J. Plasma Physics (2008), *vol.* 74, *part* 4, *pp.* 437–444. © 2007 Cambridge University Press
doi:10.1017/S0022377807006733 Printed in the United Kingdom

Large-amplitude electron plasma oscillations

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(Received 18 May 2007 and in revised form 7 June 2007, first published online 6 December 2007)

Abstract. Large-amplitude electron plasma oscillations in a one-dimensional, cold electron plasma are investigated for a spatially varying, immobile ion density. The Eulerian variables are transformed to Lagrangian variables. The problem is then treated as an effective one-dimensional particle in a potential. Two examples of spatially varying ion densities that lead to analytic functions for the electron position, velocity and electric field are found by Gauss' law to have zero electron density, an unphysical result. A generic solution that includes the two examples is shown to have a spatially homogeneous ion density. A nonlinear ordinary differential equation is the condition for the appropriate form of the electron density and is solved by Lie group symmetries. A more general form of a solution is presented that possesses a spatially varying ion density, but when the necessary conditions are specified it has either zero electron density or secular terms in the electron density.

1. Introduction

Exact solutions of plasmas have been few in number. However, these exact solutions are important if perturbation expansions fail and these solutions furnish a useful benchmark for numerical simulations. Analysis of nonlinear plasma oscillations in a one-dimensional, cold electron plasma with immobile ions is an old problem [1–4]. Nonlinear plasma oscillations in warm plasmas were also considered [2,4,5], as were envelope electron plasma solitons in warm plasmas [6]. More recently time variations of the densities have been included [7], bounded plasmas have been treated [8–11] and spatial inhomogeneities of ion density have been included [12–16].

Here we treat a one-dimensional, cold electron plasma with an ion background at rest in a fluid approximation. The Eulerian independent variables are transformed to the Lagrangian variables [2–5,7,11–14,16]. The aim is to determine what spatial dependence the ion density may possess. Also, unlike many previous analyses the problem is transformed first to the problem of an effective particle in a potential although one analysis has a similar approach [12]. Then the Lagrangian orbital time is found in terms of an integral over electron position variable. Inversion of an implicit function of the electron position as a function of the Lagrangian variables then leads to expressions for the electron fluid velocity, the electric field and the electron density where the fluid and Maxwell equations are checked.

The classic example of the large-amplitude electron plasma oscillations with a constant ion density is derived with our approach. Two examples of spatial dependencies of the ion density that involve hyperbolic and Jacobian elliptic functions are

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not allowed where Gauss' law (Poisson's equation) imposes crucial restraints. A generic solution that includes the two examples is proved to allow only spatially homogeneous ion densities. An example of a spatially inhomogeneous ion density is presented that illustrates the necessary constraints but has either a zero electron density or secular terms in the electron density.

2. Exact solution for uniform ion density

We begin with the one-dimensional, cold electron plasma fluid and Maxwell equations:

$$\frac{\partial n}{\partial t} + \frac{\partial (nv)}{\partial x} = 0, \tag{1a}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{e}{m}E, \tag{1b}$$

$$\frac{\partial E}{\partial x} = -\frac{e}{\varepsilon_0}(n - n_{\rm i}), \tag{1c}$$

$$\frac{\partial E}{\partial t} = \frac{e}{\varepsilon_0} nv, \tag{1d}$$

in SI units. Here n is the electron density, v is the magnitude of the electron fluid velocity, E is the magnitude of the electric field, e is the charge, m is the electron mass, ε_0 is the electric permittivity, x is the spatial variable and t is the time in Eulerian variables. We assume that the ion density $n_{\rm i} = n_0 N_{\rm i}(x)$ with n_0 a constant.

