual crossing, we cannot assign a sign to it, so we will simply assign it the number 0. Figure 4 shows such an example.

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

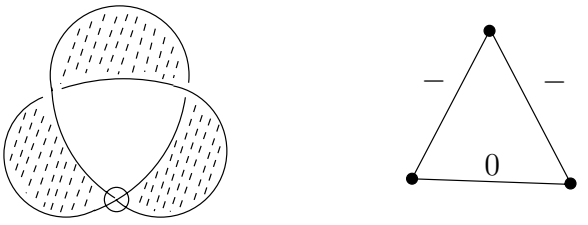


Figure 4. The virtual trefoil knot (which is not checkerboard-colourable) and its face graph.

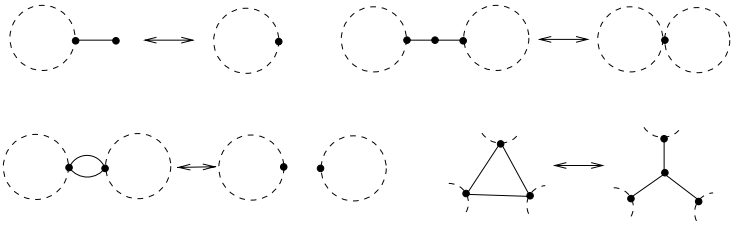


Figure 5. The 0-operations.

5.2. Zero order of a plane graph

As we have seen from the above discussion, a face graph of a virtual link diagram will contain two kinds of edges: the edges that correspond to classic crossings (the regular edges) and the edges that correspond to virtual crossings (the zero edges). Since we cannot carry out the typical crossing/splitting operations at a virtual crossing (which are used in defining all the knot polynomials), it means that we cannot perform the typical contraction/deletion operation on the zero edges (which is our motivation for introducing the relative Tutte polynomials). In order to define an appropriate relative Tutte polynomial (which can lead us to the Kauffman bracket polynomial) for a face graph of a virtual link diagram, we have to choose a proper ψ defined on graphs with only the zero edges (such graphs are face graphs of virtual link diagrams with only virtual crossings). Let G' be such a plane graph (it is not necessarily connected). Since it is the face graph of a (virtual) link diagram, its number of components (*i.e.*, the number of components of the link) is a well-defined number. We call this number the *zero order* of the graph G' and denote it by $|G'|_0$. By a result due to Las Vergnas [24], $|G'|_0 = \log_2 |T_{G'}(-1,-1)| + 1$, where $T_{G'}(x,y)$ is the ordinary (non-coloured) Tutte polynomial of G'. $|G'|_0$ can also be determined using the simplification operations shown in Figure 5. These are called *zero edge operations* or simply 0-operations.

Lemma 5.1. *For any given plane graph G, there exists a finite sequence S of* 0*-operations that leads to a graph G_S with only vertices. Furthermore, the number of vertices in G_S is equal to $|G'|_0$.*

Proof. The 0-operations, translated into the link diagrams, are simply the Reidemeister moves. These moves never change the number of components in the link diagram and there exists a sequence of Reidemeister moves that will take the original diagram to disjoint

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

circles (since there are no restrictions on the moves in this case), which is equivalent to a plane graph with only vertices. □

We now choose $\psi(G) = d^{n-1}$ where $n = |G|_0$. Notice that ψ so defined is invariant under vertex pivots, since a vertex pivot in G, when translated in terms of the knot diagram, is equivalent to taking a connected sum of two link diagrams at two different places, which of course does not affect the number of components in the connected sum (which is always the sum of the numbers of components in each link diagram minus one). Under this choice of ψ, the Tutte polynomial for a disconnected (face) graph will then have the form stated in the following proposition.

