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Period doubling in area-preserving maps: an associated one-dimensional problem

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Abstract. It has been observed that the famous Feigenbaum–Coullet–Tresser period-doubling universality has a counterpart for area-preserving maps of $\mathbb{R}^2$. A renormalization approach has been used in a computer-assisted proof of existence of an area-preserving map with orbits of all binary periods in Eckmann *et al* [Existence of a fixed point of the doubling transformation for area-preserving maps of the plane. *Phys. Rev. A* **26**(1) (1982), 720–722; A computer-assisted proof of universality for area-preserving maps. *Mem. Amer. Math. Soc.* **47** (1984), 1–121]. As is the case with all non-trivial universality problems in non-dissipative systems in dimensions more than one, no analytic proof of this period-doubling universality exists to date. We argue that the period-doubling renormalization fixed point for area-preserving maps is almost one dimensional, in the sense that it is close to the following Hénon-like (after a coordinate change) map:

$$H_*(x, u) = (\phi(x) - u, x - \phi(\phi(x) - u)),$$

where ϕ solves

$$\phi(x) = \frac{2}{\lambda}\phi(\phi(\lambda x)) - x.$$

We then give a 'proof' of existence of solutions of small analytic perturbations of this one-dimensional problem, and describe some of the properties of this solution. The 'proof' consists of an analytic argument for factorized inverse branches of ϕ together with verification of several inequalities and inclusions of subsets of $\mathbb{C}$ numerically. Finally, we suggest an analytic approach to the full period-doubling problem for area-preserving maps based on its proximity to the one-dimensional case. In this respect, the paper is an exploration of possible analytic machinery for a non-trivial renormalization problem in a conservative two-dimensional system.

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1. *Introduction*

Following the pioneering discovery of the Feigenbaum–Coullet–Tresser period-doubling universality in unimodal maps [**Fei1, Fei2, TC**], universality has been demonstrated to be a rather generic phenomenon in dynamics.

To prove universality, one usually introduces a *renormalization* operator on a functional space, and demonstrates that this operator has a hyperbolic fixed point.

Such renormalization approach to universality has been very successful in one-dimensional dynamics, and has led to explanation of universality in unimodal maps [**Eps1, Eps2, Lyu**], critical circle maps [**dF1, dF2, Ya1, Ya2**], and holomorphic maps with a Siegel disk [**GaiYa, McM, Ya3**].

Universality has been abundantly observed in higher dimensions, in particular in two and more dimensional dissipative systems (cf. [**CEK1, Spa**]), in area-preserving maps, as both the period-doubling universality [**BCGG, CEK2, DP, EKW1, EKW2, Hel**] and the universality associated with the break-up of invariant surfaces [**McK1, McK2, ME, SK**], and in Hamiltonian flows [**AK, AKW, ED, Gai1, GK, Koch1, Koch2, Koch3, Kocić**]. It has been established that the universal behavior in dissipative and conservative higher-dimensional systems is fundamentally different. The case of dissipative systems is often reducible to the one-dimensional Feigenbaum–Coullet–Tresser universality [**CEK1, dCLM**]. The case of conservative systems is very different, and at present there is no deep understanding of universality in conservative systems, other than in the 'trivial' case of the universality for systems 'near integrability' [**Gai1, KLDM, Koch1, Koch2, Kocić**]. The study of the interesting cases of universality for maps and flows far from dissipative or linear is at present confined to numerics (for instance [**CEK1, GK, McK1**]) or computer-assisted proofs [**EKW2, Koch2, Koch3**]. The latter approach turned out to be quite

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(a) (b) (c)

FIGURE 1. Bifurcation at $a = 4$ of a stable fixed point $((1 + \sqrt{a-3})/a, (1 - \sqrt{a-3})/a)$ of period two into a stable period-four orbit in the area-preserving Hénon family H_a: (a) $a = 3.8$; (b) $a = 3.98$; (c) $a = 4.02$ (from [**Bou**]).

powerful, albeit rather specialized and time and effort consuming. Very little analytic machinery exists for higher-dimensional maps and flows far from dissipative or linear.

In this paper we will consider a period-doubling universality for area-preserving maps of the plane—an analogue of Feigenbaum–Coullet–Tresser universality in higher dimensions.

An infinite period-doubling cascade in families of area-preserving maps was observed by several authors in the early 1980s [**BCGG, Bou, CEK2, DP, Hel**]. A typical period-doubling scenario can be illustrated with the area-preserving Hénon family (cf. [**Bou**]):

$$H_a(x, y) = (-y + 1 - ax^2, x).$$

Maps in this family possess a fixed point $((-1 + \sqrt{1+a})/a, (-1 + \sqrt{1+a})/a)$, which is stable for $-1 < a < 3$. When $a_1 = 3$, this fixed point becomes unstable; at the same time, an orbit of period two is born with $H_a(x_\pm, x_\mp) = (x_\mp, x_\pm)$, $x_\pm = (1 \pm \sqrt{a-3})/a$. This orbit, in turn, becomes unstable at $a_2 = 4$, giving birth to a period-four stable orbit (Figure 1). Generally, there exists a sequence of parameter values a_k at which the orbit of period 2^{k-1} turns unstable, while at the same time a stable orbit of period 2^k is born. The parameter values a_k accumulate on some a_∞. The crucial observation is that the accumulation rate

$$\lim_{k\to\infty} \frac{a_k - a_{k-1}}{a_{k+1} - a_k} = 8.721\ldots$$

is universal for a large class of families, not necessarily Hénon.

Furthermore, the 2^k periodic orbits scale asymptotically with two scaling parameters

$$\lambda = -0.249\ldots, \quad \mu = 0.061\ldots.$$

To explain how orbits scale with λ and μ, we will follow [**Bou**]. Consider an interval (a_k, a_{k+1}) of parameter values in a 'typical' family F_a. For any value $\alpha \in (a_k, a_{k+1})$, the map F_α possesses a stable periodic orbit of period 2^k. We fix some α_k within the interval (a_k, a_{k+1}) in some consistent way; for instance, by requiring that the restriction of $F_{\alpha_k}^{2^k}$ to a neighborhood of a stable periodic point in the 2^k-periodic orbit is conjugate via a diffeomorphism H_k to a rotation with some fixed rotation number r. Let p_k' be some

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unstable periodic point in the 2^{k-1}-periodic orbit, and let p_k be the further (from p'_k) of the two stable 2^k-periodic points that bifurcated from p'_k. Then,

$$\frac{1}{\lambda} = -\lim_{k\to\infty} \frac{|p_k - p'_k|}{|p_{k+1} - p'_{k+1}|}, \qquad \frac{\lambda}{\mu} = -\lim_{k\to\infty} \frac{\rho_k}{\rho_{k+1}},$$

where ρ_k is the ratio of the eigenvalues of $DH_k(p_k)$.

This universality can be explained rigorously if one shows that the *renormalization* operator

$$R[F] = \Lambda_F^{-1} \circ F \circ F \circ \Lambda_F, \tag{1.1}$$

where Λ_F is some F-dependent coordinate transformation, has a fixed point, and the derivative of this operator is hyperbolic at this fixed point.

It has been argued in [**CEK2**] that Λ_F is a diagonal linear transformation. Furthermore, such Λ_F has been used in [**EKW1, EKW2**] in a computer-assisted proof of existence of the renormalization fixed point. This is the strongest result to date.

In this paper we will first present some numerical and qualitative evidence that the fixed point for the renormalization operator (1.1) is, in some appropriate sense, very close to the area-preserving map

$$H_*(x, u) = (\phi(x) - u, x - \phi(\phi(x) - u)),$$

where ϕ solves the following one-dimensional problem:

$$\phi(y) = \frac{2}{\lambda}\phi(\phi(\lambda y)) - y. \tag{1.2}$$

The map H_* is Hénon-like after a change of coordinates $I \circ N$,

$$I(x, u),\ N(x, u) = (x, \phi(x) - u). \tag{1.3}$$

Furthermore, we will consider a more general functional equation

$$\phi(y) = \frac{2}{\lambda}\phi(\phi(\lambda y)) - y + \tau(y), \tag{1.4}$$

where τ is some small analytic 'perturbation'.

Problem (1.4) will be reformulated as a fixed-point problem for the diffeomorphic part of the factorized inverse branches of ϕ. Specifically, we will restate the problem as a fixed-point problem for a continuous operator on the diffeomorphic part, and demonstrate that there is a choice of a compact functional space which is left invariant by this operator. Existence of a fixed point will follow from the Schauder–Tikhonov theorem. Thus, the proof relies on an analytic argument; however, several rather technical conditions will be verified on a computer numerically.

Furthermore, we will suggest an approach to the full problem $R[F] = F$ based on a 'factorization' of F into a dominant 'one-dimensional' Hénon-like part plus a small two-dimensional correction. Qualitatively, the corrective term is contracted by the renormalization operator in a neighborhood of the 'one-dimensional' approximate fixed point. An important remark is that this contraction is not shown by brute force (which would be the case with a 'hard' computer-assisted proof); instead, the period-doubling operator essentially turns into a contraction after the above-mentioned one-dimensional problem is 'factored out'.

We will briefly outline further steps that are necessary to complete the proof of existence of a period-doubling fixed point.

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We should note that construction of an approximate renormalization fixed point is generally an easy task. What we are trying to do here is (i) construct an approximate fixed point which lies in a compact functional space and (ii) turn the period-doubling operator for area-preserving maps into a contraction using the *a-priori* bounds from this compact space. The dominant 'one-dimensional' part of the fixed point of the full problem (1.1) is expected to lie in the same space and obey similar *a-priori* bounds.

2. *A renormalization operator for generating functions*

Recall that a symplectic manifold $(\mathcal{M}, \omega)$ is called exact if its symplectic form ω satisfies

$$\omega = d\theta.$$

A transformation F acting on a symplectic manifold is called symplectic if it preserves the symplectic form:

$$F^*\omega - \omega = 0,$$

where F^* denotes the pullback of F. A symplectic transformation acting on an exact symplectic manifold is called *exact symplectic* if it changes the one-form θ only by a differential of a scalar function:

$$F^*\theta - \theta = d\mathcal{S}. \tag{2.1}$$

An 'area-preserving map' will mean an exact symplectic diffeomorphism of a subset of $\mathbb{R}^2$ onto its image.

If one denotes the components of such F as $F(x, u) = (y(x, u), v(x, u))$, then (2.1) can be written in the explicit coordinate notation:

$$v(x, u)\, dy(x, u) - u\, dx = d\mathcal{S}(x, u).$$

If u can be expressed as a function of x and y, $u = u(x, y)$, on some subset of $\mathbb{R}^2$, then one defines the generating function of F as

$$S(x, y) \equiv \mathcal{S}(x, u(x, y)).$$

Clearly,

$$u = -S_1(x, y), \quad v = S_2(x, y), \tag{2.2}$$

where S_i denotes a partial derivative. If the hypothesis of the implicit function theorem holds for some S, then (2.2) can be solved for y and v as functions of x and u. Clearly, the implicit function theorem will also apply in some C^2 open neighborhood of S and one, therefore, obtains a parametrization of an open set of maps of $\mathbb{R}^2$ by *scalar functions*.

Our immediate goal will be to rewrite the period-doubling renormalization operator (1.1) in terms of generating functions. We will, therefore, assume that there exists an open set of renormalizable area-preserving maps F analytic on some subset $\mathcal{D}$ of $\mathbb{R}^2$ which are uniquely representable by their generating functions $S = S(x, y)$. We will not describe the subset $\mathcal{D}$ here (see [**GJ1, GJ2**] for details), since below we will only have to deal with the domain of analyticity of the generating functions.

We write F as

$$\begin{pmatrix} x \\ -S_1(x, y) \end{pmatrix} \overset{F}{\mapsto} \begin{pmatrix} y \\ S_2(x, y) \end{pmatrix}, \quad S_i \equiv \partial_i S. \tag{2.3}$$

Furthermore, we will assume that F is reversible, that is,

$$T \circ F \circ T = F^{-1} \quad \text{where } T(x, u) = (x, -u).$$

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For such maps, it follows from (2.3) that

$$S_1(y, x) = S_2(x, y) \equiv s(x, y)$$

and

$$\begin{pmatrix} x \\ -s(y, x) \end{pmatrix} \overset{F}{\mapsto} \begin{pmatrix} y \\ s(x, y) \end{pmatrix}.$$

It is this 'little' s that will be referred to below as 'the generating function'.

We will now derive an equation for the generating function of the renormalized map $\Lambda_F \circ F \circ F \circ \Lambda_F^{-1}$.

Applying a reversible F twice, we get

$$\begin{pmatrix} x' \\ -s(z', x') \end{pmatrix} \overset{F}{\mapsto} \begin{pmatrix} z' \\ s(x', z') \end{pmatrix} = \begin{pmatrix} z' \\ -s(y', z') \end{pmatrix} \overset{F}{\mapsto} \begin{pmatrix} y' \\ s(z', y') \end{pmatrix}.$$

It has been argued in [**CEK2**] that

$$\Lambda(x, u) = (\lambda x, \mu u).$$

We therefore set

$$(x', y') = (\lambda x, \lambda y), \quad z'(\lambda x, \lambda y) = z(x, y)$$

to obtain

$$\begin{pmatrix} x \\ -\dfrac{1}{\mu} s(z, \lambda x) \end{pmatrix} \overset{\Lambda}{\mapsto} \begin{pmatrix} \lambda x \\ -s(z, \lambda x) \end{pmatrix} \overset{F \circ F}{\mapsto} \begin{pmatrix} \lambda y \\ s(z, \lambda y) \end{pmatrix} \overset{\Lambda^{-1}}{\mapsto} \begin{pmatrix} y \\ \dfrac{1}{\mu} s(z, \lambda y) \end{pmatrix}, \tag{2.4}$$

where $z(x, y)$ solves

$$s(\lambda x, z(x, y)) + s(\lambda y, z(x, y)) = 0. \tag{2.5}$$

If the solution of (2.5) is unique, then $z(x, y) = z(y, x)$, and it follows from (2.4) that the generating function of the renormalized F is given by

$$\tilde{s}(x, y) = \mu^{-1} s(z(x, y), \lambda y).$$

Furthermore, it is possible to fix some normalization conditions for $\tilde{s}$ and z which serve to determine scalings λ and μ as functions of s. Notice that the normalization

$$s(1, 0) = 0$$

is reproduced for $\tilde{s}$ as long as

$$z(1, 0) = z(0, 1) = 1.$$

In particular, this implies that

$$s(\lambda, 1) + s(0, 1) = 0.$$

Furthermore, the condition

$$\partial_1 s(1, 0) = 1 \tag{2.6}$$

is reproduced as long as

$$\mu = \partial_1 z(1, 0).$$

We will now summarize the above discussion in the following definition of the renormalization operator acting on generating functions originally due to the authors of [**EKW1, EKW2**].