The relation between the Eulerian (x, t) and Lagrangian (x_0, τ) variables is

$$\tau = t, \quad x_0 = x - \int_0^\tau d\tau' v(x_0, \tau'). \tag{2}$$

The convective derivative following the electron fluid is

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}. \tag{3}$$

Then (1b) can be rewritten as

$$\frac{\partial v}{\partial \tau} = -\frac{e}{m}E \tag{4}$$

and (1c) and (1d) can be combined to

$$\frac{\partial E}{\partial \tau} = \frac{e}{\varepsilon_0} n_{\rm i} v. \tag{5}$$

In addition

$$\frac{\partial}{\partial x} = \left(\frac{\partial x}{\partial x_0}\right)^{-1} \frac{\partial}{\partial x_0}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - v\left(\frac{\partial x}{\partial x_0}\right)^{-1} \frac{\partial}{\partial x_0}. \tag{6}$$

It has been shown [3, 4] that

$$n(x_0, \tau) = n(x_0, 0)\left(\frac{\partial x}{\partial x_0}\right)^{-1}. \tag{7}$$

By eliminating the electric field from (4) and (5) one finds a differential equation for v. Since the ion density is assumed a function of x, we replace v by $\partial x/\partial \tau$ from

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(2) and find the equation

$$\frac{\partial^3 x}{\partial \bar{\tau}^3} + N_{\rm i}(x)\frac{\partial x}{\partial \bar{\tau}} = 0 \tag{8}$$

where $\bar{\tau} = \omega_{\rm p0}\tau$, $\omega_{\rm p0} = (e^2 n_0/m\varepsilon_0)^{1/2}$ is the plasma frequency. If $x(x_0, \tau)$ can be found from (8), then v, E and n can be determined. The substitution for v has produced an equation (8) that can be integrated to give

$$\frac{\partial^2 x}{\partial \bar{\tau}^2} + \int_{x_0}^{x} dx'\, N_{\rm i}(x') = \frac{C_1(x_0)}{2} \tag{9}$$

where we have the equation of motion of an effective particle in one dimension. Next (9) is reduced to quadratures by the usual techniques to

$$\int_{x_0}^{x} \frac{dx'}{[C_1(x_0)x' + C_2(x_0) - 2g(x')]^{1/2}} = \bar{\tau} \tag{10}$$

with $g(x) = \int_{x_0}^{x} dx' \int_{x_0}^{x'} dx''\, N_{\rm i}(x'')$ and we have an implicit function $x(x_0, \tau)$. Here $C_1(x_0)$ and $C_2(x_0)$ depend on the initial value of $x(x_0, 0) = x_0$.

Analytic expressions for the integral in (10) are numerous where there should be turning points for oscillations with $\bar{\tau}$ the orbital time. We first consider a uniform ion density with $N_i(x) = 1$. Then $2g(x) = (x - x_0)^2$ and the integral is an inverse sine function. Inversion of the sine function leads to

$$x = x_0 + \frac{C_1(x_0)}{2} + (C_2(x_0) + C_1(x_0)x_0)^{1/2}\sin\bar{\tau} - \frac{C_1(x_0)}{2}\cos\bar{\tau}. \tag{11}$$

The result for x is equal to the result in the text by Davidson [4] if

$$C_1(x_0) = 2X(x_0), \quad (C_2(x_0) + C_1(x_0)x_0)^{1/2} = \frac{V(x_0)}{\omega_{\rm p0}}. \tag{12}$$

From (11) v, E and n can be derived and all of the equations checked. The approach here has been to replace the differential equation for v by (8) and then to reduce it to quadratures in (10).

3. Spatially inhomogeneous ion density with zero electron density

Two examples of possible spatially inhomogeneous ion density are presented here. One is

$$N_i(x) = -(2ac + b^2) - 6bcx - 6c^2x^2 \tag{13}$$

where a, b and c are constants since the ion density is a function of position but not of time. The relation for x is

$$x = -\frac{b}{2c} + \frac{\sqrt{-q}}{2c}\tanh\left[\frac{\sqrt{-q}}{2}(\bar{\tau} - \alpha)\right] \tag{14}$$