Proposition 5.2. *Let G be a graph with connected components $G_1, G_2, \ldots, G_m$ and assume that ψ is as defined above. Assume that $\mathcal{H}_j$ is a subgraph of G_j and let $\mathcal{H} = \cup_{1 \leqslant j \leqslant m} \mathcal{H}_j$; then $T_{\mathcal{H}}(G)$ is*

$$T_{\mathcal{H}}(G) = d^{m-1} \prod_{1 \leqslant j \leqslant m} T_{\mathcal{H}_j}(G_j), \tag{5.1}$$

where d is the same variable as in the definition of ψ.

Proof. Each contracting set $\mathcal{C}$ of G can be uniquely written as $\mathcal{C} = \cup_{1 \leqslant j \leqslant m} \mathcal{C}_j$, where $\mathcal{C}_j$ is a contracting set in G_j. Assume that $|\mathcal{H}_{\mathcal{C}_j}|_0 = k_j$; then $|\mathcal{H}_{\mathcal{C}}|_0 = \sum_{1 \leqslant j \leqslant m} k_j$. Thus the total contribution of $\mathcal{H}_{\mathcal{C}}$ to $T_{\mathcal{H}}(G)$ is $d^{-1+\sum_{1 \leqslant j \leqslant m} k_j}$. On the other hand, each $\mathcal{H}_{j_{\mathcal{C}_j}}$ contributes a factor to $d^{k_j - 1}$ to $T_{\mathcal{H}_j}(G_j)$. The proposition statement now follows from the equality $d^{-1+\sum_{1 \leqslant j \leqslant m} k_j} = d^{m-1} d^{k_1 - 1} d^{k_2 - 1} \cdots d^{k_m - 1}$. □

Remark 5.3. It is worth noting that the zero order is defined for *plane graphs only*, and thus the relative Tutte polynomial introduced in this section is also *defined on the class of plane graphs only*. This should not present a problem, since the class of plane graphs is closed not only under deletion and contraction, frequently used in all Tutte polynomial calculations, but also under the vertex pivot operation, introduced in connection with the map ψ. A generalization of the Bollobás–Riordan polynomial to minor-closed classes of matroids was developed by Ellis-Monaghan and Traldi [15]. One would need to start with the class of plane graphs and such a generalized Bollobás–Riordan polynomial, and then adapt the reasoning of the preceding sections.

5.3. Converting the relative Tutte polynomial to the Kauffman bracket polynomial

The following theorem is the main motivation of this paper.

Theorem 5.4. *Let K be a virtual link diagram and let G be its face graph obtained from K, where a virtual crossing in K corresponds to a zero edge in G. Let $\mathcal{H}$ be the subgraph of G that contains all the zero edges, then $T_{\mathcal{H}}(G)$, as defined in* (5.1)*, equals the Kauffman bracket polynomial through the following variable substitution:*

$$X_+ \to -A^{-3},\ X_- \to -A^3,\ Y_+ \to -A^3,\ Y_- \to -A^{-3}$$

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

$$x_+ \to A,\ x_- \to A^{-1},\ y_+ \to A^{-1},\ y_- \to A,$$
$$d \to -(A^2 + A^{-2}).$$

In the following proof we will use only very basic properties of the Kauffman bracket of virtual links. For details of these properties, see Kauffman [21].

Proof. Let K be any virtual link diagram with n regular crossings and let G be its face graph. We will prove the theorem by induction on n.

For $n = 0$, $\langle K \rangle = d^{m-1}$ by definition, where $d = -(A^2 + A^{-2})$ and m is the number of components in the link diagram K. In this case, G is a plane graph with only zero edges and $T_{\mathcal{H}}(G) = \psi(\mathcal{H}) = d^{m-1}$ as well, by the definition of ψ. So we have $\langle K \rangle = T_{\mathcal{H}}(G)$.

Assume now that we have $\langle K \rangle = T_{\mathcal{H}}(G)$ for any K with $n \geqslant 0$ regular crossings. We would like to show that for any K with $n+1$ regular crossings, this is still the case.