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Definition 1.

$$\mathcal{R}_{\mathrm{EKW}}[s](x, y) = \mu^{-1} s(z(x, y), \lambda y) \quad \text{where}$$
$$0 = s(\lambda x, z(x, y)) + s(\lambda y, z(x, y)), \tag{2.7}$$
$$0 = s(\lambda, 1) + s(0, 1), \quad \text{and} \quad \mu = \partial_1 z(1, 0). \tag{2.8}$$

As we have already mentioned, the following has been proved with the help of a computer in [**EKW1, EKW2**].

THEOREM 2.1. *There is an s_* in some Banach space of functions analytic on a bi-disk in $\mathcal{C}_7^2$, such that the operator $\mathcal{R}_{\mathrm{EKW}}$ is well defined, analytic, and compact on some neighborhood of s_*, and $\mathcal{R}_{\mathrm{EKW}}[s_*] = s_*$. Furthermore, the scalings λ_* and μ_* corresponding to the fixed point s_* satisfy*

$$-0.2492 < \lambda < -0.2485, \tag{2.9}$$
$$0.0606 < \mu < 0.0616. \tag{2.10}$$

Here, we will quote for reference purposes approximations of several first coefficients of the fixed point s_*, obtained in our numerical computations, and the corresponding mid-point function z_*, together with the set of relations between them, all of which are obtained by differentiation of the fixed-point equation $s(x, y) = \mu^{-1} s(z(x, y), \lambda y)$:

$$s_*(x, y) = (x - 1) + ay + \frac{b}{2}(x - 1)^2 + c(x - 1)y + \frac{d}{2}y^2 + O((x - 1)^i y^j), \quad j + i = 3, \tag{2.11}$$

$$z_*(x, y) = 1 + \mu(x - 1) + \theta y + \frac{\upsilon}{2}(x - 1)^2 + \iota(x - 1)y + \frac{\nu}{2}y^2 + O((x - 1)^i y^j), \quad j + i = 3, \tag{2.12}$$

$$a = 0.1948 \ldots, \quad b = -0.0523 \ldots, \quad c = -0.0497, \quad d = 2.11 \ldots,$$

$$a = \frac{\theta}{\mu - \lambda}, \quad b\mu = \frac{\upsilon}{1 - \mu}, \quad c\mu = \frac{\upsilon\theta + \iota(1 - \mu)}{(1 - \lambda)(1 - \mu)}, \quad d = a(2c - ba) + \frac{2\iota a - \upsilon a^2 - \nu}{\lambda^2 - \mu}.$$

3. *A λ-manifold-based approach. Nonlinear scalings*

The fixed-point equation $\mathcal{R}_{\mathrm{EKW}}[s] = s$ together with the tautological identity $\lambda y = \lambda y$ can be written simultaneously as

$$AS = S \circ G, \quad A = \begin{bmatrix} \mu & 0 \\ 0 & \lambda \end{bmatrix}, \tag{3.1}$$

where

$$G(x, y) = (z(x, y), \lambda y) \quad \text{and} \quad S(x, y) = (s(x, y), y).$$

Given G, this is an equation for a linearizer S that conjugates G to a linear map A. Notice that the point $(1, 0)$ is fixed under G; the line $y = 0$ is, locally, an invariant manifold

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for G associated with the eigenvalue $\mu = \partial_1 z(1, 0)$. Next, suppose that the function $\phi(y)$ defined on some neighborhood of 0 is such that $s(\phi(y), y) = 0$ (necessarily, $\phi(0) = 1$). We will assume that $(\phi(y), y)$ is the only zero-level of s in its domain. Then, the first equation of (3.1), evaluated at $(\phi(y), y)$, becomes

$$s(G(\phi(y), y)) = s(z(\phi(y), y), \lambda y) = 0,$$

which implies that

$$z(\phi(y), y) = \phi(\lambda y).$$

That is,

$$G(\phi(y), y) = (\phi(\lambda y), \lambda y),$$

and the curve $(\phi(y), y)$ is an invariant manifold for G associated with the eigenvalue λ.

We will refer to the two invariant manifolds as the λ- and the μ-manifolds.

A solution S of the Schröder equation (3.1) maps the fixed point of G to the origin, the μ-manifold to the x-axis, and the λ-manifold to the y-axis. Such (normalized) solution necessarily satisfies

$$s(x, y) = (x - \phi(y))(1 + \varepsilon(x, y)) \equiv x - \phi(y) + \epsilon(x, y), \tag{3.2}$$

where ϕ is the parametrization of the λ-manifold of G.

Notice that the fixed-point problem for the renormalization operator (2.8) is equivalent to the solution of the Schröder equation (3.1) together with the mid-point equation (2.7).

The mid-point equation can be readily solved on some neighborhood of $(1, 0)$ for a rather specific $s(x, y) = x - \phi(y)$ ($\epsilon(x, y) = 0$), where ϕ is some function invertible in a neighborhood of zero (with no further assumptions on ϕ at this point). The mid-point equation for such s becomes

$$x + y = \frac{2}{\lambda}\phi(z(x, y)).$$

Evaluate this mid-point equation at $(\phi(y), y)$:

$$\phi(y) + y = \frac{2}{\lambda}\phi(z(\phi(y), y)).$$

Clearly, if $\phi(y)$ is also the parametrization of the λ-manifold for $G(x, y) = (z(x, y), \lambda y)$, then it satisfies the following functional equation:

$$\phi(y) = \frac{2}{\lambda}\phi(\phi(\lambda y)) - y. \tag{3.3}$$

We would like to reiterate that a solution ϕ of (3.3) is the parametrization of the λ-manifold for $G(x, y) = (z(x, y), \lambda y)$, where z solves the mid-point equation for $s(x, y) = x - \phi(y)$, but such s is not yet the solution of the Schröder equation nor of the fixed-point problem for $\mathcal{R}$.

Now, suppose that $\epsilon(x, y)$ in (3.2) is non-zero. Then, the mid-point equation for such s reads

$$x + y = \frac{2}{\lambda}\phi(z) - \frac{1}{\lambda}\epsilon(\lambda x, z) - \frac{1}{\lambda}\epsilon(\lambda y, z).$$

Again, suppose that ϕ is also a parametrization for the λ-manifold of G; then, it satisfies

$$\phi(y) = \frac{2}{\lambda}\phi(\phi(\lambda y)) - y - \omega_{\epsilon,\phi}(y), \tag{3.4}$$

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where

$$\omega_{\epsilon,\phi}(y) \equiv \frac{1}{\lambda}\epsilon(\lambda\phi(y), \phi(\lambda y)) + \frac{1}{\lambda}\epsilon(\lambda y, \phi(\lambda y)).$$

At this point, we will modify the operator $\mathcal{R}_{\mathrm{EKW}}$ by introducing a nonlinear scaling $\mu = \mu(y)$ and turning it into an operator for the 'corrective term' ϵ. The reasons for this modification will be clear momentarily.

Definition 2. Assume that given ϵ there is a solution φ_ϵ of (3.4) defined on some neighborhood of the interval (0, 1). Set, formally,

$$\mathcal{RG}[\epsilon](x, y) \equiv \mu^{-1}(y)(z(x, y) - \varphi_\epsilon(\lambda y) + \epsilon(z(x, y), \lambda y)) - (x - \varphi_\epsilon(y)), \tag{3.5}$$

where the mid-point function z solves

$$\lambda x - \varphi_\epsilon(z(x, y)) + \lambda y - \varphi_\epsilon(z(x, y)) + \epsilon(\lambda x, z(x, y)) + \epsilon(\lambda y, z(x, y)) = 0, \tag{3.6}$$

and the scalings λ and $\mu(y)$ satisfy

$$\lambda = 2\varphi_\epsilon(1) - \epsilon(\lambda, 1) - \epsilon(0, 1),$$
$$\mu(y) = \partial_1 z(\phi(y), y).$$

Suppose that there is a fixed point ϵ_* such that $\mathcal{RG}[\epsilon_*] = \epsilon_*$, the corresponding scaling being λ_* and μ_*, and the mid-point function being z_*. Then, it is immediate that the function

$$s_*(x, y) = x - \phi_{\epsilon_*}(y) + \epsilon_*(x, y)$$

satisfies

$$s_* = \frac{s_* \circ G_*}{\mu_*} \quad \text{where } G_*(x, y) \equiv (z_*(x, y), \lambda_* y),$$

and the corresponding reversible F_* satisfies

$$F_* = \Lambda_*^{-1} \circ F_* \circ F_* \circ \Lambda_*,$$
$$\Lambda_*(x, u) = (\lambda_* x, \mu_*(x)u),$$
$$\Lambda_*^{-1}(y, v) = \left(\frac{y}{\lambda_*}, \frac{v}{\mu_*(y/\lambda_*)}\right).$$

At this point, Definition 2 is purely formal. The standard properties of being well defined and analytic (in a neighborhood of the fixed point) should be verified after a functional space for the ϵ's is appropriately chosen.

Notice that $s(x, y) = x - \varphi_\epsilon(y) + \epsilon(x, y)$ is an approximate renormalization fixed point in a neighborhood of the λ-manifold $(\varphi_\epsilon(y), y)$ for any sufficiently small ϵ, in the sense that

$$\begin{aligned}
\mathcal{R}_{\mathrm{EKW}}[s](x, y) &= \mu^{-1}(y)(z(x, y) - \varphi_\epsilon(\lambda y) + \epsilon(z(x, y), \lambda y)) \\
&= \mu^{-1}(y)(z(x, y) - z(\varphi_\epsilon(y), y) + \epsilon(z(x, y), \lambda y)) \\
&= \frac{\partial_1 z(\varphi_\epsilon(y), y)}{\mu(y)}(x - \varphi_\epsilon(y) + O((x - \varphi_\epsilon(y))^2)) + \mu^{-1}(y)\epsilon(z(x, y), \lambda y) \\
&= x - \varphi_\epsilon(y) + O((x - \varphi_\epsilon(y))^2) + \mu^{-1}(y)\epsilon(z(x, y), \lambda y).
\end{aligned}$$

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We would like to emphasize that the coefficients of O are proportional to third and higher-order derivatives of z, which are 'tiny' (cf. (2.12)). Therefore, O is expected to be small in any reasonable norm on a sufficiently small neighborhood of the λ-manifold $(\varphi_\epsilon(y), y)$.

Furthermore, the operator $\mathcal{RG}$ is expected to be a contraction in a neighborhood of the λ-manifold. This can be seen if one writes

$$\begin{aligned}\epsilon(x, y) &\equiv (x - \varphi_\epsilon(y))\varepsilon(x, y),\\ \varepsilon(x, y) &= f(y) + O((x - \varphi_\epsilon(y))),\\ s(x, y) &= (x - \varphi_\epsilon(y))(1 + f(y)) + O((x - \varphi_\epsilon(x, y))^2);\end{aligned}$$

then, the condition (2.6) implies that

$$1 + f(0) = 1 \implies f(y) = O(y) \implies \varepsilon(x, y) = O((x - \varphi_\epsilon(y))^i y^j), \quad i + j = 1.$$

In particular,

$$\varepsilon(1, 0) = 0,$$

and the operator

$$\varepsilon(x, y) \mapsto \varepsilon(G(x, y))$$

is a contraction in a neighborhood of $(1, 0)$. Then,

$$\begin{aligned}\mathcal{RG}[\epsilon](x, y) &= O((x - \varphi_\epsilon(y))^2) + \mu^{-1}(y)\epsilon(z(x, y), \lambda y)\\ &= O((x - \varphi_\epsilon(y))^2) + \mu^{-1}(y)(z(x, y) - \varphi_\epsilon(\lambda y))\varepsilon(G(x, y))\\ &= O((x - \varphi_\epsilon(y))^2) + \frac{\partial_1 z(\varphi_\epsilon(y), y)}{\mu(y)}\\ &\quad \times (x - \varphi_\epsilon(y) + O((x - \varphi_\epsilon(y))^2))\varepsilon(G(x, y))\\ &= (x - \varphi_\epsilon(y))\varepsilon(G(x, y)) + O((x - \varphi_\epsilon(y))^2)\end{aligned}$$

is also a contraction on a sufficiently small neighborhood of the curve $(\varphi_\epsilon(y), y)$.

Any rigorous formulation of the above informal discussion should start with, first, solving (3.4) and, second, identifying a suitable functional space for the ϵ's. In this paper, we will concentrate on studying (3.4) and its special case (3.3). In §6, we will suggest an iterative procedure for the operator $\mathcal{RG}$ which can be used to prove the existence of a fixed point for this operator. Its implementation will be a subject of an upcoming work.

4. *Notation. Some facts about Herglotz–Pick functions*

We will proceed with some definitions.

We will use the standard notation for the lower and upper half planes:

$$\mathbb{C}_\pm \equiv \{z \in \mathbb{C} : \pm\Im(z) > 0\}.$$

Let $J = (l, r) \subset \mathbb{R}$. Define $D_+(J, \theta)$ to be an open subset of $\mathbb{C}_+$ bounded by a circular arc intersecting $\mathbb{R}$ at the end points of J at an angle θ, and let $D_-(J, \theta) = D_+(J, \theta)^*$, where * stands for the complex conjugation. A Poincaré neighborhood is defined as

$$D(J, \theta) = D_+(J, \theta) \cup D_-(J, \theta) \cup J.$$

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Given an interval $J \subset \mathbb{R}$, denote

$$\mathbb{C}(J) \equiv \mathbb{C}_+ \cup \mathbb{C}_- \cup J, \quad \mathbb{C}_1 \equiv \mathbb{C}((-1, 1)).$$

We will denote by $\mathcal{F}(\mathcal{D})$ the Frechet space of functions holomorphic on a domain $\mathcal{D}$ equipped with the topology of uniform convergence on compact sets. A subset of functions in $\mathcal{F}$ assuming their values in a set $\mathcal{E}$ will be denoted by $\mathcal{O}(\mathcal{D}, \mathcal{E})$.