with $q = 4ac - b^2$ and $\alpha = 2/\sqrt{-q}\tanh^{-1}[(2cx_0 + b)/\sqrt{-q}]$. It is required that $ac < 0$ so that $n_i > 0$ for some range of x. This ion density was chosen for simplicity but is not an oscillation. The second example has an ion density that produces an oscillation. It is

$$N_i(x) = 1 - b^2\delta^2 + 6\delta^2(x - K)^2 \tag{15}$$

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where b, δ and K are constants. The relation for x is

$$x = K + bcn\left[\frac{\delta}{h}(\bar{\tau} - \alpha), k\right] \tag{16}$$

where

$$\frac{\delta}{h} = \sqrt{1 + b^2\delta^2}, \quad k = \frac{b\delta}{\sqrt{1 + b^2\delta^2}} \quad \text{and} \quad \alpha = -\frac{h}{\delta} cn^{-1}\left(\frac{x_0 - K}{b}\right).$$

The Jacobian elliptic function $cn(u, k)$ (see [17]) was chosen because it is doubly periodic and in the limit $\delta \to 0$, it becomes a cosine function.

The difficulty with both of these examples is that the numerator, $n(x_0, 0)$, in the electron density in (7) is zero when Gauss' law (1c) is solved for these two examples, an unphysical result. We next look at a generic solution that includes the two examples for the general form of x if the modulus of the Jacobian elliptic function is a constant. We start with the assumption that the solution of (10) can be inverted. Then a generic form that includes the two examples is

$$x(x_0, \tau) = A(x_0) + B(x_0) f(\bar{\tau} - \alpha(x_0)). \tag{17}$$

The variables $\bar{\tau} - \alpha(x_0)$ are not multiplied by a function of x_0 because otherwise secular terms in the Lagrangian time arise in the denominator of the electron density. We use (1c) with

$$\frac{\partial E}{\partial x} = \left(\frac{\partial x}{\partial x_0}\right)^{-1} \frac{\partial E}{\partial x_0}. \tag{18}$$

For the numerator $n(x_0, 0)$ of (7) to be independent of $\bar{\tau}$ we find that the following equation must hold

$$y'y'' - yy''' = \bar{n}y'. \tag{19}$$

Here

$$y' = \frac{dy}{d\bar{\tau}}, \text{ etc.}, \quad y = f + \frac{\partial A}{\partial x_0}\left(\frac{\partial B}{\partial x_0}\right)^{-1}, \quad \bar{n} = \frac{n(x_0, \tau)}{n_0}\left(\frac{\partial B}{\partial x_0}\right)^{-1}$$

and we use (8) for $N_i(x)$. In addition

$$\frac{\partial f}{\partial x_0} = -\frac{\partial f}{\partial \bar{\tau}} \frac{\partial \alpha}{\partial x_0}.$$

Now (19) is a nonlinear ordinary differential equation (x_0 is a parameter) that can be solved by use of Lie group symmetries (see [18] where many references on Lie symmetries can also be found). There are two Lie point group symmetries represented by the group generators

$$U_1 = \frac{\partial}{\partial \bar{\tau}}, \quad U_2 = \bar{\tau}\frac{\partial}{\partial \bar{\tau}} + 2y\frac{\partial}{\partial y}, \tag{20}$$

that insures that the differential equation can be reduced at least to a first-order ordinary differential equation. The new variables are first chosen from the translation symmetry of U_1 as it represents a normal subgroup. The new variables are $w = y'$, $z = y$ and we find

$$w^2\frac{dw}{dz} - wz\frac{d}{dz}\left(w\frac{dw}{dz}\right) = \bar{n}w. \tag{21}$$