So let K be any given virtual link diagram with $n+1$ regular crossings and let G be its face graph. Since $n + 1 \geqslant 1$, there exists at least one regular edge in G. Let e be a regular edge. In the following proofs, we will assume that e is positive. The case of e being negative can be proved similarly and is left to the reader. Again $\mathcal{H}$ is the set of all the zero edges of G.

Case 1: e is neither a bridge nor a loop in G. By the recursive formula (3.12), we have

$$T_{\mathcal{H}}(G) = x_+ T_{\mathcal{H}}(G/e) + y_+ T_{\mathcal{H}}(G \setminus e) = A T_{\mathcal{H}}(G/e) + A^{-1} T_{\mathcal{H}}(G \setminus e).$$

On the other hand, a similar recursive formula of the Kauffman bracket for the crossing in K that corresponds to e gives

$$\langle K \rangle = \langle \text{⋊⋉} \rangle = A\langle \text{⌒} \rangle + A^{-1}\langle \,)\ (\, \rangle.$$

But the face graph of the link diagram corresponding to the ⌒ split is G/e and the face graph of the link diagram corresponding to the $)\ ($ split is $G \setminus e$. Thus, by our induction hypothesis, we have

$$T_{\mathcal{H}}(G) = A T_{\mathcal{H}}(G/e) + A^{-1} T_{\mathcal{H}}(G \setminus e) = A\langle \text{⌒} \rangle + A^{-1}\langle \,)\ (\, \rangle = \langle \text{⋊⋉} \rangle = \langle K \rangle.$$

Case 2: e is a loop. By (3.12), we have

$$T_{\mathcal{H}}(G) = Y_+ T_{\mathcal{H}}(G \setminus e) = -A^3 T_{\mathcal{H}}(G \setminus e).$$

On the other hand, we have

$$\langle K \rangle = \langle \text{◯} \rangle = A\langle \,) \text{◯} \rangle + A^{-1}\langle\ )\ \rangle = (Ad + A^{-1})\langle\ )\ \rangle = -A^3\langle\ )\ \rangle.$$

But the diagram $)$ obtained from K (by deleting the loop) has face graph $G \setminus e$, and the result follows from the induction hypothesis again.

Case 3: e is a bridge. By (3.12) and our induction hypothesis, we have

$$T_{\mathcal{H}}(G) = X_+ T_{\mathcal{H}}(G/e) = -A^{-3} T_{\mathcal{H}}(G/e) = -A^{-3}\langle K_0 \rangle,$$

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

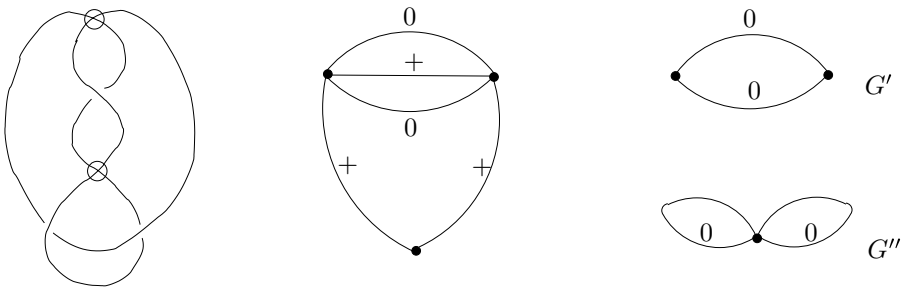


Figure 6. A virtual knot diagram with two virtual crossings and its face graph.

where K_0 is the link diagram obtained from K by the $\asymp$ split. On the other hand, we have

$$\langle K\rangle = \langle \bowtie \rangle = A\langle \asymp \rangle + A^{-1}\langle)\ \ (\rangle. \tag{5.2}$$

Since e is a bridge, the $)\ \ ($ split above creates two disjoint link diagrams $K_1 =)$ and $K_2 = (.$ Furthermore, $K_0 = K_1 \# K_2$. It is well known that $\langle K_1 \# K_2\rangle = \langle K_1\rangle \cdot \langle K_2\rangle$. Thus it follows that

$$\langle)\ \ (\rangle = \langle K_1 \sqcup K_2\rangle = d \cdot \langle K_1\rangle \cdot \langle K_2\rangle = d \cdot \langle K_1 \# K_2\rangle = d \cdot \langle \asymp \rangle.$$