Suppose that $\mathcal{E}$ is simply connected and open, $\mathcal{D}$ and $\mathcal{E}$ are real symmetric, the boundary of $\mathcal{E} \cap \mathbb{R}$ is $\{L, R\}$, and let $\mathfrak{c} = \{\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4\}$ be a quadruple of real numbers, such that $\{\mathfrak{c}_1, \mathfrak{c}_2\} \in \mathcal{D}$ and $\{\mathfrak{c}_3, \mathfrak{c}_4\} \in \mathcal{E}$. We will further define

$$\begin{aligned}\mathcal{A}(\mathcal{D}, \mathcal{E}; \mathfrak{c}) \equiv \{u \in \mathcal{O}(\mathcal{D}, \mathcal{E}) : u(z) = u(z^*)^*, u(\mathcal{D} \cap \mathbb{C}_\pm) \\ \subset \overline{\mathcal{E} \cap \mathbb{C}_\pm}, u(\mathfrak{c}_1) = \mathfrak{c}_3, u(\mathfrak{c}_2) = \mathfrak{c}_4\}, \\ \mathcal{A}_1(\mathbf{c}) \equiv \mathcal{A}(\mathbb{C}_1, \mathbb{C}_1; \mathbf{c}).\end{aligned}$$

Clearly, a function u in $\mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$ is isomorphic to some $f \in \mathcal{A}_1(\mathbf{c})$ through unique conformal isomorphisms Φ_1 and Φ_2:

$$u = \Phi_2^{-1} \circ f \circ \Phi_1,$$

normalized so that

$$\Phi_1(\ell) = -1, \quad \Phi_1(r) = 1, \quad \Phi_1(a_1) = b_1$$

and

$$\Phi_2(L) = -1, \quad \Phi_2(R) = 1, \quad \Phi_2(a_2) = b_2.$$

Here, a_k and b_k are some points, and

$$c_{1,2} = \Phi_1(\mathfrak{c}_{1,2}), \quad c_{3,4} = \Phi_2(\mathfrak{c}_{3,4}).$$

Functions in $\mathcal{A}_1(\mathbf{c})$ admit the following Cauchy integral representation (cf. [**Val**]):

$$f(z) - c_3 = a(z - c_1) + \int d\nu(t)\left(\frac{1}{t - z} - \frac{1}{t - c_1}\right), \tag{4.1}$$

where ν is a measure supported in $\mathbb{R}\backslash(-1, 1)$.

This integral representation can be used to obtain the following *a-priori* bounds for functions in $\mathcal{A}_1(\mathbf{c})$:

$$\frac{c_4 - c_3}{c_2 - c_1}\frac{1 + c_2}{1 + x} \geq \frac{f(x) - c_3}{x - c_1} \geq \frac{c_4 - c_3}{c_2 - c_1}\frac{1 - c_2}{1 - x}, \quad x \in (-1, c_2), \tag{4.2}$$

$$\frac{c_4 - c_3}{c_2 - c_1}\frac{1 + c_2}{1 + x} \leq \frac{f(x) - c_3}{x - c_1} \leq \frac{c_4 - c_3}{c_2 - c_1}\frac{1 - c_2}{1 - x}, \quad x \in (c_2, 1), \tag{4.3}$$

$$\frac{1 + c_1}{(x - c_1)(1 + x)} \leq \frac{f'(x)}{f(x) - c_3} \leq \frac{1 - c_1}{(x - c_1)(1 - x)}, \quad x \in (-1, 1), \tag{4.4}$$

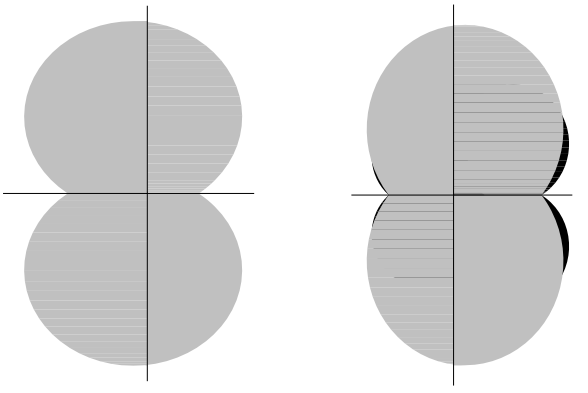
$$\frac{-2f'(x)}{1 + x} \leq f''(x) \leq \frac{2f'(x)}{1 - x}, \quad x \in (-1, 1). \tag{4.5}$$

If $\Phi|_{\mathbb{R}}$ is a monotone function, then one can transfer the bounds (4.2)–(4.5) to $\mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$.

Finally, we will mention the following version of the Schwarz lemma, which will play an important role in our results below (cf. [**Eps2, LY, Sul**]).

LEMMA 3. *Let $u : \mathbb{C}_J \mapsto \mathbb{C}_{J'}$ be a holomorphic map such that $u(J) \subset J'$. Then, for any $\theta \in (0, \pi)$, $u(D_\pm(J, \theta)) \subset D_\pm(J', \theta)$.*

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(a) (b)

FIGURE 2. Domain (a) and range (b) of functions in $\mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$. In (b) the Poincaré neighborhood $D(I_2, \theta_2)$ is given in light color, $D(I_3, \theta_3)\backslash D(I_2, \theta_2)$ and $D(I_4, \theta_4)\backslash D(I_2, \theta_2)$ in dark.

5. *Main results and observations*

We will now summarize the main results of the paper.

Define

$$\mathcal{K}_\epsilon[\phi](y) = \frac{2}{\lambda}\phi(\phi(\lambda y)) - y - \omega_{\epsilon,\phi}(y), \tag{5.1}$$

where ω_ϕ is as in (3.4).

Below we will denote the solution of $\mathcal{K}_\epsilon[\phi] = \phi$ by φ_ϵ, and that of (1.4) by ϕ_τ.

MAIN CLAIM 1. *Set*

$$I_1 = (-1.62,\ 1.04), \quad \theta_1 = 0.207\pi, \tag{5.2}$$

$$I_2 = (-2.69,\ 3.40), \quad \theta_2 = 0.295\pi, \tag{5.3}$$

$$I_3 = (-2.69,\ -1.60), \quad \theta_3 = 0.165\pi, \tag{5.4}$$

$$I_4 = (1.40,\ 3.40), \quad \theta_4 = 0.175\pi, \tag{5.5}$$

and

$$\mathcal{D} = D(I_1, \theta_1), \quad \mathcal{E} = D(I_2, \theta_2) \cup D(I_3, \theta_3) \cup D(I_4, \theta_4).$$

There exist $\delta > 0$ and $\kappa > 0$ such that, given any τ holomorphic in $\mathcal{E}$, real valued on $\mathcal{E} \cap \mathbb{R}$, and satisfying

$$\sup_{z\in\mathcal{E}} |\tau(z)| < \delta, \quad \sup_{z\in\mathcal{E}} |\tau'(z)| \le \kappa, \quad \tau(0) = 0,$$

there exists a function ϕ_τ, holomorphic on some complex neighborhood $\mathcal{U}$ of $(-1, 1)$ and satisfying $\phi_\tau(0) = 1$, and a number λ, such that the following holds:

(1) *ϕ_τ solves (1.4) on $\mathcal{U}$;*

(2) *ϕ_τ has a unique quadratic critical point on $\mathcal{U}$ at some real c: $\phi_\tau(c + z) = O(z^2)$;*

(3) *the two inverse branches η and ζ of ϕ_τ can be factorized as*

$$\eta(z) = u(T(-\sqrt{L(z)})), \quad \zeta(z) = u(T(\sqrt{L(z)})), \tag{5.6}$$

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T and L are affine, and

$$u \in \mathcal{A}(\mathcal{D}, \mathcal{E}; \mathfrak{c}), \quad \mathfrak{c} = (-\tfrac{1}{2}, 0, 0, 1);$$

(4) $-0.2809 < \lambda < -0.2465$*;*

(5) *if $\tau \equiv 0$, then $\lambda < c < 0$, and there are points*

$$0 < x_+ < \phi_0(\lambda) < 1, \quad \frac{x_+}{\lambda} < x_- < -1, \quad \textit{and} \quad x_*$$

such that $\phi_0(x_\pm) = 0$ and $\phi_0(x_*) = x_*$;

(6) *additionally,*

$$b \equiv \phi_0(c) < 1 + |\lambda|.$$

In this paper, we will present a 'proof' of this claim which is not rigorous in two respects.

First, it involves uniformization (construction of a conformal isomorphism to the unit circle) of the sets $\mathcal{D}$ and $\mathcal{E}$ (Figure 2). This uniformization is straightforward for $\mathcal{D}$, but cannot be put in a simple form in the case of $\mathcal{E}$. In this paper, we will approximate the restriction of the uniformizing coordinate of $\mathcal{E}$ to a set compactly contained in $D(I_2, \theta_2)$ by that of $D(I_2, \theta_2)$. A rigorous construction of the uniformizing coordinate of $\mathcal{E}$ would require implementing computer bounds on a numerical approximation using, for instance, the Schwarz–Christoffel formula, which is an undertaking in itself.

Second, the 'proof' relies on verification of several inequalities of the form $f(\mathbf{x}) > 0$, where f is some explicit function and $\mathbf{x}$ ranges over a set in $\mathbb{R}^d$ ($d = 4$ or 5). At this time this verification is done only numerically; its interval arithmetic implementation will be a subject of a future work.

We have also implemented the renormalization operator $\mathcal{RG}$ as well as $\mathcal{K}_\epsilon$ numerically, and have found good approximations of their fixed points by iterating Newton maps for these operators. The following summarizes our findings.

NUMERICAL OBSERVATION 1.

(1) *The following is an approximate Taylor series for the ϕ_0:*

$$\begin{aligned}\phi_0(y) \approx\ & 1.0 - 1.9719 \times 10^{-1} y - 9.2103 \times 10^{-1} y^2 - 3.1550 \times 10^{-2} y^3 \\ & + 2.5252 \times 10^{-2} y^4 - 5.6774 \times 10^{-4} y^5 + 1.6209 \times 10^{-5} y^6 \\ & + 5.5554 \times 10^{-6} y^7 - 2.5832 \times 10^{-6} y^8 + 5.1783 \times 10^{-8} y^9 \cdots, \\ \lambda \approx\ & -0.250\,14 \ldots.\end{aligned}$$

(2) *The operator $\mathcal{K}_0$ is hyperbolic at ϕ_0 with a local stable manifold of codimension two and the two eigenvalues outside of the unit circle given by*

$$\delta_1 = 8.700\,52 \ldots, \quad \delta_2 = \frac{1}{\lambda}.$$

δ_2 is the eigenvalue of the expanding eigenvector corresponding to coordinate translations.

(3) *The operator $\mathcal{K}_\epsilon$ has a fixed point φ_ϵ for all ϵ in some neighborhood of zero.*

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(4) *The operator $\mathcal{RG}$ has a fixed point ϵ_* whose approximate power series is given by*

$$\begin{aligned}\epsilon_* = &-2.668\times 10^{-2} + 5.477\times 10^{-2}y - 1.385\times 10^{-2}y^2 - 3.543\times 10^{-2}y^3\\ &+ 2.187\times 10^{-2}y^4 + 5.504\times 10^{-2}x - 6.029\times 10^{-2}xy\\ &+ 5.780\times 10^{-3}xy^2 - 5.777\times 10^{-4}xy^3 - 3.015\times 10^{-2}x^2\\ &+ 6.096\times 10^{-3}x^2y - 9.384\times 10^{-4}x^2y^2 + 1.927\times 10^{-3}x^3\\ &- 6.256\times 10^{-4}x^3y - 1.444\times 10^{-5}x^4 + \cdots,\end{aligned}$$

while

$$\begin{aligned}\phi_{\epsilon_*} = &\ 1.0 - 1.9468\times 10^{-1}y - 9.1492\times 10^{-1}y^2 - 3.6943\times 10^{-2}y^3\\ &+ 2.2002\times 10^{-2}y^4 - 5.9077\times 10^{-4}y^5 + 1.4033\times 10^{-5}y^6\\ &+ 7.5534\times 10^{-6}y^7 - 4.8304\times 10^{-6}y^8 + 8.5963\times 10^{-8}y^9\ldots,\\ \lambda_* = &-0.248\,88\ldots.\end{aligned}$$

(5) *The operator $\mathcal{RG}$ is a contraction in a neighborhood of ϵ_*.*

(6) *The operator $\mathcal{K}_{\epsilon_*}$ is hyperbolic at its fixed point φ^*_ϵ with the local stable manifold of codimension two, and the two expanding eigenvalues are given by*

$$\delta_1 = 8.662\,47\ldots, \quad \delta_2 = \frac{1}{\lambda_*}.$$

δ_2 is the eigenvalue of the expanding eigenvector corresponding to coordinate translations.

6. *Towards a complete proof of existence of a renormalization fixed point for area-preserving maps*

Let φ_ϵ be the solutions of (3.4), and let $s_{\phi,\epsilon}$ be a generating function of the form (3.2) and $\tilde{s}_{\phi,\epsilon}$ be its renormalization. Ultimately, we would like to show that the map

$$\epsilon(x,\,y) \mapsto \tilde{\epsilon}(x,\,y) \equiv \tilde{s}_{\varphi_\epsilon,\epsilon}(x,\,y) - x + \varphi_\epsilon(y)$$

is a *contraction* for small ϵ.

We propose the following scheme to achieve this goal.

Choose two intervals $I_x \supset (\lambda/2,\, b)$ and $I_y \supset (c,\, 1)$, and set $\Omega \equiv I_x \times I_y$. Let $\mathcal{O}(\Sigma)$, $\Sigma = I_y$ or $\Sigma = \Omega$, be the space of real analytic functions on the open set Σ. Let $\|\cdot\|_\Sigma$ signify the sup-norm in this space. The so-called *Epstein* class of functions ϕ factorizable as in part (3) of Claim 1 will be denoted by $Ep(\mathcal{D},\,\mathcal{E},\,\mathfrak{c})$.

(1) Let $\epsilon_0 \in \mathcal{O}(\Omega)$ be such that $\|\epsilon_0\| \le \nu_0 < (|\lambda|\delta)/2$. The last inequality implies that $\|\omega_{\epsilon_0,\varphi}\|_{I_y} \le \delta$ for all $\varphi \in Ep(\mathcal{D},\,\mathcal{E},\,\mathfrak{c})$. Since $Ep(\mathcal{D},\,\mathcal{E},\,\mathfrak{c})$ is compact, the operator

$$\varphi \mapsto \phi_{\omega_{\epsilon_0,\varphi}},$$

where, as before, ϕ_τ is the solution of (1.4), has a fixed point φ_{ϵ_0} which solves (3.4) for $\epsilon \equiv \epsilon_0$.

(2) Set

$$z_0(x,\,y) \equiv \psi_{\epsilon_0}\left(\frac{\lambda}{2}(x+y)\right),$$

where ψ_{ϵ_0} is the inverse of φ_{ϵ_0} on $I_y \cap \mathbb{R}_+$.