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The factor of w cancels (one solution is for $y = D(x_0)$) and if $W = w^2$, $Z = z^2$, we find

$$\frac{d^2W}{dZ^2} = -\frac{\bar{n}}{2Z^{3/2}}. \tag{22}$$

This equation is invariant under one Lie group symmetry, but because it arises from a variational condition it can be reduced by two orders. It is also an example of an ordinary differential equation with a Type II hidden symmetry [18]. The equation is integrated twice and the variables changed to give

$$y' = [2\bar{n}z + K_1 z^2 + K_2]^{1/2} \tag{23}$$

where K_1 and K_2 depend on x_0 in general. Then as $z = y$, we can integrate again, invert and find

$$y = -\frac{\sqrt{-q}}{K_1}\sin[\sqrt{-K_1}(\bar{\tau} - \alpha)] - \frac{\bar{n}}{K_1} \tag{24}$$

with $q = K_1K_2(x_0) - \bar{n}^2$. In addition we have

$$\alpha(x_0) = \frac{1}{\sqrt{-K_1}}\sin^{-1}\left(\frac{K_1 y + \bar{n}}{\sqrt{-q}}\right).$$

We use (8) to show that $N_i(x)$ is a constant in x. It is equal to $-K_1$ where K_1 cannot depend on x_0 because otherwise secular terms arise in the electron density. Then $N_i(x)$ is set equal to 1 or $K_1 = -1$. We transform (25) back to x noting that $x_0 = A(x_0) + B(x_0)f(-\alpha(x_0))$. Finally, we find

$$\begin{aligned} x = x_0 &+ \sqrt{-q}B(x_0)\sin\alpha(x_0) + \sqrt{-q}B(x_0)\cos\alpha(x_0)\sin\bar{\tau} \\ &- \sqrt{-q}B(x_0)\sin\alpha(x_0)\cos\bar{\tau} \end{aligned} \tag{25}$$

with $q = K_2(x_0) - \bar{n}^2$. Comparing (25) with (11) and (12), we find that they are identical once the coefficients are equated. As a result the ion density must be spatially independent to insure that the numerator of the electron density is a function of x_0 and non-zero.

4. Spatially varying ion density with secular terms

An example of a spatially varying ion density is next developed. Various conditions need to be specified or likely cases such as the two examples in Sec. 3 may not be valid solutions. The first condition is that the ion density is a function of x but is not a function of time and therefore not a function of x_0. The electron density should be finite and positive. The presence of secular terms in the Lagrangian time destroys the coherent oscillation. We start with the denominator of the expression of the electron density in (7),

$$\frac{\partial x}{\partial x_0} > 0 \tag{26}$$

where it is positive because it equals one at $\tau = 0$ and remaining positive insures that the electron density is finite. This condition was required for the spatially homogeneous ion density previously [4]. The numerator of the electron density is then positive for a physical solution. We equate the function inside the square root in the denominator of the integral in (10) to a polynomial that could result from

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the form of the ion density or be a Taylor expansion:

$$C_1(x_0)x + C_2(x_0) - 2g(x) = \sum_{l=0}^{n} a_l x^l \tag{27}$$

where n is an integer. A simple condition to insure a non-zero electron density is

$$\frac{n(x_0,0)}{n_0} = \frac{1}{2}\frac{\partial C_1(x_0)}{\partial x_0} + N(x_0) = \frac{\partial a_1}{\partial x_0} > 0, \quad \frac{\partial a_i}{\partial x_0} = 0, l \geqslant 2 \tag{28}$$

that is found from Gauss' law with (1c), (6), (9) and (27). The second inequality holds because the ion density may be a function of x but not of x_0. This condition failed for both examples in Sec. 3.

We next introduce another Jacobian elliptic function. The integral that is equivalent to that in (10) is

$$\int_{x_2}^{x} \frac{dx'}{[(x_1 - x')(x' - x_2)(x' - x_3)]^{1/2}} = \bar{g} sn^{-1} = \bar{\tau}. \tag{29}$$