Combining this with (5.2), we have

$$\langle K\rangle = (A + dA^{-1})\langle \asymp \rangle = -A^{-3}\langle \asymp \rangle = T_{\mathcal{H}}(G).$$

This finishes our proof. □

Example 5.5. Figure 6 shows a simple virtual knot diagram K with two virtual crossings marked (which are circled in the diagram), together with its face graph. The edges marked with 0 correspond to the virtual crossings. At the far right are the remaining graphs at the end of the contraction/deletion process of the regular edges. It is easy to see that $\psi(G') = d$ and $\psi(G'') = 1$. Thus we have

$$\begin{aligned} T_{\mathcal{H}}(G) &= y_+^2(X_+ + x_+)\psi(G') + (x_+y_+X_+ + x_+^2y_+ + x_+^2Y_+)\psi(G'') \\ &= y_+^2(X_+ + x_+)d + (x_+y_+X_+ + x_+^2y_+ + x_+^2Y_+) \\ &= (y_+^2 d + x_+y_+)(X_+ + x_+) + x_+^2Y_+ \\ &= (-(A^2 + A^{-2})A^{-2} + AA^{-1})(-A^{-3} + A) - A^5 \\ &= -A^{-3} + A^{-7} - A^5 = \langle K\rangle. \end{aligned}$$

Since the writhe of the diagram is 3, it follows that the Jones polynomial of K is $J_K(t) = (-A^{-3})^3(-A^{-3} + A^{-7} - A^5)|_{A=t^{-1/4}} = t + t^3 - t^4$.

Remark 5.6. The introduction and development of the relative Tutte polynomial and its connection to the Kauffman bracket (and hence Jones) polynomial of virtual links will make the generalization of some existing results in classical knot theory to virtual

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

knot theory possible. For instance, the results of the authors on the coloured Tutte polynomials of coloured graphs through repeated tensor product operation will generalize to the relative Tutte polynomials without much difficulty [14]. Consequently, the Jones polynomials of virtual knots and links obtained through repeated tangle replacement operation (as discussed in [12]) can be computed in polynomial time. The authors intend to further explore these and other applications of the relative Tutte polynomials in the near future.

Acknowledgements

The authors wish to thank Professor Louis Kauffman for introducing them to virtual knot theory, Professor Douglas West for providing some useful information on graph operations, Professors Xian'an Jin and Fuji Zhang for bringing the results of [24] to their attention, and Professor Sergei Chmutov for advice and many useful comments. Finally, the authors wish to express their gratitude to the anonymous referee for providing many thoughtful suggestions to improve the paper.