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(3) Notice that the mid-point equation (3.6) can be solved by iteration:

$$z_{k+1}(x, y) = \varphi_\epsilon^{-1}\left(\frac{\lambda}{2}(x+y) + \frac{1}{2}(\epsilon(\lambda x, z_k(x, y)) + \epsilon(\lambda y, z_k(x, y)))\right).$$

Therefore, set

$$z_1(x, y) = \psi_{\epsilon_0}\left(\frac{\lambda}{2}(x+y) + \frac{1}{2}(\epsilon_0(\lambda x, z_0(x, y)) + \epsilon_0(\lambda y, z_0(x, y)))\right).$$

(4) Set

$$\epsilon_1(x, y) \equiv s_{\varphi_{\epsilon_0}, \epsilon_0}(z_1(x, y), \lambda y) - x + \varphi_{\epsilon_0}(y).$$

(5) Generally, at the kth step, set

$$z_{k+1}(x, y) \equiv \psi_{\epsilon_k}\left(\frac{\lambda}{2}(x+y) + \frac{1}{2}(\epsilon_k(\lambda x, z_k(x, y)) + \epsilon_k(\lambda y, z_k(x, y)))\right),$$
$$\epsilon_{k+1}(x, y) \equiv s_{\varphi_{\epsilon_k}, \epsilon_k}(z_{k+1}(x, y), \lambda y) - x + \varphi_{\epsilon_k}(y).$$

(6) Use *a-priori* bounds on ψ_{ϵ_k} and its derivative to estimate

$$\|z_{k+1} - z_k\|_\Omega \le \eta_k, \quad \|\varphi_{\epsilon_{k+1}} - \varphi_{\epsilon_k}\|_{I_y} \le \upsilon_k, \quad \|\epsilon_{k+1} - \epsilon_k\|_\Omega \le \nu_k.$$

Show that $\sum_{k=0}^\infty \eta_k \le \infty$, $\sum_{k=0}^\infty \upsilon_{k+1} \le \infty$, and, most importantly,

$$\sum_{k=0}^\infty \nu_k \le \frac{|\lambda|\delta}{2},$$

which implies that the solution φ_{ϵ_k} exists and is in $Ep(\mathcal{D}, \mathcal{E}, \mathfrak{c})$ for all k.

We should mention that an essential ingredient for step (6) of the scheme is continuity of φ_ϵ in ϵ. As will be apparent from the discussion in the following sections, obtaining an estimate of the type

$$\|\varphi_\epsilon - \varphi_{\epsilon'}\|_{I_y} \le C\|\epsilon - \epsilon'\|_\Omega$$

is rather tricky, and amounts to computing a similar bound for the diffeomorphic part u of the factorized inverse, as well as λ.

We are currently working on the details of this scheme. It should be noted that the scheme is clearly convergent numerically for initial ϵ_0 with $\|\epsilon_0\|_\Omega$ much larger than $|\lambda|\delta/2$.

7. *Factorization of inverse branches*

We will now start preparing a demonstration of Claim 1.

We will look for the solution of (1.4) within a class of functions which are unimodal on some interval $I \equiv (a, d) \ni \{0, 1\}$, that is, they have a unique critical point on I, and that this critical point c is quadratic in the sense that

$$\phi_\tau(y) = O((y-c)^2).$$

We will now proceed to derive equations that the two inverse branches of such ϕ_τ would satisfy.

Write

$$\phi_\tau(y) = b - g(y-c), \quad b \equiv \phi_\tau(c);$$

then, (1.4) can be written as

$$g = F \circ g \circ \xi + \mathrm{id} - \tau \circ (\mathrm{id} + c), \tag{7.1}$$

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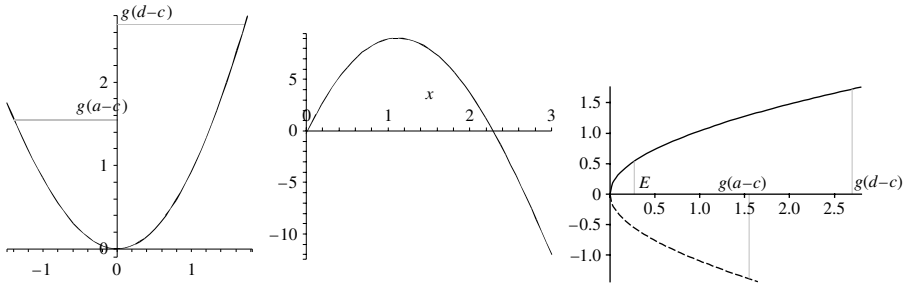


FIGURE 3. Function g (in (a)), function F (in (b)), and inverse branches h and f (in (c)) for the solution ϕ of $\phi(x) = 2\lambda^{-1}\phi(\phi(\lambda x)) - x$.

where

$$F(y) = b + c - \frac{2}{\lambda}(b - g(b + c - y)),$$
$$\xi(y) = \lambda y + c(\lambda - 1).$$

Denote by h and f the two inverse branches of g (Figure 3):

$$h : (0,\, g(d-c)) \mapsto (0,\, d-c), \quad f : (0,\, g(a-c)) \mapsto (a-c,\, 0).$$

The 'inverse' of (7.1) is the following set of equations for the inverse branches (Figure 4):

$$f \circ F^{-1} \circ (\mathrm{id} - h + \tau \circ (h + c)) = \xi \circ h \quad \text{on } (E,\, g(d-c)), \tag{7.2}$$
$$h \circ F^{-1} \circ (\mathrm{id} - h + \tau \circ (h + c)) = \xi \circ h \quad \text{on } (0,\, E), \tag{7.3}$$
$$h \circ F^{-1} \circ (\mathrm{id} - f + \tau \circ (f + c)) = \xi \circ f \quad \text{on } (0,\, g(a-c)), \tag{7.4}$$

where

$$E \equiv e^2, \quad e \equiv -\sqrt{g\left(\frac{c}{\lambda} - c\right)}.$$

We will look for solutions of (1.4) within the *Epstein class*:

$$\phi(y) = U(y)^2, \tag{7.5}$$

where U is a diffeomorphism of I onto its image, and we will write

$$h = v \circ - \circ s, \quad f = v \circ s, \tag{7.6}$$

where v is a diffeomorphism on $K \equiv (-\sqrt{g(d-c)}, \sqrt{g(a-c)})$, $s(x) \equiv \sqrt{x}$ (the principal square root) and $-(x) \equiv -x$. A similar factorization has been used in [**LY, Sul**]; however, unlike the authors of those works, we will not make any assumptions on the univalence of v in some neighborhood of K; below we will choose a space of v's as functions holomorphic in a neighborhood of K, and we will point out a specific obstruction to univalence.

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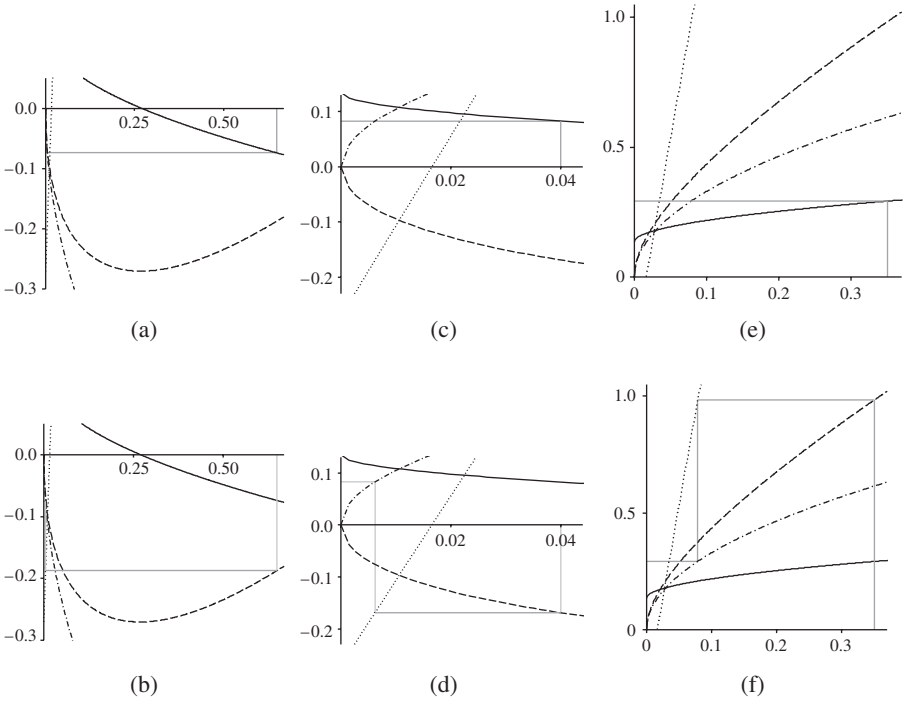


FIGURE 4. Combinatorics in the equalities (7.2)–(7.4). *Equality* (7.2)*, (a) and (b):* function $\xi \circ h$ is given in a solid line, id $- h$—in dash, F—in dot, f—in dash–dot; the image of the point under the right-hand side of the equality is shown in (a), under the left-hand side in (b). *Equality* (7.3)*, (c) and (d):* function $\xi \circ h$ is given in solid, id $- h$—in dash, F—in dot, h—in dash–dot; the image of the point under the right-hand side of the equality is shown in (c), under the left-hand side in (d). *Equality* (7.4)*, (e) and (f):* function $\xi \circ f$ is given in solid, id $- f$—in dash, F—in dot, h—in dash–dot; the image of the point under the right-hand side of the equality is shown in (e), under the left-hand side in (f).

With this factorization, (7.2)–(7.4) become

$$\xi \circ v = v \circ V, \tag{7.7}$$

$$V(x) = -[F^{-1}(x^2 - v(x) + \tau(v(x) + c))]^{\frac{1}{2}}, \quad x \in [e, \sqrt{g(a-c)}), \tag{7.8}$$

$$V(x) = [F^{-1}(x^2 - v(x) + \tau(v(x) + c))]^{\frac{1}{2}}, \quad x \in (-\sqrt{g(d-c)}, e). \tag{7.9}$$

Below, we will choose a compact functional space for the $v's$, and we will demonstrate that the operator $v \mapsto \xi^{-1} v \circ V$ is defined and is continuous on this functional space; existence of a fixed point v_* (and, hence, existence of ϕ_τ) will follow from the Schauder–Tikhonov theorem, which guarantees existence of a fixed point for a continuous operator on a compact set.

8. *An operator on a compact space*

We will now formally introduce an operator which will be later shown to be defined on $\mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$ for some choice of $\mathcal{D}$, $\mathcal{E}$, and

$$\mathfrak{c} = (-\tfrac{1}{2}, 0, 0, 1).$$

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The operator is defined through the following sequence of steps.

(i) Given $u \in \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$, and a function τ, holomorphic on $\mathcal{E} \ni 0$, real valued on $\mathbb{R}$, and satisfying $\tau(0) = 0$, find b, λ, and e from the following set of equations:

$$-2e = \alpha(b, \lambda)(1 - \tau'(u(T_{b,\lambda}(e))))u'(T_{b,\lambda}(e)), \tag{8.1}$$

$$\lambda = u\left(T_{b,\lambda}\left(\left[b - u\left(T_{b,\lambda}\left[-\sqrt{b - \frac{\lambda}{2} - \frac{\lambda^2}{4} - \tau(1)}\right]\right)\right]^{\frac{1}{2}}\right)\right), \tag{8.2}$$

$$b = u\left(T_{b,\lambda}\left[\sqrt{b - \frac{\lambda}{2}(b - e^2 + u(T_{b,\lambda}(e)) - \tau(u(T_{b,\lambda}(e)))}\right]\right), \tag{8.3}$$

where α, and additional functions β and γ, are given by

$$\alpha(b, \lambda) = \frac{1}{2\beta(b, \lambda) - 2\gamma(b)}, \quad \beta(b, \lambda) = \sqrt{b - \frac{\lambda}{2}}, \quad \gamma(b) = \sqrt{b - 1},$$

and

$$T_{b,\lambda}(x) = -\alpha(b, \lambda)(x + \beta(b, \lambda)).$$

(ii) Define for all $x \in T_{b,\lambda}^{-1}(\mathcal{D} \cap \mathbb{R})$

$$V_{u,\tau}(x) = \operatorname{sign}(e - x)[b - u(T_{b,\lambda}(-[w(T_{b,\lambda}(x))]^{\frac{1}{2}}))]^{\frac{1}{2}}, \tag{8.4}$$

where

$$w(z) = b - \frac{\lambda}{2}(b - T_{b,\lambda}^{-1}(x)^2 + u(x) + \tau(u(x))). \tag{8.5}$$

We will demonstrate that there is a choice of $\mathcal{D}$ and $\mathcal{E}$ such that $V_{u,\tau}$ extends to a holomorphic function on $T_{b,\lambda}^{-1}(\mathcal{D})$.

(iii) Set

$$\mathcal{T}_\tau[u](T_{b,\lambda}(z)) \equiv \lambda^{-1} u(T_{b,\lambda}(V_{u,\tau}(z))). \tag{8.6}$$

The operator $\mathcal{T}_0$ will also be denoted by $\mathcal{T}$.

Remark 4. Notice that $\gamma = -\sqrt{b - 1} \in (e, 0)$ is the fixed point of $V_{u,\tau}$.

Remark 5. The normalization conditions (8.1)–(8.3) ensure that $V_{u,\tau}$ is differentiable at e, and that

$$\mathcal{T}_\tau[u](-1/2) = 0, \quad \mathcal{T}_\tau[u](0) = 1.$$

Remark 6. The function u is related to functions v, ψ, h, and f appearing in §7 through the following equations:

$$v(x) = u(-\alpha(x + \beta)) - c,$$
$$h(x) \equiv = \psi(b - x) - c = u(\alpha(\sqrt{x} - \beta)) - c, \quad x \in (0, [T_{b,\lambda}^{-1}(r)]^2),$$
$$f(x) \equiv u(\alpha(-\sqrt{x} - \beta)) - c, \quad x \in (0, [T_{b,\lambda}^{-1}(l)]^2).$$

We will show that for small τ, there is a choice of $\mathcal{D}$ and $\mathcal{E}$ such that $\mathcal{T}_\tau[u] \in \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$ whenever $u \in \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$. By compactness of the set $\mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$, there is a function $u_* \in \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$ such that $\mathcal{T}_\tau[u_*] = u_*$, which is equivalent to the set of equations (7.2)–(7.4). In particular, u_* is the 'factorized inverse' (in the sense of Remark 6) of a solution of (1.4).

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Remark 7. Before we proceed with the proofs, we would like to emphasize two crucial difficulties that have forced us to modify the standard techniques that are commonly used to control inverse branches of unimodal maps (cf. [**Eps1, Eps2, LY, Sul**]).