Here

$$x_1 > x > x_2 > x_3, \quad k^2 = \frac{x_1 - x_2}{x_1 - x_3} > 0, \quad sn(\bar{g}^{-1}\bar{\tau}, k)^2 = \frac{(x_1 - x_3)(x - x_2)}{(x_1 - x_2)(x - x_3)},$$

where k is the modulus and

$$\bar{g} = \frac{2}{\sqrt{x_1 - x_3}}.$$

A spatially inhomogeneous ion density can be computed for this Jacobian elliptic function where (28) is obeyed. One example is to choose $x_2 = x_0$ and $N_i(x) = a + bx$ where a and b are positive constants. The two factors x_1, x_3 are determined with care that a and b are positive constants and the inequalities above are obeyed. The factor $\bar{g}$ is chosen to be constant for this example to eliminate secular terms in the electron density. Then the electron density numerator can be found from

$$\frac{n(x_0,0)}{n_0} = \frac{1}{2}\frac{\partial C_1(x_0)}{\partial x_0} + N(x_0) = \frac{3x_0 + a}{4} \tag{30}$$

that is positive for a range of x_0. The requirement that there be no secular terms requires that

$$\frac{\partial sn(g^{-1}\bar{\tau}, k)}{\partial x_0} = 0. \tag{31}$$

If we were to choose $\bar{g}$ to be a function of x_0, then a factor of $\bar{\tau}$ would appear in the derivative of the argument. If as above we choose $\bar{g}$ to be a constant, but choose the modulus k to be a function of x_0, a factor of $\bar{\tau}$ would appear in the derivative of the modulus. However, the requirement that $\bar{g}$, k and a_2 be constants leads to three linear algebraic equations in x_1, x_2 and x_3 with the result that all are constant. Then a_1 is a constant and not a function of x_0 as required by (28) for a non-zero electron density. We conclude that either the electron density is zero or there are secular terms in time in the electron density. The analysis can be repeated for a Jacobian elliptic function where there is a fourth-order polynomial in the square root of the denominator in the integral of (29). The results are the same for the example just analyzed but the three algebraic equations are replaced by four more complicated algebraic equations.

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5. Conclusion

The one-dimensional, cold electron plasma fluid and Maxwell equations have been solved in Lagrangian variables. The differential equations have been replaced first by the equation of motion of an effective one-dimensional particle in a potential. This equation has been reduced to quadratures where an implicit function is found for the electron position as a function of the Lagrangian time. For certain integrals the implicit function can be inverted. First, the classic solution with a uniform ion density at rest has been derived. Then two solutions of spatially dependent ion densities that involve hyperbolic and Jacobian elliptic functions have been derived. However, in these two cases the allowed form of the electron density from Gauss' law (Poisson's equation) was not found. The electron density was zero.

Next a generic form of the electron position in Lagrangian variables was assumed that included the two examples for possible spatially inhomogeneous ion density. The electron velocity and electric field were determined and then the constraint of Gauss' law was imposed. The correct condition for the electron density led to the requirement for a spatially independent ion density. The condition also resulted in a nonlinear ordinary differential equation that was solved by Lie group symmetries whose solution recapitulated the solution for the electron position with a spatially independent ion density.

A different spatially inhomogeneous ion density was analyzed where a number of constraints were imposed on the solution for the electron position, fluid speed and electric field to insure a positive, finite electron density. The electron density has been found to have secular terms in time or to be zero for this case. Dawson [1] argued that spatially inhomogeneous ions would lead to fine-scale mixing such that coherent oscillations could not endure in the plasma. Our analysis supports his argument but it has not been shown for all cases. Previous cases have introduced spatially inhomogeneous ion densities, but our results demonstrate that care must be taken to insure a physical solution. The examples considered here with spatially inhomogeneous ion densities seemed fine until the constraint of Gauss' law was imposed. The analysis of previous solutions is complicated by perturbative solutions that may include moving ions, but our analysis does not include moving ions. In addition the transformation from Eulerian to Lagrangian variables does restrict possible solutions, but removing these restrictions leads to wave breaking and multi-stream flow. The analysis reported here does have the advantage over many previous reports in that within the model the solutions are exact.

Acknowledgement

The research was partially supported by the Professional Development Fund of the School of Engineering and Applied Science of Washington University.

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