References

[1] Bollobás, B. (1998) *Modern Graph Theory*, Graduate Texts in Mathematics, Springer.
[2] Bollobás, B. and Riordan, O. (1999) A Tutte polynomial for coloured graphs. *Combin. Probab. Comput.* **8** 45–93.
[3] Bollobás, B. and Riordan, O. (2001) A polynomial of graphs on orientable surfaces. *Proc. London Math. Soc.* **83** 513–531.
[4] Bollobás, B. and Riordan, O. (2002) A polynomial of graphs on surfaces. *Math. Ann.* **323** 81–96.
[5] Brylawski, T. (1971) A combinatorial model for series-parallel networks. *Trans. Amer. Math. Soc.* **154** 1–22.
[6] Brylawski, T. (1972) A decomposition for combinatorial geometries. *Trans. Amer. Math. Soc.* **171** 235–282.
[7] Chaiken, S. (1989) The Tutte polynomial of a ported matroid. *J. Combin. Theory Ser. B* **46** 96–117.
[8] Chmutov, S. Personal communication.
[9] Chmutov, S. (2009) Generalized duality for graphs on surfaces and the signed Bollobás–Riordan polynomial. *J. Combin. Theory Ser. B* **99** 617–638.
[10] Chmutov, S. and Pak, I. (2007) The Kauffman bracket of virtual links and the Bollobás–Riordan polynomial. *Moscow Math. J.* **7** 409–418.
[11] Chmutov, S. and Voltz, J. (2008) Thistlethwaite's theorem for virtual links. *J. Knot Theory Ramification* **17** 1189–1198.
[12] Diao, Y., Ernst, C. and Ziegler, U. (2009) Jones polynomial of knots formed by repeated tangle replacement operations. *Topology Appl.* **156** 2226–2239.
[13] Diao, Y., Hetyei, G. and Hinson, K. (2009) Tutte polynomials of tensor products of signed graphs and their applications in knot theory. *J. Knot Theory Ramification* **18** 561–590.
[14] Diao, Y., Hetyei, G. and Hinson, K. (2009) Invariants of composite networks arising as a tensor product. *Graphs Combin.* **25** 273–290.
[15] Ellis-Monaghan, J. A. and Traldi, L. (2006) Parametrized Tutte polynomials of graphs and matroids. *Combin. Probab. Comput.* **15** 835–854.

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press

[16] Fortuin, C. M. and Kasteleyn, P. W. (1972) On the random-cluster model I: Introduction and relation to other models. *Physica* **57** 536–564.
[17] Jaeger, F., Vertigan, D. L. and Welsh, D. J. A. (1990) On the computational complexity of the Jones and Tutte polynomials. *Math. Proc. Cambridge Phil. Soc.* **108** 35–53.
[18] Kamada, N. (2002) On the Jones polynomials of checkerboard colorable virtual links. *Osaka J. Math.* **39** 325–333.
[19] Kamada, N. (2004) Span of the Jones polynomial of an alternating virtual link. *Algebraic & Geometric Topology* **4** 1083–1101.
[20] Kauffman, L. H. (1989) A Tutte polynomial for signed graphs. *Discrete Appl. Math.* **25** 105–127.
[21] Kauffman, L. H. (1999) Virtual knot theory. *Europ. J. Combin.* **20** 663–690.
[22] Las Vergnas, M. (1977/78) Acyclic and totally cyclic orientations of combinatorial geometries. *Discrete Math.* **20** 51–61.
[23] Las Vergnas, M. (1981) Eulerian circuits of 4-valent graphs imbedded in surfaces. In *Algebraic Methods in Graph Theory*, Vol. I, II (Szeged 1978), Vol. 25 of *Colloq. Math. Soc. János Bolyai*, pp. 451–477.
[24] Las Vergnas, M. (1979) On Eulerian partitions of graphs. In *Graph Theory and Combinatorics*, Vol. 34 of *Research Notes in Mathematics* (R. J. Wilson, ed.), Pitman Advanced Publishing Program, pp. 62–75.
[25] Las Vergnas, M. (1984) The Tutte polynomial of a morphism of matroids II: Activities of orientations. In *Progress in Graph Theory* (Waterloo, Ont., 1982), Academic Press, pp. 367–380.
[26] Las Vergnas, M. (1999) The Tutte polynomial of a morphism of matroids I: Set-pointed matroids and matroid perspectives. *Ann. Inst. Fourier* **49** 973–1015.
[27] Thistlethwaite, M. B. (1987) A spanning tree expansion for the Jones polynomial. *Topology* **26** 297–309.
[28] Traldi, L. (2004) A subset expansion of the coloured Tutte polynomial. *Combin. Probab. Comput.* **13** 269–275.
[29] Tutte, W. T. (1954) A contribution to the theory of chromatic polynomials. *Canad. J. Math.* **6** 80–91.
[30] Welsh, D. J. A. (1976) *Matroid Theory*, Academic Press, London.

https://doi.org/10.1017/S0963548309990484 Published online by Cambridge University Press