(i) The terms y and $\tau(y)$ in (1.4) are responsible for the appearance of the terms $u(T_{b,\lambda}(x))$ and $\tau(u(T_{b,\lambda}(x)))$ in (8.4). The effect of these terms is that one loses the benefit of needing to estimate u, every time it enters the expression for $V_{u,\tau}$, only on a compact subset of its domain where one can use *a-priori* bounds. These terms do not appear in the Feigenbaum case, where this difficulty is absent. *In the current situation, one has to make assumptions on the range of u, and show that these assumptions are reproduced.*

(ii) Another effect of the terms $u(T_{b,\lambda}(x))$ and $\tau(u(T_{b,\lambda}(x)))$ in (8.4) is that the derivative

$$\mathcal{T}_\tau[u]'(z) = -\lambda^{-1}u'(T_{b,\lambda}(V_{u,\tau}(T_{b,\lambda}^{-1}(z))))\alpha V'_{u,\tau}(T_{b,\lambda}^{-1}(z))$$

can become zero since

$$V_{u,\tau}(T_{b,\lambda}(z))' = \cdots \times \frac{1}{V_{u,\tau}(T_{b,\lambda}^{-1}(z))}\left(\frac{2}{\alpha}T_{b,\lambda}^{-1}(z) + u'(z) - \tau'(u(z))u'(z)\right)$$

can be zero. Notice that $V_{u,\tau}(T_{b,\lambda}(z))'$ is not zero at e, where an application of L'Hopital's rule shows that the derivative is finite. However, it can be zero at other points on the real line where $2\alpha^{-1}T_{b,\lambda}^{-1}(z) + u'(z) - \tau'(u(z))u'(z)$ is zero. This would totally destroy the argument, since a function $\tilde{u} \equiv \mathcal{T}_\tau[u]$ whose derivative is zero somewhere in the real slice of its domain generally is not in $\mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$; in particular, $\tilde{u}(\mathcal{D} \cap \mathbb{C}_\pm) \not\subseteq \overline{\mathcal{E} \cap \mathbb{C}_\pm}$.

We will deal with this problem by assuming an upper bound on the derivative u' in the 'problematic' subinterval of the real slice of $\mathcal{D}$ so that

$$2\alpha^{-1}T_{b,\lambda}^{-1}(z) + u'(z) - \tau'(u(z))u'(z)$$

is guaranteed to be non-zero, and we will demonstrate that this bound is reproduced.

9. *Detailed statement of the main claim*

Let $S_{-1,\vartheta}$ denote the sector in $\mathbb{C}$ of angle 2ϑ with the vertex at point -1, containing $(-\infty, -1)$ and symmetric with respect to the real axis, and let $S_{1,\vartheta}$ be a similar sector with the vertex at point 1, containing $(1, \infty)$. We denote by

$$\Pi_\vartheta(z) \equiv -2^{\vartheta/(\pi-\vartheta)}(1-z)^{\pi/(\pi-\vartheta)} + 1,$$
$$\Omega_\vartheta(z) \equiv 2^{\vartheta/(\pi-\vartheta)}(1+z)^{\pi/(\pi-\vartheta)} - 1$$

the conformal isomorphisms of, respectively, $\mathbb{C}_1 \setminus S_{-1,\vartheta}$ and $\mathbb{C}_1 \setminus S_{1,\vartheta}$, with $\mathbb{C}_1$.

In what follows, we will make the following choices:

$$\mathcal{D} = D(I_1, \theta_1), \quad \tilde{\mathcal{E}} = q_2(\Pi_{\vartheta_2}^{-1}(\Omega_{\vartheta_1}^{-1}(q_2^{-1}(\tilde{\Theta}_2(\mathbb{C}_1))))) \subset D(I_2, \theta_2),$$

where $I_k = (l_k, r_k)$ and θ_k are as in (5.2)–(5.4), $\vartheta_1 = 0.03$, $\vartheta_2 = 0.04$, q_2 is the linear map

$$q_2(z) \equiv \frac{r_2 - l_2}{2}z + \frac{r_2 + l_2}{2},$$

and $\tilde{\Theta}_2$ is a normalized conformal isomorphism of $\mathbb{C}_1$ and $D(I_2, \theta_2)$ (Figure 5). We will consider the corresponding space $\mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$.

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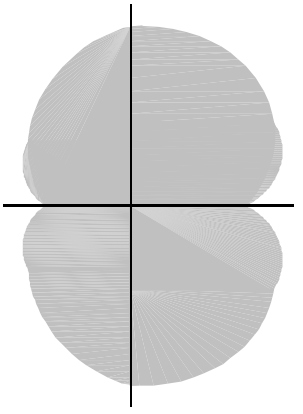


FIGURE 5. Set $\tilde{\mathcal{E}}$.

The point of considering such rather peculiar subset of a Poincaré neighborhood as the target set is that the factorized inverse u of the solution ϕ_τ of (1.4) generally maps a symmetric domain into a non-symmetric one. In our numerical experiments we have determined that the above choice $\mathcal{E}$ seems to be a relatively good approximation of $\mathcal{E} = u(\mathcal{D})$, being, at the same time, the set for which one can write the conformal isomorphism with $\mathbb{C}_1$ exactly.

Denote

$$\mathcal{D}_\pm \equiv \mathcal{D} \cap \mathbb{C}_\pm,$$

and let

$$\theta' \mapsto \partial\mathcal{D}_+(\theta'), \quad \theta' \in \left(\theta_1 - \frac{\pi}{2}, \frac{3}{2}\pi - \theta_1\right)$$

be the standard parametrization of the circle arc $\partial D_+(I_1, \theta_1)$. In this paper, we will use an affine rescaling of θ' to the interval $(0, \pi)$:

$$\theta \mapsto \partial\mathcal{D}_+(\theta), \quad \theta = \frac{1}{2\pi - 2\theta_1}\left(\theta' - \theta_1 + \frac{\pi}{2}\right)$$

as our parametrization of the boundary of $\mathcal{D}_+$. In a similar way, $\mathcal{D}_-$ is parametrized by $\theta \in (-\pi, 0)$.

The double slit plane $\mathbb{C}_1$ is isomorphic to Poincaré neighborhoods $D(I_k, \theta_k)$ via conformal isomorphisms

$$\tilde{\Theta}_k \equiv q_k \circ \sigma_k \circ m_k \circ \zeta, \tag{9.1}$$

where

$$\begin{aligned}
\zeta(z) &\equiv \frac{\sqrt{1+z} - \sqrt{1-z}}{\sqrt{1+z} + \sqrt{1-z}}, \\
m_k(z) &\equiv \frac{z + a_k}{1 + a_k z}, \\
\sigma_k(z) &\equiv \frac{(c_k+1)(1+z)^\kappa - (c_k-1)i^\kappa(1-z)^{\kappa_k}}{(c_k+1)(1+z)^{\kappa_k} + (c_k-1)i^{\kappa_k}(1-z)^{\kappa_k}}, \\
q_k(z) &\equiv \frac{r_k - l_k}{2} z + \frac{r_k + l_k}{2},
\end{aligned}$$

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l_k and r_k are the left and the right end points of intervals I_k, and

$$c_k \equiv \Im\left(\frac{e^{i\kappa_k\pi} + 1 + 2\,\mathrm{sign}(1-\kappa_k)\sqrt{e^{i\kappa_k\pi}}}{e^{i\kappa_k\pi} - 1}\right), \quad \kappa_k \equiv 2 - \frac{2\theta_k}{\pi}.$$

With a little bit of work, one can check that the transformation ζ maps $\mathbb{C}_1$ onto the unit disk, m_k is the normalizing Moebius transformation, σ_k maps the unit disk onto $D((-1, 1), \theta_k)$, and, finally, q_k maps $D((-1, 1), \theta_k)$ onto $D(I_k, \theta_k)$.

Finally, we set

$$\Theta_1 \equiv \tilde{\Theta}_1, \quad \Theta_2 \equiv q_2 \circ \Pi_{\vartheta_2}^{-1} \circ \Omega_{\vartheta_1}^{-1} \circ q_2^{-1} \circ \tilde{\Theta}_2, \quad \Phi_1 \equiv \Theta_1^{-1}, \quad \Phi_2 \equiv \Theta_2^{-1}.$$

The constants a_k in the normalizing Moebius transformations m_k are defined through the conditions

$$\Theta_1(0) = -1/2, \quad \Theta_2(0) = 0.$$

A function u in $\mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$ can now be factorized approximately as

$$u = \Theta_2 \circ f \circ \Phi_1,$$

where $f \in \mathcal{A}_1(\mathfrak{c})$. We emphasize that $\tilde{\mathcal{E}}$ is only an approximation of $\mathcal{E} = u(\mathcal{D})$. We will, however, ignore this fact in our notation from now on, and use $\mathcal{E}$ in place of $\tilde{\mathcal{E}}$ below.

According to the Schwarz Lemma 3, if $f \in \mathcal{A}_1(\mathfrak{c})$ and an interval J are such that $g(J) \subset J'$, then

$$u(\Theta_1(D(J, \theta))) \subset \Theta_2(D(J', \theta)).$$

Furthermore, one can use the fact that $\Theta_k|_{\mathbb{R}}$ are monotone functions to transfer the improved Herglotz bounds (A.6) from $\mathcal{A}_1(\mathfrak{c})$ to $\mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$:

$$\mathfrak{U}(x; t, s) \equiv \Theta_2(\mathfrak{F}(\Phi_1(x); t, s)), \tag{9.2}$$

$$\mathfrak{u}(x; t, s) \equiv \Theta_2(\mathfrak{f}(\Phi_1(x); t, s)). \tag{9.3}$$

We have implemented bounds (9.2)–(9.3) on the computer (see [**Prog**]), and used them for a numerical verification of various conditions in our proofs.

We shall now proceed to describe a set $\mathcal{S}$ of realizable derivatives $(u'(-\frac{1}{2}), u'(0))$.

LEMMA 8. *There is a convex open set $\mathcal{S} \subset \mathbb{R}^2$, such that*

$$(u'(-\tfrac{1}{2}), u'(0)) \in \mathcal{S},$$

whenever $u \in \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$.

Proof. See §A.2 of the appendix. □

The following result shows that a set of functions $u \in \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$ that satisfy a certain set of conditions on the distortion of $\partial\mathcal{D}$ is invariant under $\mathcal{T}_\tau$.

MAIN CLAIM 2.† *Suppose that $u \in \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$ satisfies the following two sets of conditions.*

† The proof of this claim relies on several verifications of inclusion of a domain in $\mathbb{C}$ that continuously depends on four real parameters in another domain in $\mathbb{C}$, and verification of several functional inequalities. These containments and inequalities have been verified numerically on a computer; however, presently, we have not made this computer verification rigorous (that is, the programs do not use interval arithmetics yet).

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(1) *The distortion of the boundary of the domain by u is bounded:*

$$u(\partial\mathcal{D}(\theta)) \in \mathcal{J}(\theta), \quad 0 \le \theta \le \theta_* = \frac{2.7}{2\pi - 2\theta_1}, \tag{9.4}$$

where θ_1 is the angle of the Poincaré neighborhood $D(I_1, \theta_1)$, and

$$\mathcal{J}(\theta) = \{z \in \mathcal{E} : \Im(z) < C_\Im |\Im(\partial\mathcal{D}(\theta))| \text{ and } r_2 - c_\Re |\Im(\partial\mathcal{D}(\theta))| < \Re(z)\}, \tag{9.5}$$

and $C_\Im = C_\Im(\theta)$ and $c_\Re = c_\Re(\theta)$ are piecewise linear functions of θ†.

(2) *The derivative of u on a subinterval of the real slice of the domain is bounded:*

$$u'(x) \le \omega + \sigma x \quad \text{where } \omega = 15, \quad \sigma = 30, \tag{9.6}$$

for all $x \in (0, r_1)$.

Then, there are $\delta > 0$ and $\kappa > 0$, and four linear functions ‡:

$$\mathcal{B}_-(t, s) \equiv b_1(t - t^*) + b_2(s - s^*) + b_3,$$
$$\mathcal{B}_+(t, s) \equiv B_1(t - t^*) + B_2(s - s^*) + B_3,$$

and

$$\mathcal{L}_-(t, s) \equiv a_1(t - t^*) + a_2(s - s^*) + a_3,$$
$$\mathcal{L}_+(t, s) \equiv A_1(t - t^*) + A_2(s - s^*) + A_3,$$

such that for any τ, holomorphic on $\mathcal{E}$, and satisfying

$$\sup_{z\in\mathcal{E}} |\tau(z)| \le \delta, \quad \sup_{z\in\mathcal{E}} |\tau'(z)| \le \kappa, \quad \tau(0) = 0, \tag{9.7}$$

the following holds:

(i) *there is a continuous branch of solutions of (8.1)–(8.2); that is, a triple $\mathfrak{S} \equiv (e, b, \lambda)$ that solves (8.1)–(8.2) and is such that the map $(u, \tau) \mapsto \mathfrak{S}$ is continuous for all $u \in \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$ and τ as in (9.7). Furthermore, it satisfies*

$$-\gamma(b) \ge e \ge -\beta(b, \lambda), \tag{9.8}$$
$$\mathcal{L}_+(t, s) \ge \lambda \ge \mathcal{L}_-(t, s), \tag{9.9}$$
$$\mathcal{B}_+(t, s) \ge b \ge \mathcal{B}_-(t, s); \tag{9.10}$$

(ii) *$\mathcal{T}_\tau[u]'$ also admits the bound (9.6);*

(iii) *the function $V_{u,\tau}$ extends to a conformal function on $T_{b,\lambda}^{-1}(\mathcal{D})$ that maps $T_{b,\lambda}^{-1}(\mathcal{D}) \cup \mathbb{C}_\pm$ compactly into $T_{b,\lambda}^{-1}(\mathcal{D}) \cup \mathbb{C}_\mp$;*

(iv) *$\mathcal{T}_\tau[u] \in \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$;*

(v) *$\mathcal{T}_\tau[u]$ satisfies condition (9.4).*

A rather technical demonstration of this result is given in the appendix.

† Values of $C_\Im(\theta)$ and $c_\Re(\theta)$ are linear interpolations of the following values.

$\theta(2\pi - 2\theta_1)$	0.0	0.5	0.8	1.1	1.4	1.7	2.2	2.7
$c_\Re(\theta)$	1.4	1.5	1.5	1.7	1.7	2.0	2.0	2.7
$C_\Im(\theta)$	2.2	4.5	4.2	3.8	2.5	2.2	1.2	1.0

‡ Values of b_i and B_i, a_i and A_i are as follows:

$b_1 = 0.008\,00$, $b_2 = 0.004\,50$, $b_3 = 1.010\,45$, $B_1 = 0.016\,00$, $B_2 = 0.000\,00$, $B_3 = 1.010\,83$,
$a_1 = -0.2900$, $a_2 = 0.0200$, $a_3 = -0.2560$, $A_1 = -0.2100$, $A_2 = -0.0800$, $A_3 = -0.2484$.

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Remark 9. To prove bounds (9.8)–(9.10) in the appendix, we construct a subset $\mathcal{A}$ in the (λ, b)-plane and an interval $\mathcal{I} \subset \mathbb{R}_+$, and demonstrate that the right-hand sides of (8.2)–(8.3) map $\mathcal{A}$ into itself for all values of $e \in \mathcal{I}$, while the right-hand side of (8.1) maps the interval $\mathcal{I}$ into itself for all values of $(\lambda, b) \in \mathcal{A}$. This guarantees existence of a solution (e, b, λ) in $\mathcal{I} \times \mathcal{A}$. To demonstrate uniqueness, one would have to control (rather cumbersome) derivatives of the right-hand sides of (8.1)–(8.3), and use a version of a contraction mapping principle. We have not yet been able to successfully control these derivatives with the help of the *a-priori* bounds and, therefore, we do not demonstrate uniqueness of the solution (e, b, λ) at this time.

Convexity of the set $\mathcal{J}$ (see (9.5)) and linearity if the condition (9.6) imply that the subset $\mathcal{A}' \subset \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$ of functions satisfying (9.4) and (9.6) is convex.

We fix a branch of solutions $\mathfrak{S}$ (there is at least one such branch by part (i) of the Main Claim 2). Together with the definition of the operator $\mathcal{T}_\tau[u]$, this implies that this operator is *continuous* on the subset $\mathcal{A}'$. By the Schauder–Tikhonov theorem, there exists a fixed point $u_\infty \in \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$ of $\mathcal{T}_\tau[u]$:

$$\mathcal{T}_\tau[u_*] = u_*.$$

Part (iv) together with the fact that Φ_k' and Θ_k' are positive on $\mathbb{R}$ implies that $u_*'|_{\mathbb{R}} > 0$, and therefore u_* is injective on some neighborhood of I_1. No conclusion about the injectivity of u_* can be made on all of $\mathcal{D}$, since the equality

$$u_* = \lambda_*^{-1} u_* \circ T_{b_*,\lambda_*} \circ V_{u_*,\tau_*} \circ T_{b_*,\lambda_*}^{-1}$$

implies that $u_*' = 0$ whenever $\partial V_{u_*,\tau} \circ T_{b_*,\lambda_*}^{-1} = 0$, which might happen at points where $-2\alpha^{-1} T_{b_*,\lambda_*}^{-1}(z) + u_*'(z) - \tau'(u_*(z)) u_*'(z) = 0$ outside of the real line.

We conclude that

$$\eta(z) = u_*(T_{b_*,\lambda_*}(-\sqrt{b_* - z})), \quad \zeta(z) = u_*(T_{b_*,\lambda_*}(\sqrt{b_* - z}))$$

are the factorized inverses of a solution ϕ_τ of (1.4) on some complex neighborhood of

$$u_*(I_1) \supset \left(\max_{(t,s)\in\mathcal{S}} \mathfrak{U}(l_1; t, s), \min_{(t,s)\in\mathcal{S}} \mathfrak{u}(r_1; t, s) \right) \supset (-1, 1).$$

10. *Properties of ϕ_τ*

In this section we will complete our 'proof' of the Main Claim.

Demonstration of the Main Claim 1. For a demonstration of parts (1), (2), (3), and (4), see §9 and the appendix.

Proof of parts (5) and (6). Clearly, since $\phi_\tau(0) = 1$ and $\phi_\tau(1) = \lambda/2 < 0$, there is $0 < x_+ < 1$ such that $\phi_\tau(x_+) = 0$.

Next, let x be in $[x_+/\lambda, 0]$; then, $\phi_0(\lambda x)$ assumes all values in $[0, 1]$ and

$$\phi_0(x) \equiv 2\phi_0(\phi_0(\lambda x))/\lambda - x$$

assumes all values in $[(2 - x_+)/\lambda, 1]$. Since $x_+ < 1$, we have $(2 - x_+)/\lambda < 0$ and the equation

$$\phi_0(x) = 0$$

has at least one solution $x_- \in [x_+/\lambda, 0]$.

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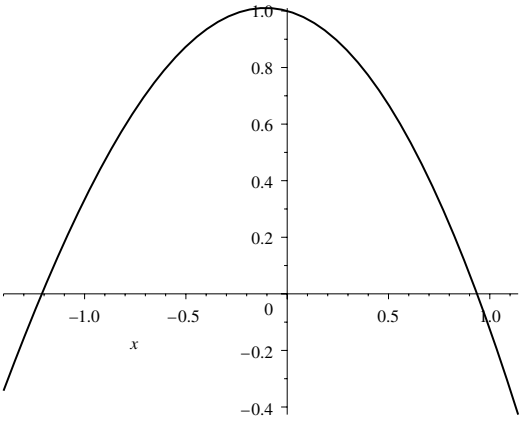


FIGURE 6. The graph of ϕ_0.

The existence of x_* is obvious.

We have

$$\phi_0(1) = \frac{2}{\lambda}\phi_0(\phi_0(\lambda)) - 1 \Rightarrow \frac{\lambda}{2} < \phi_0(\phi_0(\lambda)) \Rightarrow \phi_0(\lambda) < 1.$$

Furthermore,

$$\frac{2}{\lambda}\phi_0(\phi_0(\lambda)) = \frac{\lambda}{2} + 1 > 0 \Rightarrow \phi_0(\phi_0(\lambda)) < 0 \Rightarrow \phi_0(\lambda) > x_+.$$

Next, we demonstrate that $c < 0$.

Since ϕ_0 is smooth on $I = (-1, 1)$, and since $\phi_0(\lambda) < 1$, $\phi_0(0) = 1$, and $\phi_0(1) < 0$, we must have $c \in (\lambda, 1)$.

Suppose that $c \geq 0$. Then $\phi_0'(0) \geq 0$, while $\phi_0'(1) < 0$. On the other hand, we have from (3.3)

$$\phi_0'(0) = 2\phi_0'(1)\phi_0'(0) - 1,$$

or

$$\phi_0'(0) = \frac{-1}{1 - 2\phi_0'(1)} < 0,$$

a contradiction.

Finally, to show part (6), we first notice that

$$1 < \phi_0(c) = \frac{2}{\lambda}\phi_0(\phi_0(\lambda c)) - c \Rightarrow \phi_0(\phi_0(\lambda c)) < 0 \Rightarrow \phi_0(\lambda c) > x_+,$$

and, therefore, $\phi_0(\lambda x) \in (x_+, 1)$ and $\phi_0'(\phi_0(\lambda x)) < 0$ for all $x \in (c, 0)$. For $x \in (c, 0)$ we also have that $\phi_0'(\lambda x) < 0$, therefore, for all such x we get

$$\phi_0'(x) = 2\phi_0'(\phi_0(\lambda x))\phi_0'(\lambda x) - 1 > -1,$$

and

$$\phi_0(c) < 1 + c \min_{x \in (c,0)} \phi_0'(x) < 1 + |c| < 1 + |\lambda|. \qquad \square$$

The graphs of ϕ_0 on two different domains are shown in Figures 6 and 7.

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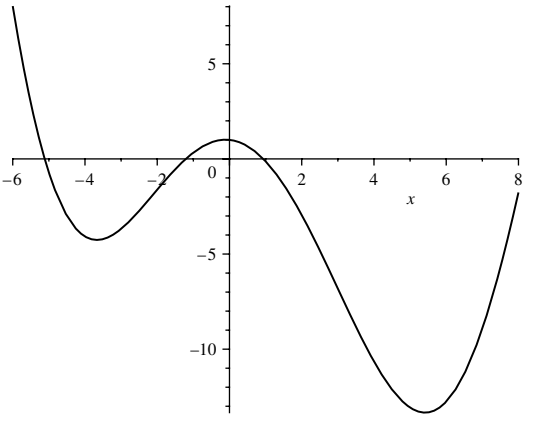


FIGURE 7. The graph of ϕ_0 on a larger domain.

A. *Appendix*

A.1. *New a-priori bounds on* $\mathbb{R}$. In this subsection we will use *a-priori* bounds on $\mathcal{A}_1(\mathbf{c})$ to produce quite better bounds on a subset of functions bounded on $(-1, 1)$ by a constant.

As before, we denote $(t, s) = (u'(-1/2), u'(0))$ for a function $u \in \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$, $\mathfrak{c} = (-1/2, 0, 0, 1)$. Recall that $u = \Theta_2 \circ f \circ \Phi_1$, where

$$f \in \mathcal{A}_1(\mathbf{c}), \quad \mathbf{c} = (\Phi_1(\mathfrak{c}_1), \Phi_1(\mathfrak{c}_2), \Phi_2(\mathfrak{c}_3), \Phi_2(\mathfrak{c}_4)).$$

Therefore, the following are the derivatives of f at points c_1 and c_2:

$$T(t) = \frac{t}{\Theta'(c_3)\Phi_1'(-1/2)}, \quad S(s) = \frac{s}{\Theta'(c_4)\Phi_1'(0)}.$$

Now, recall that $f'(x)$ is convex and, therefore, using (4.5),

$$\min_{x\in[c_1,c_2]} f''(x) \geq -2\frac{(c_2 - x)T(t) + (x - c_1)S(s)}{(c_2 - c_1)(1 + c_1)} \equiv m(x, t, s), \tag{A.1}$$

$$\max_{x\in[c_1,c_2]} f''(x) \leq 2\frac{(c_2 - x)T(t) + (x - c_1)S(s)}{(c_2 - c_1)(1 - c_2)} \equiv M(x, t, s). \tag{A.2}$$

Now, fix t and s, and consider the function $y(x) = T(t) + \int_{c_1}^{x} m(z, t, s)\, dz$. Suppose that the line $w(x) = S(s) + n(t, s)(x - c_2)$ intersects $(x, y(x))$ at the point $x(t, s)$, and $n(t, s)$ is such that the following holds:

$$c_4 - c_3 = \int_{c_1}^{c_2} \mathfrak{y}(z)\, dz,$$

$$\mathfrak{y}(x) = \begin{cases} y(x), & c_1 \leq x \leq x(t, s), \\ w(x), & x(t, s) \leq x \leq c_2. \end{cases}$$

First, notice that any curve $(x, f'(x))$ on (c_1, c_2) with end points (c_1, t) and (c_2, s) cannot intersect $(x, y(x))$, and has to intersect $(x, w(x))$ somewhere on $(x(t, s), c_2)$ once $(f'(x)$

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is convex), for, if it does not, then $\int_{c_1}^{c_2} f'(z)\,dz \neq c_4 - c_3$. It is also clear that

$$f(x) \geq c_3 + \int_{c_1}^{x} \mathfrak{y}(z)\,dz \equiv f_2(x; t, s), \quad x \in [c_1, c_2].$$

One can repeat a similar argument for $Y(x) = S(s) + \int_{c_2}^{x} M(z, t, s)\,dz$ and $W(x) = T(t) + N(t, s)(x - c_1)$ that intersect at $X(t, s)$ to get

$$f(x) \leq c_4 - \int_{x}^{c_2} \mathfrak{Y}(z)\,dz \equiv F_2(x; t, s), \quad x \in [c_1, c_2],$$

$$\mathfrak{Y}(x) = \begin{cases} Y(x), & X(t, s) \leq x \leq c_2, \\ W(x), & c_1 \leq x \leq X(t, s). \end{cases}$$

To obtain an upper bound on $(-1, c_2)$ and a lower bound on $(c_2, 1)$, we recall that the positivity of the Schwarzian derivative for functions in $\mathcal{A}_1(\mathbf{c})$ together with the positivity of all $f^{(n)}$ for odd n implies that for all $x \in (-1, 1)$,

$$f'''(x) \geq \frac{3f''(x)^2}{2f'(x)} \tag{A.3}$$

and, consequently,

$$f''(x) \leq f''(c_1) + \frac{3}{2}\int_{c_1}^{x} \frac{f''(y)^2}{f'(y)}\,dy,$$

for all $x \in (-1, c_1)$, the equality being realized by the solution

$$f'(x) = \frac{4f'(c_1)^3}{(-f''(c_1)(x - c_1) + 2f'(c_1))^2}$$

of (A.3). Therefore,

$$f(x) \leq c_3 + \int_{c_1}^{x} \frac{4T(t)^3}{(-f''(c_1)(x - c_1) + 2T(t))^2},$$

for all $x \in (-1, c_1)$, the maximum of the right-hand side being realized by the maximum admissible $f''(c_1)$ which can be obtained from the condition

$$\frac{4T(t)^3}{(-f''(c_1)(c_2 - c_1) + 2T(t))^2} = S(s). \tag{A.4}$$

We denote by $Z(t, s)$ the solution $f''(c_1)$ of this equation; then,

$$f(x) \leq c_3 + \frac{4T(t)^3}{Z(t, s)}\left(\frac{1}{2T(t)} - \frac{1}{2T(t) + Z(t, s)(c_1 - x)}\right) \equiv F_1(x; t, s),$$

for all $x \in (-1, c_1)$.

In a similar way,

$$f(x) \geq c_4 + \frac{4S(s)^3}{X(t, s)}\left(\frac{1}{2S(s) + X(t, s)(c_2 - x)} - \frac{1}{2S(s)}\right) = f_3(x; t, s), \quad x \in (c_2, 1);$$

here, $X(t, s)$ solves

$$\frac{4S(s)^3}{(-X(t, s)(c_1 - c_2) + 2S(s))^2} = T(t). \tag{A.5}$$

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Finally, suppose that $\mathfrak{m} \le f(x) \le \mathfrak{M}$ on the real slice of its domain (this is certainly true if $f \in \mathcal{A}_1(\mathbf{c})$). Consider the line $(x,\, S(s) + \mathfrak{K}(x - c_2))$, where $\mathfrak{K}$ is such that

$$\int_{c_2}^{1} S(s) + \mathfrak{K}(x - c_2)\, dx = \mathfrak{M} - c_4;$$

that is,

$$\mathfrak{K} = 2\frac{\mathfrak{M} - c_4}{(1 - c_2)^2} - \frac{S(s)}{1 - c_2}.$$

Since $f'(x)$ is convex, the curve $(x,\, f'(x))$ intersects the line $(x,\, S(s) + \mathfrak{K}(x - c_2))$ strictly once on $(c_2,\, 1)$. Convexity of $f'(x)$ implies that

$$\int_{c_2}^{x} f'(y)\, dy < \int_{c_2}^{x} S(s) + \mathfrak{K}(y - c_2)\, dy, \quad x \in (c_2,\, 1);$$

that is,

$$\begin{aligned} f(x) &\le c_4 + S(s)(x - c_2) + (\mathfrak{M} - c_4 - S(s)(1 - c_2))\frac{(x - c_2)^2}{(1 - c_2)^2} \\ &\equiv F_3(x;\, t,\, s), \quad x \in (c_2,\, 1). \end{aligned}$$

A similar argument on $(-1,\, c_1)$ demonstrates that

$$\begin{aligned} f(x) &\ge c_3 - T(t)(c_1 - x) + (T(t)(1 + c_1) + \mathfrak{m} - c_3)\frac{(x - c_1)^2}{(1 + c_1)^2} \\ &\equiv f_1(x;\, t,\, s), \quad x \in (-1,\, c_1). \end{aligned}$$

Finally, $\mathfrak{f}(x;\, t,\, s) \le f(x) \le \mathfrak{F}(x;\, t,\, s)$ on $(-1,\, 1)$, where

$$\mathfrak{f}(x;\, t,\, s) = \begin{cases} f_1(x;\, t,\, s), & x \in (-1,\, c_1), \\ f_2(x;\, t,\, s)), & x \in (c_1,\, c_2), \\ f_3(x;\, t,\, s)), & x \in (c_2,\, 1), \end{cases} \qquad \mathfrak{F}(x;\, t,\, s) = \begin{cases} F_1(x;\, t,\, s), & x \in (-1,\, c_1), \\ F_2(x;\, t,\, s)), & x \in (c_1,\, c_2), \\ F_3(x;\, t,\, s)), & x \in (c_2,\, 1). \end{cases} \tag{A.6}$$

Bounds (A.6) transferred to the space $\mathcal{A}(\mathcal{D},\, \mathcal{E};\, \mathfrak{c})$ will be denoted by $\mathfrak{u}$ and $\mathfrak{U}$:

$$\mathfrak{u}(x;\, t,\, s) \equiv \Theta_2(\mathfrak{f}_2(\Phi_1(x);\, t,\, s)), \tag{A.7}$$

$$\mathfrak{U}(x;\, t,\, s) \equiv \Theta_2(\mathfrak{F}_2(\Phi_1(x);\, t,\, s)). \tag{A.8}$$

A.2. *Set of realizable* $(u'(-1/2),\, u'(0))$. In this subsection we will describe the set $\mathcal{S}$ of realizable $t = u'(-1/2)$ and $s = u'(0)$ whenever $u \in \mathcal{A}(\mathcal{D},\, \mathcal{E},\, \mathfrak{c})$. We write $u = \Theta_2 \circ f \circ \Phi_1$, $f \in \mathcal{A}_1(\mathbf{c})$ as before.

Since $f(x) < F_1(x;\, t,\, s)$ on $(-1,\, c_1)$ (see §A.1), we have

$$-1 + \int_{-1}^{c_1} F_1(x;\, t,\, s)\, dx \le c_3. \tag{A.9}$$

The relevant (positive) solution $s = s(t)$ of (A.9) will be denoted by $\mathcal{Z}(t)$.

Similarly,

$$\int_{c_2}^{1} f_3(x;\, t,\, s)\, dx + c_4 \le 1.$$

The relevant solution $s = s(t)$ of this equation will be denoted by $\mathcal{C}(t)$.

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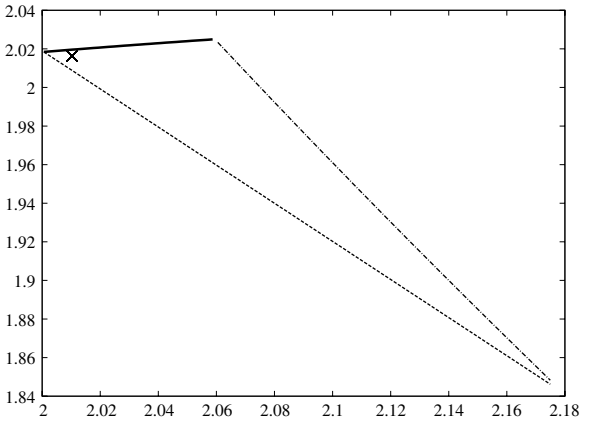


FIGURE A.1. Set $\mathcal{S}$ bounded by curves $\mathcal{Z}$ (solid lines), $\mathcal{C}$ (dash line), and $\mathcal{Q}$ (dash–dot line). The cross marks the pair (t^*, s^*) corresponding to the numerical approximation of the fixed point u.

Finally, the set of realizable (t, s) has to be such that $F_2(x; t, s) \geq f_2(x; t, s)$ at any point $x \in (c_1, c_2)$, for example at the mid point x_m of c_1 and c_2. We denote the solution $s = s(t)$ of the equation $F_2(x_m; t, s) = f_2(x_m; t, s)$ by $\mathcal{Q}(t)$.

The curves $\mathcal{Z}(t)$, $\mathcal{C}(t)$, and $\mathcal{Q}(t)$ bound a convex set $\mathcal{S}$, depicted in Figure A.1.

A.3. *Demonstration of part (i) of Main Claim 2.* We will demonstrate existence of a solution (e, b, λ) of (8.1)–(8.2) for the case $\tau \equiv 0$. Specifically, we will show that the set defined by conditions (9.8)–(9.10) is mapped compactly into itself. This, together with the continuity of $\mathcal{T}_\tau$ in τ at $\tau = 0$, implies that there exist $\delta > 0$ and $\kappa > 0$ such that the same is true for the τ's, holomorphic on $\mathcal{E}$, whose norm and the norm of whose derivative are bounded by δ and κ, respectively.

To demonstrate (9.8), we introduce a function

$$\mathfrak{E}(x; \lambda, b) \equiv -u'(T_{b,\lambda}(x))\frac{\alpha(b, \lambda)}{2}.$$

Notice that

$$\mathfrak{E}(-\gamma(b); \lambda, b) \equiv -u'(T_{b,\lambda}(-\gamma(b)))\frac{\alpha(b, \lambda)}{2} = -t\frac{\alpha(b, \lambda)}{2},$$
$$\mathfrak{E}(-\beta(b, \lambda); \lambda, b) \equiv -u'(T_{b,\lambda}(-\beta(b, \lambda)))\frac{\alpha(b, \lambda)}{2} = -s\frac{\alpha(b, \lambda)}{2}.$$

We verify numerically that $\mathfrak{E}$ maps the interval $(-\beta, -\gamma)$ into itself for all $\mathcal{L}_-(t, s) \leq \lambda \leq \mathcal{L}_+(t, s)$, $\mathcal{B}_-(\lambda, t, s) \leq b \leq \mathcal{B}_+(\lambda, t, s)$, and $(t, s) \in \mathcal{S}$.

To show (9.9)–(9.10), we consider two functions

$$\mathfrak{L}(\lambda, b) \equiv \mathcal{T}[u](0) = u\left(T_{b,\lambda}\left(\sqrt{b - u\left(T_{b,\lambda}\left(-\sqrt{b - \frac{\lambda}{2} - \frac{\lambda^2}{4}}\right)\right)}\right)\right),$$
$$\mathfrak{B}(\lambda, b; e) \equiv u\left(T_{b,\lambda}\left(-\sqrt{b - \frac{\lambda}{2}(b - e^2 + u(T_{b,\lambda}(e)))}\right)\right),$$

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and demonstrate that the map

$$(\lambda, b) \mapsto (\mathfrak{L}(\lambda, b), \mathfrak{B}(\lambda, b; e))$$

maps the parallelogram (9.9)–(9.10) in the (λ, b)-plane into itself for all $(t, s) \in \mathcal{S}$ and all e as in (9.8).

To this end, we first show that $\mathfrak{L}(\mathcal{L}_+(t, s), b) < \mathcal{L}_+(t, s)$ and $\mathfrak{L}(\mathcal{L}_-(t, s), b) > \mathcal{L}_-(t, s)$ for all $\mathcal{B}_-(t, s) \le b \le \mathcal{B}_+(t, s)$. For this, we define

$$\mathfrak{L}_+(\lambda, b; t, s) \equiv \mathfrak{U}\left(T_{b,\lambda}\left(\sqrt{b - \mathfrak{U}\left(T_{b,\lambda}\left(-\sqrt{b - \frac{\lambda}{2} - \frac{\lambda^2}{4}}\right); t, s\right)}; t, s\right)\right),$$

$$\mathfrak{L}_-(\lambda, b; t, s) \equiv \mathfrak{u}\left(T_{b,\lambda}\left(\sqrt{b - \mathfrak{u}\left(T_{b,\lambda}\left(-\sqrt{b - \frac{\lambda}{2} - \frac{\lambda^2}{4}}\right); t, s\right)}; t, s\right)\right),$$

and verify that, first,

$$\mathcal{L}_+(t, s) > \mathfrak{L}_+(\mathcal{L}_+(t, s), b; t, s) \tag{A.10}$$

for all $\mathcal{B}_-(t, s) \le b \le \mathcal{B}_+(t, s)$ and $(t, s) \in \mathcal{S}$, and, second,

$$\mathcal{L}_-(t, s) < \mathfrak{L}_-(\mathcal{L}_-(t, s), b; t, s) \tag{A.11}$$

for all $\mathcal{B}_-(t, s) \le b \le \mathcal{B}_+(t, s)$ and $(t, s) \in \mathcal{S}$.

Inequalities (A.10) and (A.11) have been verified on a computer.

Next, we check that

$$\mathcal{B}_+(t, s) > \mathfrak{B}(\lambda, \mathcal{B}_+(t, s); e)$$

and

$$\mathcal{B}_-(t, s) < \mathfrak{B}(\lambda, \mathcal{B}_-(t, s); e),$$

for all $\mathcal{L}_-(t, s) \le \lambda \le \mathcal{L}_+(t, s)$. We consider

$$\mathfrak{B}_+(\lambda, b; e) \equiv \mathfrak{U}\left(T_{b,\lambda}\left(-\sqrt{b - \frac{\lambda}{2}(T_{b,\lambda}^{-1}(e)^2 + \mathfrak{U}(e; t, s))}\right); t, s\right),$$

$$\mathfrak{B}_-(\lambda, b; e) \equiv \mathfrak{u}\left(T_{b,\lambda}\left(-\sqrt{b - \frac{\lambda}{2}(T_{b,\lambda}^{-1}(e)^2 + \mathfrak{u}(e; t, s))}\right); t, s\right),$$

and check the following inequalities numerically:

$$\mathcal{B}_+(t, s)) > \mathfrak{B}_+(\lambda, \mathcal{B}_+(t, s); e),$$
$$\mathcal{B}_-(t, s) < \mathfrak{B}_-(\lambda, \mathcal{B}_-(t, s); e),$$

for all $\mathcal{L}_-(t, s) \le \lambda \le \mathcal{L}_+(t, s)$, e as in (9.8) and $(t, s) \in \mathcal{S}$.

Since the maps

$$u \mapsto (\mathfrak{E}(x; \lambda, b), \mathfrak{L}(\lambda, b), \mathfrak{B}(\lambda, d; e)) \quad \text{and} \quad \tau \mapsto (\mathfrak{E}(x; \lambda, b), \mathfrak{L}(\lambda, b), \mathfrak{B}(\lambda, d; e))$$

are clearly continuous, we have that there exists at least one fixed point of the map $(\mathfrak{E}, \mathfrak{L}, \mathfrak{B})$ in the set (9.8)–(9.10) which moves continuously with u and τ. □

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A.4. *Demonstration of part (ii) of Main Claim 2.* Differentiate $\mathcal{T}[u]$ with respect to x:

$$\mathcal{T}[u]'(x) = \frac{\alpha^2}{2} u'(T_{b,\lambda}(V_{u,\tau}(T_{b,\lambda}^{-1}(x)))) \frac{u'(T_{b,\lambda}(-\sqrt{w(x)}))}{4V_{u,\tau}(T_{b,\lambda}^{-1}(x))} \frac{w'(x)}{\sqrt{w(x)}},$$

where w is the function defined in (8.5) with $\tau \equiv 0$.

On the real line, $\mathfrak{w} \le w \le \mathfrak{W}$ and $\mathfrak{v} \le V_{u,\tau} \circ T_{b,\lambda}^{-1} \le \mathfrak{V}$, where

$$\begin{aligned}
\mathfrak{W}(x; t, s) &= b - \frac{\lambda}{2}(b - T_{b,\lambda}^{-1}(x)^2 + \mathfrak{U}(x; t, s)),\\
\mathfrak{w}(x; t, s) &= b - \frac{\lambda}{2}(b - T_{b,\lambda}^{-1}(x)^2 + \mathfrak{u}(x; t, s)),\\
\mathfrak{V}(x; ts) &= [b - \mathfrak{u}(T_{b,\lambda}(-[\mathfrak{w}(x; t, s)]^{\frac{1}{2}}); t, s)]^{\frac{1}{2}},\\
\mathfrak{v}(x; ts) &= [b - \mathfrak{U}(T_{b,\lambda}(-[\mathfrak{W}(x; t, s)]^{\frac{1}{2}}); t, s)]^{\frac{1}{2}}
\end{aligned}$$

are upper and lower bounds on the corresponding functions.

Notice that

$$u'(x) \le \Theta_2'(\mathfrak{F}(\Phi_1(x); t, s))\mathfrak{Df}(\Phi_1(x); t, s)\Phi_1'(x) \equiv \mathfrak{Du}(x; t, s),$$

where $\mathfrak{F}$ and $\mathfrak{f}$ are as in (A.6) and

$$\mathfrak{Df}(x; t, s) \equiv \eta(x - c_1)\mathfrak{F}(x, t, s)\frac{(1 - c_1)}{(x - c_1)(1 - x)} + \eta(c_1 - x)\mathfrak{F}(x, t, s)\frac{(1 + c_1)}{(x - c_1)(1 + x)}$$

is an upper bound on derivatives on $\mathcal{A}_1(\mathbf{c})$ that follows from (4.4) (η is the Heaviside function).

Therefore,

$$\begin{aligned}
\mathcal{T}[u]'(x) \le{}& \frac{\alpha^2}{2}\mathfrak{Du}(T_{b,\lambda}(\mathfrak{v}(x; t, s)))\frac{\mathfrak{Du}(T_{b,\lambda}(-\sqrt{\mathfrak{W}(x; t, s)}))}{4\mathfrak{v}(x; t, s)}\\
&\times \frac{2\alpha^{-1}T_{b,\lambda}^{-1}(x) + (\omega + \sigma x)(1 + \kappa)}{\sqrt{\mathfrak{w}(x; ts)}}.
\end{aligned} \tag{A.12}$$

We finally verify on the computer (see [**Prog**]) that the right-hand side of (A.12) for $\kappa = 0$ is strictly less than $\omega + \sigma x$ for all $x \in (0, r_1)$. As in the previous part, this implies existence of $\delta > 0$ and $\kappa > 0$ such that

$$\mathcal{T}_\tau[u]'(x) < \omega + \sigma x, \quad x \in (0, r_1)$$

whenever $\sup_{z\in\mathcal{E}} |\tau(z)| < \delta$ and $\sup_{z\in\mathcal{E}} |\tau'(z)| < \kappa$. □

A.5. *Demonstration of parts (iii) and (iv) of Main Claim 2.* Let $\tau \equiv 0$. We consider a bound on

$$\mathcal{W}(\theta) \equiv w(\partial\mathcal{D}(\theta))$$

for $0 \le \theta \le \pi$. Suppose that the boundary of $\mathcal{E} \cap \mathbb{C}_+$ is parametrized by $p \in (0, \pi)$. Then, for every fixed $\theta \in [0, \pi]$, $\mathcal{W}(\theta)$ is bounded by the curves

$$\begin{aligned}
\mathcal{W}_J(\theta, p) &= b - \frac{\lambda}{2}(b - T_{b,\lambda}^{-1}(\partial\mathcal{D}(\theta))^2 + (p\mathcal{E}(0) + (1 - p)\mathcal{E}(\pi))), \quad p \in (0, 1),\\
\mathcal{W}_\mathcal{E}(\theta, p) &= b - \frac{\lambda}{2}(b - T_{b,\lambda}^{-1}(\partial\mathcal{D}(\theta))^2 + \partial\mathcal{E}(p)), \quad p \in (0, \pi).
\end{aligned}$$

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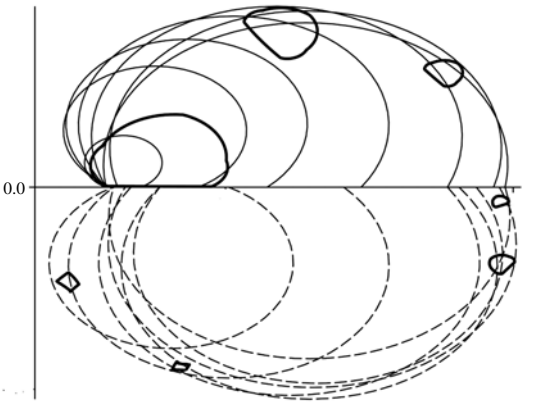


FIGURE A.2. Orbit of the set $\mathcal{F}(\theta)$ for several values of θ. The collection of Poincaré neighborhoods $\bigcup_k \mathcal{D}_+(I_k, \theta_k)$ is given in solid lines, the collection $\bigcup_j \mathcal{D}_-(I_j, \theta_j)$ in dashed lines.

Hypothesis (1) of the Main Claim 2 implies that $\mathcal{W}(\theta)$ is additionally bounded by the following curves (parametrized by p):

$$\mathcal{W}_{c_\Re}(\theta, p) = b - \frac{\lambda}{2}(b - T_{b,\lambda}^{-1}(\partial\mathcal{D}(\theta))^2 + r_2 - c_\Re(\theta)|\Im(\partial\mathcal{D}(\theta))| + ip),$$
$$\mathcal{W}_{C_\Im}(\theta, p) = b - \frac{\lambda}{2}(b - T_{b,\lambda}^{-1}(\partial\mathcal{D}(\theta))^2 + iC_\Im(\theta)|\Im(\partial\mathcal{D}(\theta))| + p),$$

for all $0 \le \theta \le \theta_*$ (see assumption (9.4) on bounded distortion).

Recall that $u = \Theta_2 \circ f \circ \Phi_1$. We first cover the set

$$\mathcal{F}(\theta) = \Phi_1(T_{b,\lambda}(-\sqrt{\mathcal{W}(\theta)})), \quad 0 \le \theta \le \pi, \tag{A.13}$$

by a collection of Poincaré half-neighborhoods

$$\mathcal{P} = \left(\bigcup_k \mathcal{D}_+(I_k, \theta_k)\right) \cup \left(\bigcup_j \mathcal{D}_-(I_j, \theta_j)\right),$$

for some appropriately chosen $I_k = (l_k, r_k)$, $I_j = (l_j, r_j)$ and θ_k, θ_j (see Figure A.2); then, according to Lemma 3, the set $f(\mathcal{F}(\theta))$, $0 \le \theta \le \pi$, is contained in

$$\tilde{\mathcal{P}}(t, s) = \left(\bigcup_k \mathcal{D}_+(\tilde{I}_k, \theta_k)\right) \cup \left(\bigcup_j \mathcal{D}_-(\tilde{I}_j, \theta_j)\right),$$

where

$$\tilde{I}_m = (\mathfrak{f}(l_m; t, s), \mathfrak{F}(r_m; t, s)).$$

Set $\mathcal{V}(t, s) \equiv \Theta_2(\tilde{\mathcal{P}}(t, s))$. We construct the set

$$\mathcal{M}(t, s) = -\text{sign}(\Im(b - \mathcal{V}(t, s)))\sqrt{b - \mathcal{V}(t, s)},$$

which is a bound on $V_{u,0}(T_{b,\lambda}^{-1}(\mathcal{D}_+))$, and verify that it is contained compactly in $T_{b,\lambda}^{-1}(\mathcal{D}_-)$.

Notice that

$$\bar{H} \equiv (b - \mathcal{V}(t, s)) \cup \mathbb{R}_-$$

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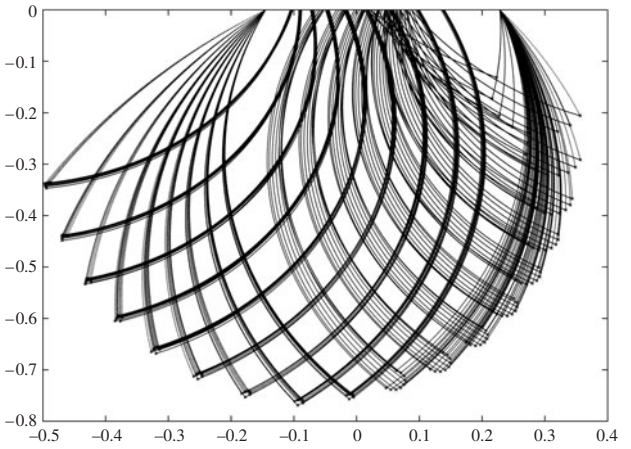


FIGURE A.3. Intersections of the Poincaré neighborhoods $\mathcal{D}_+(K_{\hat{t}}, \alpha_{\hat{t}})$ and $\mathcal{D}_+(J_{\hat{t}}, \phi_{\hat{t}})$ for some points $p \in \partial\mathcal{N}$.

is non-empty. Therefore, $\sqrt{\ }$ is not defined on $\bar{H}$, and neither is $V_{u,0}$ on

$$H \equiv q^{-1}(\bar{H})), \quad q \equiv b - u(T_{b,\lambda}(-\sqrt{w(T_{b,\lambda}(z))})).$$

It is easily checked however that it is continuous across H, and holomorphic in both components of $T_{b,\lambda}^{-1}(\mathcal{D}_+)\backslash H$. Therefore, by Morera's theorem, it is holomorphic in all of $T_{b,\lambda}^{-1}(\mathcal{D}_+)$. Analyticity on $T_{b,\lambda}^{-1}(\mathcal{D}_-)$ follows in a similar way.

We next construct the set

$$\mathcal{N}(t, s) = \Phi_1(T_{b,\lambda}(\mathcal{M}(t, s); t, s)).$$

Now, suppose that $\hat{t} \mapsto \partial\mathcal{N}(\hat{t})$ is some parametrization of the boundary of this set. For each point $p(\hat{t}) \in \partial\mathcal{N}$, we construct a pair of Poincaré half-neighborhoods (see Figure A.3):

$$\mathcal{D}_+(K_{\hat{t}}, \alpha_{\hat{t}}), \quad K_{\hat{t}} = (k_{\hat{t}}, j_{\hat{t}}) \quad \text{and} \quad \mathcal{D}_+(J_{\hat{t}}, \phi_{\hat{t}}), \quad J_{\hat{t}} = (m_{\hat{t}}, n_{\hat{t}}),$$

in such a way that

$$p(\hat{t}) = \partial\mathcal{D}_+(K_{\hat{t}}, \alpha_{\hat{t}}) \cap \partial\mathcal{D}_+(J_{\hat{t}}, \phi_{\hat{t}}).$$

Then,

$$\mathcal{H} = \bigcup_{\hat{t}} (\mathcal{D}_+(K_{\hat{t}}, \alpha_{\hat{t}}) \cap \mathcal{D}_+(J_{\hat{t}}, \phi_{\hat{t}}))$$

is a cover of $\mathcal{N}$.

Set

$$\tilde{\mathcal{H}}(t, s) = \bigcup_{\hat{t}} (\mathcal{D}_+(\tilde{K}_{\hat{t}}, \alpha_{\hat{t}}) \cap \mathcal{D}_+(\tilde{J}_{\hat{t}}, \phi_{\hat{t}})),$$
$$\tilde{K}_{\hat{t}} = (\mathfrak{f}(k_{\hat{t}}; t, s), \mathfrak{F}(j_{\hat{t}}; t, s)),$$
$$\tilde{J}_{\hat{t}} = (\mathfrak{f}(m_{\hat{t}}; t, s), \mathfrak{F}(n_{\hat{t}}; t, s)).$$

Finally, the set

$$\mathcal{X}(t, s) = \lambda^{-1}\Theta_2(\tilde{\mathcal{H}}(t, s))$$

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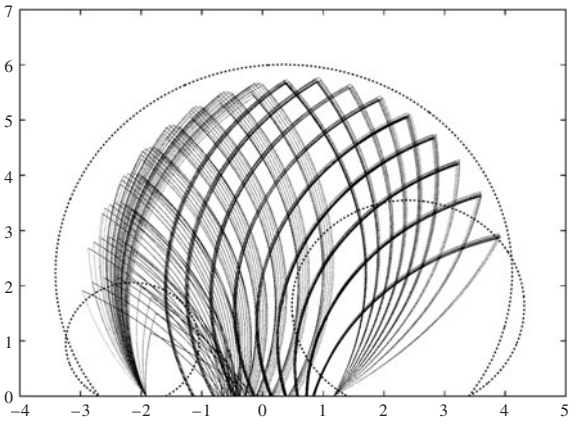


FIGURE A.4. Images under $\lambda^{-1}\Theta_2$ of intersections of some of the Poincaré neighborhoods $\mathcal{D}_+(\tilde{K}_{\hat{t}}, \alpha_{\hat{t}})$ and $\mathcal{D}_+(\tilde{J}_{\hat{t}}, \phi_{\hat{t}})$ (solid lines) are contained in $\mathcal{E}$ (dash lines).

is verified numerically to be contained compactly in $\mathcal{E}$ (see Figure A.4) for all $(t, s) \in \mathcal{S}$. This shows that $\mathcal{T}[u]$ is in $\mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$ whenever $u \in \mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$.

The claims of parts (iii) and (iv) in the case of non-zero τ follow from the fact that all set containments verified in the case of $\tau \equiv 0$ were compact. □

A.6. *Demonstration of part (v) of Main Claim 2: invariance of bounded boundary distortion.* As can be seen from the previous proofs, to demonstrate the invariance of the set $\mathcal{A}(\mathcal{D}, \mathcal{E}, \mathfrak{c})$, one needs to assume a bound on the distortion of $\partial\mathcal{D}$ by u. We, therefore, need to reproduce this bound for $\mathcal{T}[u]$.

Suppose that $\theta \mapsto \partial\mathcal{D}(\theta)$ is a parametrization of $\partial\mathcal{D}$ as described in §A.5. Recall that we have made the following assumptions (see (9.4)):

$$u(\partial\mathcal{D}(\theta)) \in \mathcal{J}(\theta) \equiv \{z \in \mathcal{E} : \Im(z) < C_{\Im}|\Im(\partial\mathcal{D}(\theta))| \text{ and } r_2 - c_{\Re}|\Im(\partial\mathcal{D}(\theta))| < \Re(z)\}$$

for $0 \le \theta \le \theta_*$.

To show that these assumptions are reproduced for $\mathcal{T}[u]$, we first construct four continuous families of Poincaré half-neighborhoods

$$D_-(L_i(\theta; b, \lambda), \vartheta_i(\theta; b, \lambda)),$$
$$L_i(\theta; b, \lambda) = (w_i(\theta; b, \lambda), v_i(\theta; b, \lambda)) \Subset (-1, 1) \quad (i = 1, \ldots, 4)$$

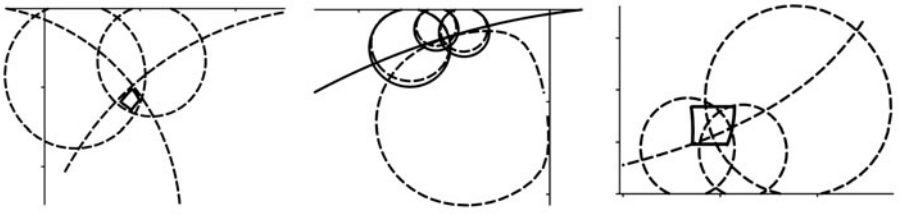
(below, we will suppress dependence on b and λ) that contain the set

$$\mathcal{F}(\theta) \equiv \Phi_1(T_{b,\lambda}(-\sqrt{w(\mathcal{J}(\theta))}))$$

(see (A.13)) in their intersection (Figure A.5(a)). Set

$$\mathcal{K}(\theta) = \Theta_2\Big(\bigcap_{i=1\ldots4} D_-(\tilde{L}_i(\theta), \vartheta_i(\theta))\Big), \quad \tilde{L}_i(\theta) = (\mathfrak{f}(w_i(\theta); t, s), \mathfrak{F}(v_i(\theta); t, s)).$$

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(a) (b) (c)

FIGURE A.5. (a). Set $\mathcal{F}(\theta)$ (solid lines), $\theta = 0.3$, is contained in the intersection of four Poincaré neighborhoods (dash lines); (b). set $\mathcal{Q}(\theta)$, the intersection of the sets given by dash lines, is contained in the intersection of four Poincaré neighborhoods (solid lines); (c) set $\mathcal{G}(\theta)$, the intersection of four sets given by the dash lines, is inside the set $\mathcal{J}(\theta)$ (solid lines).

By the Schwarz lemma,

$$u(T_{b,\lambda}(-\sqrt{\mathcal{W}(\theta)})) \subset \mathcal{K}(\theta).$$

Next, we set

$$\mathcal{Q}(\theta) = \Phi_1(T_{b,\lambda}(-\mathrm{sign}(\Im(b - \mathcal{K}(\theta)))\sqrt{b - \mathcal{K}(\theta)})),$$

and construct four more families of Poincaré neighborhoods

$$D_-(M_i(\theta), \varphi_i(\theta)), \quad i = 1, \ldots, 4, \quad M_i = (h_i(\theta), q_i(\theta)) \Subset (-1, 1)$$

that contain $\mathcal{Q}(\theta)$ in their intersection (see Figure A.5(b)). Again, by the Schwarz lemma,

$$\mathcal{T}[u](\partial\mathcal{D}(\theta)) \subset \lambda^{-1}\Theta_2\left(\bigcap_{i=1,\ldots,4} D_-((\mathfrak{f}(h_i(\theta); t, s), \mathfrak{F}(q_i(\theta); t, s)), \varphi_i(\theta))\right) \equiv \mathcal{G}(\theta)$$

(we suppress dependence on t and s in $\mathcal{G}$).

Finally, we verify numerically that

$$\mathcal{G}(\theta) \subset \mathcal{J}(\theta)$$

for all

$$0 \le \theta \le \theta_*, \quad \mathcal{L}_-(t, s) \le \lambda \le \mathcal{L}_+(t, s), \quad \mathcal{B}_-(t, s) \le b \le \mathcal{B}_+(t, s)$$

and $(t, s) \in \mathcal{S}$ (see Figure A.5(c)). □

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